A NEUMANN TYPE PROBLEM ON AN UNBOUNDED DOMAIN IN THE HEISENBERG GROUP

Shivani Dubey, Mukund Madhav Mishra, and Ashutosh Pandey

Abstract. We discuss the wellposedness of the Neumann problem on a half-space for the Kohn-Laplacian in the Heisenberg group. We then construct the Neumann function and explicitly represent the solution of the associated inhomogeneous problem.

1. Introduction

1.1. General background and motivation

The Heisenberg group $\mathbb{H}_n$ being one of the simplest example of a non-abelian and a non-compact Lie group finds a special place in the study of harmonic analysis and potential theory. The potential theoretic aspect demands a notion of harmonic functions which, on the Heisenberg group, are given by a Laplace like operator, known as the Kohn-Laplacian of $\mathbb{H}_n$. The Kohn-Laplacian can either be visualized as the generalization of the well-known Laplace-Beltrami operator on Riemannian spaces or it can be seen as the unique (up to a multiplicative constant) homogeneous differential operator of degree two that is left-invariant and rotation invariant [20]. A fundamental solution for this operator was obtained by Folland [4]. This existence of the fundamental solution for the Kohn-Laplacian ensures the hypoellipticity [8] of this operator. The Dirichlet problem and its wellposedness on $\mathbb{H}_n$ have been discussed in [7,9,10]. An integral kernel, called the Green's function, which is used to solve the associated inhomogeneous Dirichlet problem, was first obtained by Korányi [13] when the boundary data enjoys certain symmetry properties. For similar boundary data and for various domains in the generalised Heisenberg groups and in particular the Heisenberg group, the Green's functions have been discussed in [2,6,14,15].

An equally interesting problem *viz.* the Neumann problem on $\mathbb{H}_n$ is represented by

(1.1)
$$\begin{cases} \Delta_{\mathbb{H}_n} u = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial n} u = g & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open domain in $\mathbb{H}_n$ and n denotes the outward unit normal at the boundary $\partial\Omega$. The above problem for a bounded domain in $\mathbb{H}_n$ is discussed in [3] and in a more generalised set up in [16].

However the study of the Neumann problem on $\mathbb{H}_n$ so far has been very much confined to bounded domains only. The analysis of a similar problem on an unbounded set up involves many improper integrals and the corresponding approximations, tackling which poses a different challenge. Hence, in order to establish the wellposedness of the problem, the situation demands certain decay conditions on the boundary data. For example, a similar problem in the classical case requires the boundary data to have a compact support [21].

In this article, we start with a model unbounded domain, namely the upper half-space $\Omega = \{(\zeta, t) \in \mathbb{H}_n : t > 0\}$ and analyse the Neumann problem. We then propose an additional condition on the boundary data so that the integrals converge near infinity and hence obtain the necessary and sufficient conditions for the solvability of the Neumann problem. Later we construct a Green's type function that solves the associated inhomogeneous problem for a circular data.

1.2. Definitions and preliminary results

Consider the set $\mathbb{C}^n \times \mathbb{R} = \{(\zeta, t) : \zeta \in \mathbb{C}^n, \ t \in \mathbb{R}\}$ and the following composition law

(1.2)
$$(\zeta, t)(\eta, s) = (\zeta + \eta, \ t + s + 2\Im(\zeta \cdot \bar{\eta})),$$

where $\zeta \cdot \bar{\eta}$ is the usual Hermitian inner product in $\mathbb{C}^n$. It can be easily checked that Equation (1.2) turns $\mathbb{C}^n \times \mathbb{R}$ into a Lie group, known as the *Heisenberg group* and denoted by $\mathbb{H}_n$. Let $\mathfrak{h}$ denote the vector space of left-invariant vector fields on $\mathbb{H}_n$. The space $\mathfrak{h}$ is closed with respect to the bracket operation $[\alpha, \beta] = \alpha\beta - \beta\alpha$. With this bracket, $\mathfrak{h}$ is referred to as the Lie algebra of $\mathbb{H}_n$ which is well discussed in [19]. If $\mathfrak{z}$ denotes the center in $\mathfrak{h}$, then we have the following stratification

$$\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{z},$$

where $\mathfrak{v} = \mathfrak{z}^\perp$, and called the horizontal layer in $\mathfrak{h}$. We choose the spanning set for $\mathfrak{v}$ and $\mathfrak{z}$, respectively, denoted by $\{X_j, Y_j; 1 \leq j \leq n\}$ and $\{T\}$, where X_j, Y_j and T are defined as

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \ \ Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \ T = \frac{\partial}{\partial t},$$

and $\zeta_j = x_j + \iota y_j$. We define the complex vector fields

$$Z_j = \frac{1}{2}(X_j - \iota Y_j) = \frac{\partial}{\partial \zeta_j} + \iota \bar{\zeta}_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n,$$

$$\bar{Z}_j = \frac{1}{2}(X_j + \iota Y_j) = \frac{\partial}{\partial \bar{\zeta}_j} - \iota \zeta_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n.$$

Explicitly, the Kohn-Laplacian is given by

$$\Delta_{\mathbb{H}_n} = -\sum_{j=1}^n (X_j^2 + Y_j^2).$$

An infinitesimal metric that is consistent with the group structure of $\mathbb{H}_n$, qualifies to be a sub-Riemannian metric which is thoroughly discussed in [11]. As in [13], this metric is given by an inner product $(\cdot, \cdot)_0$ on $W = \text{span}\{X_j, Y_j; 1 \le j \le n\}$ and turns W into an orthonormal system. All vectors in W are called *horizontal* and any vector that is not in W is said to have infinite length.

The *horizontal gradient* of a smooth function F on $\mathbb{H}_n$ is defined as the unique horizontal vector $\nabla_0 F$ such that

$$(\nabla_0 F, v)_0 = v \cdot F$$

for all horizontal vectors v. Equivalently, we have

$$\nabla_0 F = \sum_{j=1}^n \{(X_j F) X_j + (Y_j F) Y_j\} = 2 \sum_{j=1}^n \{(\bar{Z}_j F) Z_j + (Z_j F) \bar{Z}_j\}.$$

A *horizontal normal* unit vector pointing outwards for a domain $\{F < 0\}$, where $\{F = 0\}$ is a hypersurface in $\mathbb{H}_n$, is defined as

$$(1.3) \qquad \frac{\partial}{\partial n_0} = \frac{1}{||\nabla_0 F||_0} \nabla_0 F.$$

For the half-space Ω in particular, $F(\zeta, t) = t$.

From [13], we have

$$(1.4) \qquad \frac{\partial}{\partial n_0} = \iota \frac{E - \bar{E}}{|\zeta|}$$

for $(\zeta, t) \in \partial \Omega$ such that $|\zeta| \neq 0$ and $E = \sum_{j=1}^p \zeta_j Z_j$.

In rest of the paper, the points α and β in $\mathbb{H}_n$ will denote (ζ, t) and (ζ', t'), respectively.

Theorem 1.1 (Folland [4]). *There exists a positive constant c such that*

$$g(\alpha) := c \, \mathfrak{p}(\alpha)^{-2n},$$

where $\mathfrak{p}$ is the homogeneous norm on $\mathbb{H}_n$ and is given by $\mathfrak{p}(\alpha) = (|\zeta|^4 + t^2)^{\frac{1}{4}}$. This g is the fundamental solution for the operator $\Delta_{\mathbb{H}_n}$, that is,

$$\Delta_{\mathbb{H}_n} g = -\delta.$$

From [12], the fundamental solution with pole at β can be expressed as

$$(1.5) \qquad g_\beta(\alpha) = a_0 \big| C(\beta, \alpha) - Q(\beta, \alpha) \big|^{-n},$$

where $C(\beta, \alpha) = |\zeta|^2 + |\zeta'|^2 + \iota(t' - t)$ and $Q(\beta, \alpha) = 2\zeta \cdot \bar{\zeta}'$.

The average of an integrable function f on $\mathbb{H}_n$ is defined as

$$\bar{f}([\zeta, t]) = \frac{1}{2\pi} \int_0^{2\pi} f([e^{\iota\theta}\zeta, t]) \, d\theta.$$

When $f([\zeta, t]) = \bar{f}([\zeta, t])$ for $[\zeta, t] \in \mathbb{H}_n$, we say that f is circular. Again from [12], we have

$$\bar{g}_\beta(\alpha) = a_0 \big|C(\beta, \alpha)\big|^{-n} F\left(\frac{n}{2}; \frac{n}{2}; n; \frac{|Q(\beta, \alpha)|^2}{|C(\beta, \alpha)|^2}\right),$$

where F denotes the Gaussian hypergeometric function [17].

1.3. Main results

The interior homogeneous Neumann problem on Ω is about looking for a function u in a suitable class $\mathfrak{C}_\Omega$ (to be defined later), that satisfies

(1.6)
$$\begin{cases} \Delta_{\mathbb{H}_n} u = 0, & \text{in } \Omega, \\ \partial^\perp u = g, & \text{on } \partial\Omega, \end{cases}$$

where $g \in C(\partial\Omega)$. The operator $\partial^\perp$ is similar to the normal operator $\frac{\partial}{\partial n_0}$ and is defined later as a remedy to deal with the characteristic points which we encounter on our way in this article. Our first task is to prove the following theorem.

Theorem 1.2. *For $k \geq 1$, if $g \in C(\partial\Omega)$ such that $g(\alpha) = O\left(\frac{1}{\zeta^k}\right)$ as α nears infinity, then the interior Neumann problem* (1.6) *is solvable if and only if*

$$\int_{\partial\Omega} g \, d\sigma = 0.$$

In Section 4, we consider an inhomogeneous Neumann problem for Ω and obtain a Green's type function (or a Neumann function) G, by means of the fundamental solution for the Kohn-Laplacian. Finally we look to establish the necessary and sufficient conditions for the solvability of the following problem

(1.7)
$$\begin{cases} \Delta_{\mathbb{H}_n} u = f, & \text{in } \Omega, \\ \frac{\partial}{\partial n_0} u = g, & \text{on } \partial\Omega, \end{cases}$$

where f and g are circular functions.

2. Formulation of the problem and the uniqueness of solution

From here onwards, for the convenience of calculations, we use a slightly modified operator *viz.* $\Delta_0 = -\frac{1}{4}\Delta_{\mathbb{H}_n}$ and a slightly modified kernel $\Psi(\beta, \alpha) = 2g_\beta(\alpha)$. Unless otherwise specified, for functions involving more than one variable, the differentiation and integration will be with respect to α. Before we move on to formulate the main problem, it is important to look at the points where the horizontal normal vector is not defined, *i.e.*, the points where $\nabla_0 F$

vanishes. These are called the *characteristic points.* For smooth F, the set of characteristic points forms a submanifold of dimension at most n.

Let $\partial\Omega$ be given as the level set of a smooth function ρ, that is, $\partial\Omega = \{\alpha \in \mathbb{H}_n : \rho(\alpha) = 0\}$. Define

$$\mathfrak{C}_\Omega := \{f \in C^2(\Omega) \cap C(\bar{\Omega}) : \lim_{\alpha\to\alpha_0} \frac{\partial}{\partial n_0} f(\alpha) \text{ exists for all characteristic points}$$
$$\alpha_0 \in \partial\Omega\},$$

where the limit taken is consistent with the relative topology in $\bar{\Omega}$.

Define the operator $\partial^\perp : \mathfrak{C}_\Omega \to C(\Omega)$ as

$$\partial^\perp f(\alpha_0) = \begin{cases} \lim_{\alpha\to\alpha_0} \frac{\partial f}{\partial n_0}(\alpha), & \text{if } \alpha_0 \text{ is a characteristic point on } \partial\Omega, \\ \frac{\partial f}{\partial n_0}(\alpha_0), & \text{if } \alpha_0 \text{ is a non-characteristic point on } \partial\Omega. \end{cases}$$

The following version of Gaveau's Green's formula [7] will be useful for our analysis. Using classical arguments, it can be verified that this formula holds good for the fundamental solution and the Green's function. For further details, one can refer to [1, 18].

Proposition 2.1. *Let* $f_1, f_2 \in \mathfrak{C}_\Omega$. *Then*

$$\int_\Omega (f_1\Delta_0 f_2 - f_2\Delta_0 f_1) \ d\nu = \int_{\partial\Omega} (f_1\partial^\perp f_2 - f_2\partial^\perp f_1) \ d\sigma,$$

where

$$(2.1) \qquad d\sigma = \frac{||\nabla_0\rho||_0}{||\nabla\rho||} ds,$$

and ds is the surface element on $\partial\Omega$, *determined by the Euclidean measure.*

Theorem 2.2. *A solution of the problem* (1.6), *if exists, is unique up to additive constants.*

Proof. As Ω is a $\mathbb{H}$-Caccioppoli set [5], hence using the substitution $v\nabla_0 u$ in the divergence theorem [5, Corollary 7.7], we get the following Green's first identity

$$(2.2) \qquad \int_{\partial\Omega} v\partial^\perp u \ d\sigma = \int_\Omega (v\Delta_0 u - \nabla_0 v \cdot \nabla_0 u) \ d\nu,$$

where $u, v \in C^1(\bar{\Omega})$. Now for any two solutions u_1, u_2 of Equation (1.6), the difference $u = u_1 - u_2$ is harmonic in Ω and continuous up to boundary. Also $\partial^\perp u = 0$. Using Equation (2.2),

$$\int_\Omega |\nabla_0 u|^2 \ d\nu = \int_{\partial\Omega} u\partial^\perp u \ d\sigma - \int_\Omega u(\Delta_0 u) \ d\nu = 0,$$

which means $\nabla_0 u = 0$. Using [16, Lemma 4.3], it can be easily proved that u is a constant. $\qquad\square$

3. The surface potentials and the existence of solution

For $k \geq 1$, define $C_*(\partial\Omega) = \{\psi \in C(\partial\Omega) : \psi(\alpha) = O\left(\frac{1}{\zeta^k}\right)$ as $\zeta \to \infty\}$.

Definition 3.1. For $\psi \in C_*(\partial\Omega)$ and $\beta \in \mathbb{H}_n \setminus \partial\Omega$, define

$$V(\beta) := \int_{\partial\Omega} \psi(\alpha)\Psi(\beta, \alpha) \, d\sigma(\alpha) \text{ and } \tilde{V}(\beta) := \int_{\partial\Omega} \psi(\alpha)\partial^\perp \Psi(\beta, \alpha) \, d\sigma(\alpha).$$

Both V and $\tilde{V}$ are Δ_0-harmonic and respectively called the single- and double-layer potentials with density ψ.

Lemma 3.2. *For $\beta \in \partial\Omega$ and $\psi \in C_*(\partial\Omega)$, the integral*

$$V(\beta) = \int_{\partial\Omega} \psi(\alpha)\Psi(\beta, \alpha) \, d\sigma(\alpha)$$

exists and V is continuous throughout $\mathbb{H}_n$.

Proof. We have $\Psi(\beta, \alpha) = 2c \, \mathfrak{p}(\beta^{-1}\alpha)^{-2n}$. For each $\beta \in \partial\Omega$ and some $\epsilon > 0$, let $\Omega_\beta(\epsilon) = \{\alpha \in \partial\Omega : \mathfrak{p}(\beta^{-1}\alpha) \leq \epsilon\}$. As $\Omega_\beta(\epsilon)$ is bounded and $\psi \in L^\infty(\partial\Omega)$, we have

$$\left| \int_{\Omega_\beta(\epsilon)} \psi(\alpha)\Psi(\beta, \alpha) \, d\sigma \right| \leq 2c \sup_{\alpha \in \Omega_\beta(\epsilon)} \left| \psi(\alpha) \right| \int_{\Omega_\beta(\epsilon)} \mathfrak{p}(\beta^{-1}\alpha)^{-2n} \, d\sigma.$$

As $\Psi(\beta, \alpha)$ admits a pole at $\alpha = \beta$, hence by taking a sufficiently small Korányi-like ball around β and using the polar coordinates for $\mathbb{H}_n$ [13], it can be easily verified that the integral exists on $\Omega_\beta(\epsilon)$.

Using Equation (2.1) and [12, Eq. (3.7)], we obtain

$$d\sigma = \frac{|\zeta|}{2} ds.$$

The following expression for the gauge norm follows from Equation (1.5).

(3.1) $$\mathfrak{p}(\beta^{-1}\alpha) = \left| C(\beta, \alpha) - Q(\beta, \alpha) \right|^{\frac{1}{2}}.$$

Set $\Omega' = \partial\Omega \setminus \Omega_\beta(\epsilon)$ and consider

$$\int_{\Omega'} \psi(\alpha)\Psi(\beta, \alpha) \, d\sigma = c \int_{\Omega'} \psi(\alpha) \left| |\zeta|^2 + |\zeta'|^2 + \iota(t' - t) - 2\zeta \cdot \bar{\zeta}' \right|^{-n} |\zeta| \, ds.$$

On the boundary, $t = 0$ and therefore,

$$\int_{\Omega'} \psi(\alpha)\Psi(\beta, \alpha) \, d\sigma = c \int_{\Omega'} \frac{\psi(\alpha) \, |\zeta|}{\left| |\zeta|^2 + |\zeta'|^2 - 2\zeta \cdot \bar{\zeta}' + \iota t' \right|^n} \, ds.$$

As $\psi \in C_*(\partial\Omega)$, the integral exists. It is to note that ds is a Radon measure and hence the uniform continuity of convolutions of two integrable functions can be established through a routine proof. As a particular case, V is continuous. □

Lemma 3.3. *The kernel Ψ satisfies the following.*

$$\int_{\partial\Omega} \partial^\perp \Psi(\beta,\alpha) \, d\sigma(\alpha) = \begin{cases} -2, & \beta \in \Omega, \\ -1, & \beta \in \partial\Omega, \\ 0, & \beta \in \mathbb{H}_n \setminus \bar{\Omega}. \end{cases}$$

Proof. For $\beta \in \partial\Omega$ and $\Omega_\beta(\epsilon)$ as defined in Lemma 3.2, using Proposition 2.1 on $\Omega \setminus \Omega_\beta(\epsilon)$ and substituting $f_1 = \Psi(\cdot,\beta)$, $f_2 = 1$, we get

$$\int_{\partial(\Omega\setminus\Omega_\beta(\epsilon))} \partial^\perp \Psi(\beta,\alpha) \, d\sigma(\alpha) = 0,$$

i.e.,

$$\int_{\partial\Omega\setminus\Omega_\beta(\epsilon)} \partial^\perp \Psi(\beta,\alpha) \, d\sigma(\alpha) = -\lim_{\epsilon\to 0} \int_{\Omega\cap\partial\Omega_\beta(\epsilon)} \partial^\perp \Psi(\beta,\alpha) \, d\sigma(\alpha)$$

$$= -\frac{1}{2}\lim_{\epsilon\to 0} \int_{\partial\Omega_\beta(\epsilon)} \partial^\perp \Psi(\beta,\alpha) d\sigma(\alpha).$$

From [12, Eq. (1.15)], we get $\int_{\partial\Omega} \partial^\perp \Psi(\beta,\alpha) \, d\sigma(\alpha) = -1$. Further using appropriate substitutions in Proposition 2.1, the results can be proved for $\beta \in \Omega$ and $\mathbb{H}_n \setminus \Omega$. $\qquad\square$

Corollary 3.4. *For $\psi \in C_*(\partial\Omega)$ and $\beta \in \partial\Omega$,*

$$\int_{\partial\Omega} \psi(\alpha)\partial^\perp \Psi(\beta,\alpha) \, d\sigma(\alpha) < \infty.$$

We now proceed to probe the double-layer potential $\tilde{V}$ for its continuity around the boundary $\partial\Omega$. For that, we consider a neighbourhood $N_{h_0}(\partial\Omega)$ of $\partial\Omega$ for a sufficiently small $h_0 > 0$ such that

$$N_{h_0}(\partial\Omega) = \{\gamma + h\hat\gamma : \gamma \in \partial\Omega \text{ and } h \in [-h_0, h_0]\}.$$

Lemma 3.5. *Define*

$$u(\beta) = \int_{\partial\Omega} \{\psi(\alpha) - \psi(\gamma)\}\partial^\perp \Psi(\beta,\alpha) \, d\sigma(\alpha)$$

for $\beta \in N_{h_0}(\partial\Omega) \setminus \partial\Omega$. For $\gamma \in \partial\Omega$, as $h \to 0^+$, we have $u(\gamma + h\hat\gamma) \to u(\gamma)$ uniformly over compact neighbourhoods of γ in $N_{h_0}(\partial\Omega)$.

Proof. For $\alpha = (\zeta, t)$ such that $|\zeta| \neq 0$, let

$$K(\beta,\alpha) = 2\zeta^2\zeta'^2 - 3\zeta\bar\zeta'|\zeta|^2 - \zeta\bar\zeta'\left|\zeta'\right|^2 + \iota|\zeta|^2(t - t').$$

With a certain amount of work using Equation (1.4), we obtain

$$(3.2) \qquad \frac{\partial}{\partial n_0}\Psi(\beta,\alpha) = -\iota\frac{4nc\,\mathfrak{p}(\beta^{-1}\alpha)^{-2(n+2)}}{|\zeta|}K(\beta,\alpha).$$

Let $B_r(\gamma)$ denote the Korányi-like ball in $\mathbb{H}_n$, centered at γ and having radius r. Set

$$\Omega_1 = \partial\Omega \cap B_r(\gamma), \ \Omega_2 = \partial\Omega \setminus \Omega_1,$$

and let $r < \mathfrak{p}(\beta^{-1}\gamma) = \lambda$ (say). For $\beta \neq \alpha$, using Equation (3.2)

$$\left| \int_{\Omega_1} \partial^\perp \Psi(\beta, \alpha) \, d\sigma(\alpha) \right| \leq 4nc \int_{\Omega_1} \frac{K(\beta, \alpha) \, \mathfrak{p}(\beta^{-1}\alpha)^{-2(n+2)}}{|\zeta|} \, d\sigma(\alpha).$$

As $\lambda - r \leq \mathfrak{p}(\beta^{-1}\alpha)$, we get

$$\left| \int_{\Omega_1} \partial^\perp \Psi(\beta, \alpha) \, d\sigma(\alpha) \right| \leq 4nc \ \sup_{\alpha \in \Omega_1} \left| K(\beta, \alpha) \right| \int_{\Omega_1} \frac{1}{r(\lambda - r)^{2(n+2)}} \, d\sigma(\alpha),$$

$$(3.3) \qquad\qquad \leq \frac{4nc \ \sup_{\alpha \in \Omega_1} \left| K(\beta, \alpha) \right|}{(\lambda - 1)^{2(n+2)}} |\Omega_1|,$$

where $|\Omega_1|$ denotes the surface measure of Ω_1. Using mean value theorem, we have

$$\left| \partial^\perp \Psi(\beta, \alpha) - \partial^\perp \Psi(\gamma, \alpha) \right| \leq c_1 \left| \nabla_\gamma(\partial^\perp \Psi(\beta, \alpha)) \right| \, \mathfrak{p}(\gamma^{-1}\beta),$$

$$\leq c_2 \frac{\mathfrak{p}(\gamma^{-1}\beta)}{(\mathfrak{p}(\gamma^{-1}\alpha))^{2(n+2)}}$$

for some suitable constants c_1 and c_2. Now,

$$(3.4) \qquad \int_{\Omega_2} \left| \partial^\perp \Psi(\beta, \alpha) - \partial^\perp \Psi(\gamma, \alpha) \right| d\sigma(\alpha) \leq c_2 \int_{\Omega_2} \frac{\mathfrak{p}(\gamma^{-1}\beta)}{(\mathfrak{p}(\gamma^{-1}\alpha))^{2(n+2)}} d\sigma(\alpha).$$

Using Equation (3.1), we obtain

$$\frac{\mathfrak{p}(\gamma^{-1}\beta)}{(\mathfrak{p}(\gamma^{-1}\alpha))^{2(n+2)}} = \left| \frac{\left|\zeta'\right|^2 + \left|\zeta''\right|^2 + \iota(t'' - t') - 2\zeta' \cdot \bar\zeta''}{\left(|\zeta''|^2 + |\zeta|^2 + \iota(t - t'') - 2\zeta'' \cdot \bar\zeta \right)^{2(n+2)}} \right|,$$

where $\gamma = (\zeta'', t'')$. Clearly the term on the right-hand side in Equation (3.4) remains bounded on Ω_2. Combining Equation (3.3) and Equation (3.4), we get

$$\left| u(\beta) - u(\gamma) \right| = \int_{\partial\Omega} \{\psi(\alpha) - \psi(\gamma)\} \left(\partial^\perp \Psi(\beta, \alpha) - \partial^\perp \Psi(\gamma, \alpha) \right) \, d\sigma(\alpha),$$

$$\leq c_3 \left(\max_{\alpha \in \Omega_1} \left| \psi(\alpha) - \psi(\gamma) \right| + \mathfrak{p}(\gamma^{-1}\beta) \int_{\Omega_2} \frac{1}{r^{2(n+2)}} d\sigma(\alpha) \right).$$

For any $\epsilon > 0$, ψ being uniformly continuous gives us the liberty to choose a $\delta > 0$ so that $B_\delta(\gamma) \subseteq \Omega_1$ and

$$\max_{\alpha \in \Omega_1} \left| \psi(\alpha) - \psi(\gamma) \right| < \frac{\epsilon}{2c_3}.$$

Choosing $\delta < \frac{\epsilon}{2c_3 f(r)}$, where $f(r) = \int_{\Omega_2} \frac{1}{r^{2n+2}} d\sigma(\alpha)$, we observe that

$$\left| u(\beta) - u(\gamma) \right| < \epsilon,$$

whenever $\mathfrak{p}(\gamma^{-1}\beta) < \delta$. $\square$

Theorem 3.6. *For $\beta \in \partial\Omega$, the double layer potential $\tilde{V}$ takes following limiting values*:

$$\lim_{\gamma \to \beta} \tilde{V}(\gamma) = \begin{cases} \int_{\partial\Omega} \psi(\alpha) \ \partial^\perp \Psi(\beta, \alpha) \ d\sigma(\alpha) - \psi(\beta), & \gamma \in \Omega, \\ \int_{\partial\Omega} \psi(\alpha) \ \partial^\perp \Psi(\beta, \alpha) \ d\sigma(\alpha) + \psi(\beta), & \gamma \in \mathbb{H}_n \setminus \bar{\Omega}. \end{cases}$$

Proof. Using Corollary 3.4, the integral defined above is a continuous function on $\partial\Omega$. We can write $\tilde{V}$ as

$$\tilde{V}(\beta) = u(\beta) + \psi(\gamma)\omega(\beta), \quad \beta = \gamma + h\hat{\gamma} \in N_{h_0}(\partial\Omega),$$

where u is as defined in Lemma 3.5 and $\omega(\beta) = \int_{\partial\Omega} \partial^\perp \Psi(\beta, \alpha) \ d\sigma(\alpha)$. The remaining part of the proof now follows using Lemma 3.5. $\square$

Corollary 3.7. *The double-layer potential $\tilde{V}$ can be extended in a continuous manner from Ω to $\bar{\Omega}$ and from $\mathbb{H}_n \setminus \bar{\Omega}$ to $\mathbb{H}_n \setminus \Omega$.*

Theorem 3.8. *For $\beta \in \partial\Omega$, the single layer potential V satisfies the following*:

$$\lim_{\gamma \to \beta} \partial^\perp V(\gamma) = \begin{cases} \int_{\partial\Omega} \psi(\alpha) \ \partial^\perp \Psi(\beta, \alpha) \ d\sigma(\alpha) - \psi(\beta), & \gamma \in \Omega, \\ \int_{\partial\Omega} \psi(\alpha) \ \partial^\perp \Psi(\beta, \alpha) \ d\sigma(\alpha) + \psi(\beta), & \gamma \in \mathbb{H}_n \setminus \bar{\Omega}. \end{cases}$$

Proof. As $\nabla_\beta \Psi(\beta, \alpha) = \nabla_\alpha \Psi(\beta, \alpha)$, we have

$$(\nabla V(\beta)) \cdot \hat{\gamma} + \tilde{V}(\beta) = \int_{\partial\Omega} \psi(\alpha) \ (\nabla_\alpha \Psi(\beta, \alpha)) \cdot \{\hat{\gamma} + \hat{\beta}\} \ d\sigma(\alpha),$$

where $\beta = \gamma + h\hat{\gamma} \in N_{h_0}(\partial\Omega)$. Using Theorem 3.6, analogous to the double-layer potential $\tilde{V}$, the right-hand side can be shown to be continuous on $N_{h_0}(\partial\Omega)$. The proof now follows from Theorem 3.6. $\square$

Theorem 3.9. *The following limit holds uniformly for all β in $\partial\Omega$*:

$$\lim_{\epsilon \to 0^+} \{\nabla \tilde{V}(\beta + \epsilon\hat{\beta}) - \nabla \tilde{V}(\beta - \epsilon\hat{\beta})\} \cdot \hat{\beta} = 0.$$

Proof. The theorem can be proved along similar lines to the proof of Theorem 3.6. $\square$

Define the integral operators $W, \tilde{W} : C_*(\partial\Omega) \to C_*(\partial\Omega)$ as

$$(W\psi)(\beta) := \int_{\partial\Omega} \psi(\alpha)(\partial^\perp \Psi(\beta, \alpha))_\alpha \ d\sigma(\alpha),$$

$$(\tilde{W}\phi)(\beta) := \int_{\partial\Omega} \phi(\alpha)(\partial^\perp \Psi(\beta, \alpha))_\beta \ d\sigma(\alpha),$$

where $\beta \in \partial\Omega$. We define the dual system $\langle C_*(\partial\Omega), C_*(\partial\Omega) \rangle$ as

$$\langle \phi, \psi \rangle := \int_{\partial\Omega} \phi\psi \ d\sigma.$$

As Ω has a smooth boundary, Corollary 3.4 implies that W and $\tilde{W}$ are compact operators. Also, W and $\tilde{W}$ are adjoint with respect to the above dual system.

Theorem 3.10. *Nullity of each of the operators $I + W$ and $I + \tilde{W}$ is one.*

Proof. The proof follows along similar lines as that of [3, Theorem 3.9]. □

Theorem 3.11. *If $g \in C_*(\partial\Omega)$, then any solution ϕ of the integral equation*

$$\phi(\beta) + \int_{\partial\Omega} \phi(\alpha)\partial^{\perp}\Psi(\beta, \alpha) \, d\sigma = g(\beta), \quad \beta \in \partial\Omega,$$

is also in $C_*(\partial\Omega)$. For such ϕ, the single-layer potential

$$V(\beta) = \int_{\partial\Omega} \phi(\alpha)\Psi(\beta, \alpha) \, d\sigma, \quad \beta \in \Omega,$$

acts as a solution for the interior Neumann problem (1.6).

Proof. As $g \in C_*(\partial\Omega)$, the single-layer potential V is well defined. The proof now follows from Theorem 3.8. □

Now we find ourselves in a position to prove the first part of our main result in this section.

Proof of Theorem 1.2. (*Necessity*) This can be proved using Proposition 2.1 by substituting $f_1 = 1$ and for a solution u of the problem (1.6).

(*Sufficiency*) The Fredholm's theorem indicates that the inhomogeneous problem $\phi + \tilde{W}\phi = g$ admits a solution if and only if g is orthogonal to a solution of $\psi + W\psi = 0$. Using Theorem 3.10, it is equivalent to saying

$$\int_{\partial\Omega} g \, d\sigma = 0.$$

Finally using Theorem 3.11, the Neumann problem (1.6) has a solution. □

4. The Neumann function and explicit representation of the solution

Definition 4.1. The Neumann function for the pair (Δ_0, Ω) is defined as a function G that satisfies

$$\begin{cases} \Delta_0 G(\beta, \alpha) = \delta_\beta, & \text{in } \Omega, \\ \partial^{\perp} G(\beta, \alpha) = 0, & \text{on } \partial\Omega. \end{cases}$$

Lemma 4.2. *Let β^* denote the reflection of the point β with respect to the boundary of the half-space Ω, i.e., $\beta^* = [\zeta', -t']$. Then for $\alpha \neq \beta$, $G(\beta, \alpha) = \bar{g}_\beta(\alpha) + \bar{g}_{\beta^*}(\alpha)$ acts as the Neumann function when applied to circular functions.*

Proof. As $\bar{g}_{\beta^*}(\alpha)$ is harmonic in Ω, we first observe

$$\Delta_0 G(\beta, \alpha) = \Delta_0(\bar{g}_\beta(\alpha)) = \delta_\beta.$$

Next we have

$$\bar{g}_\beta(\alpha) = a_0 \big|C(\beta, \alpha)\big|^{-n} F\left(\frac{n}{2}; \frac{n}{2}; n; \frac{\big|Q(\beta, \alpha)\big|^2}{\big|C(\beta, \alpha)\big|^2}\right),$$

$$\bar{g}_{\beta^*}(\alpha) = a_0 \big|C(\beta^*,\alpha)\big|^{-n} F\left(\frac{n}{2}; \frac{n}{2}; n; \frac{|Q(\beta^*,\alpha)|^2}{|C(\beta^*,\alpha)|^2}\right).$$

$$E\big(|C(\beta,\alpha)|^2\big)$$
$$= \sum_{j=1}^{n} \zeta_j \left(\frac{\partial}{\partial \zeta_j} + \iota \bar{\zeta}_j \frac{\partial}{\partial t}\right)\left(\zeta_j{}^2 \bar{\zeta}_j{}^2 + |\zeta'|^4 + t^2 + t'^2 + 2\zeta_j\bar{\zeta}_j|\zeta'|^2 - 2tt'\right)$$
$$= 2|\zeta|^2 \left(|\zeta|^2 + |\zeta'|^2 + \iota(t - t')\right).$$

Similarly, $\bar{E}\big(|C(\beta,\alpha)|^2\big) = 2|\zeta|^2 \left(|\zeta|^2 + |\zeta'|^2 - \iota(t - t')\right)$ and therefore Equation (1.4) implies that

$$\partial^\perp\big(|C(\beta,\alpha)|^2\big) = -4|\zeta|\,(t - t').$$

Using elementary relations of Gaussian hypergeometric functions, we get

$$\partial^\perp(\bar{g}_\beta(\alpha)) = 2n|\zeta|\,a_0(t - t')\big|C(\beta,\alpha)\big|^{-(n+2)} F\left(\frac{n}{2} + 1; \frac{n}{2}; n; \frac{|Q(\beta,\alpha)|^2}{|C(\beta,\alpha)|^2}\right).$$

Going through similar steps,

$$\partial^\perp(\bar{g}_{\beta^*}(\alpha)) = 2n|\zeta|\,a_0(t + t')\big|C(\beta^*,\alpha)\big|^{-(n+2)} F\left(\frac{n}{2} + 1; \frac{n}{2}; n; \frac{|Q(\beta^*,\alpha)|^2}{|C(\beta^*,\alpha)|^2}\right).$$

As $\big|C(\beta,\alpha)\big|^2 = \big|C(\beta^*,\alpha)\big|^2$ on the boundary $\partial\Omega$, we get

$$\partial^\perp\big(\bar{g}_\beta(\alpha) + \bar{g}_{\beta^*}(\alpha)\big) = 0 \quad \text{on } \partial\Omega. \qquad \square$$

Theorem 4.3. *The inhomogeneous Neumann boundary value problem*

(4.1)
$$\begin{cases} \Delta_0 u = f, & in\ \Omega, \\ \partial^\perp u = g, & on\ \partial\Omega, \end{cases}$$

where f and g are circular functions such that f is bounded and $g \in C_*(\partial\Omega)$, is solvable if and only if

(4.2)
$$\int_\Omega f(\alpha)\,d\nu(\alpha) = \int_{\partial\Omega} g(\alpha)\,d\sigma(\alpha).$$

The solution is given by following representation formula

(4.3)
$$u(\beta) = \int_\Omega G(\beta,\alpha)f(\alpha)\,d\nu(\alpha) - \int_{\partial\Omega} G(\beta,\alpha)g(\alpha)\,d\sigma(\alpha).$$

Proof. (*Necessity*) Using Proposition 2.1 with a solution u of Equation (4.1) and $f_2 = 1$, we have

$$\int_\Omega f(\alpha)\,d\nu(\alpha) = \int_{\partial\Omega} g(\alpha)\,d\sigma(\alpha).$$

(*Sufficiency*) Firstly, consider the following inhomogeneous boundary value problem

$$(4.4) \qquad \begin{cases} \Delta_0 u_1 = f, & \text{in } \Omega, \\ \quad u_1 = 0, & \text{on } \partial\Omega, \end{cases}$$

where $f \in C(\Omega)$. This problem clearly admits a solution and hence $\partial^\perp u_1$ exists on $\partial\Omega$. Here, it can be observed that $\partial^\perp u_1 \in C_*(\partial\Omega)$.

Now consider the following homogeneous Neumann problem, where $u_2 \in \mathfrak{C}_\Omega$

$$(4.5) \qquad \begin{cases} \Delta_0 u_2 = 0, & \text{in } \Omega, \\ \partial^\perp u_2 = \tilde{g}, & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{g} = g - \partial^\perp u_1$. As one can easily show that this problem possesses a solution, we have

$$\int_{\partial\Omega} \tilde{g} = 0,$$

which gives $\int_{\partial\Omega} g = \int_{\partial\Omega} \partial^\perp u_1$. Using Proposition 2.1, we finally get

$$\int_\Omega f(\alpha) \, d\nu(\alpha) = \int_{\partial\Omega} g(\alpha) \, d\sigma(\alpha).$$

(*Representation*) From the above discussion, we can conclude that the following problems

$$(4.6) \qquad \begin{cases} \Delta_0 u_1 = f, & \text{in } \Omega, \\ \partial^\perp u_1 = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$(4.7) \qquad \begin{cases} \Delta_0 u_2 = 0, & \text{in } \Omega, \\ \partial^\perp u_2 = g, & \text{on } \partial\Omega, \end{cases}$$

admit a solution. Let $\tilde{u_1}$ and $\tilde{u_2}$, respectively, denote the solutions of problem (4.6) and problem (4.7). Using the substitutions $f_1 = \tilde{u_1}(\alpha)$ and $f_2 = G(\beta, \alpha)$ in Proposition 2.1, we get

$$\tilde{u_1} = \int_\Omega G(\beta, \alpha) f(\alpha) \, d\nu(\alpha).$$

Similarly $\tilde{u_2} = -\int_{\partial\Omega} G(\beta, \alpha) g(\alpha) \, d\sigma(\alpha)$. It can be checked now that $u = \tilde{u_1} + \tilde{u_2}$ is a solution of the problem (4.1). $\qquad\square$

References

[1] L. Capogna, N. Garofalo, and D.-M. Nhieu, *Mutual absolute continuity of harmonic and surface measures for Hörmander type operators*, in *Perspectives in partial differential equations, harmonic analysis and applications*, 49–100, Proc. Sympos. Pure Math., 79, Amer. Math. Soc., Providence, RI, 2008. https://doi.org/10.1090/pspum/079/2500489

[2] S. Dubey, A. Kumar, and M. M. Mishra, *Green's function for a slice of the Korányi ball in the Heisenberg group* $\mathbb{H}_n$, Int. J. Math. Math. Sci. **2015** (2015), Art. ID 460461, 7 pp. https://doi.org/10.1155/2015/460461

[3] S. Dubey, A. Kumar, and M. M. Mishra, *The Neumann problem for the Kohn-Laplacian on the Heisenberg group* $\mathbb{H}_n$, Potential Anal. **45** (2016), no. 1, 119–133. https://doi.org/10.1007/s11118-016-9538-1

[4] G. B. Folland, *A fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc. **79** (1973), 373–376. https://doi.org/10.1090/S0002-9904-1973-13171-4

[5] B. Franchi, R. Serapioni, and F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann. **321** (2001), no. 3, 479–531. https://doi.org/10.1007/s002080100228

[6] N. Garofalo, M. Ruzhansky, and D. Suragan, *On Green functions for Dirichlet sub-Laplacians on H-type groups*, J. Math. Anal. Appl. **452** (2017), no. 2, 896–905. https://doi.org/10.1016/j.jmaa.2017.03.031

[7] B. Gaveau, *Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents*, Acta Math. **139** (1977), no. 1-2, 95–153. https://doi.org/10.1007/BF02392235

[8] L. Hörmander, *The analysis of linear partial differential operators. III*, reprint of the 1994 edition, Classics in Mathematics, Springer, Berlin, 2007. https://doi.org/10.1007/978-3-540-49938-1

[9] D. S. Jerison, *The Dirichlet problem for the Kohn Laplacian on the Heisenberg group. I*, J. Functional Analysis **43** (1981), no. 1, 97–142. https://doi.org/10.1016/0022-1236(81)90040-9

[10] D. S. Jerison, *The Dirichlet problem for the Kohn Laplacian on the Heisenberg group. II*, J. Funct. Anal. **43** (1981), no. 2, 224–257. https://doi.org/10.1016/0022-1236(81)90031-8

[11] A. Korányi, *Geometric aspects of analysis on the heisenberg group*, Topics in Modern Harmonic Analysis **1** (1983), 209–258.

[12] A. Korányi, *Poisson formulas for circular functions on some groups of type H*, Sci. China Ser. A **49** (2006), no. 11, 1683–1695. https://doi.org/10.1007/s11425-006-2065-6

[13] A. Korányi and H. M. Reimann, *Horizontal normal vectors and conformal capacity of spherical rings in the Heisenberg group*, Bull. Sci. Math. (2) **111** (1987), no. 1, 3–21.

[14] A. Kumar and M. M. Mishra, *Green functions and related boundary value problems on the Heisenberg group*, Complex Var. Elliptic Equ. **58** (2013), no. 4, 547–556. https://doi.org/10.1080/17476933.2012.693482

[15] M. M. Mishra, A. Kumar, and S. Dubey, *Green's function for certain domains in the Heisenberg group* $\mathbb{H}_n$, Bound. Value Probl. **2014** (2014), 182, 16 pp. https://doi.org/10.1186/s13661-014-0182-7

[16] M. M. Mishra and A. Pandey, *Well-posedness of a Neumann-type problem on a gauge ball in H-type groups*, Bound. Value Probl. **2020** (2020), Paper No. 92, 14 pp. https://doi.org/10.1186/s13661-020-01389-2

[17] E. D. Rainville, *Elementary Differential Equations*, The Macmillan Company, New York, 1958.

[18] M. Ruzhansky and D. Suragan, *Layer potentials, Kac's problem, and refined Hardy inequality on homogeneous Carnot groups*, Adv. Math. **308** (2017), 483–528. https://doi.org/10.1016/j.aim.2016.12.013

[19] E. M. Stein, *Harmonic analysis*: *real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43, Princeton University Press, Princeton, NJ, 1993.

[20] S. Thangavelu, *Harmonic analysis on the Heisenberg group*, Progress in Mathematics, 159, Birkhäuser Boston, Inc., Boston, MA, 1998. `https://doi.org/10.1007/978-1-4612-1772-5`

[21] K. Wang, *On the Neumann problem for harmonic functions in the upper half plane*, J. Math. Anal. Appl. **419** (2014), no. 2, 839–848. `https://doi.org/10.1016/j.jmaa.2014.04.076`

SHIVANI DUBEY
DEPARTMENT OF MATHEMATICS
VIVEKANANDA COLLEGE
UNIVERSITY OF DELHI
DELHI, INDIA
Email address: `shvndb@gmail.com`

MUKUND MADHAV MISHRA
DEPARTMENT OF MATHEMATICS
HANSRAJ COLLEGE
UNIVERSITY OF DELHI
DELHI, INDIA
Email address: `mukund.math@gmail.com`

ASHUTOSH PANDEY
DEPARTMENT OF MATHEMATICS
FACULTY OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELHI
DELHI, INDIA
Email address: `ashutoshpandey4521@gmail.com`