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THE MODULI SPACES OF LORENTZIAN LEFT-INVARIANT METRICS ON THREE-DIMENSIONAL UNIMODULAR SIMPLY CONNECTED LIE GROUPS

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Abstract. Let G be an arbitrary, connected, simply connected and unimodular Lie group of dimension 3. On the space $\mathfrak{M}(G)$ of left-invariant Lorentzian metrics on G, there exists a natural action of the group Aut(G) of automorphisms of G, so it yields an equivalence relation $\backsimeq$ on $\mathfrak{M}(G)$, in the following way: $h_1 \backsimeq h_2 \Leftrightarrow h_2 = \phi^*(h_1)$ for some $\phi \in \mathrm{Aut}(G)$.

In this paper a procedure to compute the orbit space $\mathrm{Aut}(G)/\mathfrak{M}(G)$ (so called moduli space of $\mathfrak{M}(G)$) is given.

1. Introduction

The study of the geometry of left-invariant Lorentzian metrics on three-dimensional Lie groups G goes back to Cordero and Parker [5] and is still far from complete. There are important and interesting unresolved issues, one of which is the problem of determining the moduli space defined as the orbit space of the action of Aut(G) on the space $\mathfrak{M}(G)$ of left-invariant Lorentzian metrics on such Lie groups. A complete study on the computation of moduli spaces is carried out for the Riemannian case by K. Y. Ha and J. B. Lee in [7]. It is known from the work of Vukmirović in [14] that the Lorentzian left-invariant metrics on Heisenberg groups H_{2n+1} of dimension $2n+1$, admit three non-isometric classes which generalize the metrics of Rahmani in [12,13]. Later in [9], Kubo, Onda, Taketomi and Tamaru study the moduli space of left-invariant pseudo-Riemannian metrics on the Lie groups of real hyperbolic spaces. Also it is determined in a recent paper [8] of Kondo and Tamaru the moduli space up to isometry and scaling of Lorentzian left-invariant metrics on certain nilpotent Lie groups. In the literature to date, the moduli space for Lorentzian left-invariant metrics on unimodular three-dimensional Lie group has not been explicitly presented. The above observation motivates us to study

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this moduli space and it is interesting to notice that the resulting classification in this work can be seen as natural generalization of the work in [7].

The main objectives of the present paper are:

(1) to complete the classification in [5] realized in terms of Lie algebras with an orthonormal basis, by classifying for each three-dimensional, connected, simply connected, unimodular and non abelian Lie group G all the Lorentzian left-invariant metrics on G, up to automorphism of G, and hence achieve a similar study to the one done in the Riemannian case in [7],

(2) to compute, for each class of these metrics, their curvatures. Moreover, we provide our more explicit formulas for each Lorentzian left-invariant generalized Ricci solitons on such Lie groups [4].

There are five three-dimensional, connected, simply connected, unimodular and non abelian Lie groups characterized by the signature of their Killing form: the nilpotent Lie group Nil with signature $(0,0,0)$, the special unitary group SU(2) with signature $(-,-,-)$, the universal covering group $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the special linear group with signature $(+,+,-)$, the solvable Lie group Sol with signature $(+,0,0)$ and the universal covering group $\widetilde{\mathrm{E}_0}(2)$ of the connected component of the Euclidean group with signature $(-,0,0)$. Let (G,h) be one of these Lie groups endowed with a Lorentzian left-invariant metric and $\mathfrak{g}$ its Lie algebra with a fixed orientation. Denote by $\langle\ ,\ \rangle$ the value of h at the neutral element. The study in [5] is based on a remark first made by Milnor in [10]. Depending only on $\langle\ ,\ \rangle$ and the orientation, there exist a product[1] $\times : \mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$ and a symmetric endomorphism $L:\mathfrak{g}\longrightarrow\mathfrak{g}$ such that, for any $u,v\in\mathfrak{g}$, the Lie bracket on $\mathfrak{g}$ is given by

$$[u,v]=L(u\times v). \tag{1.1}$$

Note that L changes to $-L$ when we change the orientation. It is well-known (see [11]) that there are four types of symmetric endomorphisms on a Lorentzian vector space. Relying on the type of L, there exists $\mathbb{B}_0=(e_1,e_2,e_3)$ an orthonormal basis [2] of $\mathfrak{g}$ with $\langle e_1,e_1\rangle=\langle e_2,e_2\rangle=1$ and $\langle e_3,e_3\rangle=-1$ such that (1.1) gives one of the following forms: (see [2])

Type diag$\{a,b,c\}$**:** We say that L is of type diag$\{a,b,c\}$ if it is diagonalizable with eigenvalues $[a,b,c]$ with respect to $\mathbb{B}_0$. In terms of these eigenvalues we deduce that

$$[e_1,e_2]=-c\,e_3,\ [e_2,e_3]=a\,e_1\ \text{ and }\ [e_3,e_1]=b\,e_2. \tag{1.2}$$

In this case the eigenvalues of the matrix of the Killing form in $\mathbb{B}_0$ are $[-2ab,2ac,2bc]$.

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[1] the product $\times$ designates the cross product of two vectors u, v in the Lorentzian Lie algebra $\mathfrak{g}$ such that $\langle u\times v,w\rangle=\det[u\ v\ w]$, $\forall w\in\mathfrak{g}$.

[2] if $\mathbb{B}_0=(e_1,e_2,e_3)$ is a positively oriented and orthonormal basis with e_3 timelike, then $e_1\times e_2=-e_3$, $e_2\times e_3=e_1$ and $e_3\times e_1=e_2$.

Type {azz̄}**:** L is called of type {azz̄} if it has one real and two complex conjugate eigenvalues. Then there exist $\mathrm{a}, \alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$ such that [3]

$$\mathcal{M}_{\mathbb{B}_0}(L) = \begin{pmatrix} \mathrm{a} & 0 & 0 \\ 0 & \alpha & -\beta \\ 0 & \beta & \alpha \end{pmatrix}, \ \beta \neq 0.$$

The corresponding Lie algebra is given by:

$$[e_1, e_2] = -\beta\, e_2 - \alpha\, e_3, \ [e_2, e_3] = \mathrm{a}\, e_1 \ \text{and} \ [e_3, e_1] = \alpha\, e_2 - \beta\, e_3. \tag{1.3}$$

In this case the eigenvalues of the matrix of the Killing form in $\mathbb{B}_0$ are $[2(\alpha^2+\beta^2), 2\mathrm{a}\sqrt{\alpha^2+\beta^2}, -2\mathrm{a}\sqrt{\alpha^2+\beta^2}]$.

Type {ab2}**:** L is said to be of type {ab2} if its spectrum consists of the two eigenvalues a and b (one of which has multiplicity two), each associated to a one-dimensional eigenspace. In the orthonormal basis $\mathbb{B}_0$ the matrix of L takes the form

$$\mathcal{M}_{\mathbb{B}_0}(L) = \begin{pmatrix} \mathrm{a} & 0 & 0 \\ 0 & \mathrm{b}+1/2 & -1/2 \\ 0 & 1/2 & \mathrm{b}-1/2 \end{pmatrix}$$

and the corresponding Lie algebra is given by:

$$\begin{aligned} &[e_1, e_2] = \frac{1}{2}e_2 + \left(\frac{1}{2} - \mathrm{b}\right) e_3, \ [e_2, e_3] = \mathrm{a}e_1 \ \text{and} \\ &[e_3, e_1] = \left(\mathrm{b} + \frac{1}{2}\right) e_2 + \frac{1}{2}e_3, \end{aligned} \tag{1.4}$$

which gives the eigenvalues of the matrix of the Killing form in $\mathbb{B}_0$: $[2\mathrm{b}^2, (\sqrt{4\mathrm{b}^2+1}-1)\mathrm{a}, -(\sqrt{4\mathrm{b}^2+1}+1)\mathrm{a}]$.

Type {a3}**:** L is called of type {a3} if it has three equal eigenvalues associated to a one-dimensional eigenspace. Then there exists $\mathrm{a} \in \mathbb{R}$ such that

$$\mathcal{M}_{\mathbb{B}_0}(L) = \begin{pmatrix} \mathrm{a} & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \mathrm{a} & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \mathrm{a} \end{pmatrix}$$

the corresponding Lie algebra is given by:

$$\begin{aligned} &[e_1, e_2] = \frac{1}{\sqrt{2}}e_2 - \mathrm{a}e_3, \ [e_2, e_3] = \mathrm{a}e_1 + \frac{1}{\sqrt{2}}e_2 \ \text{and} \\ &[e_3, e_1] = \frac{1}{\sqrt{2}}e_1 + \mathrm{a}e_2 + \frac{1}{\sqrt{2}}e_3, \end{aligned} \tag{1.5}$$

In this case the eigenvalues of $\mathcal{M}_{\mathbb{B}_0}(K)$ are complicated. But

$$\det(\mathcal{M}_{\mathbb{B}_0}(K)) = -8\mathrm{a}^6 \ \text{and} \ \mathrm{tr}(\mathcal{M}_{\mathbb{B}_0}(K)) = 2(\mathrm{a}^2+1)$$

which can be used to determine the signature of the Killing form K.

[3] here and below $\mathcal{M}_{\mathbb{B}_0}(L)$ denotes the matrix of L with respect to the basis $\mathbb{B}_0$.

Moreover, according to Proposition 2.2 the type of the symmetric endomorphism L is an invariant of Lorentzian left-invariant metrics on three-dimensional unimodular Lie group. Indeed, let h_1 and h_2 be two Lorentzian left-invariant metrics on G. Let L_1 and L_2 be the associated endomorphisms to h_1 and h_2, respectively, for a fixed orientation on $\mathfrak{g}$. If there exists an automorphism of the Lie group $\phi : \mathrm{G} \longrightarrow \mathrm{G}$ such that $h_1 = \phi^*(h_2)$, then there exists a linear isometry ψ from $(\mathfrak{g}, \langle\ ,\ \rangle_1)$ onto $(\mathfrak{g}, \langle\ ,\ \rangle_2)$ satisfying $L_2 = \pm\psi \circ L_1 \circ \psi^{-1}$, where $\langle\ ,\ \rangle_1$ (resp. $\langle\ ,\ \rangle_2$) denotes the inner product on $\mathfrak{g}$ induced by the metric h_1 (resp. h_2). In particular, one can construct an orthonormal basis $\mathcal{B}$ of $(\mathfrak{g}, \langle\ ,\ \rangle_1)$ and an orthonormal basis $\mathcal{C}$ of $(\mathfrak{g}, \langle\ ,\ \rangle_2)$ such that $\mathcal{M}_{\mathcal{B}}(L_1) = \pm\mathcal{M}_{\mathcal{C}}(L_2)$. In other words, L_1 and $\pm L_2$ are represented by the same matrix.

Having this consideration in mind and the fact that two similar matrix have the same complex spectrum, we see that if we have any of the following four cases:

case 1: the type of L_1 is different from that of $\pm L_2$.

case 2: L_1 and $\pm L_2$ are diagonalizable but have different sets of eigenvalues of spacelike eigenvectors. More in detail: $(a_1, b_1) \neq \pm\,(a_2, b_2)$ and $(a_1, b_1) \neq \pm\,(b_2, a_2)$, where a_1 and b_1 (resp. a_2 and b_2) given as in (1.2), and related to an orthonormal basis with respect $\langle\ ,\ \rangle_1 = h_1(e)$ (resp. $\langle\ ,\ \rangle_2 = h_2(e)$).

case 3: L_1 and $\pm L_2$ are diagonalizable and have the same sets of eigenvalues of spacelike eigenvectors but the eigenvalue of timelike eigenvector of L_1 is different from that of $\pm L_2$, since this eigenvalue depends on the causal character of its eigenvector.

case 4: L_1 and $\pm L_2$ are not diagonalizable and have the same type given as in (1.3), (1.4) and (1.5) but the set of eigenvalues of L_1 is different from that of $\pm L_2$.

Then h_1 and h_2 are not equivalent up to automorphism.

For convenience of the reader, we now provide an extended discussion of the main steps of the approach used as well as an outline of the paper. The overall task in order to achieve our goals mentioned above splits into eight steps.

step 1: We take a three-dimensional unimodular Lie group G and we fix a natural basis $\mathbb{B}_{_N}$ of its Lie algebra $\mathfrak{g}$ (see Paragraph 2 (**Natural basis** in Section 2) where these basis are given).

step 2: We endow G with a Lorentzian left-invariant metric h_0 and we denote by L its associated endomorphism.

step 3: We have to determine what are the possible types and parameters of $\pm L$ that are subject to the signature of the Killing form.

step 4: For each possible type of $\pm L$, there exists a positively oriented and orthonormal basis $\mathbb{B}_0 = (e_1, e_2, e_3)$ with e_3 timelike such that the Lie bracket of $\mathfrak{g}$ has one of the forms (1.2), (1.3), (1.4), (1.5).

step 5: We find a basis $\mathbb{B}_1 = (xe_1 + ye_2 + ze_3, ue_1 + ve_2 + we_3, pe_1 + qe_2 + re_3)$ in which structure constants of the Lie algebra $\mathfrak{g}$ are the same structure

constants described in Paragraph 2 (**Natural basis** in Section 2) and related to the fixed basis $\mathbb{B}_{_N}$. Using this basis $\mathbb{B}_1$ we calculate the automorphism of Lie algebra $\rho : \mathfrak{g} \longrightarrow \mathfrak{g}$ which sends the orthonormal basis $\mathbb{B}_0$ to $\mathbb{B}_1$. Let ϕ_0 be the automorphism of G associated to ρ. We put $h_1 = \phi_0^*(h_0)$.

step 6: In some cases, we use another automorphism ϕ_1 of G such that $\phi_1^*(h_1)$ has a more reduced form than h_1.

step 7: Finally, we get the matrix of $\phi_1^*(h_1)$ in the natural basis $\mathbb{B}_{_N}$. The matrix obtained $\mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0))$ or $\mathcal{M}_{\mathbb{B}_{_N}}(\phi_1^*(h_1))$ constitute a list of Lorentzian left-invariant metrics on G depending on a reduced number of parameters and each Lorentzian left-invariant metric on G is isometric to one in this list. We find twenty one non-isometric classes of such metrics which shows that the situation is far more rich than the Riemannian case [7].

step 8: We compute for each metric in the list the Ricci curvature and the scalar curvature which determine all the curvature since we are in dimension 3. Finally, we exhibit some metrics with distinguished curvature properties.

We note that the fifth and sixth step, namely the finding for automorphisms ϕ_0 and ϕ_1 represent one of the most difficult tasks of our work, especially when $\mathrm{G} = \widetilde{\mathrm{PSL}}(2,\mathbb{R})$. For that reason we have used the software Maple and the expression of the group of automorphisms of the three-dimensional unimodular Lie algebra given in [7]. All the computations in the entire article have been checked by Maple.

The paper is organized as follows. In Section 2, we precise the models of the three-dimensional unimodular Lie group G we will use in this paper. In Section 3, we perform for each Lie group G the steps mentioned above and we give its list of Lorentzian left-invariant metrics. For Nil the list contains three non-isometric classes of metrics, for SU(2) one class, for $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ seven non-isometric classes, for Sol seven non-isometric classes and for $\widetilde{\mathrm{E}_0}(2)$ three non-isometric classes. These non-isometric classes are expressed by the twenty-one formulas from (nil-) to (ee3) and the classification calculated within this study is therefore summarised in the tables below. In Section 4, we provide for each class of metrics found in Section 3 its Ricci curvature, scalar curvature and the signature of the Ricci curvature. We provide Table 1 describing the possible signature of the Ricci curvature and the metrics realizing these signatures. Finally, the paper concludes in Section 5 with a comparison to related work and a discussion of the constant curvature, Einstein, locally symmetric, semi-symmetric not locally symmetric and generalized Ricci soliton metrics on G which have been determined in [1, 3, 4] by giving their Lie algebras as in (1.2), (1.3), (1.4), (1.5). Here we supply their corresponding metrics in the resulting list (see from Theorems 5.1 to 5.7).

Lie algebras	$\mathfrak{n}$		$\mathfrak{su}(2)$	$\mathfrak{sol}$						
Natural basis	$[X,Y]=Z,$ $[Z,Y]=0,$ $[Z,X]=0.$		$[\sigma_x,\sigma_y]=2\sigma_z,$ $[\sigma_y,\sigma_z]=2\sigma_x,$ $[\sigma_z,\sigma_x]=2\sigma_y.$	$[X_1,X_2]=X_2,$ $[X_1,X_3]=-X_3,$ $[X_2,X_3]=0.$						
Lie groups	Nil		SU(2)	Sol						
	nil+	nil−	su	sol1	sol2	sol3	sol4	sol5	sol6	sol7
Lorentzian left-invariant metrics (modulo automorphism)	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&\lambda\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-\lambda\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}\frac{4}{u^2-v^2}&0&0\\0&1&\frac{u}{v}\\0&\frac{u}{v}&1\end{pmatrix}$	$\begin{pmatrix}\frac{4}{v^2-u^2}&0&0\\0&\frac{u}{v}&-1\\0&-1&\frac{u}{v}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{u+v}&0&0\\0&-\frac{v}{u}&1\\0&1&1\end{pmatrix}$	$\begin{pmatrix}\frac{1}{u}&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&-\frac{2}{u}\\0&1&1\\-\frac{2}{u}&1&1\end{pmatrix}$	$\begin{pmatrix}u^2&0&0\\0&u&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$
Parameters	$\lambda>0$	$\lambda>0$	$\mu_1\geq\mu_2>0$ $\mu_3>0$	$-v<u<v$ $v>0$	$-v<u<v$ $v>0$	$u>0$ $v>0$	$u>0$	$u>0$	$u\neq 0$	
Symmetric	diag	diag	ab2	diag	diag	azz̄	azz̄	ab2	ab2	a3
endomorphisms L	$(a,0,0)$ $a>0$	$(0,0,c)$ $c>0$	a = 0 b = 0	(a,b,c) $a>0,\ b>0$ $c<0$	$(a,b,0)$ $a>0$ $b<0$	$(a,0,c)$ $a>0$ $c>0$	a = 0 $\mathfrak{Re}(z)>0$ $\mathfrak{Im}(z)>0$	a = 0 $\mathfrak{Re}(z)=0$ $\mathfrak{Im}(z)>0$	a < 0 b = 0	a = 0 b ≠ 0	a = 0
Signature of Ricci curvature	$(+,-,-)$	$(+,+,+)$	$(0,0,0)$	$(+,+,+)$ if $\mu_1<\mu_2+\mu_3$ $(+,-,-)$ if $\mu_1>\mu_2+\mu_3$ $(+,0,0)$ if $\mu_1=\mu_2+\mu_3$	$(+,-,-)$ if $u\neq 0$ $(-,0,0)$ if $u=0$	$(+,-,-)$ if $u>0$ $(+,+,+)$ if $u<0$ $(0,0,0)$ if $u=0$	$(+,-,-)$	$(-,0,0)$	$(+,-,-)$	$(-,0,0)$ if $u>0$ $(+,0,0)$ if $u<0$	$(-,0,0)$
Signature of scalar curvature	$\mathfrak{s}=\lambda/2$ $\mathfrak{s}>0$	$\mathfrak{s}=\lambda/2$ $\mathfrak{s}>0$	$\mathfrak{s}=0$	$\mathfrak{s}>0$	$\mathfrak{s}=v^2/2$ $\mathfrak{s}>0$	$\mathfrak{s}=u^2/2$ $\mathfrak{s}>0$ if $u\neq 0$ $\mathfrak{s}=0$ if $u=0$	$\mathfrak{s}=-2v$ $\mathfrak{s}<0$	$\mathfrak{s}=-2u$ $\mathfrak{s}<0$	$\mathfrak{s}=u^2/2$ $\mathfrak{s}>0$	$\mathfrak{s}=0$	$\mathfrak{s}=0$

Lie algebras	$\mathfrak{sl}(2,\mathbb{R})$							$\mathfrak{e}_0(2)$		
Natural basis	$[X_1,X_2]=2X_3,$ $[X_3,X_1]=2X_2,$ $[X_3,X_2]=2X_1.$							$[X_1,X_2]=X_3,$ $[X_1,X_3]=-X_2,$ $[X_3,X_2]=0.$		
Lie groups	$\widetilde{\mathrm{PSL}}(2,\mathbb{R})$							$\widetilde{\mathrm{E}_0(2)}$		
Lorentzian	sll1	sll2	sll3	sll4	sll5	sll6	sll7	ee1	ee2	ee3
left-inv. metrics (modulo automor-phism)	$\begin{pmatrix}-\mu_1&0&0\\0&\mu_2&0\\0&0&\mu_3\end{pmatrix}$	$\begin{pmatrix}\mu_1&0&0\\0&-\mu_2&0\\0&0&\mu_3\end{pmatrix}$	$K\begin{pmatrix}M&\beta&0\\\beta&N&0\\0&0&\frac{\alpha^2M}{N}\end{pmatrix}$	$K\begin{pmatrix}-N&0&\beta\\0&\frac{\alpha^2M}{N}&0\\\beta&0&M\end{pmatrix}$	$K\begin{pmatrix}u&0&v\\0&M&0\\v&0&u\end{pmatrix}$	$K\begin{pmatrix}M&-\mathrm{a}&0\\-\mathrm{a}&N&0\\0&0&\frac{8\mathrm{a}}{\mathrm{b}}\end{pmatrix}$	$K\begin{pmatrix}M&N&0\\N&S&R\\0&R&4\mathrm{a}^4\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&u&0\\0&0&v\end{pmatrix}$	$\begin{pmatrix}0&-1&0\\-1&u&0\\0&0&v\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&u\end{pmatrix}$
Parameters	$\mu_1>0$ $\mu_2\geq\mu_3>0$	$\mu_1>0$ $\mu_2>0,\ \mu_3>0.$	$K=\frac{4}{\alpha^2\sqrt{\alpha^2+\beta^2}}$ $M=\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{2}$ $N=\sqrt{\alpha^2+\beta^2}$ $\alpha>0,\ \beta>0$	$\alpha<0,\ \beta>0$	$K=\frac{16}{v^2-u^2}$ $M=2(u+v)$ $-v<u<v$ $v>0$	a ≠ 0 b ≠ 0 $K=\frac{1}{2aN}$ $M=\mathrm{a}-8$ $N=\mathrm{a}+8$	$K=\frac{2}{a^6(1+2a^2)}$ $M=1-4a^4$ $N=(1+2a^2)^{\frac{1}{2}}$ $S=4a^4+6a^2+1$ $R=2a^3\sqrt{2},\ \mathrm{a}>0$	$u\geq v$ $v>0$	$u<0$ $v>0$	$u>0$
Symmetric	diag	diag	a²zz̄	a²zz̄	a²zz̄	ab2	a3	diag	diag	ab2
endomor-phisms L	(a,b,c) $a>0,\ b>0$ $c>0$	(a,b,c) $a<0,\ b>0$ $c<0$	a ≠ 0 $\mathfrak{Re}(z)>0$ $\mathfrak{Im}(z)>0$	a ≠ 0 $\mathfrak{Re}(z)<0$ $\mathfrak{Im}(z)>0$	a ≠ 0 $\mathfrak{Re}(z)=0$ $\mathfrak{Im}(z)>0$	a ≠ 0 b ≠ 0	a ≠ 0	$(a,b,0)$ $a>0,\ b>0$	$(a,0,c)$ $a>0,\ c<0$	a > 0 b = 0
Signature of Ricci curvature	$(+,+,+)$ if $\mu_3<\mu_1-\mu_2$ if $\mu_1<\mu_2-\mu_3$		$(+,-,-)$ $\mathrm{a}^2\neq 2\,\mathfrak{Re}(z)$	$(+,-,-)$	$(+,-,-)$	$(+,-,-)$ if a ≠ 2b	$(+,-,-)$	$(0,0,0)$ if $u=v$	$(+,+,+)$ if $u<-v$	$(+,-,-)$
	$(+,-,-)$ if $\mu_3>\mu_1-\mu_2$ if $\mu_1>\mu_2-\mu_3$		$(-,0,0)$ $\mathrm{a}^2=2\,\mathfrak{Re}(z)$			$(-,0,0)$ if a = 2b		$(+,-,-).$ if $u\neq v$	$(+,-,-)$ if $u>-v$	
	$(+,0,0)$ if $\mu_3=\mu_1-\mu_2$ if $\mu_1=\mu_2-\mu_3$								$(+,0,0)$ if $u=-v$	
	$(-,0,0)$ if $\mu_3=-\mu_1+\mu_2$ if $\mu_1=\mu_2+\mu_3$									
Scalar curvature	$\mathfrak{s}=\frac{2[(\sqrt{\mu_1}+\sqrt{\mu_2})^2+\epsilon\mu_3][(\sqrt{\mu_1}-\sqrt{\mu_2})^2+\epsilon\mu_3]}{\mu_1\mu_2\mu_3}$ $\epsilon=-1$	$\epsilon=1$	$\mathfrak{s}=\frac{1}{2}\mathrm{a}^4-2\mathrm{a}^2\alpha-2\beta^2$		$\mathfrak{s}=\frac{u}{2}$	$\mathfrak{s}=\frac{1}{2}\mathrm{a}(\mathrm{a}-4\mathrm{b})$	$\mathfrak{s}=-\frac{3}{2}\mathrm{a}^2$ $\mathfrak{s}<0$	$\mathfrak{s}=\frac{(u-v)^2}{2v}$ $\mathfrak{s}\geq 0$	$\mathfrak{s}=\frac{(u-v)^2}{2v}$ $\mathfrak{s}>0$	$\mathfrak{s}=\frac{u}{2}$ $\mathfrak{s}>0$

2. Preliminaries

Throughout this paper, G will always be a connected, simply connected, unimodular, three-dimensional and non abelian Lie group. Denote by h, a left-invariant Lorentzian metric on G, determined by inner product $\langle\ ,\ \rangle = h(e)$ of

signature $(2,1)$ on the Lie algebra $\mathfrak{g}$ of G. Let $\nabla : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ be the Levi-Civita connection associated to (G, h). From Koszul formula we get

$$2\langle \nabla_u v, w\rangle = \langle [u,v], w\rangle + \langle [w,u], v\rangle + \langle [w,v], u\rangle. \tag{2.1}$$

For any $u, v \in \mathfrak{g}$, $\nabla_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ is skew-symmetric and $[u,v] = \nabla_u v - \nabla_v u$. Let R be the Riemannian curvature tensor of (G, h) defined for tangent vectors u, v by $\mathrm{R}(u,v) = \nabla_{[u,v]} - [\nabla_u, \nabla_v]$. The Ricci curvature is the symmetric tensor ric defined, for any tangent vectors u and v as the trace of the map: $w \mapsto \mathrm{R}(u,w)v$ and the Ricci operator $\mathrm{Ric} : \mathfrak{g} \longrightarrow \mathfrak{g}$ is given by the relation $\langle \mathrm{Ric}(u), v\rangle = \mathrm{ric}(u,v)$. The scalar curvature is given by $\mathfrak{s} = \mathrm{tr}(\mathrm{Ric})$.
Recall that:

(1) (G, h) is called flat if $\mathrm{R} = 0$;
(2) (G, h) has constant sectional curvature if there exists a constant λ such that, for any $u, v, w \in \mathfrak{g}$,

$$\mathrm{R}(u,v)w = \lambda\left(\langle v, w\rangle u - \langle u, w\rangle v\right).$$

(3) (G, h) is called Einstein if there there exists a constant λ such that $\mathrm{Ric} = \lambda \mathrm{Id}_{\mathfrak{g}}$.
(4) (G, h) is called locally symmetric if, for any $u, v, w \in \mathfrak{g}$,

$$\nabla_u(\mathrm{R})(v,w) := [\nabla_u, \mathrm{R}(v,w)] - \mathrm{R}(\nabla_u v, w) - \mathrm{R}(v, \nabla_u w) = 0.$$

(5) (G, h) is called semi-symmetric if, for any $u, v, a, b \in \mathfrak{g}$,

$$[\mathrm{R}(u,v), \mathrm{R}(a,b)] = \mathrm{R}(\mathrm{R}(u,v)a, b) + \mathrm{R}(a, \mathrm{R}(u,v)b).$$

(6) (G, h) is said to be generalized Ricci soliton (see [4] and references therein) if there exist $u \in \mathfrak{g}$ and real constants $\alpha_0, \beta_0, \lambda_0$ such that

$$\mathcal{L}_u \langle\ ,\ \rangle + 2\alpha_0\, u^\flat \odot u^\flat - 2\,\beta_0\, ric = 2\,\lambda_0 \langle\ ,\ \rangle,$$

$\mathcal{L}_u$ denotes the Lie derivative in the direction of u and $u^\flat$ denotes a 1-form such that $u^\flat(v) = \langle u, v\rangle$.
Adopting the notation in [4], one has:
(K) the Killing vector field equation when $\alpha_0 = \beta_0 = \lambda_0 = 0$;
(H) the homothetic vector field equation when $\alpha_0 = \beta_0 = 0$;
(RS) the Ricci soliton equation when $\alpha_0 = 0$, $\beta_0 = 1$;
(E-W) a special case of the Einstein-Weyl equation in conformal geometry when $\alpha_0 = 1$, $\beta_0 = -1$;
(PS) the equation for a metric projective structure with a skew-symmetric Ricci tensor representative in the projective class when $\alpha_0 = 1$, $\beta_0 = \frac{-1}{2}$, and $\lambda_0 = 0$;
(VN-H) the vacuum near-horizon geometry equation of a spacetime when $\alpha_0 = 1$, $\beta_0 = \frac{1}{2}$, with λ_0 playing the role of the cosmological constant.

Natural basis. Before focusing our attention on the classification of metrics, we should devote a few lines to the description of G as well as to structure constants of its Lie algebra related to the natural basis denoted $\mathbb{B}_{_N}$.

I– *The nilpotent Lie group* Nil known as Heisenberg group whose Lie algebra will be denoted by $\mathfrak{n}$. We have

$$\text{Nil}=\left\{\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}:x,y,z\in\mathbb{R}\right\}\ \text{and}\ \mathfrak{n}=\left\{\begin{pmatrix}0&x&z\\0&0&y\\0&0&0\end{pmatrix}:x,y,z\in\mathbb{R}\right\}$$

and the non-vanishing Lie brackets in the canonical basis $\mathbb{B}_{_N}=\{X,Y,Z\}$ are given by $[X,Y]=Z$. The Killing form is trivial.

II– *The special unitary group*

$$\text{SU}(2)=\left\{\begin{pmatrix}a+ib&-c+id\\c+id&a-ib\end{pmatrix}:a^2+b^2+c^2+d^2=1\right\}$$

whose Lie algebra $\mathfrak{su}(2)=\left\{\begin{pmatrix}iz&y+ix\\-y+xi&-zi\end{pmatrix}:x,y,z\in\mathbb{R}\right\}$.
In the natural basis

$$\mathbb{B}_{_N}=\left\{\sigma_x=\begin{pmatrix}0&i\\i&0\end{pmatrix},\sigma_y=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\sigma_z=\begin{pmatrix}-i&0\\0&i\end{pmatrix}\right\},$$

we have

$$[\sigma_x,\sigma_y]=2\sigma_z,\ [\sigma_y,\sigma_z]=2\sigma_x\ \text{and}\ [\sigma_z,\sigma_x]=2\sigma_y. \tag{2.2}$$

The Killing form has signature $(-,-,-)$.

III– *The universal covering group* $\widetilde{\text{PSL}}(2,\mathbb{R})$ *of* $\text{SL}(2,\mathbb{R})$ whose Lie algebra is $\text{sl}(2,\mathbb{R})$. In the natural basis

$$\mathbb{B}_{_N}=\left\{X_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},X_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},X_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\},$$

we have

$$[X_1,X_2]=2X_3,\ [X_3,X_1]=2X_2\ \text{and}\ [X_3,X_2]=2X_1. \tag{2.3}$$

The Killing form has signature $(+,+,-)$.

IV– *The solvable Lie group* $\text{Sol}=\left\{\begin{pmatrix}e^x&0&y\\0&e^{-x}&z\\0&0&1\end{pmatrix}:x,y,z\in\mathbb{R}\right\}$ whose Lie algebra is $\mathfrak{sol}=\left\{\begin{pmatrix}x&0&y\\0&-x&z\\0&0&0\end{pmatrix}:x,y,z\in\mathbb{R}\right\}$. In the natural

basis

$$\mathbb{B}_{_N} = \left\{ X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

we have

$$[X_1, X_2] = X_2,\ [X_1, X_3] = -X_3 \text{ and } [X_2, X_3] = 0. \tag{2.4}$$

The Killing form has signature $(+, 0, 0)$.

V– *The universal covering group* $\widetilde{\mathrm{E}_0(2)}$ *of the Lie group*

$$\mathrm{E}_0(2) = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) & x \\ -\sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{pmatrix} : \theta, x, y \in \mathbb{R} \right\}.$$

Its Lie algebra is

$$\mathrm{e}_0(2) = \left\{ \begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : \theta, x, y \in \mathbb{R} \right\}.$$

In the natural basis

$$\mathbb{B}_{_N} = \left\{ X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

we have

$$[X_1, X_2] = X_3,\ [X_1, X_3] = -X_2 \text{ and } [X_2, X_3] = 0. \tag{2.5}$$

The Killing form has signature $(-, 0, 0)$.

Remark 2.1. Obviously, we can see from the above calculus that two three-dimensional unimodular non abelian Lie algebras are isomorphic if and only their Killing forms have the same signature.

We are now ready to define the following equivalence relations.

Definition 1. Let $\mathfrak{g}$ be a unimodular Lie algebra of dimension 3. Define an equivalence relation on the set of Lorentzian inner products on $\mathfrak{g}$ that $(\mathfrak{g}, \langle\ ,\ \rangle_1) \backsimeq (\mathfrak{g}, \langle\ ,\ \rangle_2)$ if and only if there exists a Lie algebra automorphism $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\phi^*(\langle\ ,\ \rangle_1) = \langle\ ,\ \rangle_2$, i.e., $\left\langle \phi^{-1}(u), \phi^{-1}(v) \right\rangle_1 = \langle u, v \rangle_2$ or equivalently $\langle u, v \rangle_1 = \langle \phi(u), \phi(v) \rangle_2$ for all $u, v \in \mathfrak{g}$.

Definition 2. Let $\mathfrak{l} = (\mathfrak{g}, \langle\ ,\ \rangle_1)$ and $\mathfrak{p} = (\mathfrak{g}, \langle\ ,\ \rangle_2)$ be two oriented Lorentzian three-dimensional unimodular Lie algebras. Let L_1 and L_2 be symmetric endomorphisms of $\mathfrak{l}$ and $\mathfrak{p}$, respectively. We say that L_1 and L_2 are equivalent that we will denote as $L_1 \equiv L_2$ if there exists a pseudo-orthogonal matrix $\Phi \in \mathrm{O}(2, 1)$ such that $\mathcal{M}_{\mathbb{B}_2}(L_2) = \det(\Phi) \cdot \Phi\, \mathcal{M}_{\mathbb{B}_1}(L_1) \Phi^{-1}$, where $\mathbb{B}_1 = (e_1, e_2, e_3)$ (resp. $\mathbb{B}_2 = (v_1, v_2, v_3)$) is a positively oriented and orthonormal basis of $\mathfrak{l}$ (resp. $\mathfrak{p}$) with e_3 (resp. v_3) timelike.

This last equivalence relation $\equiv$ does not depend on the basis $\mathbb{B}_1$ and $\mathbb{B}_2$ and we see that $\equiv$ follows from the fact that $\mathrm{O}(2,1)$ is a group. More in detail, we find that

(1) Reflexivity arises from the fact that $\mathrm{Id} \in \mathrm{O}(2,1)$.
(2) Symmetry arises from the fact that if $\phi \in \mathrm{O}(2,1)$, then $\phi^{-1} \in \mathrm{O}(2,1)$.
(3) Transitivity arises from the fact that if $\phi \in \mathrm{O}(2,1)$ and $\psi \in \mathrm{O}(2,1)$, then $\phi\psi \in \mathrm{O}(2,1)$.

The symmetric endomorphism L contains the complete information about the Lorentzian left-invariant metrics on Lie group. More precisely the following reduces the problem of determining the isometric classes of metrics to that of finding the equivalence classes of a set of symmetric endomorphisms.

Proposition 2.2. *Let $\mathfrak{g}$ be an oriented three-dimensional unimodular Lie algebra. Then the equivalence classes of the Lorentzian Lie algebras $(\mathfrak{g}, \langle\ ,\ \rangle)$ correspond bijectively to the equivalence classes of L the associated symmetric endomorphisms.*

Proof. For each Lorentzian inner product $\langle\ ,\ \rangle$, let

$$[(\mathfrak{g}, \langle\ ,\ \rangle)] = \{(\mathfrak{g}, \phi^*\langle\ ,\ \rangle) : \phi \in \mathrm{Aut}(\mathfrak{g})\}.$$

And for each L the symmetric endomorphism of $(\mathfrak{g}, \langle\ ,\ \rangle)$, let

$$[L] = \left\{\det(\Phi)\cdot \Phi\, \mathcal{M}_{\mathbb{B}}(L)\, \Phi^{-1} : \Phi \in \mathrm{O}(2,1)\right\},$$

where $\mathbb{B} = (e_1, e_2, e_3)$ is a positively oriented and orthonormal basis with respect to the inner product $\langle\ ,\ \rangle$ with e_3 timelike. The bijection is

$$\{[(\mathfrak{g}, \langle\ ,\ \rangle)] : \langle\ ,\ \rangle \text{ Lorentzian inner product}\} \to \{[L] : L \text{ symmetric endomorphism of } (\mathfrak{g}, \langle\ ,\ \rangle)\}$$
$$[(\mathfrak{g}, \langle\ ,\ \rangle)] \mapsto [L].$$

Indeed, the following assertions are equivalent

(i) there exists a Lie algebra automorphism $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $\langle\ ,\ \rangle_2 = \phi^*(\langle\ ,\ \rangle_1)$.
(ii) there exists an isomorphism $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $L_2 = \det(\phi)\cdot\phi L_1 \phi^{-1}$ and $\langle\ ,\ \rangle_2 = \phi^*(\langle\ ,\ \rangle_1)$.
(iii) if $\mathbb{B}_1$ (resp. $\mathbb{B}_2$) is a positively oriented and orthonormal basis with respect to $\langle\ ,\ \rangle_1$ (resp. $\langle\ ,\ \rangle_2$), then there exists a matrix $\Phi \in \mathrm{O}(2,1)$ such that $\mathcal{M}_{\mathbb{B}_2}(L_2) = \det(\Phi)\cdot\Phi\, \mathcal{M}_{\mathbb{B}_1}(L_1)\Phi^{-1}$.

(i)$\Leftrightarrow$(ii)
Let $\times_1$ and $\times_2$ be two cross products (see footnote in 1) associated to $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$, respectively.

$$\begin{aligned}
&\phi \in \mathrm{Aut}(\mathfrak{g}) \text{ and } \langle\ ,\ \rangle_2 = \phi^*(\langle\ ,\ \rangle_1)\\
&\Leftrightarrow \phi([u, v]) = [\phi(u), \phi(v)] \text{ and } \langle\ ,\ \rangle_2 = \phi^*(\langle\ ,\ \rangle_1)\\
&\Leftrightarrow \phi(L_1(u \times_1 v)) = L_2(\phi(u) \times_2 \phi(v)) \text{ and } \langle\ ,\ \rangle_2 = \phi^*(\langle\ ,\ \rangle_1)\\
&\Leftrightarrow \phi(L_1(u \times_1 v)) = \det(\phi)\cdot L_2(\phi(u \times_1 v)) \text{ and } \langle\ ,\ \rangle_2 = \phi^*(\langle\ ,\ \rangle_1),
\end{aligned}$$

where the last line follows by the fact that,

$$\begin{aligned}\langle \phi(u) \times_2 \phi(v), \phi(w)\rangle_2 &= \det[\phi(u)\ \phi(v)\ \phi(w)] = \det(\phi)\det[u\ v\ w]\\ &= \det(\phi)\,\langle u \times_1 v, w\rangle_1 = \det(\phi)\,\langle \phi(u\times_1 v), \phi(w)\rangle_2,\end{aligned}$$

hence

$$\begin{aligned}&\phi \in \mathrm{Aut}(\mathfrak{g}) \text{ and } \langle\ ,\ \rangle_2 = \phi^*(\langle\ ,\ \rangle_1)\\ \Leftrightarrow\ &\phi L_1 = \det(\phi)\cdot L_2\phi \text{ and } \langle\ ,\ \rangle_2 = \phi^*(\langle\ ,\ \rangle_1)\\ \Leftrightarrow\ &L_2 = \det(\phi)\cdot \phi L_1\phi^{-1} \text{ and } \langle\ ,\ \rangle_2 = \phi^*(\langle\ ,\ \rangle_1).\end{aligned}$$

(ii)⇔(iii)
Let $\mathbb{B}_1 = (e_1, e_2, e_3)$ with e_3 timelike, and $\mathbb{B}_2 = (v_1, v_2, v_3)$ with v_3 timelike. Then

$$L_2 = \det(\phi)\cdot\phi L_1\phi^{-1} \Leftrightarrow \mathcal{M}_{\mathbb{B}_2}(L_2) = \det(\Phi)\cdot\Phi\,\mathcal{M}_{\mathbb{B}_1}(L_1)\,\Phi^{-1}, \tag{2.6}$$

where $\Phi = \mathcal{M}_{\mathbb{B}_1,\mathbb{B}_2}(\phi)$ and $\Phi^{-1} = \mathcal{M}_{\mathbb{B}_2,\mathbb{B}_1}(\phi^{-1})$.

To clarify this point, it is enough to see here that the left-hand side of equation (2.6) transforms into a matrix form with respect to the two basis $\mathbb{B}_1$, $\mathbb{B}_2$ and vice versa.

On the other hand, we set $\phi(e_j) = \sum_{i=1}^3 a_{ij}v_i$, $\forall j = 1,2,3$ and let δ_{ij} denotes the Kronecker symbol corresponding to the Minkowski metric.

$$\begin{aligned}\Phi = \mathcal{M}_{\mathbb{B}_1,\mathbb{B}_2}(\phi) \in \mathrm{O}(2,1) &\Leftrightarrow (a_{ij})_{1\leqslant i,j\leqslant 3} \in \mathrm{O}(2,1)\\ &\Leftrightarrow \left\langle \sum_{k=1}^3 a_{ki}v_k,\ \sum_{k=1}^3 a_{kj}v_k \right\rangle_2 = \delta_{ij},\ \forall i,j = 1,2,3\\ &\Leftrightarrow \langle\phi(e_i),\ \phi(e_j)\rangle_2 = \langle e_i,\ e_j\rangle_1,\ \forall i,j = 1,2,3\\ &\Leftrightarrow \langle\ ,\ \rangle_2 = \phi^*(\langle\ ,\ \rangle_1).\end{aligned}$$

□

3. Lorentzian left-invariant metrics

The aim of this section is give an overview of our main tool we employ to provide a classification of metrics for each G keeping in mind the description of main steps given in Introduction (see step 1), the structure constants of its Lie algebra in the natural basis $\mathbb{B}_{_N}$ and Definition 2.

3.1. Lorentzian left-invariant metrics on Nil

We have exactly three forms of non-equivalent symmetric endomorphisms L associated with Nil.

I– L is of type $\mathrm{diag}(0,0,c)$ with $c>0$. In the orthonormal basis $\mathbb{B}_0$ given in (1.2), we have $[e_1,e_2] = -c\,e_3$. We consider the automorphism of the Lie algebra $\rho : \mathfrak{n} \longrightarrow \mathfrak{n}$ given by

$$\rho(X) = e_1,\ \rho(Y) = e_2 \text{ and } \rho(Z) = -c\,e_3.$$

Let $\phi_0 : \mathrm{G} \longrightarrow \mathrm{G}$ be the automorphism of the Lie group associated to ρ^{-1}. The matrix of $\phi_0^*(h_0)$ in the natural basis $\mathbb{B}_{_N}$ is specified by

$$\mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \quad \lambda = c^2 > 0. \tag{nil-}$$

II– L is of type $\mathrm{diag}(a, 0, 0)$ with $a > 0$. In the orthonormal basis $\mathbb{B}_0$ given in (1.2), we have $[e_2, e_3] = a\, e_1$. We consider the automorphism of the Lie algebra $\rho : \mathfrak{n} \longrightarrow \mathfrak{n}$ given by

$$\rho(X) = e_2,\ \rho(Y) = e_3 \text{ and } \rho(Z) = a\, e_1.$$

Let ϕ_0 be the automorphism of the Lie group associated to ρ^{-1}. The matrix of $\phi_0^*(h_0)$ in the natural basis $\mathbb{B}_{_N}$ is specified by

$$\mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \lambda = a^2 > 0. \tag{nil+}$$

III– L is of type $\{\mathrm{ab2}\}$ with $\mathrm{a} = \mathrm{b} = 0$. In the orthonormal basis $\mathbb{B}_0$ given in (1.4), we have $[e_1, e_2] = [e_3, e_1] = \frac{1}{2}(e_2 + e_3)$.
We consider the automorphism of the Lie algebra $\rho : \mathfrak{n} \longrightarrow \mathfrak{n}$ given by

$$\rho(X) = e_1,\ \rho(Y) = \frac{\sqrt{2}}{2}(e_2 - e_3) \text{ and } \rho(Z) = \frac{\sqrt{2}}{2}(e_2 + e_3).$$

Let ϕ_0 be the automorphism of the Lie group associated to ρ^{-1}. The matrix of $\phi_0^*(h_0)$ in the natural basis $\mathbb{B}_{_N}$ is specified by

$$\mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{nil0}$$

Theorem 3.1. *The left-invariant Lorentzian metric on* Nil *is equivalent up to automorphism to the metric whose associated matrix in* $\mathbb{B}_{_N}$ *is exactly one of the three forms given by* (nil-), (nil+) *and* (nil0).

Remark 3.2. Theorem 3.1 confirms the results in [13], which states that up homothety, there are exactly three non-isometric metrics. These metrics are represented in matrix form in [14].

3.2. Lorentzian left-invariant metric on SU(2)

We have exactly one form of non-equivalent symmetric endomorphisms L associated with SU(2).

L is of type $\mathrm{diag}\{a, b, c\}$ with $a > 0$, $b > 0$ and $c < 0$.

Notice that we assume here that $a \geqslant b$ since $\mathrm{diag}\{a,b,c\} \equiv \mathrm{diag}\{b,a,c\}$ in view of Definition 2. In the orthonormal basis $\mathbb{B}_0$ given in (1.2), we have $[e_1,e_2] = -ce_3,\ [e_2,e_3] = ae_1$ and $[e_3,e_1] = be_2$.

We consider the automorphism of the Lie algebra $\rho : \mathfrak{su}(2) \longrightarrow \mathfrak{su}(2)$ given by

$$\rho(\sigma_x) = \frac{2}{\sqrt{-cb}}e_1 = \sqrt{\mu_1}e_1,\ \rho(\sigma_y) = \frac{2}{\sqrt{-ca}}e_2 = \sqrt{\mu_2}e_2 \text{ and}$$
$$\rho(\sigma_z) = \frac{2}{\sqrt{ab}}e_3 = \sqrt{\mu_3}e_3.$$

Let ϕ_0 be the automorphism of the Lie group associated to ρ^{-1}. The matrix of $\phi_0^*(h_0)$ in $\mathbb{B}_{_N}$ is given by

$$\text{(su)} \qquad \mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0)) = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & -\mu_3 \end{pmatrix}, \quad \mu_1 \geqslant \mu_2 > 0,\ \mu_3 > 0.$$

Theorem 3.3. *The left-invariant Lorentzian metric on* $\mathrm{SU}(2)$ *is equivalent up to automorphism to the metric whose associated matrix in* $\mathbb{B}_{_N}$ *is exactly of the form given by* (su).

3.3. Lorentzian left-invariant metrics on $\widetilde{\mathbf{PSL}}(\mathbf{2},\mathbb{R})$

Notice that the set of equivalence class representatives of symmetric endomorphisms associated with $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ admits exactly five forms.

I– L is of type $\mathrm{diag}\{a,b,c\}$ with $a>0$, $b>0$ and $c>0$, where in addition we assume that $a \geqslant b$ since $\mathrm{diag}\{a,b,c\} \equiv \mathrm{diag}\{b,a,c\}$ in view of Definition 2. In the orthonormal basis $\mathbb{B}_0$ given in (1.2), we have

$$[e_1,e_2] = -ce_3,\ [e_2,e_3] = ae_1 \text{ and } [e_3,e_1] = be_2.$$

We consider the automorphism of the Lie algebra $\rho : \mathrm{sl}(2,\mathbb{R}) \longrightarrow \mathrm{sl}(2,\mathbb{R})$ given by

$$\rho(X_1) = \frac{2}{\sqrt{ab}}e_3 = \sqrt{\mu_1}e_3,\ \rho(X_2) = \frac{2}{\sqrt{ca}}e_2 = \sqrt{\mu_2}e_2 \text{ and}$$
$$\rho(X_3) = -\frac{2}{\sqrt{cb}}e_1 = -\sqrt{\mu_3}e_1.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0. The matrix of $\phi_0^*(h_0)$ in $\mathbb{B}_{_N}$ is specified by

$$\text{(sll1)} \qquad \mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0)) = \begin{pmatrix} -\mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \mu_1 > 0,\ \mu_2 \geqslant \mu_3 > 0.$$

II– L is of type $\mathrm{diag}\{a,b,c\}$ with $a<0$, $b>0$ and $c<0$. We remember that

$$[e_1,e_2]=-ce_3,\ [e_2,e_3]=ae_1 \text{ and } [e_3,e_1]=be_2.$$

We consider the automorphism of the Lie algebra $\rho : \mathrm{sl}(2,\mathbb{R}) \longrightarrow \mathrm{sl}(2,\mathbb{R})$ given by

$$\rho(X_1)=\frac{2}{\sqrt{-cb}}e_1=\sqrt{\mu_1}e_1,\ \rho(X_2)=\frac{2}{\sqrt{-ab}}e_3=\sqrt{\mu_2}e_3 \text{ and}$$
$$\rho(X_3)=-\frac{2}{\sqrt{ac}}e_2=-\sqrt{\mu_3}e_2.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0. The matrix of $\phi_0^*(h_0)$ in $\mathbb{B}_{_N}$ is specified by

$$\mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0))=\begin{pmatrix}\mu_1 & 0 & 0\\ 0 & -\mu_2 & 0\\ 0 & 0 & \mu_3\end{pmatrix},\quad \mu_1>0,\ \mu_2>0,\ \mu_3>0. \tag{sll2}$$

III– L is of type $\{\mathrm{az}\bar{\mathrm{z}}\}$ with $\mathrm{a}\neq 0$. We remark that, from Definition 2:

$$\mathcal{M}_{\mathbb{B}_0}(L)=\begin{pmatrix}\mathrm{a} & 0 & 0\\ 0 & \alpha & -\beta\\ 0 & \beta & \alpha\end{pmatrix}\equiv\begin{pmatrix}-\mathrm{a} & 0 & 0\\ 0 & -\alpha & -\beta\\ 0 & \beta & -\alpha\end{pmatrix}\equiv\begin{pmatrix}\mathrm{a} & 0 & 0\\ 0 & \alpha & \beta\\ 0 & -\beta & \alpha\end{pmatrix},$$

where $\mathrm{z}=\alpha+\mathrm{i}\beta$.
In terms of the parameters a, α and β, this translate into $(\mathrm{a},\alpha,\beta)\equiv(-\mathrm{a},-\alpha,\beta)\equiv(\mathrm{a},\alpha,-\beta)$. We therefore assume that $\mathrm{a}>0$ and $\beta>0$. Thereby we get L is of type $\{\mathrm{a}^2z\bar{z}\}$ with $\mathrm{a}\neq 0$ and $\beta>0$.
In the orthonormal basis $\mathbb{B}_0$ given in (1.3), we have

$$[e_1,e_2]=-\beta e_2-\alpha e_3,\ [e_2,e_3]=\mathrm{a}^2e_1 \text{ and } [e_3,e_1]=\alpha e_2-\beta e_3,\ \mathrm{a}\neq 0,\ \beta>0.$$

We distinguish three cases:

i. If $\alpha>0$. We consider the automorphism of the Lie algebra $\rho : \mathrm{sl}(2,\mathbb{R}) \longrightarrow \mathrm{sl}(2,\mathbb{R})$ given by

$$\rho(X_1)=\frac{2}{\mathrm{a}\sqrt{\alpha(\alpha^2+\beta^2)}}(\beta e_2+\alpha e_3),\ \rho(X_2)=\frac{2}{\mathrm{a}\sqrt{\alpha}}e_2 \text{ and}$$
$$\rho(X_3)=-\frac{2}{\sqrt{\alpha^2+\beta^2}}e_1.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0. The matrix of $\phi_0^*(h_0)$ in $\mathbb{B}_{_N}$ is specified by

$$\mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0))=\frac{4}{\mathrm{a}^2\alpha\sqrt{\alpha^2+\beta^2}}\begin{pmatrix}\frac{\beta^2-\alpha^2}{\sqrt{\alpha^2+\beta^2}} & \beta & 0\\ \beta & \sqrt{\alpha^2+\beta^2} & 0\\ 0 & 0 & \frac{\mathrm{a}^2\alpha}{\sqrt{\alpha^2+\beta^2}}\end{pmatrix}, \tag{sll3}$$

$$\alpha > 0,\ \beta > 0.$$

ii. If $\alpha < 0$. We consider the automorphism of the Lie algebra $\rho : \mathrm{sl}(2,\mathbb{R}) \longrightarrow \mathrm{sl}(2,\mathbb{R})$ given by

$$\rho(X_1) = \frac{2}{\mathrm{a}\sqrt{|\alpha|}} e_2,\ \rho(X_2) = -\frac{2}{\sqrt{\alpha^2+\beta^2}} e_1 \text{ and}$$
$$\rho(X_3) = \frac{2}{\mathrm{a}\sqrt{|\alpha|(\alpha^2+\beta^2)}}(-\beta e_2 - \alpha e_3).$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0. The matrix of $\phi_0^*(h_0)$ in $\mathbb{B}_{_N}$ is specified by

$$\text{(sll4)}\quad \mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0)) = \frac{4}{\mathrm{a}^2\alpha\sqrt{\alpha^2+\beta^2}} \begin{pmatrix} -\sqrt{\alpha^2+\beta^2} & 0 & \beta \\ 0 & \frac{\mathrm{a}^2\alpha}{\sqrt{\alpha^2+\beta^2}} & 0 \\ \beta & 0 & \frac{\alpha^2-\beta^2}{\sqrt{\alpha^2+\beta^2}} \end{pmatrix},$$
$$\alpha < 0,\ \beta > 0.$$

iii. If $\alpha = 0$. We consider the automorphism of the Lie algebra $\rho : \mathrm{sl}(2,\mathbb{R}) \longrightarrow \mathrm{sl}(2,\mathbb{R})$ given by

$$\rho(X_1) = \frac{1}{\beta} e_2 + \frac{2}{\mathrm{a}^2} e_3,\ \rho(X_2) = \frac{2}{\beta} e_1 \text{ and } \rho(X_3) = \frac{1}{\beta} e_2 - \frac{2}{\mathrm{a}^2} e_3.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0. The matrix of $\phi_0^*(h_0)$ is given

$$\mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0)) = \frac{1}{\mathrm{a}^4\beta^2} \begin{pmatrix} \mathrm{a}^4 - 4\beta^2 & 0 & \mathrm{a}^4 + 4\beta^2 \\ 0 & 4\mathrm{a}^4 & 0 \\ \mathrm{a}^4 + 4\beta^2 & 0 & \mathrm{a}^4 - 4\beta^2 \end{pmatrix}.$$

By putting $u = \mathrm{a}^4 - 4\beta^2$ and $v = \mathrm{a}^4 + 4\beta^2$, therefore we get the desired matrix

$$\text{(sll5)}\quad \mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0)) = \frac{16}{v^2-u^2} \begin{pmatrix} u & 0 & v \\ 0 & 2(u+v) & 0 \\ v & 0 & u \end{pmatrix}, \quad v > 0,\ -v < u < v.$$

IV– L is of type $\{\mathrm{ab2}\}$ with $\mathrm{a} \neq 0$ and $\mathrm{b} \neq 0$. In the orthonormal basis $\mathbb{B}_0$ given in (1.4), we have

$$[e_1, e_2] = \frac{1}{2} e_2 + \left(\frac{1}{2} - \mathrm{b}\right) e_3,\ [e_2, e_3] = \mathrm{a} e_1 \text{ and } [e_3, e_1] = \left(\mathrm{b} + \frac{1}{2}\right) e_2 + \frac{1}{2} e_3.$$

We consider the automorphism of the Lie algebra $\rho : \mathrm{sl}(2,\mathbb{R}) \longrightarrow \mathrm{sl}(2,\mathbb{R})$ given by

$$\rho(X_1) = \left(\frac{1}{2} + \frac{1}{4\mathrm{b}} - \frac{2}{\mathrm{ab}}\right) e_2 + \left(-\frac{1}{2} + \frac{1}{4\mathrm{b}} - \frac{2}{\mathrm{ab}}\right) e_3,$$

$$\rho(X_2)=\left(-\frac{1}{2}-\frac{1}{4\mathrm{b}}-\frac{2}{\mathrm{ab}}\right)e_2+\left(\frac{1}{2}-\frac{1}{4\mathrm{b}}-\frac{2}{\mathrm{ab}}\right)e_3,$$
$$\rho(X_3)=-\frac{2}{\mathrm{b}}e_1.$$

Let ϕ_0 be the automorphism of the Lie group associated to ρ^{-1}. The matrix of $\phi_0^*(h_0)$ in $\mathbb{B}_{_N}$ is specified by

$$\mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0))=\frac{1}{2\mathrm{ab}}\begin{pmatrix}\mathrm{a}-8 & -\mathrm{a} & 0\\ -\mathrm{a} & \mathrm{a}+8 & 0\\ 0 & 0 & \frac{8\mathrm{a}}{\mathrm{b}}\end{pmatrix},\quad \mathrm{a}\neq 0,\mathrm{b}\neq 0. \tag{sll6}$$

V− L is of type {a3} with $\mathrm{a}\neq 0$. We observe directly from Definition 2 that

$$\mathcal{M}_{\mathbb{B}_0}(L)=\begin{pmatrix}\mathrm{a} & \sqrt{2}/2 & 0\\ \sqrt{2}/2 & \mathrm{a} & -\sqrt{2}/2\\ 0 & \sqrt{2}/2 & \mathrm{a}\end{pmatrix}\equiv\begin{pmatrix}-\mathrm{a} & \sqrt{2}/2 & 0\\ \sqrt{2}/2 & -\mathrm{a} & -\sqrt{2}/2\\ 0 & \sqrt{2}/2 & -\mathrm{a}\end{pmatrix}$$

which we can write as $\mathrm{a}\equiv(-\mathrm{a})$. From this point, we assume below that L is of type {a3} with $\mathrm{a}>0$.
In the orthonormal basis $\mathbb{B}_0$ given in (1.5), we have

$$[e_1,e_2]=\frac{1}{\sqrt{2}}e_2-\mathrm{a}e_3,\ [e_2,e_3]=\mathrm{a}e_1+\frac{1}{\sqrt{2}}e_2\ \text{ and }\ [e_3,e_1]=\frac{1}{\sqrt{2}}e_1+\mathrm{a}e_2+\frac{1}{\sqrt{2}}e_3.$$

We consider the automorphism of the Lie algebra $\rho:\mathrm{sl}(2,\mathbb{R})\longrightarrow\mathrm{sl}(2,\mathbb{R})$ given by

$$\rho(X_1)=\frac{1}{\mathrm{a}^2}\left(\sqrt{2}e_2-2\mathrm{a}e_3\right),\ \rho(X_2)=\frac{1}{\mathrm{a}^2\sqrt{2\mathrm{a}^2+1}}\left(2\mathrm{a}e_1+\sqrt{2}(2\mathrm{a}^2+1)e_2\right),$$
$$\rho(X_3)=\frac{2\sqrt{2}}{\sqrt{2\mathrm{a}^2+1}}e_1.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0. The matrix of $\phi_0^*(h_0)$ in $\mathbb{B}_{_N}$ is specified by

$$\mathcal{M}_{\mathbb{B}_{_N}}(\phi_0^*(h_0))=\frac{2}{\mathrm{a}^4(1+2\mathrm{a}^2)}\begin{pmatrix}1-4\mathrm{a}^4 & (1+2\mathrm{a}^2)^{\frac{3}{2}} & 0\\ (1+2\mathrm{a}^2)^{\frac{3}{2}} & 4\mathrm{a}^4+6\mathrm{a}^2+1 & 2\mathrm{a}^3\sqrt{2}\\ 0 & 2\mathrm{a}^3\sqrt{2} & 4\mathrm{a}^4\end{pmatrix},\quad \mathrm{a}>0. \tag{sll7}$$

Theorem 3.4. *The left-invariant Lorentzian metric on* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ *is equivalent up to automorphism to the metric whose associated matrix in* $\mathbb{B}_{_N}$ *is exactly one of the seven forms given by* (sll1),...,(sll7).

3.4. Lorentzian left-invariant metrics on Sol

We have five possibilities:

I– L is of type $\mathrm{diag}(a,b,0)$ with $a>0$, $b<0$. In the orthonormal basis $\mathbb{B}_0$ given in (1.2), we have

$$[e_1,e_2]=0,\ [e_2,e_3]=ae_1 \text{ and } [e_3,e_1]=be_2,\quad a>0,\, b<0.$$

We consider the automorphism of the Lie algebra $\rho:\mathfrak{sol}\longrightarrow\mathfrak{sol}$ given by

$$\rho(X_1)=-\frac{1}{\sqrt{-ab}}e_3,\ \rho(X_2)=e_1-\frac{b}{\sqrt{-ab}}e_2 \text{ and } \rho(X_3)=e_1+\frac{b}{\sqrt{-ab}}e_2.$$

Let ϕ_0 be the automorphism of the Lie group associated to ρ^{-1} and we put $h_1=\phi_0^*(h_0)$. We have

$$\mathcal{M}_{\mathbb{B}_N}(h_1)=\frac{1}{a}\begin{pmatrix}\frac{1}{b}&0&0\\0&a-b&a+b\\0&a+b&a-b\end{pmatrix}.$$

We can reduce this metric by considering the automorphism of the Lie algebra $Q:\mathfrak{sol}\longrightarrow\mathfrak{sol}$ given by

$$\mathcal{M}_{\mathbb{B}_N}(Q)=\begin{pmatrix}-1&0&0\\0&0&\sqrt{\frac{a}{a-b}}\\0&\sqrt{\frac{a}{a-b}}&0\end{pmatrix}.$$

Consider ϕ_1 the automorphism of Sol associated to Q^{-1}. The matrix of $\phi_1^*(h_1)$ in $\mathbb{B}_N$ is specified by

$$\mathcal{M}_{\mathbb{B}_N}(\phi_1^*(h_1))=\mathcal{M}_{\mathbb{B}_N}(Q)^T\,\mathcal{M}_{\mathbb{B}_N}(h_1)\,\mathcal{M}_{\mathbb{B}_N}(Q)=\begin{pmatrix}\frac{1}{ab}&0&0\\0&1&\frac{a+b}{a-b}\\0&\frac{a+b}{a-b}&1\end{pmatrix}.$$

We put $u=a+b$ and $v=a-b$, therefore we get the desired matrix

$$\mathcal{M}_{\mathbb{B}_N}(\phi_1^*(h_1))=\begin{pmatrix}\frac{4}{u^2-v^2}&0&0\\0&1&\frac{u}{v}\\0&\frac{u}{v}&1\end{pmatrix},\quad v>0,\,-v<u<v. \tag{sol1}$$

II– L is of type $\mathrm{diag}(a,0,b)$ with $a>0$ and $b>0$. In the orthonormal basis $\mathbb{B}_0$ given in (1.2), we have

$$[e_1,e_2]=ae_3,\ [e_2,e_3]=be_1 \text{ and } [e_3,e_1]=0,\ a<0,b>0.$$

We consider the automorphism of the Lie algebra $\rho:\mathfrak{sol}\longrightarrow\mathfrak{sol}$ given by

$$\rho(X_1)=-\frac{1}{\sqrt{-ab}}e_2,\ \rho(X_2)=e_3-\frac{b}{\sqrt{-ab}}e_1 \text{ and } \rho(X_3)=e_3+\frac{b}{\sqrt{-ab}}e_1.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0 and we put $h_1 = \phi_0^*(h_0)$. We have

$$\mathcal{M}_{\mathbb{B}_N}(h_1) = \frac{1}{a}\begin{pmatrix} -\frac{1}{b} & 0 & 0 \\ 0 & -a-b & -a+b \\ 0 & -a+b & -a-b \end{pmatrix}.$$

We can reduce this metric by considering the automorphism of the Lie algebra $Q : \mathfrak{sol} \longrightarrow \mathfrak{sol}$ given by

$$\mathcal{M}_{\mathbb{B}_N}(Q) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{a}{a-b}} \\ 0 & \sqrt{\frac{a}{a-b}} & 0 \end{pmatrix}.$$

Consider ϕ_1 the automorphism of Sol associated to Q^{-1}. The matrix of $\phi_1^*(h_1)$ in $\mathbb{B}_N$ is specified by

$$\mathcal{M}_{\mathbb{B}_N}(\phi_1^*(h_1)) = \mathcal{M}_{\mathbb{B}_N}(Q)^T\, \mathcal{M}_{\mathbb{B}_N}(h_1)\, \mathcal{M}_{\mathbb{B}_N}(Q) = \begin{pmatrix} -\frac{1}{ab} & 0 & 0 \\ 0 & \frac{a+b}{b-a} & -1 \\ 0 & -1 & \frac{a+b}{b-a} \end{pmatrix}.$$

We put $u = a+b$ and $v = b-a$ and therefore we get the desired matrix

$$\mathcal{M}_{\mathbb{B}_N}(\phi_1^*(h_1)) = \begin{pmatrix} \frac{4}{v^2-u^2} & 0 & 0 \\ 0 & \frac{u}{v} & -1 \\ 0 & -1 & \frac{u}{v} \end{pmatrix}, \quad v > 0,\ -v < u < v. \tag{sol2}$$

III– L is of type $\{\mathrm{az\bar{z}}\}$ with $\mathrm{a} = 0$. A same argument as was used in item III– for $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ in Subsection 3.3 shows that $\alpha > 0$ and $\beta > 0$.
In the orthonormal basis $\mathbb{B}_0$ given in (1.3), we have

$$[e_1, e_2] = -\beta e_2 - \alpha e_3,\ [e_2, e_3] = 0 \text{ and } [e_3, e_1] = \alpha e_2 - \beta e_3, \quad \alpha > 0,\ \beta > 0.$$

We distinguish two cases:

i. If $\alpha \neq 0$. We consider the automorphism of the Lie algebra $\rho : \mathfrak{sol} \longrightarrow \mathfrak{sol}$ given by

$$\rho(X_1) = \frac{1}{\sqrt{\alpha^2+\beta^2}} e_1,\ \rho(X_2) = e_3 + \frac{\beta - \sqrt{\alpha^2+\beta^2}}{\alpha} e_2 \text{ and}$$
$$\rho(X_3) = e_3 + \frac{\beta + \sqrt{\alpha^2+\beta^2}}{\alpha} e_2.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0 and we put $h_1 = \phi_0^*(h_0)$. We have

$$\mathcal{M}_{\mathbb{B}_N}(h_1) = \begin{pmatrix} \frac{1}{\alpha^2+\beta^2} & 0 & 0 \\ 0 & -\frac{2\beta(-\beta+\sqrt{\alpha^2+\beta^2})}{\alpha^2} & -2 \\ 0 & -2 & \frac{2\beta(\beta+\sqrt{\alpha^2+\beta^2})}{\alpha^2} \end{pmatrix}.$$

We can reduce this metric by considering the automorphism of the Lie algebra $Q : \mathfrak{sol} \longrightarrow \mathfrak{sol}$ given by

$$\mathcal{M}_{\mathbb{B}_N}(Q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}\sqrt{\beta(\beta+\sqrt{\beta^2+\alpha^2})}}{2\alpha} & 0 \\ 0 & 0 & \frac{\alpha\sqrt{2}}{2\sqrt{\beta(\beta+\sqrt{\beta^2+\alpha^2})}} \end{pmatrix}.$$

Consider ϕ_1 the automorphism of Sol associated to Q^{-1} and we lead to the following relation.

$$\mathcal{M}_{\mathbb{B}_N}(\phi_1^*(h_1)) = \begin{pmatrix} \frac{1}{u+v} & 0 & 0 \\ 0 & -\frac{v}{u} & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad u = \alpha^2 > 0,\ v = \beta^2 > 0. \tag{sol3}$$

ii. If $\alpha = 0$. Then we take the automorphism of the Lie algebra $\rho : \mathfrak{sol} \longrightarrow \mathfrak{sol}$ given by

$$\rho(X_1) = \frac{1}{\beta}e_1,\ \rho(X_2) = e_3,\ \text{and}\ \rho(X_3) = e_2 \quad \text{with}\ \beta > 0.$$

Let ϕ_0 be the automorphism of the Lie group associated to ρ^{-1}. We have

$$\mathcal{M}_{\mathbb{B}_N}(\phi_0^*(h_0)) = \begin{pmatrix} \frac{1}{u} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ u = \beta^2 > 0. \tag{sol4}$$

IV– L is of type {ab2} with $\mathrm{a} < 0$ and $\mathrm{b} = 0$. In the orthonormal basis $\mathbb{B}_0$ given in (1.4), we have

$$[e_1, e_2] = \frac{1}{2}e_2 + \frac{1}{2}e_3,\ [e_2, e_3] = -\mathrm{a}e_1\ \text{and}\ [e_3, e_1] = \frac{1}{2}e_2 + \frac{1}{2}e_3\ \text{ with } \mathrm{a} > 0$$

We consider the automorphism of the Lie algebra $\rho : \mathfrak{sol} \longrightarrow \mathfrak{sol}$ given by

$$\rho(X_1) = \frac{\sqrt{2\mathrm{a}}}{\mathrm{a}}\, e_3,\ \rho(X_2) = \sqrt{2\mathrm{a}}\, e_1 + e_2 + e_3\ \text{and}\ \rho(X_3) = -\sqrt{2\mathrm{a}}\, e_1 + e_2 + e_3.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0 and we put $h_1 = \phi_0^*(h_0)$. We have

$$\mathcal{M}_{\mathbb{B}_N}(h_1) = \frac{1}{\mathrm{a}}\begin{pmatrix} -2 & -\sqrt{2\mathrm{a}} & -\sqrt{2\mathrm{a}} \\ -\sqrt{2\mathrm{a}} & 2\,\mathrm{a}^2 & -2\,\mathrm{a}^2 \\ -\sqrt{2\mathrm{a}} & -2\,\mathrm{a}^2 & 2\,\mathrm{a}^2 \end{pmatrix},\ \mathrm{a} > 0.$$

We can reduce this metric by considering the automorphism of the Lie algebra $Q : \mathfrak{sol} \longrightarrow \mathfrak{sol}$ given by

$$\mathcal{M}_{\mathbb{B}_N}(Q) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\sqrt{2}(3-2\,\mathrm{a})}{8\mathrm{a}\sqrt{\mathrm{a}}} & \frac{-\sqrt{2\mathrm{a}}}{2\mathrm{a}} & 0 \\ \frac{-\sqrt{2}(2\,\mathrm{a}+1)}{8\mathrm{a}\sqrt{\mathrm{a}}} & 0 & \frac{\sqrt{2\mathrm{a}}}{2\mathrm{a}} \end{pmatrix}.$$

Consider ϕ_1 the automorphism of Sol associated to Q^{-1}. The matrix of $\phi_1^*(h_1)$ in $\mathbb{B}_N$ is specified by

$$\mathcal{M}_{\mathbb{B}_N}(\phi_1^*(h_1)) = \begin{pmatrix} 0 & 0 & -\frac{2}{\mathrm{a}} \\ 0 & 1 & 1 \\ -\frac{2}{\mathrm{a}} & 1 & 1 \end{pmatrix}, \quad \mathrm{a} > 0. \tag{sol5}$$

V– L is of type {ab2} with $\mathrm{a} = 0$ and $\mathrm{b} \neq 0$. In the orthonormal basis $\mathbb{B}_0$ given in (1.4), we have

$$[e_1, e_2] = \frac{1}{2}e_2 + \left(\frac{1}{2} - \mathrm{b}\right)e_3,\ [e_2, e_3] = 0 \text{ and } [e_3, e_1] = \left(\mathrm{b} + \frac{1}{2}\right)e_2 + \frac{1}{2}e_3.$$

We set the automorphism of the Lie algebra $\rho : \mathfrak{sol} \longrightarrow \mathfrak{sol}$ to be:

$$\rho(X_1) = \frac{1}{\mathrm{b}}e_1,\ \rho(X_2) = (2\mathrm{b}+1)e_2 + (1-2\mathrm{b})e_3 \text{ and } \rho(X_3) = e_2 + e_3.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0 and we put $h_1 = \phi_0^*(h_0)$. We have

$$\mathcal{M}_{\mathbb{B}_N}(h_1) = \begin{pmatrix} \frac{1}{\mathrm{b}^2} & 0 & 0 \\ 0 & 8\mathrm{b} & 4\mathrm{b} \\ 0 & 4\mathrm{b} & 0 \end{pmatrix}.$$

Taking the following automorphism of the Lie algebra $Q : \mathfrak{sol} \longrightarrow \mathfrak{sol}$

$$\mathcal{M}_{\mathbb{B}_N}(Q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4\mathrm{b}} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

We can reduce h_1 to h_2 in the following way. Consider ϕ_1 the automorphism of Sol associated to Q^{-1}.

The matrix of $\phi_1^*(h_1)$ in $\mathbb{B}_{_N}$ is specified by

$$\mathcal{M}_{\mathbb{B}_{_N}}(\phi_1^*(h_1)) = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda = \frac{1}{\mathrm{b}} \neq 0. \tag{sol6}$$

VI– L is of type {a3} with a = 0. In the orthonormal basis $\mathbb{B}_0$ given in (1.5), we have

$$[e_1, e_2] = \frac{1}{\sqrt{2}} e_2,\ [e_2, e_3] = \frac{1}{\sqrt{2}} e_2 \text{ and } [e_3, e_1] = \frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_3.$$

We can then define the automorphism of the Lie algebra $\rho : \mathfrak{sol} \longrightarrow \mathfrak{sol}$ as

$$\rho(X_1) = \sqrt{2} e_1,\ \rho(X_2) = e_2 \text{ and } \rho(X_3) = e_3 + e_1.$$

Let ϕ_0 be the automorphism of the Lie group associated to ρ^{-1} and we put $h_1 = \phi_0^*(h_0)$. We have

$$\mathcal{M}_{\mathbb{B}_{_N}}(h_1) = \begin{pmatrix} 2 & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}.$$

We can reduce this metric by considering the automorphism of the Lie algebra $Q : \mathfrak{sol} \longrightarrow \mathfrak{sol}$ given by

$$\mathcal{M}_{\mathbb{B}_{_N}}(Q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Consider ϕ_1 the automorphism of Sol associated to Q^{-1}. The matrix of $\phi_1^*(h_1)$ in $\mathbb{B}_{_N}$ is specified by

$$\mathcal{M}_{\mathbb{B}_{_N}}(\phi_1^*(h_1)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{sol7}$$

Theorem 3.5. *The left-invariant Lorentzian metric on* Sol *is equivalent up to automorphism to the metric whose associated matrix in* $\mathbb{B}_{_N}$ *is exactly one of the seven forms given by* (sol1),...,(sol7).

Remark 3.6. The metric (sol7) defines in [6] a maximum Lorentzian and non-Riemannian geometry designated by *Lorentz-SOL*.

3.5. Lorentzian left-invariant metrics on $\widetilde{\mathbf{E}_0}(2)$

There are three possibilities:

I– L is of type $\mathrm{diag}(a,b,0)$ with $a>0$ and $b>0$. Applying the relation $\mathrm{diag}(a,b,0)\equiv\mathrm{diag}(b,a,0)$, we further deduce that $a\geqslant b$.
In the orthonormal basis $\mathbb{B}_0$ given in (1.2), we have

$$[e_1,e_2]=0,\ [e_2,e_3]=ae_1 \text{ and } [e_3,e_1]=be_2,\quad a\geqslant b>0.$$

As usual we can then define the automorphism of the Lie algebra $\rho:\mathrm{e}_0(2)\longrightarrow\mathrm{e}_0(2)$ as

$$\rho(X_1)=\frac{1}{\sqrt{ab}}e_3,\ \rho(X_2)=e_1-\frac{b}{\sqrt{ab}}e_2 \text{ and } \rho(X_3)=e_1+\frac{b}{\sqrt{ab}}e_2.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0 and we put $h_1=\phi_0^*(h_0)$. We have

$$\mathcal{M}_{\mathbb{B}_N}(h_1)=\frac{1}{a}\begin{pmatrix}-\frac{1}{b}&0&0\\0&a+b&a-b\\0&a-b&a+b\end{pmatrix},\ a\geqslant b>0.$$

We can reduce this metric by considering the automorphism of the Lie algebra $Q:\mathrm{e}_0(2)\longrightarrow\mathrm{e}_0(2)$ given by

$$\mathcal{M}_{\mathbb{B}_N}(Q)=\begin{pmatrix}1&0&0\\\frac{1}{2\sqrt{ab}}&\frac{\sqrt{ab}}{2}&-\frac{\sqrt{ab}}{2}\\\frac{1}{2\sqrt{ab}}&\frac{\sqrt{ab}}{2}&\frac{\sqrt{ab}}{2}\end{pmatrix}.$$

Consider ϕ_1 the automorphism of $\widetilde{\mathrm{E}_0}(2)$ associated to Q^{-1}. The matrix of $\phi_1^*(h_1)$ in $\mathbb{B}_N$ is specified by

$$\mathcal{M}_{\mathbb{B}_N}(\phi_1^*(h_1))=\begin{pmatrix}0&1&0\\1&u&0\\0&0&v\end{pmatrix},\quad u=ab\geqslant v=b^2>0. \tag{ee1}$$

II– L is of type $\mathrm{diag}(a,0,b)$ with $a>0$ and $b<0$. In the orthonormal basis $\mathbb{B}_0$ given in (1.2), we have

$$[e_1,e_2]=ae_3,\ [e_2,e_3]=be_1 \text{ and } [e_3,e_1]=0,\ a>0,b>0.$$

We consider the automorphism of the Lie algebra $\rho:\mathrm{e}_0(2)\longrightarrow\mathrm{e}_0(2)$ given by

$$\rho(X_1)=\frac{1}{\sqrt{ab}}e_2,\ \rho(X_2)=e_3-\frac{b}{\sqrt{ab}}e_1 \text{ and } \rho(X_3)=e_3+\frac{b}{\sqrt{ab}}e_1.$$

ρ^{-1} induces an automorphism of the Lie group ϕ_0 and we put $h_1=\phi_0^*(h_0)$. We have

$$\mathcal{M}_{\mathbb{B}_N}(h_1)=\frac{1}{a}\begin{pmatrix}\frac{1}{b}&0&0\\0&-a+b&-a-b\\0&-a-b&-a+b\end{pmatrix}.$$

We can reduce this metric by considering the automorphism of the Lie algebra $Q : \mathrm{e}_0(2) \longrightarrow \mathrm{e}_0(2)$ given by

$$\mathcal{M}_{\mathbb{B}_N}(Q) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2\sqrt{ab}} & \frac{\sqrt{ab}}{2} & -\frac{\sqrt{ab}}{2} \\ \frac{1}{2\sqrt{ab}} & \frac{\sqrt{ab}}{2} & \frac{\sqrt{ab}}{2} \end{pmatrix}.$$

Consider ϕ_1 the automorphism of $\widetilde{\mathrm{E}_0}(2)$ associated to Q^{-1}. The matrix of $\phi_1^*(h_1)$ in $\mathbb{B}_N$ is specified by

$$\mathcal{M}_{\mathbb{B}_N}(\phi_1^*(h_1)) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -u & 0 \\ 0 & 0 & v \end{pmatrix}, \quad u = ab > 0,\ v = b^2 > 0. \tag{ee2}$$

III– L is of type $\{\mathrm{ab2}\}$ with $\mathrm{a} > 0$ and $\mathrm{b} = 0$. In the orthonormal basis $\mathbb{B}_0$ given in (1.4), we have

$$[e_1, e_2] = \frac{1}{2}e_2 + \frac{1}{2}e_3,\ [e_2, e_3] = \mathrm{a}e_1 \text{ and } [e_3, e_1] = \frac{1}{2}e_2 + \frac{1}{2}e_3.$$

We consider the automorphism of the Lie algebra $\rho : \mathrm{e}_0(2) \longrightarrow \mathrm{e}_0(2)$ given by

$$\rho(X_1) = \frac{\sqrt{2\mathrm{a}}}{\mathrm{a}}e_3,\ \rho(X_2) = \sqrt{2\mathrm{a}}e_1 + e_2 + e_3 \text{ and } \rho(X_3) = -\sqrt{2\mathrm{a}}e_1 + e_2 + e_3.$$

Let ϕ_0 be the automorphism of Lie group associated to ρ^{-1} and we put $h_1 = \phi_0^*(h_0)$. We have

$$\mathcal{M}_{\mathbb{B}_N}(h_1) = -\frac{1}{\mathrm{a}}\begin{pmatrix} 2 & \sqrt{2\mathrm{a}} & \sqrt{2\mathrm{a}} \\ \sqrt{2\mathrm{a}} & -2\mathrm{a}^2 & 2\mathrm{a}^2 \\ \sqrt{2\mathrm{a}} & 2\mathrm{a}^2 & -2\mathrm{a}^2 \end{pmatrix}.$$

We can reduce this metric by considering the automorphism of the Lie algebra $Q : \mathrm{e}_0(2) \longrightarrow \mathrm{e}_0(2)$ given by

$$\mathcal{M}_{\mathbb{B}_N}(Q) = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\sqrt{2\mathrm{a}}}{4\mathrm{a}} & -\frac{\sqrt{2\mathrm{a}}}{4} & -\frac{\sqrt{2\mathrm{a}}}{4} \\ -\frac{\sqrt{2\mathrm{a}}}{4\mathrm{a}} & -\frac{\sqrt{2\mathrm{a}}}{4} & -\frac{\sqrt{2\mathrm{a}}}{4} \end{pmatrix}.$$

Consider ϕ_1 the automorphism of $\widetilde{\mathrm{E}_0}(2)$ associated to Q^{-1}. The matrix of $\phi_1^*(h_1)$ in $\mathbb{B}_N$ is specified by

$$\mathcal{M}_{\mathbb{B}_N}(\phi_1^*(h_1)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & u \end{pmatrix}, \quad u = \mathrm{a}^2 > 0. \tag{ee3}$$

Theorem 3.7. *The left-invariant Lorentzian metric on* $\widetilde{\mathrm{E}_0}(2)$ *is equivalent up to automorphism to the metric whose associated matrix in* $\mathbb{B}_{_N}$ *is exactly one of the three forms given by* (ee1), (ee2) *and* (ee3).

4. Curvature of Lorentzian left-invariant metrics

In this section, we give for each metric whose matrix is given by one of the formulas (nil-), ..., (ee3) its Ricci tensor, its signature and the scalar curvature.

4.1. Curvature of Lorentzian left-invariant metrics on Nil

There are three classes of metrics on Nil given by the formulas (nil-), (nil+) and (nil0). Here are their Ricci curvature and scalar curvature.

Proposition 4.1. (1) *The Ricci curvature and the scalar curvature of the metric* (nil-) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_{_N}}(\mathrm{ric}) = \frac{1}{2}\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda^2\end{pmatrix}, \quad \mathfrak{s} = \frac{1}{2}\lambda.$$

In particular, $\mathrm{ric} > 0$ *and* $\mathfrak{s} > 0$.

(2) *The Ricci curvature and the scalar curvature of the metric* (nil+) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_{_N}}(\mathrm{ric}) = \frac{1}{2}\begin{pmatrix}\lambda & 0 & 0\\ 0 & -\lambda & 0\\ 0 & 0 & -\lambda^2\end{pmatrix}, \quad \mathfrak{s} = \frac{1}{2}\lambda.$$

The signature of ric *is* $(+,-,-)$ *and* $\mathfrak{s} > 0$.

(3) *The metric* (nil0) *is flat.*

4.2. Curvature of Lorentzian left-invariant metrics on SU(2)

There is one class of metrics on SU(2) given by the formula (su). Here is its Ricci curvature and scalar curvature.

Proposition 4.2. *The Ricci curvature and the scalar curvature of the metric on* SU(2) *given by* (su) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_{_N}}(\mathrm{ric}) = \mathrm{diag}\left[\sigma_1,\sigma_2,\sigma_3\right], \text{ where } \sigma_1 = -\tfrac{2(\mu_1-\mu_2-\mu_3)(\mu_1+\mu_2+\mu_3)}{\mu_2\mu_3},$$
$$\sigma_2 = \tfrac{2(\mu_1+\mu_2+\mu_3)(\mu_1-\mu_2+\mu_3)}{\mu_1\mu_3}, \text{ and } \sigma_3 = -\tfrac{2(\mu_1-\mu_2-\mu_3)(\mu_1-\mu_2+\mu_3)}{\mu_1\mu_2}$$
$$\mathfrak{s} = \tfrac{2((\sqrt{\mu_1}+\sqrt{\mu_2})^2+\mu_3)((\sqrt{\mu_1}-\sqrt{\mu_2})^2+\mu_3)}{\mu_1\mu_2\mu_3} > 0.$$

Moreover, the signature of ric *is given by*

$$\mathrm{sign}(\mathrm{ric}) = \begin{cases}(+,+,+) & \text{if} \quad \mu_1 < \mu_2+\mu_3,\\ (+,-,-) & \text{if} \quad \mu_1 > \mu_2+\mu_3,\\ (+,0,0) & \text{if} \quad \mu_1 = \mu_2+\mu_3.\end{cases}$$

4.3. Curvature of Lorentzian left-invariant metrics on $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$

There are seven classes of metrics on $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ given by the formulas (sll1), ..., (sll7). Here are their Ricci curvature and scalar curvature.

Proposition 4.3. *The Ricci curvature and the scalar curvature of the metric* (sll1) *on* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \mathrm{diag}\left[\frac{2(\mu_1^2-(\mu_2-\mu_3)^2)}{\mu_2\mu_3}, -\frac{2(\mu_2^2-(\mu_1-\mu_3)^2)}{\mu_1\mu_3}, -\frac{2(\mu_3^2-(\mu_2-\mu_1)^2)}{\mu_1\mu_2}\right],$$
$$\mathfrak{s} = \frac{2[(\sqrt{\mu_1}+\sqrt{\mu_2})^2-\mu_3][(\sqrt{\mu_1}-\sqrt{\mu_2})^2-\mu_3]}{\mu_1\mu_2\mu_3}.$$

When $\mu_1=\mu_2=\mu_3=\mu$ *then* $\mathrm{ric} = -\frac{2}{\mu}h$ *and in fact the metric has constant sectional curvature* $-\frac{1}{\mu}$. *Moreover, the signature of* ric *is given by*

$$\mathrm{sign}(\mathrm{ric}) = \begin{cases} (+,+,+) & \text{if} \quad \mu_3 < \mu_1-\mu_2, \\ (+,0,0) & \text{if} \quad \mu_3 = \mu_1-\mu_2, \\ (+,-,-) & \text{if} \quad \mu_3 > \mu_1-\mu_2,\ \mu_1 \neq \mu_2-\mu_3, \\ (-,0,0) & \text{if} \quad \mu_1 = \mu_2-\mu_3. \end{cases}$$

Proposition 4.4. *The Ricci curvature and the scalar curvature of the metric* (sll2) *on* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \mathrm{diag}\left[\sigma_1,\sigma_2,\sigma_3\right], \text{ where } \sigma_1 = -\tfrac{2(\mu_1-\mu_2-\mu_3)(\mu_1+\mu_2+\mu_3)}{\mu_2\mu_3},$$
$$\sigma_2 = -\tfrac{2(\mu_1+\mu_2+\mu_3)(\mu_1-\mu_2+\mu_3)}{\mu_1\mu_3}, \text{ and } \sigma_3 = \tfrac{2(\mu_1-\mu_2-\mu_3)(\mu_1-\mu_2+\mu_3)}{\mu_1\mu_2}$$
$$\mathfrak{s} = \tfrac{2((\sqrt{\mu_1}+\sqrt{\mu_2})^2+\mu_3)((\sqrt{\mu_1}-\sqrt{\mu_2})^2+\mu_3)}{\mu_1\mu_2\mu_3} > 0.$$

Moreover, the signature of ric *is given by*

$$\mathrm{sign}(\mathrm{ric}) = \begin{cases} (+,+,+) & \text{if} \quad \mu_1 < \mu_2-\mu_3, \\ (+,0,0) & \text{if} \quad \mu_1 = \mu_2-\mu_3, \\ (+,-,-) & \text{if} \quad \mu_1 > \mu_2-\mu_3,\ \mu_1 \neq \mu_2+\mu_3, \\ (-,0,0) & \text{if} \quad \mu_1 = \mu_2+\mu_3. \end{cases}$$

Proposition 4.5. *The Ricci curvature and the scalar curvature of the metric* (sll3) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} \frac{2(\mathrm{a}^2-2\alpha)(\mathrm{a}^2(\beta^2-\alpha^2)+4\alpha\beta^2)}{\mathrm{a}^2\alpha(\alpha^2+\beta^2)} & \frac{2(\mathrm{a}^4-4\alpha^2)\beta}{\mathrm{a}^2\alpha\sqrt{\alpha^2+\beta^2}} & 0 \\ \frac{2(\mathrm{a}^4-4\alpha^2)\beta}{\mathrm{a}^2\alpha\sqrt{\alpha^2+\beta^2}} & \frac{2(\mathrm{a}^2-2\alpha)}{\alpha} & 0 \\ 0 & 0 & -\frac{2(\mathrm{a}^4+4\beta^2)}{\alpha^2+\beta^2} \end{pmatrix},$$
$$\mathfrak{s} = \frac{1}{2}\mathrm{a}^4 - 2\mathrm{a}^2\alpha - 2\beta^2.$$

Moreover $\det(\mathrm{ric}) = 8\,\frac{\left(a^4+4\,\beta^2\right)^2\left(a^2-2\,\alpha\right)^2}{a^4(\alpha^2+\beta^2)^2}$, *which is strictly positive if* $a^2 \neq 2\,\alpha$. *Since* $-2\,\frac{a^4+4\,\beta^2}{\alpha^2+\beta^2} < 0$, *then the signature of* ric *is* $(+,-,-)$ *if* $a^2 \neq 2\,\alpha$ *and*

$(-,0,0)$ *otherwise. The operator of Ricci is of type* $\{\mathrm{az\bar{z}}\}$ *if* $\mathrm{a}^2 \neq 2\alpha$ *and otherwise is diagonalizable.*

Proposition 4.6. *The Ricci curvature and the scalar curvature of the metric* (sll4) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} \frac{2(2\alpha-\mathrm{a}^2)}{\alpha} & 0 & \frac{2(\mathrm{a}^4-4\alpha^2)\beta}{\mathrm{a}^2\alpha\sqrt{\alpha^2+\beta^2}} \\ 0 & -\frac{2(\mathrm{a}^4+4\beta^2)}{\alpha^2+\beta^2} & 0 \\ \frac{2(\mathrm{a}^4-4\alpha^2)\beta}{\mathrm{a}^2\alpha\sqrt{\alpha^2+\beta^2}} & 0 & -\frac{2(\mathrm{a}^2-2\alpha)(\mathrm{a}^2(\beta^2-\alpha^2)+4\alpha\beta^2)}{\mathrm{a}^2\alpha(\alpha^2+\beta^2)} \end{pmatrix},$$
$$\mathfrak{s} = \frac{1}{2}\mathrm{a}^4 - 2\mathrm{a}^2\alpha - 2\beta^2.$$

The signature of ric *is* $(+,-,-)$. *The operator of Ricci is of type* $\{\mathrm{az\bar{z}}\}$.

Proposition 4.7. *The Ricci curvature and the scalar curvature of the metric* (sll5) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} \frac{4v}{v-u} & 0 & \frac{4v}{v-u} \\ 0 & \frac{16v}{u-v} & 0 \\ \frac{4v}{v-u} & 0 & \frac{4(v-2u)}{u-v} \end{pmatrix}, \quad \mathfrak{s} = \frac{u}{2}.$$

The signature of ric *is* $(+,-,-)$.

Proposition 4.8. *The Ricci curvature and the scalar curvature of the metric* (sll6) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \frac{1}{4\mathrm{b}}\begin{pmatrix} (\mathrm{a}+2\mathrm{b}-8)(\mathrm{a}-2\mathrm{b}) & 4\mathrm{b}^2-\mathrm{a}^2 & 0 \\ 4\mathrm{b}^2-\mathrm{a}^2 & (\mathrm{a}+2\mathrm{b}+8)(\mathrm{a}-2\mathrm{b}) & 0 \\ 0 & 0 & -\frac{8\mathrm{a}^2}{\mathrm{b}} \end{pmatrix},$$
$$\mathfrak{s} = \frac{1}{2}\mathrm{a}(\mathrm{a}-4\mathrm{b}).$$

The Ricci curvature has signature $(+,-,-)$ *if* $\mathrm{a} \neq 2\mathrm{b}$ *and* $(-,0,0)$ *if* $\mathrm{a} = 2\mathrm{b}$.

Proposition 4.9. *The Ricci curvature and the scalar curvature of the metric* (sll7) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} \frac{2\mathrm{a}^2-9}{\mathrm{a}^2} & \frac{-6\,\mathrm{a}^2-9}{\mathrm{a}^2\sqrt{2\,\mathrm{a}^2+1}} & -\frac{6\sqrt{2}}{\sqrt{2\,\mathrm{a}^2+1}\mathrm{a}} \\ \frac{-6\,\mathrm{a}^2-9}{\mathrm{a}^2\sqrt{2\,\mathrm{a}^2+1}} & \frac{-4\,\mathrm{a}^4-14\,\mathrm{a}^2-9}{2\,\mathrm{a}^4+\mathrm{a}^2} & -\frac{\left(6\,\mathrm{a}^2+6\right)\sqrt{2}}{2\,\mathrm{a}^3+\mathrm{a}} \\ -\frac{6\sqrt{2}}{\sqrt{2\,\mathrm{a}^2+1}\mathrm{a}} & -\frac{\left(6\,\mathrm{a}^2+6\right)\sqrt{2}}{2\,\mathrm{a}^3+\mathrm{a}} & \frac{-4\,\mathrm{a}^2-8}{2\,\mathrm{a}^2+1} \end{pmatrix}, \quad \mathfrak{s} = -\frac{3}{2}\mathrm{a}^2.$$

The operator Ric *is of type* $\{\mathrm{a3}\}$. *From the relations* $\det(\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric})) = 8$ *and* $\mathrm{tr}(\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric})) = -2\frac{(\mathrm{a}^2+9)}{\mathrm{a}^2}$, *we deduce that the signature of* ric *is* $(+,-,-)$.

4.4. Curvature of Lorentzian left-invariant metrics on Sol

There are seven classes of metrics on Sol given by the formulas (sol1), ..., (sol7). Here are their Ricci curvature and scalar curvature.

Proposition 4.10. *The Ricci curvature and the scalar curvature of the metric* (sol1) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} \frac{2v^2}{u^2-v^2} & 0 & 0 \\ 0 & -\frac{1}{2}u^2 & -\frac{1}{2}uv \\ 0 & -\frac{1}{2}uv & -\frac{1}{2}u^2 \end{pmatrix}, \quad \mathfrak{s} = \frac{1}{2}v^2.$$

The signature of ric *is given by*

$$\mathrm{sign}(\mathrm{ric}) = \begin{cases} (+,-,-) & \text{if} \quad u \neq 0, \\ (-,0,0) & \text{if} \quad u = 0. \end{cases}$$

Proposition 4.11. *The Ricci curvature and the scalar curvature of the metric* (sol2) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} \frac{2u^2}{v^2-u^2} & 0 & 0 \\ 0 & -\frac{1}{2}uv & \frac{1}{2}u^2 \\ 0 & \frac{1}{2}u^2 & -\frac{1}{2}uv \end{pmatrix}, \quad \mathfrak{s} = \frac{1}{2}u^2.$$

This metric is flat if $u = 0$. *For* $u \neq 0$, *the signature of* ric *is given by*

$$\mathrm{sign}(\mathrm{ric}) = \begin{cases} (+,-,-) & \text{if} \quad u > 0, \\ (+,+,+) & \text{if} \quad u < 0. \end{cases}$$

Proposition 4.12. *The Ricci curvature and the scalar curvature of the metric* (sol3) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} -\frac{2v}{v+u} & 0 & 0 \\ 0 & 2v & 2v \\ 0 & 2v & -2u \end{pmatrix}, \quad \mathfrak{s} = -2v.$$

The Ricci curvature has signature $(+,-,-)$ *and the Ricci operator is of type* $\{\mathrm{az\bar{z}}\}$.

Proposition 4.13. *The Ricci curvature and the scalar curvature of the metric* (sol4) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{s} = -2u.$$

Proposition 4.14. *The Ricci curvature and the scalar curvature of the metric* (sol5) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} -2 & \mathrm{a} & 0 \\ \mathrm{a} & -\frac{1}{2}\mathrm{a}^2 & -\frac{1}{2}\mathrm{a}^2 \\ 0 & -\frac{1}{2}\mathrm{a}^2 & -\frac{1}{2}\mathrm{a}^2 \end{pmatrix}, \quad \mathfrak{s} = \frac{1}{2}\mathrm{a}^2.$$

The Ricci operator is of type {ab2}. *From the relations* $\det(\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric})) = \frac{\mathrm{a}^4}{2}$ *and* $\mathrm{tr}(\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric})) = -2-\mathrm{a}^2$, *we deduce that the signature of* ric *is* $(+,-,-)$.

Proposition 4.15. *The Ricci curvature and the scalar curvature of the metric* (sol6) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{2}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{s} = 0.$$

We have $\mathrm{Ric}^2 = 0$ *and this metric is semi-symmetric.*

Proposition 4.16. *The Ricci curvature and the scalar curvature of the metric* (sol7) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{s} = 0.$$

We have $\mathrm{Ric}^2 = 0$ *and this metric is semi-symmetric.*

4.5. Curvature of Lorentzian left-invariant metrics on $\widetilde{\mathbf{E}_0}(2)$

There are three classes of metrics on $\widetilde{\mathrm{E}_0}(2)$ given by the formulas (ee1), (ee2) and (ee3). Here are their Ricci curvature and scalar curvature.

Proposition 4.17. *The Ricci curvature and the scalar curvature of the metric* (ee1) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} \frac{v-u}{v} & \frac{v^2-u^2}{2v} & 0 \\ \frac{v^2-u^2}{2v} & \frac{u(v^2-u^2)}{2v} & 0 \\ 0 & 0 & \frac{u^2-v^2}{2} \end{pmatrix}, \quad \mathfrak{s} = \frac{(u-v)^2}{2v}.$$

This metric is flat when $u = v$. *For* $u > v$ *the Ricci curvature has signature* $(+,-,-)$.

Proposition 4.18. *The Ricci curvature and the scalar curvature of the metric* (ee2) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} \frac{v+u}{v} & \frac{u^2-v^2}{2v} & 0 \\ \frac{u^2-v^2}{2v} & \frac{u(u^2-v^2)}{2v} & 0 \\ 0 & 0 & \frac{u^2-v^2}{2} \end{pmatrix}, \quad \mathfrak{s} = \frac{(u+v)^2}{2v}.$$

The signature of ric *is given by*

$$\begin{aligned} &\mathrm{sign}(\mathrm{ric}) = (+,-,-) \text{ if } u < v, \\ &\mathrm{sign}(\mathrm{ric}) = (+,+,+) \text{ if } u > v \text{ and} \\ &\mathrm{sign}(\mathrm{ric}) = (+,0,0) \text{ if } u = v. \end{aligned}$$

Proposition 4.19. *The Ricci curvature and the scalar curvature of the metric* (ee3) *are expressed by*

$$\mathcal{M}_{\mathbb{B}_N}(\mathrm{ric}) = \begin{pmatrix} 1 & \frac{u}{2} & 0 \\ \frac{u}{2} & 0 & 0 \\ 0 & 0 & -\frac{u^2}{2} \end{pmatrix}, \quad \mathfrak{s} = \frac{u}{2}.$$

The operator of Ricci is of type {ab2} *and the signature of* ric *is* $(+,-,-)$.

Table 1 gives the possible signatures of Ricci curvature of Lorentzian left-invariant metrics on three-dimensional unimodular Lie groups and the metrics realizing these signatures.

TABLE 1. Signatures of Ricci curvature on 3𝔻 Lorentzian unimodular Lie groups.

Signature of Ricci curvature	Metrics realizing this signature	Remarks
$(0,0,0)$	(nil0),[(sol2),$u=0$],[(ee1),$u=v$]	These metrics are flat.
$(+,+,+)$	(nil-),(su),(sll1),(sll2),(sol2),(ee2)	
$(+,-,-)$	(nil+),(su),(sll1),(sll2),(sll3),(sll4), (sll5),(sll6),(sll7),(sol1),(sol2) (sol3),(sol5),(ee1),(ee2),(ee3)	The metric (sll1) has negative constant sectional curvature for $\mu_1=\mu_2=\mu_3$. The metric (sll7) is a shrinking Ricci soliton.
$(+,0,0)$	(su),(sll1),(sll2),(sol6),(ee2)	
$(-,0,0)$	(sll1),(sll2),(sll3), (sll6),(sol1), (sol4), (sol6),(sol7)	The metrics (sol6) and (sol7) are steady Ricci soliton and semi-symmetric not locally symmetric, [(sll6),a = b $\neq 0$] is a shrinking Ricci soliton.

5. Lorentzian left-invariant generalized Ricci solitons

Having discussed the set of equivalence class representatives under $\backsimeq$, in what follows we would like to incorporate related work in [1, 3, 4]. In particular, we continue with this classification to the study of three-dimensional unimodular Lie groups which are of constant curvature, Einstein, locally symmetric, semi-symmetric not locally symmetric and generalized Ricci soliton. In summary, we have the following results:

Theorem 5.1 ([3])**.** *Let h be a Lorentzian left-invariant metric on a three-dimensional unimodular Lie group. Then the following assertions are equivalent*:

(1) *The metric h is locally symmetric.*
(2) *The metric h is Einstein.*
(3) *The metric h has constant sectional curvature.*

Moreover, a metric satisfying one of these assertions is either flat and is isometric to the metric (nil0), [(sol2), $u=0$] *or* [(ee1), $u=v$]; *or it has a negative constant sectional curvature and is isometric to* [(sll1), $\mu_1=\mu_2=\mu_3$].

Theorem 5.2 ([1,3])**.** *The metrics* (sol6) *and* (sol7) *are the unique, up to an automorphism, Lorentzian semi-symmetric not locally symmetric left-invariant metrics on a three-dimensional unimodular Lie group.*

Till the end of this section we keep the notations and use the results of [4].

Theorem 5.3 ([4])**.** *With the notation introduced in* (K), (H), (RS), (E-W), (PS) *and* (VN-H), *these metrics are the only Lorentzian left-invariant generalized Ricci solitons, up to automorphism, on* Nil.

a) *The metric* (nil-) *satisfies the relations*

$$\mathcal{L}_u\, h=0 \text{ if and only if } u\in \operatorname{span}(Z); \tag{5.1}$$

$$\mathcal{L}_u\, h+2\alpha_0\, u^\flat\odot u^\flat-2\,\beta_0\, ric(h)=-\beta_0\,\lambda\, h, \text{ where } u=\pm\tfrac{\sqrt{\alpha_0\,\beta_0}}{\alpha_0}\,X. \tag{5.2}$$

and hence Eq. (5.1) *involves* (K) *and Eq.* (5.2) *involves* (VN-H).

b) *The metric* (nil+) *satisfies the relations*

$$\mathcal{L}_u\, h=0 \text{ if and only if } u\in \operatorname{span}(Z); \tag{5.3}$$

$$\mathcal{L}_u\, h+2\alpha_0\, u^\flat\odot u^\flat-2\,\beta_0\, ric(h)=-\beta_0\,\lambda\, h, \text{ where } u=\pm\tfrac{\sqrt{-\alpha_0\,\beta_0}}{\alpha_0}\,X. \tag{5.4}$$

and hence Eq. (5.3) *involves* (K) *and Eq.* (5.4) *involves* (E-W).

c) *The metric* (nil0) *is flat such that* $\mathcal{L}_u\, h=0$, *if and only if* $u\in\operatorname{span}(Z)$; *which leads to* (K).

Theorem 5.4 ([4])**.** *With the notation above, these metrics are the only Lorentzian left-invariant generalized Ricci solitons, up to automorphism, on* SU(2). *The metric* (su) *satisfies the relations*

$$\mathcal{L}_u\, h=0 \text{ if and only if } u\in \operatorname{span}(\sigma_z); \tag{5.5}$$

$$\mathcal{L}_u\, h+2\alpha_0\, u^\flat\odot u^\flat-2\,\beta_0\, ric(h)=-4\,\tfrac{\beta_0(2\,\mu_1+\mu_3)}{{\mu_1}^2}h, \tag{5.6}$$

$$\text{where } u=\pm 2\,\tfrac{\sqrt{\alpha_0\beta_0\mu_3(\mu_1+\mu_3)}}{\alpha_0\mu_3\mu_1}\sigma_z,$$

and hence Eq. (5.5) *involves* (K) *and Eq.* (5.6) *involves* (VN-H).

Theorem 5.5 ([4])**.** *With the notation above, these metrics are the only Lorentzian left-invariant generalized Ricci solitons, up to automorphism, on* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$.

a) *The metric* [(sll1), $\mu_1=\mu_2$] *satisfies the relations*

$$\mathcal{L}_u\, h=0 \tag{5.7}$$

if and only if $u\in\mathfrak{g}$ *and* $\mu_3=\mu_2$ *or* $u\in\operatorname{span}(X_1)$ *and* $\mu_3\neq\mu_2$.

$$\mathcal{L}_u\, h+\tfrac{u_2-u_3}{2\,\beta_0 u_3}\,u^\flat\odot u^\flat-2\,\beta_0\, ric(h)=\tfrac{4\,\beta_0(2\,u_2-u_3)}{{u_2}^2}h, \tag{5.8}$$

$$\text{where } u=\pm\,4\,\tfrac{\beta_0}{u_2}X_3.$$

and hence Eq. (5.7) *involves* (K); *Eq.* (5.8) *involves* (VN-H) *if* $\mu_2 \neq \mu_3$.

b) *The metric* [(sll1), $\mu_1 = \mu_3$] *satisfies the relation*

$$\mathcal{L}_u\, h + \tfrac{\mu_2-\mu_3}{-2\,\beta_0\mu_2}\, u^\flat \odot u^\flat - 2\,\beta_0\, ric(h) = \tfrac{-4\,\beta_0(-2\,\mu_3+\mu_2)}{{\mu_3}^2} h, \tag{5.9}$$

$$\text{where } u = \pm\tfrac{4\,\beta_0}{\mu_3} X_2.$$

and hence Eq. (5.9) *involves* (E-W) *if* $\mu_2 \neq \mu_3$.

c) *The metric* [(sll1), $\mu_2 = \mu_3$] *satisfies the relations*

$$\mathcal{L}_u\, h = 0 \text{ if and only if } u \in \operatorname{span}(X_3) \text{ and } \mu_1 \neq \mu_2. \tag{5.10}$$

$$\mathcal{L}_u\, h + 2\,\alpha_0\, u^\flat \odot u^\flat - 2\,\beta_0\, ric(h) = \tfrac{-4\,\beta_0(-2\,\mu_3+\mu_1)}{{\mu_3}^2} h, \tag{5.11}$$

$$\text{where } u = \pm\tfrac{4\sqrt{\alpha_0\beta_0\mu_1(\mu_1-\mu_3)}}{\alpha_0\mu_1\mu_3} X_1.$$

and hence (5.10) *involves* (K); *Eq.* (5.11) *involves* (E-W) *if* $\mu_1 \leqslant \mu_3$ *and* (VN-H) *if* $\mu_1 \geqslant \mu_3$.

d) *The metric* [(sll4), $2\alpha = -\mathrm{a}^2$] *satisfies the relation*

$$\mathcal{L}_u\, h - \tfrac{3}{4\,\beta_0} u^\flat \odot u^\flat - 2\,\beta_0 ric(h) = \tfrac{-8\,(3\alpha^2-\beta^2)}{3}\,\beta_0 h, \tag{5.12}$$

$$\text{where } u = F X_1 + \tfrac{-2\sqrt{9\,\alpha^4+10\,\alpha^2\beta^2+\beta^4}}{3}\,\beta_0 X_2 + F\tfrac{\left(\sqrt{\alpha^2+\beta^2}\beta+\sqrt{9\,\alpha^4+10\,\alpha^2\beta^2+\beta^4}\right)\sqrt{\alpha^2+\beta^2}}{\sqrt{\alpha^2+\beta^2}(3\,\alpha^2+\beta^2)+\sqrt{9\,\alpha^4+10\,\alpha^2\beta^2+\beta^4}\beta} X_3;$$

$$\text{with } F = \pm\tfrac{4\sqrt{\beta^2(5\,\alpha^2+\beta^2)+\sqrt{\alpha^2+\beta^2}\beta\,\sqrt{9\,\alpha^4+10\,\alpha^2\beta^2+\beta^4}}}{3}\,\beta_0.$$

e) *The metric* [(sll6), $\mathrm{b} = -3\mathrm{a}$] *satisfies the relation*

$$\mathcal{L}_u\, h + 2\, u^\flat \odot u^\flat + 2\, ric(h) = \tfrac{7\,\mathrm{b}^2}{9} h, \tag{5.13}$$

$$\text{where } u = \pm\tfrac{\sqrt{6\,\mathrm{b}^3\mathrm{b}}}{24}\,(X_1 + X_2) - \tfrac{\mathrm{b}^2}{3}\, X_3.$$

and hence Eq. (5.13) *involves* (E-W).

f) *The metric* [(sll6), $\mathrm{b} = 3\mathrm{a}$] *satisfies the relation*

$$\mathcal{L}_u\, h + 2\, u^\flat \odot u^\flat - \, ric(h) = \tfrac{5\,\mathrm{b}^2}{18} h, \tag{5.14}$$

$$\text{where } u = \pm\tfrac{\sqrt{-\mathrm{b}^3\mathrm{b}}}{24}\,(X_1 + X_2) - \tfrac{\mathrm{b}^2}{6}\, X_3.$$

and hence Eq. (5.14) *involves* (VN-H).

g) *The metric* [(sll6), $\mathrm{a} = \mathrm{b}$] *satisfies the relations*

$$\mathcal{L}_u\, h + 2\,\alpha_0 u^\flat \odot u^\flat - 2\,\beta_0 ric(h) = \mathrm{a}^2\beta_0 h, \tag{5.15}$$

$$\text{where } u = \pm\tfrac{\sqrt{-2\,\alpha_0\beta_0\mathrm{a}}\mathrm{a}^2}{8\,\alpha_0}(X_1 + X_2).$$

$$\mathcal{L}_u\, h - 2\,\beta_0 ric(h) = \mathrm{a}^2\beta_0 h, \tag{5.16}$$

$$\text{where } u = t\,(X_1 + X_2) - \tfrac{\mathrm{a}^2\beta_0}{2} X_3.\ \forall t \in \mathbb{R}.$$

and hence Eq. (5.15) *involves* (E-W) *if* $\mathrm{a} > 0$ *and involves* (VN-H) *if* $\mathrm{a} < 0$. *Eq.* (5.16) *involves* (RS).

h) *The metric* [(sll6), a = 2b] *satisfies the relation*

$$\mathcal{L}_u\, h - \tfrac{1}{4\,\beta_0} u^\flat \odot u^\flat - 2\,\beta_0 ric(h) = 0, \tag{5.17}$$

$$\text{where } u = \pm\tfrac{\sqrt{-2\,\mathrm{a}}\mathrm{a}^2\beta_0}{4}(X_1 + X_2) - \tfrac{\mathrm{a}^2\beta_0}{2}\, X_3.$$

i) *The metric* (sll7) *satisfies the relation*

$$\mathcal{L}_u\, h + \mathrm{ric}(h) = -\tfrac{1}{2}\mathrm{a}^2 h, \tag{5.18}$$

$$\text{with } u = \tfrac{\mathrm{a}\sqrt{2}}{4} X_1 + \tfrac{(2\mathrm{a}^3 - \mathrm{a})\sqrt{2}}{4\sqrt{2\mathrm{a}^2+1}} X_2 - \tfrac{\mathrm{a}^2}{\sqrt{2\mathrm{a}^2+1}} X3$$

and hence Eq. (5.18) *involves* (RS).

Theorem 5.6 ([4])**.** *With the notation above, these metrics are the only Lorentzian left-invariant generalized Ricci solitons, up to automorphism, on* Sol.

a) *The metric* $[(\mathrm{sol1}), u = 0]$ *satisfies the relation*

$$\mathcal{L}_w\, h - \tfrac{3}{4\,\beta_0}\, w^\flat \odot w^\flat - 2\,\beta_0\, ric(h) = \tfrac{-2\,\beta_0 v^2}{3}\, h, \tag{5.19}$$

$$\text{where } w = \tfrac{\beta_0 v^2}{3} X_1 \pm \tfrac{4\,\beta_0 v}{3} X_3.$$

b) *The metric* (sol4) *satisfies the relation*

$$\mathcal{L}_w\, h + 2\,\alpha_0\, w^\flat \odot w^\flat + \tfrac{3}{4\,\alpha_0}\; ric(h) = \tfrac{-u}{\alpha_0}\, h, \tag{5.20}$$

$$\text{where } w = \tfrac{-u}{2\,\alpha_0} X_1 \pm \tfrac{\sqrt{u}}{\alpha_0} X_2.$$

c) *The metric* (sol5) *satisfies the relations*

$$\mathcal{L}_u\, h + 2\,\alpha_0\, u^\flat \odot u^\flat + \tfrac{1}{2\,\alpha_0}\; ric(h) = \tfrac{\mathrm{a}^2}{4\,\alpha_0}\, h, \tag{5.21}$$

$$\text{where } u = \tfrac{\left(\sqrt{3}-1\right)\mathrm{a}}{4\,\alpha_0\left(\sqrt{3}-2\right)}\, \left(X_2 + (\sqrt{3}-2)X_3\right).$$

$$\mathcal{L}_u\, h + 2\,\alpha_0\, u^\flat \odot u^\flat + \tfrac{2}{\alpha_0}\; ric(h) = \tfrac{\mathrm{a}^2}{\alpha_0}\, h, \tag{5.22}$$

$$\text{where } u = \tfrac{\mathrm{a}}{2\,\alpha_0}\, (X_2 + X_3).$$

and hence Eq. (5.22) *involves* (E-W).

d) *The metric* (sol6) *satisfies the relation*

$$\mathcal{L}_u\, h - 2\,\beta_0\, ric(h) = 0 \;\; \text{where } u = \tfrac{2\,\beta_0}{\lambda^2} X_1. \tag{5.23}$$

and hence Eq. (5.23) *involves* (RS).

e) *The metric* (sol7) *satisfies the relation*

$$\mathcal{L}_u\, h - 2\,\beta_0 \mathrm{ric}(h) = 0, \quad u = 2\,\beta_0 X_3. \tag{5.24}$$

and hence Eq. (5.24) *involves* (RS).

Theorem 5.7 ([4])**.** *With the notation above, these metrics are the only Lorentzian left-invariant generalized Ricci solitons, up to automorphism, on* $\widetilde{\mathrm{E}_0}(2)$*. The metric* (ee3) *satisfies the relation*

$$\mathcal{L}_w\, h + 2\,\alpha_0\, w^\flat \odot w^\flat + \tfrac{2}{\alpha_0}\; ric(h) = \tfrac{u}{\alpha_0}\, h, \;\; \text{where } w = \tfrac{1}{\alpha_0}\, X_3. \tag{5.25}$$

and hence Eq. (5.25) *involves* (E-W).

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