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RING ISOMORPHISMS BETWEEN CLOSED STRINGS VIA HOMOLOGICAL MIRROR SYMMETRY

Sangwook Lee

Abstract. We investigate how closed string mirror symmetry is related to homological mirror symmetry, under the presence of an explicit geometric mirror functor.

1. Introduction

Let X be a symplectic manifold and $W : \check{X} \to k$ be the Landau-Ginzburg mirror to X having only isolated singularities. We are interested in two different kinds of invariants on each side: one is *open* while the other is *closed*. For a symplectic manifold X, we have the Fukaya category $Fu(X)$ as an open string invariant and the quantum cohomology $QH^*(X)$ as a closed string invariant. For $W : \check{X} \to k$, the category $MF(W)$ of matrix factorizations of W is an open string invariant. A natural closed string invariant is the Jacobian algebra $Jac(W)$.

Mirror symmetry can be considered as a package of equivalences between symplectic invariants of X and algebro-geometric invariants of W. The closed string mirror symmetry asserts that $QH^*(X)$ and $Jac(W)$ are isomorphic as Frobenius algebras. The open string mirror symmetry, also known as homological mirror symmetry, is that $Fu(X)$ and $MF(W)$ are equivalent.

In this paper, we investigate a relation between these two layers (closed/open) of mirror symmetry, especially when homological mirror symmetry is given by a localized mirror functor (in [3, 4]). Given an A_∞-category, its Hochschild cohomology is equipped with a ring structure. For Fukaya categories and matrix factorization categories, we have natural ring homomorphisms from closed string algebras (which are $QH^*(X)$ and $Jac(W)$, respectively) to Hochschild cohomologies. Denote such ring homomorphisms as follows.

$$\mathcal{CO}_A : QH^*(X) \to HH^*(Fu(X)),\ \ \mathcal{CO}_B : Jac(W) \to HH^*(MF_{A_\infty}(W)).$$

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Our main result is the following.

Theorem. *Let X be a symplectic manifold. Suppose that we have a localized mirror functor*

$$\mathcal{F}^{\mathbb{L}} : Fu(X) \to MF_{A_\infty}(W).$$

Then there is a $Fu(X)$-$MF_{A_\infty}(W)$-bimodule $\mathcal{M}$ such that the following diagram commutes:

$$\begin{tikzcd}
QH^*(X) \arrow[r, "\mathcal{CO}_A"] \arrow[dd, "\mathfrak{ks}"] & HH^*(Fu(X)) \arrow[dr, "L^1_{\mathcal{M}}"'] & \\
& & \mathrm{Hom}(\mathcal{M}, \mathcal{M}). \\
Jac(W) \arrow[r, "\mathcal{CO}_B"] & HH^*(MF_{A_\infty}(W)) \arrow[ur, "R^1_{\mathcal{M}}"'] &
\end{tikzcd}$$

We will review the definition (due to [9]) of the closed string isomorphism $\mathfrak{ks}$. The data of mirror functor $\mathcal{F}^{\mathbb{L}}$ is contained in the $Fu(X)$-$MF_{A_\infty}(W)$-bimodule $\mathcal{M}$. The maps $L^1_{\mathcal{M}}$ and $R^1_{\mathcal{M}}$ from Hochschild cohomologies to the endomorphism space of $\mathcal{M}$ are described in [16] and will be reviewed. The main content of the proof begins with an explicit description of the map $\mathcal{CO}_B$ when $HH^*(MF_{A_\infty}(W))$ is described by the bar resolution.

Remark 1.1. More precisely, if W has a critical point with critical value λ, then the nontrivial category of matrix factorizations is given by $MF(W-\lambda)$. Also, if W has more than one critical point, then we need to consider a decomposition of the matrix factorization category into several critical points (we need to be careful of this point in the toric case). We can also consider decompositions of Fukaya category and quantum cohomology accordingly on A-side as follows: first, the quantum cohomology is a finite direct product of local rings over an index set A of nilpotent maximal ideals which give rise to critical points of the mirror potential. Considering closed-open map on each summand of the quantum cohomology, we can decompose the Fukaya category over the same index set A (see [6, Section 4]). It is now clear that the mirror functor and $\mathfrak{ks}$ preserve decompositions on both sides. Therefore, we will pretend that W has only one critical point with critical value 0, so that we only consider $MF(W)$ instead of nontrivial direct sums of categories.

2. A_∞-bimodules

In this section, we give basic preliminaries on A_∞-categories and bimodules over them. We refer readers to [10, 15, 16] for more details.

2.1. A_∞-bimodules

Recall that an A_∞-category $\mathcal{C}$ over $\boldsymbol{k}$ consists of objects $Ob(\mathcal{C})$ and the space of morphisms $\mathcal{C}(A,B)$ for each pair of objects A and B, with the following conditions:

(1) $\mathcal{C}(A,B)$ is a filtered $\mathbb{Z}/2$-graded $\boldsymbol{k}$-vector space for any $A, B \in Ob(\mathcal{C})$,
(2) for $k \geq 0$ there are multilinear maps of degree 1

$$m_k : \mathcal{C}(A_0,A_1)[1] \otimes \mathcal{C}(A_1,A_2)[1] \otimes \cdots \otimes \mathcal{C}(A_{k-1},A_k)[1] \to \mathcal{C}(A_0,A_k)[1]$$

such that they satisfy the A_∞-*relation*

$$\sum_{k_1+k_2=n+1} \sum_{i=1}^{k_1} (-1)^\epsilon m_{k_1}(x_1,\ldots,x_{i-1},m_{k_2}(x_i,\ldots,x_{i+k_2-1}),x_{i+k_2},\ldots,x_n) = 0, \tag{2.1}$$

where $\epsilon = \sum_{j=1}^{i-1}(|x_j|+1)$.

Definition 2.1. Let $(\mathcal{C},\{m_k^{\mathcal{C}}\})$ and $(\mathcal{D},\{m_k^{\mathcal{D}}\})$ be A_∞-categories. A $\mathcal{C}$-$\mathcal{D}$-*bimodule* $\mathcal{M}$ is the following data:

- For $V \in Ob(\mathcal{C})$ and $V' \in Ob(\mathcal{D})$, $\mathcal{M}(V,V')$ is a graded $\boldsymbol{k}$-vector space,
- degree $1-r-s$ multilinear maps (which are *scalar multiplication maps*)

$$\begin{aligned}\mu^{r|1|s} : \;& \mathcal{C}(V_0,V_1)\otimes\cdots\otimes\mathcal{C}(V_{r-1},V_r)\otimes\mathcal{M}(V_r,W_0)\otimes\mathcal{D}(W_0,W_1)\otimes\cdots\otimes\mathcal{D}(W_{s-1},W_s)\\ &\to \mathcal{M}(V_0,W_s)\end{aligned}$$

for any $V_i \in Ob(\mathcal{C})$ and $W_j \in Ob(\mathcal{D})$ satisfying

$$\begin{aligned}&\sum(-1)^\epsilon \mu^{i+1|1|s-j-1}(v_0,\ldots,v_i,\mu^{r-i-1|1|j+1}(v_{i+1},\ldots,v_{r-1},\underline{m},w_0,\ldots,w_j),w_{j+1},\ldots,w_{s-1})\\ &+\sum(-1)^\epsilon \mu^{i+r-j+1|1|s}(v_0,\ldots,v_i,m_{j-i}^{\mathcal{C}}(v_{i+1},\ldots,v_j),v_{j+1},\ldots,v_{r-1},\underline{m},w_0,\ldots,w_{s-1})\\ &+\sum(-1)^{\epsilon'} \mu^{r|1|i+s-l+1}(v_0,\ldots,v_{r-1},\underline{m},w_0,\ldots,w_i,m_{l-i}^{\mathcal{D}}(w_{i+1},\ldots,w_l),w_{l+1},\ldots,w_{s-1})\\ &=0,\end{aligned}$$

where

$$\epsilon = |v_0|'+\cdots+|v_i|',\;\; \epsilon' = |v_0|'+\cdots+|v_{r-1}|'+|m|+|w_0|'+\cdots+|w_i|'.$$

Definition 2.2. A *premorphism of* $\mathcal{C}$-$\mathcal{D}$-*bimodules* $\mathcal{F}:\mathcal{M}\to\mathcal{M}'$ *of degree* k is a collection of multilinear maps

$$\begin{aligned}\mathcal{F}^{r|1|s} : \;& \mathcal{C}(V_0,V_1)\otimes\cdots\otimes\mathcal{C}(V_{r-1},V_r)\otimes\mathcal{M}(V_r,W_0)\otimes\mathcal{D}(W_0,W_1)\otimes\cdots\otimes\mathcal{D}(W_{s-1},W_s)\\ &\to \mathcal{M}'(V_0,W_s)\end{aligned}$$

of degree $k-r-s$, and the *composition* $\mathcal{F}'\circ\mathcal{F}$ is defined by

$$\begin{aligned}&(\mathcal{F}'\circ\mathcal{F})(v_1,\ldots,v_r,\underline{m},w_1,\ldots,w_s)\\ :=&\sum(-1)^{|\mathcal{F}|(|v_1|'+\cdots+|v_i|')}\mathcal{F}'^{i|1|s-j}(v_1,\ldots,v_i,\mathcal{F}^{r-i|1|j}(v_{i+1},\ldots,\underline{m},w_1,\ldots,w_j),w_{j+1},\ldots,w_s).\end{aligned}$$

The *differential* δ on premorphisms is defined by

$$(\delta\mathcal{F})(v_1,\ldots,v_r,\underline{m},w_1,\ldots,w_s)$$

$$
\begin{aligned}
:= &\sum(-1)^{\epsilon_1}\mu_{\mathcal{M}'}^{i|1|s-j}(v_1,\ldots,v_i,\mathcal{F}^{r-i|1|j}(v_{i+1},\ldots,v_r,\underline{m},w_1,\ldots,w_j),w_{j+1},\ldots,w_s)\\
&-\sum(-1)^{\epsilon_2}\mathcal{F}^{i|1|s-j}(v_1,\ldots,v_i,\mu^{r-i|1|s-j}(v_{i+1},\ldots,v_r,\underline{m},w_1,\ldots,w_j),w_{j+1},\ldots,w_s)\\
&-\sum\mathcal{F}^{*|1|s}(\hat{m}^{\mathcal{C}}(v_1,\ldots,v_r),\underline{m},w_1,\ldots,w_s)\\
&-\sum(-1)^{\epsilon_3}\mathcal{F}^{r|1|*}(v_1,\ldots,v_r,\underline{m},\hat{m}^{\mathcal{D}}(w_1,\ldots,w_s)),
\end{aligned}
$$

where

$$\epsilon_1=|\mathcal{F}|(|v_1|'+\cdots+|v_i|'),\ \epsilon_2=|v_1|'+\cdots+|v_i|',\ \epsilon_3=|v_1|'+\cdots+|v_r|'+|m|$$

and $\hat{m}$ means the coderivation induced by the A_∞-structure $\{m_k\}$.

Remark 2.3. The definition of the degree of a premorphism of bimodules is motivated by the fact that the degree k premorphism is indeed given by multilinear maps

$$\mathcal{C}[1]^{\otimes r}\otimes\mathcal{M}\otimes\mathcal{D}[1]^{\otimes s}\to\mathcal{M}'$$

of degree k. Once we accept such a definition of degrees of maps, all signs obey Koszul rules and so we sometimes just write $(-1)^{\text{Koszul}}$ for signs.

The readers can easily check that $\mathcal{C}$-$\mathcal{D}$-bimodules together with premorphisms form a dg category. If $\mathcal{F}:\mathcal{M}\to\mathcal{N}$ is a premorphism such that $\delta\mathcal{F}=0$ and its cohomology level map $[\mathcal{F}^{0|1|0}]$ is an isomorphism, then $\mathcal{F}$ is called a *quasi-isomorphism*. We write $\mathcal{M}\simeq\mathcal{N}$ when they are quasi-isomorphic.

Example 2.4. For an A_∞-category $\mathcal{C}$, the *diagonal bimodule* $\mathcal{C}_\Delta$ is a $\mathcal{C}$-$\mathcal{C}$-bimodule defined by

$$\mathcal{C}_\Delta(X,Y):=\mathcal{C}(X,Y)$$

with scalar multiplication maps

$$\mu^{r|1|s}:=m^{\mathcal{C}}_{r+s+1}.$$

We recall some operations on bimodules.

Definition 2.5 (Base change)**.** Let $\mathcal{F}:\mathcal{C}_1\to\mathcal{C}_2$ and $\mathcal{G}:\mathcal{D}_1\to\mathcal{D}_2$ be A_∞-functors. Suppose that $\mathcal{M}$ is a $\mathcal{C}_2$-$\mathcal{D}_2$-bimodule. Then a $\mathcal{C}_1$-$\mathcal{D}_1$-bimodule $(\mathcal{F}\otimes\mathcal{G})^*\mathcal{M}$ is defined on objects by

$$(\mathcal{F}\otimes\mathcal{G})^*\mathcal{M}(X,Y):=\mathcal{M}(\mathcal{F}(X),\mathcal{G}(Y))$$

for $X\in Ob(\mathcal{C}_1)$, $Y\in Ob(\mathcal{D}_1)$, and the structure maps are given by

$$
\begin{aligned}
&\mu^{r|1|s}_{(\mathcal{F}\otimes\mathcal{G})^*\mathcal{M}}(v_1,\ldots,v_r,\underline{m},w_1,\ldots,w_s)\\
:=&\sum_{k,l}\mu^{k|1|l}_{\mathcal{M}}\big(\mathcal{F}_{i_1}(v_1,\ldots),\ldots,\mathcal{F}_{i_k}(\ldots,v_r),\underline{m},\mathcal{G}_{j_1}(w_1,\ldots),\ldots,\mathcal{G}_{j_l}(\ldots,w_s)\big).
\end{aligned}
$$

Definition 2.6 (Tensor product)**.** Let $\mathcal{M}$ be a $\mathcal{C}$-$\mathcal{D}$-bimodule and $\mathcal{N}$ be a $\mathcal{D}$-$\mathcal{E}$-bimodule for A_∞-categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$. Then $\mathcal{M}\otimes_{\mathcal{D}}\mathcal{N}$ is a $\mathcal{C}$-$\mathcal{E}$-bimodule such that

$$(\mathcal{M}\otimes_{\mathcal{D}}\mathcal{N})(C,E)$$

$$:= \bigoplus_{D_1,\dots,D_k\in Ob(\mathcal{D})} \mathcal{M}(C,D_1)\otimes\mathcal{D}(D_1,D_2)\otimes\cdots\otimes\mathcal{D}(D_{k-1},D_k)\otimes\mathcal{N}(D_k,E)$$

for $C\in Ob(\mathcal{C})$ and $E\in Ob(\mathcal{E})$, and the structure maps are given as follows:

$$\begin{aligned}
&\mu^{0|1|0}_{\mathcal{M}\otimes_{\mathcal{D}}\mathcal{N}}(\underline{m}\otimes d_1\otimes\cdots\otimes d_k\otimes\underline{n})\\
:= &\sum \mu^{0|1|i}_{\mathcal{M}}(\underline{m},d_1,\dots,d_i)\otimes d_{i+1}\otimes\cdots\otimes d_k\otimes\underline{n}\\
&+\sum(-1)^{\mathrm{Koszul}}\underline{m}\otimes d_1\otimes\cdots\otimes d_i\otimes m^{\mathcal{D}}_{j-i}(d_{i+1},\dots,d_j)\otimes d_{j+1}\otimes\cdots\otimes d_k\otimes\underline{n}\\
&+\sum(-1)^{\mathrm{Koszul}}\underline{m}\otimes d_1\otimes\cdots\otimes d_i\otimes\mu^{k-i|1|0}_{\mathcal{N}}(d_{i+1},\dots,d_k,\underline{n}),
\end{aligned}$$

$$\begin{aligned}
&\mu^{r|1|0}_{\mathcal{M}\otimes_{\mathcal{D}}\mathcal{N}}(c_1,\dots,c_r,\underline{m}\otimes d_1\otimes\cdots\otimes d_k\otimes\underline{n})\\
:= &\sum\mu^{r|1|p}_{\mathcal{M}}(c_1,\dots,c_r,\underline{m},d_1,\dots,d_p)\otimes d_{p+1}\otimes\cdots\otimes d_k\otimes\underline{n},\\
&\mu^{0|1|s}_{\mathcal{M}\otimes_{\mathcal{D}}\mathcal{N}}(\underline{m}\otimes d_1\otimes\cdots\otimes d_k\otimes\underline{n},e_1,\dots,e_s)\\
:= &\sum(-1)^{\mathrm{Koszul}}\underline{m}\otimes d_1\otimes\cdots\otimes d_p\otimes\mu^{k-p|1|s}_{\mathcal{N}}(d_{p+1},\dots,d_k,\underline{n},e_1,\dots,e_s)
\end{aligned}$$

and $\mu^{r|1|s}_{\mathcal{M}\otimes_{\mathcal{D}}\mathcal{N}}=0$ if r and s are both nonzero.

Definition 2.7. Two A_∞-categories $\mathcal{C}$ and $\mathcal{D}$ are *Morita equivalent* if there are a $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{M}$ and a $\mathcal{D}$-$\mathcal{C}$-bimodule $\mathcal{N}$ such that

$$\mathcal{M}\otimes_{\mathcal{D}}\mathcal{N}\simeq\mathcal{C}_\Delta,\ \ \mathcal{N}\otimes_{\mathcal{C}}\mathcal{M}\simeq\mathcal{D}_\Delta.$$

In this case, we call $\mathcal{M}$ and $\mathcal{N}$ *Morita bimodules.*

2.2. Hochschild cohomology of A_∞-bimodules

Definition 2.8. Let $\mathcal{M}$ be an A_∞-bimodule over $\mathcal{C}$. We define the *Hochschild cochain complex* of $\mathcal{M}$

$$\begin{aligned}
&CH^*(\mathcal{C},\mathcal{M})\\
:= &\prod_{X_0,\dots,X_k\in Ob(\mathcal{C})} hom^\bullet\big(\mathcal{C}(X_0,X_1)[1]\otimes\cdots\otimes\mathcal{C}(X_{k-1},X_k)[1],\mathcal{M}(X_0,X_k)\big)[-1]
\end{aligned}$$

with differential b^* defined by

$$\begin{aligned}
&b^*\phi(x_0,\dots,x_{k-1})\\
:= &\sum\phi(\hat{m}(x_0,\dots,x_{k-1}))\\
&+\sum(-1)^{|\phi|'(|x_0|'+\cdots+|x_i|')}\mu^{i+1|1|k-l-1}_{\mathcal{M}}(x_0,\dots,x_i,\phi(x_{i+1},\dots,x_l),x_{l+1},\dots,x_{k-1}).
\end{aligned}$$

Its cohomology of b^* is called the *Hochschild cohomology of $\mathcal{C}$ with coefficient in $\mathcal{M}$.* If $\mathcal{M}=\mathcal{C}_\Delta$, then we write $CH^*(\mathcal{C}):=CH^*(\mathcal{C},\mathcal{C}_\Delta)$.

Proposition 2.9 ([11])**.** *$CH^*(\mathcal{C})$ is an A_∞-algebra with A_∞-operations given by*

$$M^k(\phi_1,\dots,\phi_k)(x_1,\dots,x_n)$$

$$:= \sum(-1)^{\epsilon} m_*(\vec{x}_{i_1}, \phi_1(\vec{x}_{j_1}), \vec{x}_{i_2}, \phi_2(\vec{x}_{j_2}), \dots, \vec{x}_{i_k}, \phi_k(\vec{x}_{j_k}), \vec{x}_{i_{k+1}}).$$

with $\vec{x}_{i_1} \otimes \vec{x}_{j_1} \otimes \cdots \otimes \vec{x}_{i_{k+1}} = x_1 \otimes \cdots \otimes x_n$, $M^0 = 0$, $M^1 = b^*$, *and*

$$\epsilon = \sum_{l=1}^{k} |\phi_l|'(|\vec{x}_{i_1}|' + |\vec{x}_{j_1}|' + \cdots + |\vec{x}_{j_{l-1}}|' + |\vec{x}_{i_l}|').$$

In particular, the binary product M^2 induces the *Yoneda product* $\cup$ on the cohomology $HH^*(\mathcal{C})$ by

$$\phi \cup \psi := (-1)^{|\phi|} M^2(\phi, \psi).$$

Then the Yoneda product is associative.

Finally, we recall that Hochschild cohomology is a Morita invariant. Let $\mathcal{A}$ and $\mathcal{B}$ be Morita equivalent with Morita bimodules $\mathcal{M}$ and $\mathcal{N}$ which are over $\mathcal{A}$-$\mathcal{B}$ and $\mathcal{B}$-$\mathcal{A}$, respectively.

Lemma 2.10 ([16]). *The following are A_∞ quasi-isomorphisms*

$$L_{\mathcal{M}} : CH^*(\mathcal{A}) \to hom^*_{\mathcal{A}-\mathcal{B}}(\mathcal{M}, \mathcal{M}),$$
$$R_{\mathcal{M}} : CH^*(\mathcal{B})^{op} \to hom^*_{\mathcal{A}-\mathcal{B}}(\mathcal{M}, \mathcal{M})$$

which are defined as follows:

$$L^p_{\mathcal{M}}(\phi_1, \dots, \phi_p)(a_1, \dots, a_r, \underline{m}, b_1, \dots, b_s)$$
$$:= \sum(-1)^{\text{Koszul}} \mu^{*|1|s}_{\mathcal{M}}(\vec{a}_1, \phi_1(\vec{a}_2), \vec{a}_3, \dots, \phi_p(\vec{a}_{2p}), \vec{a}_{2p+1}, \underline{m}, b_1, \dots, b_s),$$

$$R^p_{\mathcal{M}}(\phi_1, \dots, \phi_p)(a_1, \dots, a_r, \underline{m}, b_1, \dots, b_s)$$
$$:= \sum(-1)^{\text{Koszul}} \mu^{r|1|*}_{\mathcal{M}}(a_1, \dots, a_r, \underline{m}, \vec{b}_1, \phi_p(\vec{b}_2), \vec{b}_3, \dots, \phi_1(\vec{b}_{2p}), \vec{b}_{2p+1}).$$

In particular, $L^1_{\mathcal{M}}$ and $R^1_{\mathcal{M}}$ induce ring isomorphisms on cohomology.

3. Homological mirror symmetry by localized mirror functors

First we recall A_∞-categories which are counterparts to each other in the homological mirror symmetry statement.

3.1. Fukaya categories

Given a symplectic manifold, we consider the Fukaya category which is defined over the Novikov field. So we first give the definition of Novikov field.

Definition 3.1. The *Novikov field* is

$$\Lambda := \Big\{ \sum_{i \ge 0} a_i T^{\lambda_i} \mid a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lambda_i \to \infty \text{ as } i \to \infty \Big\}.$$

We write

$$\Lambda_+ := \Big\{ \sum_{i \ge 0} a_i T^{\lambda_i} \in \Lambda \mid \lambda_i > 0 \text{ for all } i \Big\},\quad \Lambda_0 := \Big\{ \sum_{i \ge 0} a_i T^{\lambda_i} \in \Lambda \mid \lambda_i \ge 0 \text{ for all } i \Big\}.$$

Let $\mathcal{C}$ be an A_∞-category. We want to consider objects for which m_1-operations become differentials. It motivates the following definition.

Definition 3.2. Let A be an object of a unital A_∞-category $\mathcal{C}$. We say A is *weakly unobstructed* if there is a morphism $b \in \mathcal{C}(A, A)$ such that for some constant $W(b) \in \Lambda$,

$$m_0^b := m_0 + m_1(b) + m_2(b, b) + \cdots = W(b) \cdot 1_A.$$

The constant $W(b)$ is called the *superpotential* of (A, b). In this case, b is called a *weak bounding cochain* of A.

Let $\mathcal{M}_{weak}(A)$ be the set of weak bounding cochains of A (it is called a *weak Maurer-Cartan space*). Then W is a function on $\mathcal{M}_{weak}(A)$. Define a new A_∞-category $\mathcal{C}^{wo}$ consisting of (A_i, b_i) as objects (b_i is a weak bounding cochain of A_i), with morphisms and operations are defined by

$$\mathcal{C}^{wo}\big((A_i, b_i), (A_j, b_j)\big) := \mathcal{C}(A_i, A_j)$$

with the following new A_∞-structure maps

$$\begin{aligned} m_k^{b_0,\ldots,b_k} : \mathcal{C}^{wo}&\big((A_0, b_0), (A_1, b_1)\big) \otimes \cdots \otimes \mathcal{C}^{wo}\big((A_{k-1}, b_{k-1}), ((A_k, b_k)\big) \\ &\to \mathcal{C}^{wo}\big((A_0, b_0), (A_k, b_k)\big), \end{aligned}$$

$$m_k^{b_0,\ldots,b_k}(x_1, \ldots, x_k) := \sum_{l_0,\ldots,l_k} m_{k+l_0+\cdots+l_k}(b_0^{l_0}, x_1, b_1^{l_1}, \ldots, b_{k-1}^{l_{k-1}}, x_k, b_k^{l_k}),$$

where $x_i \in \mathcal{C}^{wo}\big((A_i, b_i), (A_{i+1}, b_{i+1})\big)$. We have $(m_1^{b,b})^2 = 0$ because b_i are weak bounding cochains.

Now we briefly define the Fukaya category. Let (X, ω) be a symplectic manifold, J be an almost complex structure and $L_0, \ldots, L_k$ be its transversally intersecting Lagrangian submanifolds. Let

$$CF(L_i, L_{i+1}) := \bigoplus_{p \in L_i \cap L_{i+1}} \Lambda \cdot p$$

be a $\mathbb{Z}/2$-graded vector space over Λ. The degree is defined by the Maslov index of each intersection point. Let $\beta \in \pi_2(X, L_0 \cup \cdots \cup L_k)$. Define a moduli space

$$\begin{aligned} &\widehat{\mathcal{M}}(p_0, \ldots, p_{k-1}; q; \beta; J) \\ := &\big\{u : (D^2, z_0, \ldots, z_{k-1}, z_k) \xrightarrow{J\text{-hol}} (X, p_0, \ldots, p_{k-1}, q) \\ &\qquad \mid u(\widehat{z_i z_{i+1}}) \subset L_{i+1}, u(\widehat{z_k z_0}) \subset L_0, [u] = \beta\big\}. \end{aligned}$$

and let $\mathcal{M}(p_0, \ldots, p_{k-1}; q; \beta)$ be its stable map compactification.

Definition 3.3. The *Fukaya category* $Fu(X)$ is an A_∞-category whose objects are Lagrangian submanifolds and morphism spaces are $CF(L, L')$. The A_∞-structure is given by operations $\{m_k\}_{k\geq 0}$, defined by

$$m_k(p_0, \ldots, p_{k-1}) := \sum_{\substack{\beta\in\pi_2(X, L_0\cup\cdots\cup L_k)\\ q\in L_k\cap L_0}} \#\mathcal{M}(p_0, \ldots, p_{k-1}; q; \beta) \cdot T^{\omega(\beta)} \cdot q.$$

We only count the moduli space when its virtual dimension is zero. Modulo technical assumptions (such as transversality of moduli spaces), these operations give rise to a filtered A_∞-algebra. In general, there might be nonzero m_0, which comes from holomorphic discs with one marked point. As we discussed above, we are only interested in objects whose endomorphism spaces are m_1-chain complexes. So let us still use $Fu(X)$ as the category of weakly unobstructed Lagrangians (with weak bounding cochains), instead of $Fu(X)^{wo}$.

3.2. Category of matrix factorizations

Let R be a commutative algebra and $W \in R$ be a non-zero-divisor. A *matrix factorization* of W is a $\mathbb{Z}/2$-graded R-module $E = E^0 \oplus E^1$ with a degree 1 endomorphism

$$Q = \begin{pmatrix} 0 & Q_{01} \\ Q_{10} & 0 \end{pmatrix}, \text{ where } Q_{ij} \in \mathrm{Hom}(E^j, E^i),$$

satisfying $Q^2 = W \cdot \mathrm{id}$. We denote the above data by (E, Q) for short.

Matrix factorizations of W form a differential $\mathbb{Z}/2$-graded category $MF(R, W)$ as follows: given two matrix factorizations (E, Q), (F, Q'), $\mathbb{Z}/2$-graded morphisms from (E, Q) to (F, Q') are given by homomorphisms

$$\Phi = \begin{pmatrix} \Phi_{00} & \Phi_{01} \\ \Phi_{10} & \Phi_{11} \end{pmatrix}, \text{ where } \Phi_{ij} \in \mathrm{Hom}(E^j, F^i).$$

Compositions of morphisms are defined in the obvious way. The differential on a morphism is defined as

$$\delta\Phi := Q\Phi - (-1)^{|\Phi|}\Phi Q$$

for morphisms of homogeneous degrees.

3.3. Localized mirror functors

To define a localized mirror functor, we modify the dg category $(MF(W), \delta, \circ)$ to an A_∞-category $(MF_{A_\infty}(W), m_1^{MF}, m_2^{MF})$. Objects are still matrix factorizations of W, but morphism spaces are changed:

$$\mathrm{Hom}_{MF_{A_\infty}(W)}((E, Q_E), (F, Q_F)) := \mathrm{Hom}_R(F, E).$$

m_1^{MF} and m_2^{MF} are defined as

$$m_1^{MF}(\Phi) := \delta(\Phi) = Q_E \circ \Phi - (-1)^{|\Phi|}\Phi \circ Q_F,$$
$$m_2^{MF}(\Phi, \Psi) := (-1)^{|\Phi|}\Phi \circ \Psi.$$

Then $\{m_k^{MF} \mid k \geq 1,\ m_k^{MF} = 0 \text{ for all } k \geq 3\}$ satisfy A_∞-relation (2.1), rather than usual dg relation.

Let $\mathbb{L}$ be a weakly unobstructed Lagrangian and let $W(b)$ be the superpotential function on $\mathcal{M}_{weak}(\mathbb{L})$. For any other weakly unobstructed Lagrangian L with potential λ (i.e., there is a weak bounding cochain b_0 such that $m_0^{b_0} = \lambda \cdot 1_L$), the A_∞-equation gives the following matrix factorization identity

$$(m_1^{b_0,b})^2 = (W - \lambda) \cdot \mathrm{id}.$$

Theorem 3.4 ([3])**.** *Define $\mathcal{F}^{\mathbb{L}}$ from $Fu(X)$ to $MF_{A_\infty}(W)$ as follows. $\mathcal{F}_0^{\mathbb{L}}$ sends an object $(L, b_0) \in Fu(X)$ to the matrix factorization (E, Q) by*

$$E := CF\big((L, b_0), (\mathbb{L}, b)\big),\ Q := -m_1^{b_0,b}.$$

On the level of morphisms, $\mathcal{F}_k^{\mathbb{L}}$ is defined as

$$\mathcal{F}_k^{\mathbb{L}}(x_1, \ldots, x_k)(\bullet) := m_{k+1}(x_1, \ldots, x_k, \bullet). \tag{3.1}$$

Then $\{\mathcal{F}_k^{\mathbb{L}}\}$ becomes an A_∞-functor.

Remark 3.5. In [3,4], they considered $CF\big((\mathbb{L}, b), (L, b_0)\big)$ instead. In our new convention, we do not have any sign in (3.1).

Proof. We need to show that

$$\begin{aligned} &\mathcal{F}^{\mathbb{L}}(\hat{m}^{Fu}(x_1, \ldots, x_k)) \\ &\quad= \sum_{\vec{x}_1 \otimes \vec{x}_2 = x_1 \otimes \cdots \otimes x_k} m_2^{MF}(\mathcal{F}^{\mathbb{L}}(\vec{x}_1), \mathcal{F}^{\mathbb{L}}(\vec{x}_2)) + m_1^{MF}(\mathcal{F}^{\mathbb{L}}(x_1, \ldots, x_k)). \end{aligned} \tag{3.2}$$

The left hand side can be written as

$$\sum_{\vec{x}_1 \otimes \vec{x}_2 \otimes \vec{x}_3 = x_1 \otimes \cdots \otimes x_k} (-1)^{|\vec{x}_1|'} m^{Fu}\big(\vec{x}_1 \otimes m^{Fu}(\vec{x}_2) \otimes \vec{x}_3, \bullet\big).$$

By definition of m_2^{MF},

$$\begin{aligned} &\sum_{\vec{x}_1 \otimes \vec{x}_2 = x_1 \otimes \cdots \otimes x_k} m_2^{MF}(\mathcal{F}^{\mathbb{L}}(\vec{x}_1), \mathcal{F}^{\mathbb{L}}(\vec{x}_2)) \\ &= \sum_{\vec{x}_1 \otimes \vec{x}_2 = x_1 \otimes \cdots \otimes x_k} (-1)^{|\vec{x}_1|'+1} \mathcal{F}^{\mathbb{L}}(\vec{x}_1) \circ \mathcal{F}^{\mathbb{L}}(\vec{x}_2) \\ &= \sum_{\vec{x}_1 \otimes \vec{x}_2 = x_1 \otimes \cdots \otimes x_k} (-1)^{|\vec{x}|'+1} m^{Fu}(\vec{x}_1, m^{Fu}(\vec{x}_2, \bullet)). \end{aligned}$$

The last term equals to

$$-m_1^{Fu} m_{k+1}^{Fu}(x_1, \ldots, x_k, \bullet) - (-1)^{|x_1|' + \cdots + |x_k|' + 1} m_{k+1}^{Fu}(x_1, \ldots, x_k, -m_1^{Fu}(\bullet)).$$

Summarizing, the equation (3.2) is nothing but an A_∞-relation of the Fukaya category. $\square$

A priori $\mathcal{F}^{\mathbb{L}}$ need not be an equivalence, but in various examples of localized mirror functors were indeed shown to give homological mirror symmetry. The examples include orbifold spheres or toric Fano manifolds. See [3, 4].

4. Closed string mirror symmetry

Let $\beta \in H_2(X, L)$ where L is a Lagrangian submanifold of X. Let $\mathcal{M}^{main}_{k+1,l}(\beta)$ be the moduli space of holomorphic discs of class β with $k+1$ boundary marked points respecting the cyclic order and l interior marked points. Define the evaluation map

$$ev = (ev_1^+, \dots, ev_l^+, ev_0, \dots, ev_k) : \mathcal{M}^{main}_{k+1,l}(\beta) \to X^l \times L^{k+1}$$

and consider the map [7, 8]:

$$\mathfrak{q}_{l,k;\beta} : E_l(H^*(X, \Lambda_+)) \otimes H^*(L; \Lambda)^{\otimes k} \to H^*(L; \Lambda),$$

$$\begin{aligned} &\mathfrak{q}_{l,k;\beta}([\alpha_1 \otimes \cdots \otimes \alpha_l], p_1, \dots, p_k) \\ := &(ev_0)_*(ev_1^+ \times \cdots \times ev_l^+ \times ev_0 \otimes \cdots \otimes ev_k)^*([\alpha_1 \otimes \cdots \otimes \alpha_l], p_1 \otimes \cdots \otimes p_l). \end{aligned}$$

The map $\mathfrak{q}$ is induced from the chain level and is well-defined. $E_l(H^*(X, \Lambda_+))$ means the subspace of $H^*(X, \Lambda_+)^{\otimes l}$ consisting of S_l-invariant elements. The bracket is the symmetrization as follows.

$$[\alpha_1 \otimes \cdots \otimes \alpha_l] = \sum_{\sigma \in S_l} \frac{1}{l!} \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(l)}.$$

We similarly define

$$\mathfrak{q}_{l,k;\beta}([\alpha_1 \otimes \cdots \otimes \alpha_l], p_1, \dots, p_k)$$

for $p_1, \dots, p_k$ being transverse intersections among distinct Lagrangians $L_0, \dots,$ L_k, using moduli spaces of holomorphic polygons with faces on $L_0, \dots, L_k$ together with interior marked points. We also define the following for both a single Lagrangian or a collection of transverse Lagrangians

$$\mathfrak{q}_{l,k}([\alpha_1 \otimes \cdots \otimes \alpha_l], p_1, \dots, p_k) := \sum_{\beta \in \pi_2(M, \vec{L})} \mathfrak{q}_{l,k;\beta}([\alpha_1 \otimes \cdots \otimes \alpha_l], p_1, \dots, p_k) \cdot T^{\omega(\beta)}.$$

Now, suppose that we have a Lagrangian submanifold $\mathbb{L}$ such that $\mathcal{F}^{\mathbb{L}}$ gives a mirror equivalence. The following assumption on $\mathbb{L}$ is crucial in this paper.

Assumption 4.1. For any $\mathfrak{b} \in H(X, \Lambda_+)$ and $b \in H^1(\mathbb{L}; \Lambda_+)$ the following always holds:

$$\sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \mathfrak{q}_{l,k}(\mathfrak{b}^l; b^k) \equiv 0 \mod \Lambda_+ 1_{\mathbb{L}}.$$

We denote the left hand side by $W^{\mathfrak{b}}(b) \cdot 1_{\mathbb{L}}$ and call the coefficient $W^{\mathfrak{b}}(b)$ a *bulk-deformed potential.*

Remark 4.2. The assumption holds for compact toric manifolds and orbifold spheres. See [8] and [1], respectively.

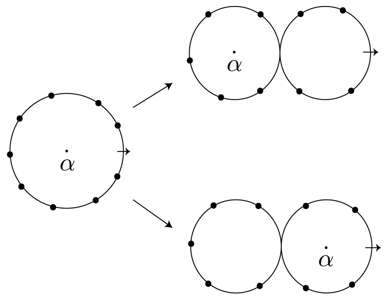

FIGURE 1. A degeneration of holomorphic discs with interior constraint α. In broken discs, components without interior constraints correspond to $m_k(\vec{p})$ and components with interior constraints correspond to $\mathfrak{q}_{1,k}(\alpha;\vec{p})$.

For $\alpha \in H^*(X,\Lambda_+)$ and t is a formal variable, let $\mathfrak{b} = t\alpha$ and $b = \sum x_i \mathbf{e}_i \in H^1(\mathbb{L},\Lambda_0)$. Then $W^{\mathfrak{b}}(b)$ is a formal power series in t. We observe the following:

$$\widetilde{\mathfrak{ks}}(\alpha) := \frac{\partial W^{\mathfrak{b}}(b)}{\partial t}\Big|_{t=0} = \sum_{k\geq 0} \mathfrak{q}_{1,k}(\alpha; b^k).$$

By Assumption 4.1, $\widetilde{\mathfrak{ks}}(\alpha)$ is a multiple of the unit $1_{\mathbb{L}}$. Again, the coefficient is a power series in $x_1,\ldots,x_n$, or a Laurent polynomial in $y_1 := e^{x_1},\ldots,y_n := e^{x_n}$. We will later employ $\widetilde{\mathfrak{ks}}$ for the statement of the closed string mirror symmetry for toric Fano manifolds.

Now we recall the bulk-deformation of A_∞-structure by ambient cohomology elements.

Definition 4.3. Let X be a symplectic manifold and $\mathfrak{b} \in H^*(X,\Lambda_+)$. Let $(L_0,b_0),\ldots,(L_k,b_k)$ be objects in $Fu(X)$. Then the following operations define a new A_∞-structure:

$$m_k^{\mathfrak{b},\vec{b}}(x_1,\ldots,x_k) := \sum_{l=0}^{\infty} \mathfrak{q}_{l,*}(\mathfrak{b}^l; e^{b_0}, x_1, e^{b_1}, x_2, \ldots, e^{b_{k-1}}, x_k, e^{b_k}).$$

We can also consider the t-derivative of A_∞-relation of $\{m_k^{t\alpha,\vec{b}}\}$ at $t=0$ as follows. We omit the decoration e^{b_i} for simplicity.

(4.1)
$$\begin{aligned} 0 = &\sum (-1)^{|x_1|'+\cdots+|x_i|'} m_{i+k-j+1}\big(x_1,\ldots,x_i,\mathfrak{q}(\alpha;x_{i+1},\ldots,x_j),x_{j+1},\ldots,x_k\big) \\ &+ \sum (-1)^{|x_1|'+\cdots+|x_i|'} \mathfrak{q}(\alpha;x_1,\ldots,x_i,m_{j-i}(x_{i+1},\ldots,x_j),x_{j+1},\ldots,x_k). \end{aligned}$$

This equation will be crucially used in Section 5. We remark that it is related to the degeneration of holomorphic discs with one interior constraint (see Figure 1).

We turn to mirror symmetry between closed strings. The following statement has been of great interest.

Conjecture 4.4. *For a symplectic manifold X and its mirror W, there is a ring isomorphism*

$$QH^*(X) \cong Jac(W).$$

There were some related results due to [2,12,13]. While these approaches are explicit and given by algebraic methods, the following Fukaya-Oh-Ohta-Ono's construction of the isomorphism is rather geometric. Their construction was also used to prove mirror symmetry of orbifold spheres by [1]. We summarize the results as follows.

Theorem 4.5 ([1, 9])**.** *When X is a compact toric manifold or an orbifold sphere, the following map $\mathfrak{ks}$ is a ring isomorphism (when X is an orbifold, $QH^*(X)$ is the orbifold quantum cohomology).*

$$\mathfrak{ks} : QH^*(X) \to Jac(W), \quad \alpha \mapsto [\widetilde{\mathfrak{ks}}(\alpha)].$$

5. Main result

Homological mirror symmetry and closed string mirror symmetry are connected by the closed-open map on each side of mirror pair.

Definition 5.1. Let X be a symplectic manifold. The *closed-open map* is

$$\mathcal{CO}_A : QH^*(X) \to HH^*(Fu(X))$$

defined by

$$\mathcal{CO}_A(\alpha)(p_1, \ldots, p_k) := \mathfrak{q}_{1,k}(\alpha, p_1, \ldots, p_k)$$

for $p_1 \in CF(L_1, L_2), \ldots, p_k \in CF(L_k, L_{k+1})$.

We give a mirror counterpart of the map $\mathcal{CO}_A$ by the following lemma.

Lemma 5.2. *The following map is a well-defined ring isomorphism*:

$$\mathcal{CO}_B : Jac(W) \to HH^*(MF_{A_\infty}(W)),\ [r] \mapsto [\phi_r],$$

where

$$\phi_r = \bigoplus_{E \in Ob(MF_{A_\infty}(W))} r \cdot \mathrm{id}_E \in CH^*(MF_{A_\infty}(W))$$

which is a Hochschild cocycle with length zero part only.

Proof. Denote $\mathcal{B} = MF_{A_\infty}(R, W)$ for convenience. Recall from [5] that a well-defined ring isomorphism

$$\gamma : Jac(W) \to HH^*(\mathcal{B})$$

is induced by the map

$$Jac(W) \to \mathbb{R}\mathrm{Hom}_{\mathcal{B}-\mathcal{B}}(\mathcal{B}_\Delta, \mathcal{B}_\Delta),\ [r] \mapsto (\mathcal{B}_\Delta \xrightarrow{r\cdot} \mathcal{B}_\Delta).$$

We reformulate γ (and rename it by $\mathcal{CO}_B$) by realizing the map $r\cdot : \mathcal{B}_\Delta \to \mathcal{B}_\Delta$ as a map from $B\mathcal{B}_\Delta$, the bar resolution of $\mathcal{B}_\Delta$ as follows:

$$\begin{tikzcd} \cdots \arrow[r] & \bigoplus_{X_0,X_1\in\mathcal{B}} \mathcal{B}(-,X_0)\otimes\mathcal{B}(X_0,X_1)\otimes\mathcal{B}(X_1,-) \arrow[r] \arrow[d] & \bigoplus_{E\in\mathcal{B}} \mathcal{B}(-,E)\otimes\mathcal{B}(E,-) \arrow[r] \arrow[d, "r\cdot m_2^{\mathcal{B}}"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathcal{B}_\Delta \arrow[r] & 0 \end{tikzcd}$$

and by the following identification

$$\begin{aligned} &hom_{\mathcal{B}-\mathcal{B}}\big(\mathcal{B}(-,X_0)\otimes\mathcal{B}(X_0,X_1)\otimes\cdots\otimes\mathcal{B}(X_{p-1},X_p)\otimes\mathcal{B}(X_p,-),\mathcal{B}_\Delta\big) \\ \simeq\ &hom_k\big(\mathcal{B}(X_0,X_1)\otimes\cdots\otimes\mathcal{B}(X_{p-1},X_p),\mathcal{B}(X_0,X_p)\big), \end{aligned}$$

the above map of complexes changes to

$$\begin{tikzcd} \cdots \arrow[r] & \bigoplus_{X_0,X_1\in\mathcal{B}} \mathcal{B}(X_0,X_1) \arrow[r] \arrow[d] & \bigoplus_{E\in\mathcal{B}} k \arrow[r] \arrow[d, "r"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \bigoplus_{E\in\mathcal{B}} \mathcal{B}(E,E) \arrow[r] & 0 \end{tikzcd}$$

where the map r sends 1 to $r\cdot \mathrm{id}_E$ for each E. □

We justify that the map $\mathcal{CO}_B$ is indeed the appropriate closed-open map on B-model as follows. For simplicity let $\mathcal{A} := Fu(X)$ (and $\mathcal{B} = MF_{A_\infty}(W)$ as well). Given the open-closed map $\mathcal{OC}_A : HH_*(Fu(X)) \to QH^*(X)$ on the A-model, let $\sigma \in HH_*(Fu(X))$ be the preimage of the unit $1 \in QH^*(X)$. For $\psi = \underline{a_0}\otimes a_1\otimes\cdots\otimes a_n \in HH_*(\mathcal{C})$ for an A_∞-category $\mathcal{C}$, recall the *cap product*

$$-\cap\psi : HH^*(\mathcal{C}) \to HH_*(\mathcal{C}),$$

$$\begin{aligned} &\phi\cap\psi \\ :=&\sum(-1)^\star m_*^{\mathcal{C}}\big(a_{l+1},\ldots,a_i,\phi(a_{i+1},\ldots,a_j)\otimes a_{j+1}\otimes\cdots\otimes a_n\otimes\underline{a_0}\otimes\cdots\otimes a_k\big) \\ &\qquad\otimes a_{k+1}\otimes\cdots\otimes a_l, \end{aligned}$$

where

$$\begin{aligned} \star =\ &|\phi|'(|a_0|'+|a_1|'+\cdots+|a_i|') \\ &+(|a_{l+1}|'+\cdots+|\phi(a_{i+1},\ldots,a_j)|'+\cdots+|a_n|')(|a_0|'+|a_1|'+\cdots+|a_l|'). \end{aligned}$$

Via the cap product, $HH_*(\mathcal{C})$ is equipped with a module structure over $HH^*(\mathcal{C})$. Then we have the following important fact:

$$\mathcal{OC}_A\circ(\cap\sigma)\circ\mathcal{CO}_A = \mathrm{id}. \tag{5.1}$$

Now consider the "open-closed map" on the B-model

$$\mathcal{OC}_B : HH_*(\mathcal{B}) \to Jac(W)$$

which is explained in [14, Section 3.1]. Since it is an isomorphism, let $\psi := \xi^{-1}(1) \in HH_*(\mathcal{B})$. Let $\psi = \underline{a_0} \otimes a_1 \otimes \cdots \otimes a_n$. Pick $r \in Jac(W)$. Since $\mathcal{CO}_B(r) = \bigoplus_{X \in Ob(\mathcal{B})} r \cdot \mathrm{id}_X$, we have the following computation of cap products:

$$\mathcal{CO}_B(r) \cap \psi = m_2(r \cdot \mathrm{id}, \underline{a_0}) \otimes a_1 \otimes \cdots \otimes a_n = r \cdot \underline{a_0} \otimes a_1 \otimes \cdots \otimes a_n,$$

so the following analogous relation as (5.1) is straightforward:

$$\mathcal{OC}_B \circ (\cap \psi) \circ \mathcal{CO}_B = \mathrm{id}.$$

Now we state the main theorem.

Theorem 5.3. *Let X be a symplectic manifold and $\mathcal{F}^{\mathbb{L}} : Fu(X) \to MF_{A_\infty}(W)$ be a localized mirror functor. Then the following diagram commutes*:

$$\begin{tikzcd}
QH^*(X) \arrow[r, "\mathcal{CO}_A"] \arrow[d, "\mathfrak{ks}"] & HH^*(Fu(X)) \arrow[dr, "{[L^1_{\mathcal{M}}]}"] & \\
 & & \mathrm{Hom}_{Fu(X)-MF_{A_\infty}(W)}(\mathcal{M},\mathcal{M}) \\
Jac(W) \arrow[r, "\mathcal{CO}_B"] & HH^*(MF_{A_\infty}(W)) \arrow[ur, "{[R^1_{\mathcal{M}}]}"'] &
\end{tikzcd}$$

where $\mathcal{M} = (\mathcal{F}^{\mathbb{L}} \otimes 1)^* MF_{A_\infty}(W)_\Delta$ is a $Fu(X)$-$MF_{A_\infty}(W)$ bimodule given by base change of $MF_{A_\infty}(W)_\Delta$ via $\mathcal{F}^{\mathbb{L}}$.

Proof. Let us denote $\mathcal{A} = Fu(X)$ and $\mathcal{B} = MF_{A_\infty}(W)$. Let

$$\Phi_\alpha := (L^1_{\mathcal{M}} \circ \mathcal{CO}_A)(\alpha) \in hom_{\mathcal{A}-\mathcal{B}}(\mathcal{M}, \mathcal{M})$$

for $\alpha \in QH^*(X)$. Let $(a_i : L_i \to L_{i+1})_{i=1,\dots,r}$ be a tuple of morphisms in $Fu(X)$. Let

$$\underline{m} \in \mathcal{M}(L_{r+1}, P) = \mathcal{B}_\Delta(\mathcal{F}^{\mathbb{L}}(L_{r+1}), P) = hom_{MF_{A_\infty}(W)}(\mathcal{F}^{\mathbb{L}}(L_{r+1}), P)$$

for some $P \in Ob(MF_{A_\infty}(W))$. Then

$$\begin{aligned}
&\Phi_\alpha^{r|1|0}(a_1, \dots, a_r, \underline{m}) \\
&= L^1_{\mathcal{M}}(\mathcal{CO}_A(\alpha))(a_1, \dots, a_r, \underline{m}) \\
&= \sum (-1)^{|a_1|' + \cdots + |a_i|'} \mu_{\mathcal{M}}^{(i+r-j+1)|1|0}\big(a_1, \dots, a_i, \mathfrak{q}(\alpha; a_{i+1}, \dots, a_j), \\
&\qquad\qquad a_{j+1}, \dots, a_r, \underline{m}\big) \\
&= \sum (-1)^{|a_1|' + \cdots + |a_i|'} m_2^{\mathcal{B}}\Big(\mathcal{F}^{\mathbb{L}}\big(a_1, \dots, a_i, \mathfrak{q}(\alpha; a_{i+1}, \dots, a_j), \\
&\qquad\qquad a_{j+1}, \dots, a_r\big), \underline{m}\Big)
\end{aligned} \tag{5.2}$$

and $\Phi_\alpha^{r|1|s} = 0$ if s is nonzero, since $\mathcal{B}$ has A_∞-operations only up to m_2.

Also, for $\Psi_\alpha := (R^1_{\mathcal{M}} \circ \gamma \circ \mathfrak{ks})(\alpha) \in hom^\bullet_{\mathcal{A}-\mathcal{B}}(\mathcal{M}, \mathcal{M})$,

$$\Psi_\alpha^{0|1|0}(\underline{m}) = R^1_{\mathcal{M}}(\gamma(\mathfrak{ks}(\alpha)))(\underline{m}) = (-1)^{|\underline{m}|} m_2^{\mathcal{B}}\big(\underline{m}, \mathfrak{ks}(\alpha) \cdot \mathrm{id}_P\big)$$

and $\Psi_\alpha^{r|1|s} = 0$ if $r \neq 0$ or $s \neq 0$. The sign $(-1)^{|\underline{m}|}$ is due to the definition of $R^p_{\mathcal{M}}$ in Lemma 2.10.

We show that

$$\Psi_\alpha - \Phi_\alpha = \delta\xi_\alpha$$

for some $\xi_\alpha \in \mathit{hom}^\bullet_{\mathcal{A}-\mathcal{B}}(\mathcal{M},\mathcal{M})$, so $[\Phi_\alpha] = [\Psi_\alpha]$ in $\mathrm{Hom}_{\mathcal{A}-\mathcal{B}}(\mathcal{M},\mathcal{M})$. For any $r \geq 0$, let

$$\xi_\alpha^{r|1|0}(a_1,\dots,a_r,\underline{m}) := m_2^{\mathcal{B}}(\mathfrak{q}(\alpha;a_1,\dots,a_r,\bullet),\underline{m})$$

and $\xi_\alpha^{r|1|s} = 0$ if $s \neq 0$. Then $|\xi_\alpha| = 0$. Here, $\mathfrak{q}(\alpha;a_1,\dots,a_r,\bullet)$ is a morphism in $\mathcal{B}$ from $CF(L_1,\mathbb{L})$ to $CF(L_{r+1},\mathbb{L})$, i.e., the bullet means an input in $CF(L_{r+1},\mathbb{L})$.

First let $r \geq 1$. Then $(\Phi_\alpha - \Psi_\alpha)^{r|1|0} = \Phi_\alpha^{r|1|0}$, and continuing from (5.2),

$$\begin{aligned}
&\Phi_\alpha^{r|1|0}(a_1,\dots,a_r,\underline{m})\\
&= \sum (-1)^{|a_1|'+\cdots+|a_i|'} m_2^{\mathcal{B}}(\mathcal{F}^{\mathbb{L}}(a_1,\dots,a_i,\mathfrak{q}(\alpha;a_{i+1},\dots,a_j),\\
&\qquad\qquad a_{j+1},\dots,a_r),\underline{m})\\
&= \sum (-1)^{|a_1|'+\cdots+|a_i|'} m_2^{\mathcal{B}}(m^{\mathcal{A}}_{i+r-j+1}(a_1,\dots,a_i,\mathfrak{q}(\alpha;a_{i+1},\dots,a_j),\\
&\qquad\qquad \dots,a_r,\bullet),\underline{m})
\end{aligned}$$

$$\begin{aligned}
&= \sum m_2^{\mathcal{B}}\Big((-1)^{|\vec{a}_1|'+1}\mathfrak{q}(\alpha;\vec{a}_1,m^{\mathcal{A}}(\vec{a}_2,\bullet)) \\
&\qquad + (-1)^{|\vec{a}'_1|'+1}\mathfrak{q}(\alpha;\vec{a}'_1,m^{\mathcal{A}}(\vec{a}'_2),\vec{a}'_3,\bullet),\underline{m}\Big)
\end{aligned} \tag{5.3}$$

$$+\sum (-1)^{|\vec{a}_1|'+1} m_2^{\mathcal{B}}\Big(m^{\mathcal{A}}(\vec{a}_1,\mathfrak{q}(\alpha;\vec{a}_2,\bullet)),\underline{m}\Big) \tag{5.4}$$

$$\pm \sum m_2^{\mathcal{B}}\Big(\mathfrak{q}(\alpha;a_1,\dots,a_r,\bullet,m_0^{\mathbb{L}}(1)),\underline{m}\Big) \tag{5.5}$$

$$\pm \sum m_2^{\mathcal{B}}\Big(m^{\mathcal{A}}_{r+2}(a_1,\dots,a_r,\bullet,\mathfrak{q}_0^{\mathbb{L}}(\alpha)),\underline{m}\Big). \tag{5.6}$$

Recall that the third identity is given by the formula (4.1). Also observe that

$$(5.3) = -(-1)^{|\xi_\alpha|}(\xi_\alpha \circ \hat{\mu}_{\mathcal{M}})(a_1,\dots,a_r,\underline{m}).$$

On the other hand, the following also holds

$$(5.4) = -(\mu_{\mathcal{M}} \circ \hat{\xi_\alpha})(a_1,\dots,a_r,\underline{m}),$$

by computations below:

$$\begin{aligned}
&\sum m_2^{\mathcal{B}}\Big((-1)^{|\vec{a}_1|'+1} m^{\mathcal{A}}(\vec{a}_1,\mathfrak{q}(\alpha;\vec{a}_2,\bullet)),\underline{m}\Big)\\
&= \sum m_2^{\mathcal{B}}\Big((-1)^{|\vec{a}_1|'+1} m^{\mathcal{A}}(\vec{a}_1,\bullet)\circ\mathfrak{q}(\alpha;\vec{a}_2,\bullet),\underline{m}\Big)\\
&= \sum m_2^{\mathcal{B}}\Big(m_2^{\mathcal{B}}\big(m^{\mathcal{A}}(\vec{a}_1,\bullet),\mathfrak{q}(\alpha;\vec{a}_2,\bullet)\big),\underline{m}\Big)\\
&= \sum (-1)^{|\vec{a}_1|'} m_2^{\mathcal{B}}\Big(m^{\mathcal{A}}(\vec{a}_1,\bullet), m_2^{\mathcal{B}}\big(\mathfrak{q}(\alpha;\vec{a}_2,\bullet),\underline{m}\big)\Big)
\end{aligned}$$

$$= \sum (-1)^{|\vec{a}_1|'+|\vec{a}_1|'+1} m^{\mathcal{A}}\Big(\vec{a}_1, m_2^{\mathcal{B}}\big(\mathfrak{q}(\alpha;\vec{a}_2,\bullet),\underline{m}\big)\Big)$$
$$= -(\mu_{\mathcal{M}} \circ \hat{\xi_\alpha})(a_1,\dots,a_r,\underline{m}).$$

Furthermore, (5.5) and (5.6) vanish since $m_0^{\mathbb{L}}(1)$ and $\mathfrak{q}_0^{\mathbb{L}}(\alpha)$ are both constant multiples of A_∞-unit. Hence,

$$(\Psi_\alpha - \Phi_\alpha)^{r|1|0} = (\delta\xi_\alpha)^{r|1|0} \text{ for } r > 0.$$

If $\underline{m} \in \mathcal{B}(\mathcal{F}^{\mathbb{L}}(L), P)$, then we have

$$\begin{aligned}(\Phi_\alpha - \Psi_\alpha)^{0|1|0}(\underline{m}) &= m_2^{\mathcal{B}}(\mathcal{F}^{\mathbb{L}}(\mathfrak{q}_0^L(\alpha)),\underline{m}) - (-1)^{|\underline{m}|} m_2^{\mathcal{B}}(\underline{m}, \mathfrak{ks}(\alpha)\cdot \mathrm{id}_P)\\ &= m_2^{\mathcal{B}}(\mathcal{F}^{\mathbb{L}}(\mathfrak{q}_0^L(\alpha)),\underline{m}) - \underline{m}\circ(\mathfrak{ks}(\alpha)\cdot \mathrm{id}_P)\end{aligned}$$
$$= m_2^{\mathcal{B}}\Big(m_2^{\mathcal{A}}\big(\mathfrak{q}_0^L(\alpha),\bullet\big) - (-1)^{|\bullet|} m_2^{\mathcal{A}}\big(\bullet,\mathfrak{q}_0^{\mathbb{L}}(\alpha)\big),\underline{m}\Big) \tag{5.7}$$

and by (4.1) again,

$$(5.7) = -m_2^{\mathcal{B}}\Big(\mathfrak{q}(\alpha; m_1^{\mathcal{A}}(\bullet)) + m_1^{\mathcal{A}}(\mathfrak{q}(\alpha;\bullet)),\underline{m}\Big) \tag{5.8}$$
$$-m_2^{\mathcal{B}}\Big(\mathfrak{q}(\alpha; m_0^L(1),\bullet) + (-1)^{|\bullet|'}\mathfrak{q}(\alpha;\bullet, m_0^{\mathbb{L}}(1)),\underline{m}\Big) \tag{5.9}$$

and since $m_0^L(1)$ and $m_0^{\mathbb{L}}(1)$ are both (multiples of) A_∞-units, (5.9) vanish. On the other hand,

$$(\delta\xi_\alpha)^{0|1|0}(\underline{m}) = \xi_\alpha(m_1^{\mathcal{B}}(\underline{m})) + m_1^{\mathcal{B}}(\xi_\alpha(\underline{m}))$$
$$= m_2^{\mathcal{B}}\Big(\mathfrak{q}(\alpha;\bullet), m_1^{\mathcal{B}}(\underline{m})\Big) + m_1^{\mathcal{B}}\Big(m_2^{\mathcal{B}}(\mathfrak{q}(\alpha;\bullet),\underline{m})\Big) \tag{5.10}$$

and we observe that

$$\mathfrak{q}(\alpha; -m_1^{\mathcal{A}}(\bullet)) - m_1^{\mathcal{A}}(\mathfrak{q}(\alpha;\bullet)) = m_1^{\mathcal{B}}(\bullet \mapsto \mathfrak{q}(\alpha;\bullet)),$$

but by A_∞-relation on $\mathcal{B}$ we have

$$m_2^{\mathcal{B}}\Big(m_1^{\mathcal{B}}(\mathfrak{q}(\alpha;\bullet)),\underline{m}\Big) + m_2^{\mathcal{B}}\Big(\mathfrak{q}(\alpha;\bullet), m_1^{\mathcal{B}}(\underline{m})\Big) + m_1^{\mathcal{B}}\Big(m_2^{\mathcal{B}}\big(\mathfrak{q}(\alpha;\bullet),\underline{m}\big)\Big) = 0,$$

hence

$$(5.8) = -(5.10) = -(\delta\xi_\alpha)^{0|1|0}(\underline{m}).$$

Finally, we show $(\delta\xi_\alpha)^{r|1|s} = 0$ for $s \neq 0$, so that in this case

$$(\Psi_\alpha - \Phi_\alpha)^{r|1|s} = (\delta\xi_\alpha)^{r|1|s},$$

where the left hand side is automatically zero by definition of Ψ_α and Φ_α.

Since $\mathcal{B}$ has no A_∞-operations $m_{\geq 3}$, we only need to compute $(\delta\xi_\alpha)^{r|1|1}$.

$$\delta\xi_\alpha(a_1,\dots,a_r,\underline{m},b)$$
$$= (-1)^{|a_1|'+\cdots+|a_r|'}\xi_\alpha(a_1,\dots,a_r,m_2^{\mathcal{B}}(\underline{m},b))$$
$$+\,\xi_\alpha(\hat{m}(a_1,\dots,a_r),\underline{m},b) \tag{5.11}$$
$$+\sum(-1)^{|a_1|'+\cdots+|a_i|'}\xi_\alpha(a_1,\dots,a_i,\mu_{\mathcal{M}}(a_{i+1},\dots,a_r,\underline{m}),b) \tag{5.12}$$
$$+\,m_2^{\mathcal{B}}(\xi_\alpha(a_1,\dots,a_r,\underline{m}),b).$$

By property $\xi_\alpha^{r|1|s} = 0$ for $s \neq 0$, (5.11) and (5.12) are zero, and

$$\begin{aligned}&(-1)^{|a_1|'+\cdots+|a_r|'}\xi_\alpha(a_1,\ldots,a_r,m_2^{\mathcal{B}}(\underline{m},b))\\&=(-1)^{|a_1|'+\cdots+|a_r|'}m_2^{\mathcal{B}}\Big(\mathfrak{q}(\alpha;a_1,\ldots,a_r,\bullet),m_2^{\mathcal{B}}(\underline{m},b)\Big),\end{aligned}$$

$$m_2^{\mathcal{B}}\Big(\xi_\alpha(a_1,\ldots,a_r,\underline{m}),b\Big)=m_2^{\mathcal{B}}\Big(m_2^{\mathcal{B}}\big(\mathfrak{q}(\alpha;a_1,\ldots,a_r,\bullet),\underline{m}\big),b\Big).$$

The sum of above two terms is zero due to the A_∞-relation of $m_2^{\mathcal{B}}$, hence

$$\delta\xi_\alpha(a_1,\ldots,a_r,\underline{m},b)=0.$$

Summarizing all above arguments, we conclude that

$$(\Psi_\alpha-\Phi_\alpha)^{r|1|s}=(\delta\xi_\alpha)^{r|1|s}$$

for any r and s, so on the cohomology level,

$$[\Phi_\alpha]=[\Psi_\alpha].$$

□

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Sangwook Lee
Department of Mathematics
Soongsil University
Seoul 06978, Korea
Email address: sangwook@ssu.ac.kr