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SYMMETRY AND MONOTONICITY OF SOLUTIONS TO FRACTIONAL ELLIPTIC AND PARABOLIC EQUATIONS

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Abstract. In this paper, we first apply parabolic inequalities and a maximum principle to give a new proof for symmetry and monotonicity of solutions to fractional elliptic equations with gradient term by the method of moving planes. Under the condition of suitable initial value, by maximum principles for the fractional parabolic equations, we obtain symmetry and monotonicity of positive solutions for each finite time to nonlinear fractional parabolic equations in a bounded domain and the whole space. More generally, if bounded domain is a ball, then we show that the solution is radially symmetric and monotone decreasing about the origin for each finite time.

We firmly believe that parabolic inequalities and a maximum principle introduced here can be conveniently applied to study a variety of nonlocal elliptic and parabolic problems with more general operators and more general nonlinearities.

1. Introduction

Symmetry of positive solutions of the local elliptic equation in unit ball was first established by Gidas, Ni and Nirenberg [16]. In recent decades, elliptic equations involving nonlocal operators, especially fractional operators, have received extensive attention and a number of results have been achieved [4, 5, 7, 10, 12, 15, 19, 21, 29–31] since the work of Caffarelli and Silvestre [3]. For other results on fractional Laplace equations, we refer readers to [2] for regularity and maximum principles, [14, 28] for existence and symmetry results of a Schrödinger type problem, [25] for regularity up to the boundary, [26] for mountain pass solutions, [6, 8] for a review, and references therein.

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For parabolic problems, first symmetry results of similar nature started to emerge much later. After a prelude [13] devoted to time periodic solutions, and symmetry of general positive solutions of parabolic equations on bounded domains was considered in [1, 17] and later in [22, 23]. Li [20] obtained symmetry of positive solutions for fully nonlinear second order parabolic equations with symmetric initial values. In 2006, Poláčik [24] studied symmetry properties of positive solutions for second order quasilinear parabolic equations in $\mathbb{R}^n$ and proved that such solutions are symmetric at each time $t < T(> 0)$. So far, the symmetry results of the solutions for parabolic equations involving nonlocal operators are still very few. Recently, Jarohs and Weth [18] established asymptotic symmetry of weak solutions for a class of nonlinear fractional reaction-diffusion equations in bounded domains. Chen, Wang, Niu and Hu [9] introduced an asymptotic method of moving planes and obtained asymptotic symmetry of solutions for fractional parabolic equations.

In this section, we present the symmetry and monotonicity of solutions for the following fractional parabolic equation

$$(1) \qquad \begin{cases} \frac{\partial u}{\partial t} + (-\triangle)^s u(x,t) = f(x,t,u(x,t)), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \phi(x,t), & (x,t) \in (\mathbb{R}^n \backslash \Omega) \times [0,T), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ which is symmetric and convex in the x_1 direction and $0 < T < +\infty$, $n \geq 2$. We call a domain Ω convex in x_1 direction if and only if $(x_1, x'), (x_1', x') \in \Omega$ imply that $(\gamma x_1 + (1-\gamma)x_1', x') \in \Omega$ for $0 < \gamma < 1$. For each fixed $t > 0$,

$$(-\triangle)^s u(x,t) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x,t) - u(y,t)}{|x-y|^{n+2s}} dy,$$

where $0 < s < 1$ and $P.V.$ stands for the Cauchy principal value.

Define

$$\mathcal{L}_{2s} = \{u(\cdot,t) \in L^1_{loc}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x,t)|}{1+|x|^{n+2s}} dx < +\infty\},$$

then it is easy to see that for $u \in C^{1,1}_{loc} \cap \mathcal{L}_{2s}$, $(-\triangle)^s u(x,t)$ is well defined.

Our main results:

Theorem 1.1. *Assume that* $u(x,t) \in \big(C^{1,1}_{loc}(\Omega) \cap C(\mathbb{R}^n)\big) \times C^1([0,T))$ *is a solution of* (1) *which is continuous on* $\bar{\Omega} \times [0,T]$, f *is Lipschitz continuous in* u *and satisfies*

$$(2) \quad f(x_1, x', t, u(x,t)) \leq f(\tilde{x}_1, x', t, u(x,t)) \text{ for } x_1 \leq \tilde{x}_1 \leq -x_1,\ (x_1, x') \in \Omega.$$

Suppose that the initial-boundary values of u *satisfy the following*:

$$(3) \quad u^0(x) = u(x,0), u^0(x_1,x') < u^0(\tilde{x}_1, x') \text{ for } x_1 < \tilde{x}_1 < -x_1,\ (x_1,x') \in \Omega.$$

$$(4) \qquad \phi(\tilde{x},t) \leq u(x,t),\ (\tilde{x},t) \in (\mathbb{R}^n \backslash \Omega) \times [0,T),\ (x,t) \in \Omega \times [0,T).$$

Then u *is monotone increasing in* x_1 *for* $x_1 < 0$ *and* $u(x_1,x',t) \leq u(-x_1,x',t)$ *for* $(x_1,x') \in \Omega,\ x_1 < 0$ *and* $0 \leq t \leq T$.

Remark 1.2. The assumptions that $u(x,t) \in (C^{1,1}_{loc}(\Omega) \cap C(\mathbb{R}^n)) \times C^1([0,T))$ and u is compact supported ensure that the fractional Laplacian $(-\triangle)^s u(x,t)$ is well defined for each fixed t. The solutions of fractional parabolic equations can be negative.

Theorem 1.3. *Under the hypotheses of Theorem 1.1, if* $u(x,t) \in \big(C^{1,1}_{loc}(\Omega) \cap C(\mathbb{R}^n)\big) \times C^1([0,T))$ *is a solution of* (1), *in addition, assume that*

$$f(x_1, x', t, u(x,t)) = f(-x_1, x', t, u(x,t)) \text{ for } (x_1, x') \in \Omega, \tag{5}$$

and the initial-boundary values of u are symmetric in $\{x_1 = 0\}$, then u is symmetric in x_1 and has only one crest. That is

$$u(x_1, x', t) = u(-x_1, x', t)$$

and

$$\frac{\partial u}{\partial x_1}(x_1, x', t) \geq 0 \text{ for } (x_1, x') \in \Omega,\ x_1 < 0,\ 0 < t \leq T.$$

Furthermore, if the initial value is not independent of x_1, we have

$$\frac{\partial u}{\partial x_1}(x_1, x', t) > 0 \text{ for } (x_1, x') \in \Omega,\ x_1 < 0,\ 0 \leq t \leq T. \tag{6}$$

When Ω is a special radial domain in $\mathbb{R}^n$, we obtain the following radially symmetry of solution for the following parabolic equation

$$\frac{\partial u}{\partial t} + (-\triangle)^s u(x,t) = f(x, t, u(x,t)),\ (x,t) \in B_1(0) \times (0,T). \tag{7}$$

Theorem 1.4. *Let* $u(x,t) \in \big(C^{1,1}_{loc}(B_1(0)) \cap C(\overline{B_1(0)})\big) \times C^1([0,T))$ *be a positive solution of* (7). *Assume that* $f(x,t,u(x,t)) = f(|x|,t,u(x,t))$ *is decreasing in* $|x|$ *and Lipschitz continuous in u for $t \geq t_0$ and some time $t_0 \geq 0$. And the boundary values of u satisfy*

$$u(x,t) = 0,\ (x,t) \in B_1^c(0) \times [t_0, T). \tag{8}$$

Suppose that

$$u(x,t_0) = u(|x|,t_0) \text{ and } u(x,t_0) \text{ is decreasing in } |x|. \tag{9}$$

Then for every time $t \in (t_0, \infty)$, u is radially symmetric and monotone decreasing about the origin, that is

$$u(x,t) = u(|x|,t),\ x \in B_1(0),\ t \in (t_0, T). \tag{10}$$

Remark 1.5. One can see that if the solution of (7) at any time $t_0 \geq 0$ is radially symmetric, then at all times $t_0 \leq t < T$, the solution is radially symmetric.

Theorem 1.6. *Suppose that* $u(x,t) \in (C^{1,1}_{loc}(\mathbb{R}^n) \cap \mathcal{L}_{2s}) \times C^1([0,T))$ *is a positive solution of*

$$\frac{\partial u}{\partial t} + (-\triangle)^s u(x,t) = f(x, t, u(x,t)),\ (x,t) \in \mathbb{R}^n \times (0,T). \tag{11}$$

Here f is Lipschitz continuous in u and satisfies

(12) $f(x_1, x', t, u(x,t)) \leq f(\tilde{x}_1, x', t, u(x,t))$ *for* $x_1 \leq \tilde{x}_1,\ x_1 \leq 0,\ (x_1, x') \in \mathbb{R}^n$.

Assume that for all $t \in [0, T)$

$$\lim_{|x| \to \infty} u(x,t) = 0. \tag{13}$$

Let $u^0(x) = u(x,0)$ *satisfies*

$$u(x,0) = u(|x|,0) \text{ and } u(x,0) \text{ is decreasing in } |x|. \tag{14}$$

Then u is monotone increasing in x_1 for $x_1 < 0$ and

$$u(x_1, x', t) \leq u(-x_1, x', t) \text{ for } x_1 < 0,\ 0 \leq t \leq T.$$

Furthermore, if $f(x,t,u) = f(|x|,t,u)$, then u is radially symmetric and monotone decreasing about the origin for each time $t \in [0, T]$.

In Section 2, we establish the maximum principle for the anti-symmetric functions to fractional parabolic equations. Then we give an application of the maximum principle for the anti-symmetric functions to the following fractional elliptic equation with a gradient term

$$\begin{cases} (-\triangle)^s u(x) = g(x, u(x), \nabla u(x)), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \Omega^c, \end{cases} \tag{15}$$

where Ω is a bounded domain in $\mathbb{R}^n$ which is convex in x_1 direction.

In the following, we denote $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n})$ by $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and prove:

Theorem 1.7. *Suppose that $u(x) \in C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})$ is a solution of fractional elliptic equation* (15) *with $g(x, u, \boldsymbol{p})$ is Lipschitz continuous in $(u, \boldsymbol{p})$ and*

$$\begin{aligned} &\text{for } x_1 < x_1',\ x_1' + x_1 < 0 \text{ and } p_1 \geq 0, \\ &g(x_1, x', u, p_1, p_2, \dots, p_n) \leq g(x_1', x', u, -p_1, p_2, \dots, p_n). \end{aligned} \tag{16}$$

Then $u(x_1, x')$ is strictly increasing in the left half of Ω in x_1-direction and

$$u(x_1, x') \leq u(x_1', x'),\ \forall\ x_1 < x_1',\ x_1 + x_1' < 0,\ (x_1, x') \in \Omega. \tag{17}$$

Furthermore if $g(x_1, x', u, p_1, p_2, \dots, p_n) = g(-x_1, x', u, -p_1, p_2, \dots, p_n)$, then u is a symmetric function of $x_1 = 0$, that is

$$u(x_1, x') = u(-x_1, x').$$

Remark 1.8. Theorem 1.7 has been obtained in [11] different from our methods.

In Section 3, we give the proof of Theorem 1.7 by a maximum principle for anti-symmetric functions and the method of moving planes. Section 4 is devoted to the proofs of symmetry and monotonicity of solutions for fractional parabolic equation in bounded domain and the whole space.

Throughout the paper, C will be the positive constant which can be different from line to line and only the relevant dependence is specified.

2. Maximum principles

In the section, we introduce a maximum principle for fractional parabolic equation.

For simplicity, we list some notations used frequently: for $\lambda \in \mathbb{R},\ t \in [0,\infty)$, denote $x = (x_1, x')$, $x^\lambda = (2\lambda - x_1, x')$, $T_\lambda = \{x \in \mathbb{R}^n \,|\, x_1 = \lambda\}$, $\Sigma_\lambda = \{x \in \mathbb{R}^n \,|\, x_1 < \lambda\}$ and $\tilde{\Sigma}_\lambda = \{x \in \mathbb{R}^n \,|\, x_1 > \lambda\}$. Set

$$u^\lambda(x,t) = u(x^\lambda, t),\ w_\lambda(x,t) = u^\lambda(x,t) - u(x,t).$$

We call a function $w_\lambda(x,t)$ is λ axial antisymmetric function, that is, for $t \geq 0$, $w_\lambda(x,t)$ is λ antisymmetric function in $\mathbb{R}^n$ if and only if

$$w_\lambda(x_1, x_2, \ldots, x_n, t) = -w_\lambda(2\lambda - x_1, x_2, \ldots, x_n, t). \tag{18}$$

Now we introduce a maximum principle of the anti-symmetric functions for fractional parabolic equation.

Theorem 2.1 (Maximum principle for anti-symmetric functions)**.** *Let Ω be a domain in Σ_λ. Assume that $w_\lambda(x,t) \in (C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_{2s}) \times C^1([0,\infty))$ is lower semi-continuous in x on $\bar{\Omega}$ and satisfies*

$$\begin{cases} \frac{\partial w_\lambda}{\partial t}(x,t) + (-\triangle)^s w_\lambda(x,t) \geq c(x,t) w_\lambda(x,t), & (x,t) \in \Omega \times (0,\infty), \\ w_\lambda(x^\lambda, t) = -w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times (0,\infty), \\ w_\lambda(x,t) \geq 0, & (x,t) \in (\Sigma_\lambda \backslash \Omega) \times [0,\infty), \\ w_\lambda(x,0) \geq 0, & x \in \Omega, \end{cases} \tag{19}$$

where $c(x,t)$ is bounded from above.

(i) *If Ω is a bounded domain, then we have*

$$w_\lambda(x,t) \geq 0 \ \textit{in}\ \Omega \times [0,T]; \tag{20}$$

(ii) *If Ω is unbounded, then the conclusion* (20) *still holds under the additional condition*: *for all $t \in (0,T]$*

$$\varliminf_{|x| \to \infty} w_\lambda(x,t) \geq 0; \tag{21}$$

(iii) *Furthermore, under the conclusion* (20), *if $w_\lambda(x,t)$ attains* 0 *at some point $(x_0, t_0) \in \Omega \times (0,T]$, then*

$$w_\lambda(x, t_0) = 0,\ \ a.e.\ x \in \mathbb{R}^n. \tag{22}$$

Remark 2.2. As we can see from the proof, if $c(x,t) = 0$ or $\Omega \subset \Sigma_\lambda$ is a bounded narrow region, the conclusions of Theorem 2.1 still hold. For more maximum principles of nonlocal parabolic equations, see [27].

Proof. (i) Let m be a determined positive constant to be chosen later and

$$\tilde{w}_\lambda(x,t) = e^{-mt} w_\lambda(x,t).$$

Then from (19) we have

$$\begin{aligned}&\frac{\partial \tilde{w}_\lambda}{\partial t}(x,t)+(-\triangle)^s\tilde{w}_\lambda(x,t)\\ &= -me^{-mt}w_\lambda(x,t)+e^{-mt}\frac{\partial w_\lambda}{\partial t}(x,t)+e^{-mt}(-\triangle)^s w_\lambda(x,t)\\ &\geq (-m+c(x,t))\tilde{w}_\lambda(x,t).\end{aligned} \tag{23}$$

We will establish (20) by proving

$$\tilde{w}_\lambda(x,t)\geq 0,\ \ (x,t)\in\Omega\times[0,T]. \tag{24}$$

If (24) does not hold, then the lower semi-continuity of $w_\lambda(x,t)$ in x on $\bar{\Omega}$ indicates that there exists a $(x^0,t^0)\in\bar{\Omega}\times(0,T]$ such that

$$\tilde{w}_\lambda(x^0,t^0):=\min_{\bar{\Omega}\times(0,T]}\tilde{w}_\lambda(x,t)<0.$$

Since $\tilde{w}_\lambda$ and w_λ have the same sign, one can further deduce from the third and fourth inequality of (19) that (x^0,t^0) is in the interior of $\Omega\times(0,T]$. Therefore $\frac{\partial \tilde{w}_\lambda}{\partial t}(x^0,t^0)\leq 0$.

From the definition of fractional Laplacian, we have

$$\begin{aligned}&\frac{\partial w_\lambda}{\partial t}(x^0,t^0)+(-\triangle)^s w_\lambda(x^0,t^0) \qquad (25)\\ &\leq C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{w_\lambda(x^0,t^0)-w_\lambda(y,t^0)}{|x^0-y|^{n+2s}}dy\\ &= C_{n,s}P.V.\left\{\int_{\Sigma_\lambda}\frac{w_\lambda(x^0,t^0)-w_\lambda(y,t^0)}{|x^0-y|^{n+2s}}dy+\int_{\tilde{\Sigma}_\lambda}\frac{w_\lambda(x^0,t^0)-w_\lambda(y,t^0)}{|x^0-y|^{n+2s}}dy\right\}\\ &= C_{n,s}P.V.\left\{\int_{\Sigma_\lambda}\frac{w_\lambda(x^0,t^0)-w_\lambda(y,t^0)}{|x^0-y|^{n+2s}}dy+\int_{\Sigma_\lambda}\frac{w_\lambda(x^0,t^0)-w_\lambda(y^\lambda,t^0)}{|x^0-y^\lambda|^{n+2s}}dy\right\}\\ &= C_{n,s}P.V.\left\{\int_{\Sigma_\lambda}\frac{w_\lambda(x^0,t^0)-w_\lambda(y,t^0)}{|x^0-y|^{n+2s}}dy+\int_{\Sigma_\lambda}\frac{w_\lambda(x^0,t^0)+w_\lambda(y,t^0)}{|x^0-y^\lambda|^{n+2s}}dy\right\}\\ &\leq C_{n,s}\int_{\Sigma_\lambda}\left\{\frac{w_\lambda(x^0,t^0)-w_\lambda(y,t^0)}{|x^0-y^\lambda|^{n+2s}}+\frac{w_\lambda(x^0,t^0)+w_\lambda(y,t^0)}{|x^0-y^\lambda|^{n+2s}}\right\}dy\\ &= C_{n,s}\int_{\Sigma_\lambda}\frac{2w_\lambda(x^0,t^0)}{|x^0-y^\lambda|^{n+2s}}dy<0.\end{aligned}$$

From (23), we derive

$$\frac{\partial \tilde{w}_\lambda}{\partial t}(x^0,t^0)+(-\triangle)^s\tilde{w}_\lambda(x^0,t^0)\geq(-m+c(x^0,t^0))\tilde{w}_\lambda(x^0,t^0). \tag{26}$$

Since $c(x,t)$ is bounded from above, we choose m such that $-m+c(x,t)<0$ to derive that the right hand side of (26) is strictly greater than 0. This contradicts (25). So we obtain (24).

Therefore, (20) must be true.

(ii) If Ω is unbounded, then (21) guarantees that the negative minimum of $w_\lambda(x,t)$ must be attained at some point. Then one can follow the same discussion as the case of (i) to arrive at a contradiction.

(iii) Next we prove (22) based on (20). Suppose that there exists $(x_0,t_0)\in\Omega\times(0,T]$ such that

$$w_\lambda(x_0,t_0)=0.$$

It is obvious that (x_0,t_0) is the minimum point of $w_\lambda(x,t)$. Hence,

$$\frac{\partial w_\lambda}{\partial t}(x_0,t_0)\le 0.$$

So we have

$$\begin{aligned}&\frac{\partial w_\lambda}{\partial t}(x_0,t_0)+(-\triangle)^s w_\lambda(x_0,t_0)-c(x_0,t_0)w_\lambda(x_0,t_0)\\ =\;&\frac{\partial w_\lambda}{\partial t}(x_0,t_0)+C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{-w_\lambda(y,t_0)}{|x_0-y|^{n+2s}}dy.\end{aligned}\tag{27}$$

If $w_\lambda(x,t_0)\not\equiv 0$ for any $x\in\Sigma_\lambda,\ t_0\in(0,T]$, then (27) implies

$$\frac{\partial w_\lambda}{\partial t}(x_0,t_0)+(-\triangle)^s w_\lambda(x_0,t_0)-c(x_0,t_0)w_\lambda(x_0,t_0)<0.$$

This contradicts (19). Hence $w_\lambda(x,t_0)\equiv 0$ in $\Sigma_\lambda,\ t_0\in(0,T]$.

Recalling $w_\lambda(x^\lambda,t)=-w_\lambda(x,t)$, we arrive at

$$w_\lambda(x,t_0)=0,\ \text{a.e. } x\in\mathbb{R}^n.$$

This completes the proof Theorem 2.1. □

3. Symmetric and monotonicity of solutions for fractional elliptic equations

We first establish a parabolic inequality, then derive symmetric and monotonicity of solutions for fractional elliptic equation by the method of moving planes and the maximum principle for antisymmetric functions to parabolic equations.

Proof of Theorem 1.7. Since Ω is a bounded domain with smooth boundary and convex in the x_1 direction, without loss of generality, we may assume

$$\Omega\subset\{|x_1|\le a\},\ a>0,\ \partial\Omega\cap\{x_1=-a\}\neq\emptyset$$

and

$$\Omega_\lambda=\{(x_1,x')\in\Omega\mid -a<x_1<\lambda\},\ \Sigma_\lambda=\{(x_1,x')\in\mathbb{R}^n\mid x_1<\lambda\}.$$

Denote $v(x)=u(x^\lambda)=u(2\lambda-x_1,x')$ and $w_\lambda(x)=v(x)-u(x)$.

From (16), if $p_1 < 0$, then by Lipschitz continuity of g with respect to $\mathbf{p}$, there exists a positive constant C such that for $x_1 < x_1',\ x_1 + x_1' < 0$, we have

$$\begin{aligned}
&g(x_1', x', u, -p_1, p_2, \ldots, p_n) - g(x_1, x', u, p_1, p_2, \ldots, p_n) \\
&= g(x_1', x', u, -p_1, p_2, \ldots, p_n) - g(x_1', x', u, 0, p_2, \ldots, p_n) \\
&\quad + g(x_1', x', u, 0, p_2, \ldots, p_n) - g(x_1, x', u, 0, p_2, \ldots, p_n) \\
&\quad + g(x_1, x', u, 0, p_2, \ldots, p_n) - g(x_1, x', u, p_1, p_2, \ldots, p_n) \\
&\geq g(x_1, x', u, 0, p_2, \ldots, p_n) - g(x_1', x', u, 0, p_2, \ldots, p_n) + Cp_1 \\
&\geq Cp_1.
\end{aligned}$$

It follows that there is an L^∞ function $\beta \geq 0$, such that for $x_1 < x_1',\ x_1 + x_1' < 0$ and all p_1,

$$g(x_1', x', u, -p_1, p_2, \ldots, p_n) - g(x_1, x', u, p_1, p_2, \ldots, p_n) \geq \beta p_1, \tag{28}$$

here β depends on $x_1,\ x_1',\ \mathbf{p}$.

Then v satisfies

$$\begin{aligned}
(-\triangle)^s v &= g(x^\lambda, v, -v_1, \nabla_{x'} v) \\
&\geq g(x, v, v_1, \nabla_{x'} v) + \beta v_1,\ x \in \Omega_\lambda,
\end{aligned}$$

with $\beta \in L^\infty$, by (28). Hence $w_\lambda(x)$ satisfies

$$\begin{aligned}
(-\triangle)^s w_\lambda(x) &= g(x^\lambda, v, -v_1, \nabla_{x'} v) - g(x, u, \nabla u) \\
&\geq g(x, v, \nabla v) - g(x, u, \nabla u) + \beta v_1.
\end{aligned}$$

Since $g(x, u, \mathbf{p})$ is Lipschitz continuous in $(u, \mathbf{p})$, it follows that for suitable bounded functions $b_j(x),\ c(x)$,

$$(-\triangle)^s w_\lambda(x) + \sum_{j=1}^{n} b_j(x)(w_\lambda)_j(x) + c(x) w_\lambda(x) - \beta v_1(x) \geq 0,$$

where $c(x) = \frac{g(x,u,\nabla u) - g(x,v,\nabla u)}{v(x) - u(x)},\ (w_\lambda)_j = \frac{\partial w_\lambda}{\partial x_j}$. But

$$\frac{\partial w_\lambda}{\partial \lambda}(x) = 2u_1(x_1^\lambda, x') = -2v_1(x).$$

Hence we derive the following parabolic inequality for $w_\lambda(x)$ as a function of x and λ,

$$\frac{\beta}{2} \frac{\partial w_\lambda}{\partial \lambda} + (-\triangle)^s w_\lambda(x) + b_j(x)(w_\lambda)_j(x) + c(x) w_\lambda(x) \geq 0. \tag{29}$$

It holds in a region V in (x, λ) space

$$V = \{(x_1, x', \lambda) \mid -a < x_1 < \lambda < \tilde{\lambda}, (x_1, x') \in \Omega_\lambda\},\ \tilde{\lambda} \leq a.$$

Next, we will use the moving plane method to divide the following two steps to prove (17).

Step 1. *Start moving the plane T_λ from $-a$ to the right in x_1-direction.*

We will show that there exists $\delta > 0$ small enough such that

$$w_\lambda(x) \geq 0, \ \forall \ x \in \Omega_\lambda, \ \lambda \in [-a, -a+\delta]. \tag{30}$$

If (30) does not hold, we set

$$A = \inf_{\substack{x \in \bar{\Omega}_\lambda \\ -a \leq \lambda \leq -a+\delta}} w_\lambda(x) < 0.$$

Obviously, $w_\lambda(x) \equiv 0, \ x_1 = \lambda$. From $u(x) > 0, x \in \Omega$ and $u(x) \equiv 0, \ x \in \Omega^c$, A can be attained for some

$$(\bar{\lambda}, \bar{x}) \in \{(\lambda, x) \mid (\lambda, x) \in [-a, -a+\delta] \times \bar{\Omega}_\lambda\}.$$

Noticing that $w_{\bar{\lambda}}(x) \geq 0, \ x \in \partial\Omega \cap \Sigma_{\bar{\lambda}}$, we have $\bar{x} \in \Omega_{\bar{\lambda}}$. From the fact that $\Sigma_{-a} \cap \Omega = \partial\Sigma_{-a} \cap \Omega$ for $\lambda = -a$, one gets $w_{-a}(x) \geq 0, \ x \in \Sigma_{-a} \cap \Omega$. This implies $\bar{\lambda} > -a$. Since $(\bar{\lambda}, \bar{t})$ is a minimizing point, we have

$$\nabla_x w_{\bar{\lambda}}(\bar{x}) = 0 \tag{31}$$

and

$$\frac{\partial w_\lambda(\bar{x})}{\partial \lambda} \mid_{\lambda=\bar{\lambda}} \leq 0. \tag{32}$$

By a direct computation, this implies that $(\partial_{x_1} u)(\bar{x}^{\bar{\lambda}}) \leq 0$. We derive

$$(\nabla_x u_{\bar{\lambda}})(\bar{x}) = (\nabla_x u)(\bar{x}).$$

So $(w_{\bar{\lambda}})_j(\bar{x}) = 0$.

We have $w_{-a}(\bar{x}) \geq 0$ and $w_\lambda(x) \geq 0, \ x \in \Sigma_\lambda \backslash \Omega_\lambda, \ \lambda \in [-a, -a+\delta]$. By (29), for δ small enough, Ω_λ is a narrow region, applying Theorem 2.1, we derive

$$w_\lambda(x) \geq 0, \ \forall \ x \in \Omega_\lambda, \ \lambda \in [-a, -a+\delta].$$

Hence (30) is proved.

Step 2. *Keep moving the planes to the right till the limiting position T_{λ_0} as long as* (30) *holds.* Define

$$\lambda_0 = \{-a \leq \lambda \leq 0 \mid w_\mu(x) \geq 0, \ \forall \ x \in \Omega_\mu, \ \forall \ \mu \leq \lambda\}.$$

By the definition of λ_0 and the continuity of $u(x)$, we have

$$w_{\lambda_0}(x) \geq 0 \ \text{ for all } x \in \Omega_{\lambda_0}.$$

We claim that $\lambda_0 = 0$ via contradiction arguments.

Suppose on the contrary that $\lambda_0 < 0$, we first show that

$$w_{\lambda_0}(x) > 0, \ x \in \Omega_{\lambda_0}. \tag{33}$$

If not, there exists $x_0 \in \Omega_{\lambda_0}$ such that $w_{\lambda_0}(x_0) = 0$. So (λ_0, x_0) is a minimizing point. Since we have $(\partial_{x_i} u)(x_0^{\lambda_0}) = (\partial_{x_i} u_{\lambda_0})(x_0) = (\partial_{x_i} u)(x_0)$ for $i = 2, \ldots, n$ and $(\partial_{x_1} u)(x_0^{\lambda_0}) = -(\partial_{x_1} u_{\lambda_0})(x_0) = -(\partial_{x_1} u)(x_0) \leq 0$ by (32) and (31). We use property (16) of $g(x, u, \mathbf{p})$ and obtain

$$(-\triangle)^s w_{\lambda_0}(x_0) = g(x_0^{\lambda_0}, u_{\lambda_0}(x_0), \nabla u_{\lambda_0}(x_0)) - g(x_0, u(x_0), \nabla u(x_0)) \geq 0,$$

which contradicts

$$(-\triangle)^s w_{\lambda_0}(x_0) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{-w_{\lambda_0}(y)}{|x_0 - y|^{n+2s}} dy < 0$$

by $w_{\lambda_0} \not\equiv 0$.

Suppose that $\lambda_0 < 0$, we claim that there exists $\varepsilon_0 > 0$ small enough such that

$$w_\lambda(x) \geq 0,\ x \in \Omega_\lambda,\ \forall\ \lambda \in (\lambda_0, \lambda_0 + \varepsilon_0). \tag{34}$$

Suppose (34) is not true, one gets

$$A_k = \inf_{\substack{x \in \bar{\Omega}_\lambda \\ \lambda_0 \leq \lambda \leq \lambda_k}} w_\lambda(x) < 0,\ \text{for a sequence of } \lambda_k \searrow \lambda_0,\ \text{as } k \to +\infty.$$

The minimum A_k can be obtained for some $\mu_k \in (\lambda_0, \lambda_k], x_k \in \Omega_{\mu_k}$ where $w_{\mu_k}(x_k) = A_k$ by the same reason as in Step 1. By the property of λ_k, we have $\mu_k \to \lambda_0$ as $k \to +\infty$. Let $\tilde{w}(x) = e^{-m\mu_k} w_{\mu_k}(x),\ m > 0$. It follows from (31) that

$$(\tilde{w})_j(x_k) = 0. \tag{35}$$

Case 1) If $\beta > 0$. From (29), we obtain

$$\frac{\beta}{2} \frac{\partial \tilde{w}}{\partial \mu_k}(x_k) + (-\triangle)^s \tilde{w}(x_k) \geq (-\frac{\beta}{2} m - c(x_k))\tilde{w}(x_k) > 0, \tag{36}$$

by choosing m such that $-\frac{\beta}{2} m - c(x_k) < 0$ since $c(x)$ and β are bounded. On the other hand, by (32), $\beta \geq 0$, $\tilde{w}(x_k) < 0$ and (25) in Theorem 2.1, we have

$$\frac{\beta}{2} \frac{\partial \tilde{w}}{\partial \mu_k}(x_k) + (-\triangle)^s \tilde{w}(x_k) < 0.$$

This contradicts (36), thus we have (34).

Case 2) If $\beta = 0$. From (29) and (35), we have

$$(-\triangle)^s \tilde{w}(x_k) + c(x_k)\tilde{w}(x_k) \geq 0.$$

It follows from (33) that for any $\delta > 0$

$$w_{\lambda_0}(x) > c_0 > 0,\ \forall\ x \in \Omega_{\lambda_0 - \delta}.$$

By the continuity of w_λ with respect to λ, for $\lambda_0 < \mu_k \leq \lambda_k$, such that

$$w_{\mu_k}(x) \geq 0,\ \forall\ x \in \Omega_{\lambda_0 - \delta},\ \forall\ \mu_k \in (\lambda_0, \lambda_0 + \varepsilon_0). \tag{37}$$

For narrow region $\Omega_{\mu_k} \backslash \Omega_{\lambda_0 - \delta}$, we apply the following Narrow region principle.

Theorem 3.1 (Narrow region principle, [5])**.** *Let Ω be a bounded narrow region such that it is contained in*

$$\{x \mid \lambda - \delta < x_1 < \lambda\} \subset \Sigma_\lambda$$

with small δ. Suppose that $w_\lambda \in C^{1,1}_{loc} \cap \mathcal{L}_{2s}$ and is lower semi-continuous on $\bar{\Omega}$. If $c(x)$ is bounded from below in Ω and

$$\begin{cases} (-\triangle)^s w_\lambda(x) + c(x) w_\lambda(x) \geq 0, & x \in \Omega, \\ w_\lambda(x^\lambda) = -w_\lambda(x), & x \in \Sigma_\lambda, \\ w_\lambda(x) \geq 0, & x \in \Sigma_\lambda \backslash \Omega, \end{cases}$$

then for sufficiently small δ, we have

$$w_\lambda(x) \geq 0, \ x \in \Omega.$$

Then we have

$$w_{\mu_k}(x) \geq 0, \ \forall \ x \in \Omega_{\mu_k} \backslash \Omega_{\lambda_0 - \delta}.$$

This together with (37) implies

$$w_{\mu_k}(x) \geq 0, \ \forall \ x \in \Omega_{\mu_k}, \ \forall \ \mu_k \in (\lambda_0, \lambda_0 + \varepsilon_0).$$

This is a contradiction with the definition of λ_0. Therefore $\lambda_0 = 0$ must be valid.

Hence

$$u(x_1, x') \leq u(-x_1, x'), \ \forall \ (x_1, x') \in \Omega, \ x_1 < 0. \tag{38}$$

Furthermore, similar to the proof of (33), we can actually deduce that

$$w_\lambda(x) > 0, \ x \in \Omega_\lambda, \ \forall \ \lambda < 0.$$

For any $(x_1, x'), (\hat{x}_1, x') \in \Omega$ with $0 > x_1 > \hat{x}_1$, one can take $\lambda = \frac{x_1 + \hat{x}_1}{2}$. Then we have

$$u(x_1, x') > u(\hat{x}_1, x')$$

and hence $u(x_1, x')$ is strictly increasing in the left half of Ω in x_1-direction.

Moreover, if $g(x_1, x', u, p_1, p_2, \ldots, p_n) = g(-x_1, x', u, -p_1, p_2, \ldots, p_n)$, then we have $\hat{u}(x_1, x') = u(-x_1, x')$ also solves (15). Thus we have derived that

$$\hat{u}(x_1, x') \leq \hat{u}(-x_1, x'), \ \forall \ (x_1, x') \in \Omega, \ x_1 < 0,$$

or equivalently,

$$u(x_1, x') \geq u(-x_1, x'), \ \forall \ (x_1, x') \in \Omega, \ x_1 < 0.$$

Combining this with (38) yields that

$$u(x_1, x') = u(-x_1, x'), \ \forall \ (x_1, x') \in \Omega, \ x_1 < 0,$$

that is, u is symmetric in the x_1 direction about $x_1 = 0$. This completes the proof of Theorem 1.7. □

4. Symmetry and monotonicity of solutions for fractional parabolic equations

In this section, we apply maximum principle for fractional parabolic equations to obtain symmetry and monotonicity of fractional parabolic equations in bounded domains and the whole space.

4.1. Bounded domain

Proof of Theorem 1.1. From (3), one knows that the initial values of u are not a constant and strictly increasing in x_1 when $x_1 < 0$, then increasing the time t, we only need to show that u is monotonicity increasing in x_1 for each time when $x_1 < 0$. Denote

$$-a := \min\{x_1 \mid x \in \Omega\},\ a > 0,\ \Omega_\lambda = \{x \in \Omega \mid -a < x_1 < 0\}$$

and

$$w_\lambda(x,t) = u(x^\lambda, t) - u(x,t).$$

We will prove

$$w_\lambda(x,t) \geq 0,\ x \in \Omega_\lambda,\ t \in (0,T),\ \lambda \in (-a,0). \tag{39}$$

Since $u(x^\lambda, t)$ is also a solution of (1), we have

$$\begin{aligned} \frac{\partial w_\lambda}{\partial t}(x,t) + (-\triangle)^s w_\lambda(x,t) &= f(x^\lambda, t, u(x^\lambda,t)) - f(x,t,u(x,t)) \\ &\geq f(x,t,u(x^\lambda,t)) - f(x,t,u(x,t)) \\ &:= c(x,t) w_\lambda(x,t),\ x \in \Omega_\lambda,\ t \in (0,T), \end{aligned} \tag{40}$$

where $c(x,t) = \frac{f(x,t,u(x^\lambda,t)) - f(x,t,u(x,t))}{u(x^\lambda,t) - u(x,t)}$ is bounded by Lipschitz continuity of f with respect to u.

Since the initial condition (3), we have

$$w_\lambda(x,0) \geq 0,\ x \in \Omega_\lambda,\ \lambda \in (-a,0).$$

Fixed λ, increasing t, we want to prove that (39) holds. From (4), we have

$$w_\lambda(x,t) \geq 0,\ (x,t) \in (\Sigma_\lambda \backslash \Omega_\lambda) \times [0,T).$$

Applying Theorem 2.1, we obtain (39). So we arrive at

$$w_\lambda(x,t) \geq 0,\ x \in \Omega_\lambda,\ t \in [0,T],\ \lambda \in (-a,0),$$

which is equivalent to the conclusions of Theorem 1.1. □

Proof of Theorem 1.3. From Theorem 1.1, we have shown that

$$u(x_1, x', t) \leq u(-x_1, x', t) \ \text{ for } (x_1, x') \in \Omega,\ x_1 < 0 \text{ and } \ 0 \leq t \leq T. \tag{41}$$

Since (5), $\bar{u}(x_1, x', t) = u(-x_1, x', t)$ is also a solution of (1). Since the initial-boundary values of u are symmetric in $\{x_1 = 0\}$, then go through the same process for the function $\bar{u}(x_1, x', t)$, we have

$$\bar{u}(x_1, x', t) \leq \bar{u}(-x_1, x', t),\ x_1 < 0,\ t \in [0,T].$$

That is

$$u(x_1, x', t) \geq u(-x_1, x', t),\ x_1 < 0,\ t \in [0,T]. \tag{42}$$

Combining (41) and (42), we obtain

$$u(x_1, x', t) = u(-x_1, x', t) \text{ for } x_1 \leq 0,\ 0 \leq t \leq T.$$

Hence we obtain

$$\frac{\partial w_\lambda}{\partial x_1}(x,t) \le 0,\ x \in \Omega_\lambda,\ t \in (0,T],\ \lambda \in (-a,0]. \tag{43}$$

By the definition of $w_\lambda(x,t)$, we have

$$\frac{\partial u}{\partial x_1}(x,t) \ge 0,\ x \in \Omega,\ x_1 < 0,\ t \in (0,T].$$

Hence

$$u(x_1,x',t) = u(-x_1,x',t) \text{ and } \frac{\partial u}{\partial x_1}(x,t) \ge 0 \text{ for } x \in \Omega, x_1 < 0,\ 0 < t \le T.$$

Next we prove

$$\frac{\partial u}{\partial x_1}(x,t) > 0,\ x_1 < 0,\ 0 < t \le T.$$

We have shown that

$$w_\lambda(x,t) \ge 0,\ x \in \Omega_\lambda,\ t \in (0,T],\ \lambda \le 0$$

and

$$w_0(x,t) = 0,\ x \in \Omega_0,\ t \in (0,T]. \tag{44}$$

For fixed $\lambda < 0$, we claim that

$$w_\lambda(x,t) > 0,\ x \in \Omega_\lambda,\ t \in (0,T],\ \lambda \in [-a,0). \tag{45}$$

Set

$$\tilde{w}_\lambda(x,t) = e^{-mt} w_\lambda(x,t),\ m > 0.$$

Since $\tilde{w}_\lambda$ and w_λ have the same sign, we claim that (45) by proving

$$\tilde{w}_\lambda(x,t) > 0,\ x \in \Omega_\lambda,\ t \in (0,T],\ \lambda \in [-a,0). \tag{46}$$

If not, there exists $\bar{x} \in \Omega_\lambda$ and the first $\bar{t} \in (0,T]$ such that

$$\tilde{w}_\lambda(\bar{x},\bar{t}) = \min_{\bar{\Omega}_\lambda \times (0,T]} \tilde{w}_\lambda(x,t) = 0.$$

And one can further deduce from conditions (3) and (4) that $(\bar{x},\bar{t})$ is in the interior of $\Omega_\lambda \times (0,T]$. On the one hand, by (40), taking $m = 2|c(x,t)|$, we obtain

$$\frac{\partial \tilde{w}_\lambda}{\partial t}(\bar{x},\bar{t}) + (-\triangle)^s \tilde{w}_\lambda(\bar{x},\bar{t}) \ge (-m + c(\bar{x},\bar{t}))\tilde{w}_\lambda(\bar{x},\bar{t}) \ge 0. \tag{47}$$

On the another hand, by $\frac{\partial \tilde{w}_\lambda}{\partial t}(\bar{x},\bar{t}) \le 0$, similar to (25) in the proof of Theorem 2.1, we have

$$\frac{\partial \tilde{w}_\lambda}{\partial t}(\bar{x},\bar{t}) + (-\triangle)^s \tilde{w}_\lambda(\bar{x},\bar{t}) < 0,$$

which contradicts (47). So (45) is true. Combining (44) and (45), we obtain

$$\frac{\partial w_\lambda}{\partial x_1}(x,t) < 0,\ x_1 < 0,\ 0 \le t \le T.$$

By the definition of w_λ, we arrive at

$$\frac{\partial u}{\partial x_1}(x,t) > 0,\ x_1 < 0,\ 0 \le t \le T.$$

Hence (6) is proven.

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. We choose a direction to be the x_1-direction. Since $f(x,t,u(x,t)) = f(|x|,t,u(x,t))$ is decreasing in $|x|$ and f is Lipschitz continuous in u for $t \ge t_0$, then employ (8) and (9), similar to the proof of Theorem 1.1 and Theorem 1.3 with initial value is $t_0 \ge 0$, we obtain

$$u(x_1,x',t) = u(-x_1,x',t),\ x_1 < 0,\ t_0 < t \le T.$$

That is

$$w_0(x,t) \equiv 0,\ x \in B_1(0), x_1 \le 0,\ t_0 < t \le T.$$

Since the direction of x_1 can be chosen arbitrarily, we have actually shown that (10). The monotonicity is similar to the proof of (45). This completes the proof of Theorem 1.4. □

4.2. The whole space

In the section, we give the proof of Theorem 1.6.

Proof of Theorem 1.6. We choose a direction to be the x_1-direction. Let

$$w_\lambda(x,t) = u(x^\lambda,t) - u(x,t)$$

and

$$H_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda < 0\}.$$

From (14), one knows that the initial values of u is not a constant and is strictly increasing in x_1 when $x_1 < 0$. So we have

$$w_\lambda(x,0) \ge 0,\ x \in H_\lambda,\ \lambda \le 0. \tag{48}$$

For the fixed λ, the assumption (13) implies that for all $t \in [0,T)$

$$u(x,t) \to 0 \text{ as } |x| \to +\infty.$$

Since $|x^\lambda| \to +\infty$, as $|x| \to +\infty$, it follows that

$$u^\lambda(x,t) = u(x^\lambda,t) \to 0,\ t \in [0,T).$$

Thus we have

$$w_\lambda(x,t) \to 0 \text{ as } |x| \to +\infty,\ t \in [0,T). \tag{49}$$

Since $u(x^\lambda,t)$ is also a solution of (11), by (12) we have

$$\begin{aligned}\frac{\partial w_\lambda}{\partial t}(x,t) + (-\triangle)^s w_\lambda(x,t) &= f(x^\lambda,t,u(x^\lambda,t)) - f(x,t,u(x,t))\\ &\ge f(x,t,u(x^\lambda,t)) - f(x,t,u(x,t))\\ &:= c(x,t)w_\lambda(x,t),\ x \in H_\lambda,\ t \in (0,T),\end{aligned} \tag{50}$$

where $c(x,t) = \frac{f(x,t,u(x^\lambda,t)) - f(x,t,u(x,t))}{u(x^\lambda,t)-u(x,t)}$ is bounded by Lipschitz continuous of f with respect to u.

Denote

$$\tilde{w}_\lambda(x,t) = e^{-mt} w_\lambda(x,t),\ m > 0,\ (x,t) \in \mathbb{R}^n \times (0,T).$$

We will prove that

$$w_\lambda(x,t) \geq 0,\ x \in H_\lambda,\ t \in (0,T). \tag{51}$$

We only need prove that

$$\tilde{w}_\lambda(x,t) \geq 0,\ x \in H_\lambda,\ t \in (0,T). \tag{52}$$

Otherwise, there exists $(x^0,t^0) \in H_\lambda \times (0,T]$ by (48), (49) and $w_\lambda(x,t) \equiv 0,\ x \in T_\lambda$ such that

$$\tilde{w}_\lambda(x^0,t^0) = \min_{H_\lambda \times (0,T]} \tilde{w}_\lambda(x,t) < 0.$$

From (50), we have

$$\frac{\partial \tilde{w}_\lambda}{\partial t}(x^0,t^0) + (-\triangle)^s \tilde{w}_\lambda(x^0,t^0) \geq (-m + c(x^0,t^0))\tilde{w}_\lambda(x^0,t^0).$$

Since $c(x,t)$ is bounded, we choose m such that $c(x^0,t^0) - m < 0$. So

$$\frac{\partial \tilde{w}_\lambda}{\partial t}(x^0,t^0) + (-\triangle)^s \tilde{w}_\lambda(x^0,t^0) > 0. \tag{53}$$

On the other hand, similar to (25) in the proof of Theorem 2.1

$$\frac{\partial \tilde{w}_\lambda}{\partial t}(x^0,t^0) + (-\triangle)^s \tilde{w}_\lambda(x^0,t^0) < 0,$$

which contradicts (53). So we obtain (52). Therefore (51) is correct, that is,

$$u(x_1,x',t) \leq u(-x_1,x',t) \text{ for } x_1 < 0,\ 0 \leq t \leq T. \tag{54}$$

Furthermore, since $f(x,t,u) = f(|x|,t,u)$, $\bar{u}(x_1,x',t) = u(-x_1,x',t)$ is also a solution of (11). Base on the initial values of u are symmetric in $\{x_1 = 0\}$, then go through the same process for the function $\bar{u}(x_1,x',t)$, we have

$$\bar{u}(x_1,x',t) \leq \bar{u}(-x_1,x',t),\ x_1 < 0,\ t \in [0,T].$$

That is

$$u(x_1,x',t) \geq u(-x_1,x',t),\ x_1 < 0,\ t \in [0,T]. \tag{55}$$

Combining (54) and (55), we obtain

$$u(x_1,x',t) = u(-x_1,x',t) \text{ for } x_1 \leq 0,\ 0 \leq t \leq T.$$

Since the direction of x_1 can be chosen arbitrarily, we obtain $u(x,t)$ is radially symmetric about the origin for each $t \in [0,T]$. The monotonicity is similar to the proof of (45). This completes the proof of Theorem 1.6. $\square$

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