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THE EXPONENTIAL GROWTH AND DECAY PROPERTIES FOR SOLUTIONS TO ELLIPTIC EQUATIONS IN UNBOUNDED CYLINDERS

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Abstract. In this paper, we classify all solutions bounded from below to uniformly elliptic equations of second order in the form of $Lu(\mathbf{x}) = a_{ij}(\mathbf{x})D_{ij}u(\mathbf{x}) + b_i(\mathbf{x})D_iu(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x})$ or $Lu(\mathbf{x}) = D_i(a_{ij}(\mathbf{x}) D_ju(\mathbf{x})) + b_i(\mathbf{x})D_iu(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x})$ in unbounded cylinders. After establishing that the Aleksandrov maximum principle and boundary Harnack inequality hold for bounded solutions, we show that all solutions bounded from below are linear combinations of solutions, which are sums of two special solutions that exponential growth at one end and exponential decay at the another end, and a bounded solution that corresponds to the inhomogeneous term f of the equation.

1. Introduction

The structure of positive solutions to elliptic equations has been studied extensively. Early in 1990 Gardiner [5] proved the cone of positive harmonic functions that vanish on boundary is generated by two minimal harmonic functions $h_{\pm}(\mathbf{x}', y) = e^{\pm\lambda y}\phi(\mathbf{x}')$ in an unbounded cylinder $B \times \mathbb{R} \subseteq \mathbb{R}^n$, where B is a unit ball in $\mathbb{R}^{n-1}(n \geq 2)$, λ is the square root of the first eigenvalue of the operator $-\Delta$ in B, and $\phi(\mathbf{x}')$ is the corresponding eigenfunction. Landis and Nadirashvili [8] considered the Dirichlet problem for strongly elliptic equations

$$a_{ij}(\mathbf{x})D_{ij}u(\mathbf{x}) = 0 \qquad \text{or} \qquad D_i(a_{ij}(\mathbf{x})D_ju(\mathbf{x})) = 0$$

in a cone. They proved that the space of the positive solutions in a cone with zero boundary value is one-dimensional. These results have been extended to the positive solutions to these equations in unbounded cylinders $\mathcal{S} \times \mathbb{R} \subseteq \mathbb{R}^n$, where $\mathcal{S}$ is a bounded Lipschitz domain in $\mathbb{R}^{n-1}$ $(n \geq 2)$, by Jun Bao and some of the authors [1]. They showed that there exist two special positive solutions with exponential growth at one end and another exponential decay

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and all positive solutions can be represented as a linear combination of these two special solutions.

Inspired by the above two papers [1] and [8], we consider the solutions bounded from below to second order elliptic equation with lower order terms and with inhomogeneous term f. We will show a similar structure theorem of these solutions.

Here we use [9] and [10] for the basic results, Harnack inequalities etc for positive functions of second-order elliptic equations. And more related works, one can see [2], [7].

Therefore, we consider the following elliptic equations

$$\begin{cases} Lu(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x}\in\mathcal{C}, \\ u(\mathbf{x}) = 0, & \mathbf{x}\in\partial\mathcal{C}, \\ u(\mathbf{x}) \text{ is uniformly bounded from below}, & \mathbf{x}\in\mathcal{C}, \end{cases} \tag{1}$$

in an unbounded cylinder $\mathcal{C}=\mathcal{S}\times\mathbb{R}\subseteq\mathbb{R}^n$, where $\mathcal{S}$ is a bounded Lipschitz domain in $\mathbb{R}^{n-1}(n\geq 2)$, and L is the second order elliptic operator in non-divergence form and in divergence form as

$$Lu(\mathbf{x}) = a_{ij}(\mathbf{x})D_{ij}u(\mathbf{x}) + b_i(\mathbf{x})D_iu(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x}\in\mathcal{C}, \tag{2}$$

$$Lu(\mathbf{x}) = D_i(a_{ij}(\mathbf{x})D_ju(\mathbf{x})) + b_i(\mathbf{x})D_iu(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x}\in\mathcal{C}. \tag{3}$$

We assume the coefficients and the inhomogeneous term satisfy

$$\begin{aligned} &a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}), a_{ij}(\mathbf{x})\in C(\bar{\mathcal{C}}), \\ &b_i(\mathbf{x}), c(\mathbf{x})\in L^\infty(\mathcal{C}), \\ &f(\mathbf{x})\in L^n_{loc}(\mathcal{C}),\ c(\mathbf{x})\leq 0,\ \forall\ \mathbf{x}\in\mathcal{C}, \end{aligned} \tag{4}$$

and L satisfies the uniform ellipticity condition:

$$\lambda|\xi|^2 \leq a_{ij}(\mathbf{x})\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad \forall\ \mathbf{x},\ \xi\in R^n, \tag{5}$$

where $0<\lambda\leq\Lambda$, and with a constant γ such that $\frac{\Lambda}{\lambda}\leq\gamma$.

Throughout the paper, we only prove the results in the non-divergence form, and all our results are valid for equations in divergence form. Here, we assume the operator L of (2) is applied to functions u in the class $W^{2,n}_{loc}(\mathcal{C})\bigcap C(\bar{\mathcal{C}})$.

Notations. $\mathbf{x}=(x_1,\dots,x_n)=(x_1,\dots,x_{n-1},y)=(\mathbf{x}',y)$ denotes a typical point of $\mathbb{R}^{n-1}\times\mathbb{R}(n\geq 2)$. For $E\subset\mathbb{R}$, $\mathcal{C}_E := \mathcal{S}\times E = \{(\mathbf{x}',y)\in\mathbb{R}^n \mid \mathbf{x}'\in\mathcal{S},\ y\in E\}$, $\partial_b\mathcal{C}_E := \partial\mathcal{S}\times E = \{(\mathbf{x}',y)\in\mathbb{R}^n \mid \mathbf{x}'\in\partial\mathcal{S},\ y\in E\}$ for any $y\in\mathbb{R}$, $\mathcal{C}_y := \mathcal{C}_{\{y\}}$, $\mathcal{C}^+_y := \mathcal{C}_{(y,+\infty)}$, $\mathcal{C}^-_y := \mathcal{C}_{(-\infty,y)}$, $\mathcal{C}^+ := \mathcal{C}^+_0$, $\mathcal{C}^- := \mathcal{C}^-_0$. $\|f\|_{L^n_*(\mathcal{C})} := \sup\limits_{y\in\mathbb{R}}\|f\|_{L^n(\mathcal{C}_{(y,y+2)})}$.

With the inhomogeneous term f, we use $\tilde{U}$ to denote the set of solutions bounded from below to the problem (1). If $f=0$, we use U to denote the positive solution set to the problem (1) (we will see $\tilde{U}=U$ with $f=0$).

For any u, $\hat{u}(y) := \sup_{\mathbf{x}'\in\mathcal{S}} u^+(\mathbf{x}', y)$, where $u^+ = \max\{u, 0\}$. Therefore, for any $u \in U$, $\hat{u}(y) = \sup_{\mathbf{x}'\in\mathcal{S}} u(\mathbf{x}', y)$, $y \in \mathbb{R}$.

For any $u \in U$, set $m(u) := \inf_{y\in\mathbb{R}} \hat{u}(y)$. $U^+ := \{u \in U \mid \lim_{y\to-\infty} u(\mathbf{x}', y) = 0\}$, $U^- := \{u \in U \mid \lim_{y\to+\infty} u(\mathbf{x}', y) = 0\}$, $U^\vee := \{u \in U \mid$ there exists $\mathbf{x}^* = (\mathbf{x}'^*, y^*) \in \mathcal{C}$ such that $u(\mathbf{x}^*) = m(u) > 0\}$.

Without loss of generality, we assume $\mathbf{0}' \in \mathcal{S}$.

One of our tool is a version of Aleksandrov-Bakelman-Pucci maximum principle in unbounded domains. This helps us to estimate and establish the existence and uniqueness of bounded solution in $\mathcal{C}$.

Theorem 1.1 (Aleksandrov-Bakelman-Pucci maximum principle)**.** *Assume $Lu(\mathbf{x}) \geq f(\mathbf{x})$, $\mathbf{x} \in \mathcal{C}$, and $u(\mathbf{x})$ is bounded from above. Then we have*

$$\sup_{\mathbf{x}\in\mathcal{C}} u^+(\mathbf{x}) \leq \sup_{\mathbf{x}\in\partial\mathcal{C}} u^+(\mathbf{x}) + C\|f\|_{L^n_*(\mathcal{C})},$$

where C only depends on $n, \gamma, diam(\mathcal{S})$.

Corollary 1.2. *Assume $Lu(\mathbf{x}) \geq f(\mathbf{x})$, $\mathbf{x} \in \mathcal{C}^+$, and $u(\mathbf{x})$ is bounded from above. Then we have*

$$\sup_{\mathbf{x}\in\mathcal{C}^+} u^+(\mathbf{x}) \leq \sup_{\mathbf{x}\in\partial\mathcal{C}^+} u^+(\mathbf{x}) + C\|f\|_{L^n_*(\mathcal{C}^+)},$$

where C only depends on $n, \gamma, diam(\mathcal{S})$.

Theorem 1.3. *Let L be given by* (2) *in $\mathcal{C} \subset \mathbb{R}^n$ and the coefficients satisfy* (4) *and* (5)*. If $f \in L^n_{loc}(\mathcal{C})$, then the Dirichlet problem*

$$\begin{cases} Lu(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{C}, \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{C}, \end{cases} \tag{6}$$

has a unique bounded solution $u \in W^{2,n}_{loc}(\mathcal{C}) \bigcap C(\bar{\mathcal{C}})$.

The next main result is about the exponential decay property of bounded solutions in $\mathcal{C}^+$.

Theorem 1.4. *Suppose $u(\mathbf{x})$ is bounded from above, and satisfies*

$$\begin{cases} Lu(\mathbf{x}) \geq f(\mathbf{x}), & \mathbf{x} \in \mathcal{C}^+, \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{S} \times (0, +\infty). \end{cases}$$

Then there exist constants $\alpha, C_0, C_1 > 0$ depending only on $n, \gamma, diam(\mathcal{S})$, such that

$$u(\mathbf{x}) \leq C_0\hat{u}(0)e^{-\alpha y} + C_1\|f\|_{L^n_*(\mathcal{C}^+)}, \qquad \mathbf{x} \in \mathcal{C}^+.$$

Followed by Theorem 1.4, we obtain a corollary in $\mathcal{C}^-$.

Corollary 1.5. *Suppose $u(\mathbf{x})$ is bounded from above, and satisfies*

$$\begin{cases} Lu(\mathbf{x}) \geq f(\mathbf{x}), & \mathbf{x} \in \mathcal{C}^-, \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{S} \times (-\infty, 0). \end{cases}$$

Then there exist constants $\alpha, C_0, C_1 > 0$ depending only on n, γ, diam($\mathcal{S}$), such that

$$u(\mathbf{x}) \le C_0 \hat{u}(0) e^{\alpha y} + C_1 \|f\|_{L_*^n(\mathcal{C}^-)}, \qquad \mathbf{x} \in \mathcal{C}^-.$$

Next, we pursue further the structure of solutions to (1). The maximum principe in Section 2 and the boundary Harnack inequality in Section 3 are proved with lower order terms. Therefore, we can prove the following theorems.

Theorem 1.6. *For the problem* (1)*, if $f = 0$, then the positive solution set U^+, U^- are well defined. And U is a linear combination of U^+ and U^-, that is, for any $u \in U^+, v \in U^-$, we have*

$$U = U^+ + U^- = \{pu + qv \,|\, p, q \ge 0,\ p + q > 0\}.$$

Theorem 1.7. *Assume $u(\mathbf{x})$ is a bounded solution of the problem*

$$\begin{cases} Lu(\mathbf{x}) & = & 0, & \mathbf{x} \in \mathcal{C}^+, \\ u(\mathbf{x}) & = & 0, & \mathbf{x} \in \partial\mathcal{C}^+ \backslash \mathcal{C}_0, \\ u(\mathbf{x}) & > & 0, & \mathbf{x} \in \mathcal{C}^+. \end{cases}$$

Then for any $v \in U^-$, there exist constants $\vartheta > 0$ depending only on $n, \gamma, \mathcal{S}$, and M, C depending only on $\hat{u}(1), n, \gamma, \mathcal{S}$, such that

$$|u(\mathbf{x}) - Mv(\mathbf{x})| \le Ce^{-\vartheta|\mathbf{x}|} v(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{C}_{(1,+\infty)}.$$

Theorem 1.8. *There exist constants $\alpha, \beta, C, C' > 0$ depending only on $n, \gamma, \mathcal{S}$, such that for any $u \in U^+$, $v \in U^-$, $w \in U^\vee$, and assume $\mathbf{x}^* = (\mathbf{x'}^*, y^*) \in \mathcal{C}$ such that $w(\mathbf{x}^*) = m(w)$, we have*

$$-C + \alpha y \le \ln(\frac{\hat{u}(y)}{\hat{u}(0)}) \le C' + \beta y, \qquad y \in (-\infty, +\infty),$$

$$-C' - \beta y \le \ln(\frac{\hat{v}(y)}{\hat{v}(0)}) \le C - \alpha y, \qquad y \in (-\infty, +\infty),$$

$$-C + \alpha|y - y^*| \le \ln(\frac{\hat{w}(y)}{\hat{w}(y^*)}) \le C' + \beta|y - y^*|, \qquad y \in (-\infty, +\infty).$$

For the inhomogeneous term f, we have the following desired structure of solutions bounded from below.

Theorem 1.9. *For the problem* (1)*, the set of solutions bounded from below can be represented by, for any $u \in U^+, v \in U^-$,*

$$\tilde{U} = U^0 + U^+ + U^- = \{u_0 + pu + qv \mid p, q \ge 0\},$$

where $U^0 = \{u_0\}$ is the bounded solution to $Lu_0 = f$ in $\mathcal{C}$ with zero boundary condition.

Theorem 1.9 has the following corollary, which has a more precise result than S. Agmon's [3].

Corollary 1.10. *Suppose $u(\mathbf{x})$ is a solution of the problem*

$$\begin{cases} Lu(\mathbf{x}) &= f(\mathbf{x}), & \mathbf{x} \in \mathcal{C}, \\ u(\mathbf{x}) &= 0, & \mathbf{x} \in \partial\mathcal{C}, \end{cases}$$

and satisfies $u(\pm\infty) = o(U^+ + U^-)$. Then we have

$$u(\mathbf{x}) \equiv u_0(\mathbf{x}), \qquad \forall\ \mathbf{x} \in \mathcal{C}.$$

Our paper is organized as follows. In Section 2, we mainly prove Aleksandrov maximum principle in an unbounded cylinder. In Section 3, we mostly demonstrate boundary Harnack inequality in a bounded cylinder. In Section 4, we mainly prove the existence and uniqueness of bounded solution and the exponential decay of bounded solutions. In the last section, we prove the structure theorems with inhomogeneous term f.

Finally we would like to remark that if we drop the requirement on the one-side boundedness on the solutions, there is no hope of classification of the growth behavior at infinity which one can observe by harmonic functions in long cylinder.

2. Aleksandrov maximum principle in an unbounded cylinder

In this section, we mainly prove Aleksandrov maximum principle in an unbounded cylinder $\mathcal{C} = \mathcal{S} \times \mathbb{R} \subseteq \mathbb{R}^n$. We notice that the diameter of $\mathcal{C}_{(k,k+2)}$ only depends on n and $diam(\mathcal{S})$, i.e., $diam(\mathcal{C}_{(k,k+2)})$ is independent of k and can be denoted by

$$diam(\mathcal{C}_{(k,k+2)}) = C(n, diam(\mathcal{S})).$$

Lemma 2.1. *There are constants $0 < \varepsilon_0, \delta < 1$, such that if $u(\mathbf{x})$ satisfies*

$$\begin{cases} Lu(\mathbf{x}) &\geq f(\mathbf{x}), & \mathbf{x} \in \mathcal{C}_{(k,k+2)}, \\ u(\mathbf{x}) &\leq 0, & \mathbf{x} \in \partial_b\mathcal{C}_{(k,k+2)}, \\ u(\mathbf{x}) &\leq 1, & \mathbf{x} \in \partial\mathcal{C}_{(k,k+2)}, \end{cases}$$

for $k \in \mathbb{Z}$ with $\|f\|_{L^n(\mathcal{C}_{(k,k+2)})} \leq \varepsilon_0$, then we have

$$u(\mathbf{x}', k+1) \leq 1 - \delta, \qquad \mathbf{x}' \in \mathcal{S}.$$

Proof. Suppose $v(\mathbf{x})$ satisfies

$$\begin{cases} Lv(\mathbf{x}) &= f(\mathbf{x}), & \mathbf{x} \in \mathcal{C}_{(k,k+2)}, \\ v(\mathbf{x}) &= \max\{u(\mathbf{x}), 0\}, & \mathbf{x} \in \partial\mathcal{C}_{(k,k+2)}. \end{cases}$$

By Aleksandrov maximum principle (Theorem 9.1) in [6], we have

$$u(\mathbf{x}) \leq v(\mathbf{x}) \leq 1 + C\|f\|_{L^n(\mathcal{C}_{(k,k+2)})} \leq 1 + C\varepsilon_0,\ \forall\ \mathbf{x} \in \mathcal{C}_{(k,k+2)},$$

where C depends only on n, γ and $\mathrm{diam}(\mathcal{S})$. With the boundary Hölder estimate (Corollary 9.29) in [6], there exists a constant $C_0 > 0$ depending only on $n, \gamma, \mathcal{S}$, such that $[v(\mathbf{x})]_{C^\alpha(\mathcal{C}_{(k+\frac{1}{2},k+\frac{3}{2})})} \leq C_0$, for any $\mathbf{x} \in \mathcal{C}_{(k+\frac{1}{2},k+\frac{3}{2})}$, with some $\alpha \in (0,1)$. Then we have $v(\mathbf{x}) \leq |v(\mathbf{x})| \leq C_0(\sigma_0)^\alpha \leq \frac{1}{2}$, if $\mathrm{dist}(\mathbf{x}, \partial_b\mathcal{C}_{(k,k+2)}) \leq \sigma_0$ is sufficiently small.

Now taking $\mathcal{C}'_{(k+\frac{1}{2},k+\frac{3}{2})} = \{\mathbf{x} \in \mathcal{C}_{(k+\frac{1}{2},k+\frac{3}{2})} : \mathrm{dist}(\mathbf{x}, \partial_b \mathcal{C}_{(k,k+2)} > \sigma_0\}$ for the σ_0 as above. Clearly, $1 + C\varepsilon_0 - v(\mathbf{x}) \geq 0$, and $L(1 + C\varepsilon_0 - v(\mathbf{x})) = -f(\mathbf{x}) + c(\mathbf{x})(1+C\varepsilon_0) \leq -f(\mathbf{x})$ in $\mathcal{C}_{(k,k+2)}$, hence $1+C\varepsilon_0 - v(\mathbf{x})$ is a nonnegative supersolution of $Lu(\mathbf{x}) = -f(\mathbf{x})$ in $\mathcal{C}_{(k,k+2)}$. We apply the weak Harnack inequality (Theorem 9.22) in [6] to $(1 + C\varepsilon_0 - v(\mathbf{x}))$ in $\mathcal{C}_{(k+\frac{1}{2},k+\frac{3}{2})}$, and we obtain for some $\eta > 0$

$$\begin{aligned} C_1 &\leq \{\frac{1}{|\mathcal{C}'_{(k+\frac{1}{2},k+\frac{3}{2})}|} \int_{\mathcal{C}'_{(k+\frac{1}{2},k+\frac{3}{2})}} (1 + C\varepsilon_0 - v)^\eta\}^{\frac{1}{\eta}} \\ &\leq C\{\inf_{\mathcal{C}'_{(k+\frac{1}{2},k+\frac{3}{2})}} (1 + C\varepsilon_0 - v) + \|f\|_{L^n(\mathcal{C}_{(k+\frac{1}{2},k+\frac{3}{2})})}\} \\ &\leq C\{\inf_{\mathcal{C}'_{(k+\frac{1}{2},k+\frac{3}{2})}} (1 + C\varepsilon_0 - v) + \varepsilon_0)\}. \end{aligned}$$

Therefore, for $\mathrm{dist}(\mathbf{x}, \partial_b \mathcal{C}_{(k,k+2)}) \geq \sigma_0, 1 - v(\mathbf{x}', k+1) \geq \frac{C_1}{C} - (1+C)\varepsilon_0 \geq \frac{C_1}{2C} > 0$ by taking $\varepsilon_0 \leq \frac{C_1}{2C(1+C)}$. Noting $1 - v(\mathbf{x}', k+1) \geq \frac{1}{2}$ for the rest of the points near the boundary, then the result follows by setting $\delta = \min(\frac{C_1}{2C}, \frac{1}{2})$. □

Remark 2.2. When L is in non-divergence form in Lemma 2.1, there is an alternative proof using barrier function. As a matter of fact, fix some constant $R > 0$ with $R \geq (3 + \mathrm{diam}(\mathcal{S}))$.

Split $u = v + w$ in $\mathcal{C}_{(0,2R)}$, where $v(\mathbf{x})$ satisfies

$$\begin{cases} Lv(\mathbf{x}) = 0, & \mathbf{x} \in \mathcal{C}_{(0,2R)}, \\ v(\mathbf{x}) = u(\mathbf{x}), & \mathbf{x} \in \partial\mathcal{C}_{(0,2R)}, \end{cases}$$

while $w(\mathbf{x})$ satisfies

$$\begin{cases} Lw(\mathbf{x}) \geq f(\mathbf{x}), & \mathbf{x} \in \mathcal{C}_{(0,2R)}, \\ w(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{C}_{(0,2R)}. \end{cases}$$

By Aleksandrov maximum principle (Theorem 9.1) in [6] respectively, we have

$$\sup_{\mathcal{C}_{(0,2R)}} v \leq 1, \qquad \sup_{\mathcal{C}_{(0,2R)}} w \leq C\|f\|_{L^n(\mathcal{C}_{(0,2R)})}. \tag{7}$$

Since $\mathcal{S}$ is a bounded Lipschitz domain, without loss of generality, we assume that there exists a unit ball $B_1(\mathbf{y}_0)$ such that $B_1(\mathbf{y}_0) \cap \mathcal{C}_{(0,2R)} = \emptyset$ and $\partial B_1(\mathbf{y}_0) \cap \partial_b \mathcal{C}_{(0,2R)} = (\mathbf{x}'_0, R)$. We also take a concentric ball $B_{\sqrt{1+R^2}}(\mathbf{y}_0)$ with $R \geq (3 + \mathrm{diam}(\mathcal{S}))$. For convenience, we assume in the sequel that $\mathbf{y}_0 = \mathbf{0}$, and write that $B_1(\mathbf{y}_0) = B_1$ and $B_{\sqrt{1+R^2}}(\mathbf{y}_0) = B_{\sqrt{1+R^2}}$.

Now, we construct a barrier function in $D = B_{\sqrt{1+R^2}} \setminus B_1$. For $p > 0$ to be determined later, we set

$$\psi(\mathbf{x}) = \frac{\frac{1}{|\mathbf{x}|^{2p}} - 1}{\frac{1}{(1+R^2)^p} - 1}, \qquad 1 \leq |\mathbf{x}| \leq \sqrt{1 + R^2}.$$

Clearly, $\psi(\mathbf{x}) = 0$ on ∂B_1; $\psi(\mathbf{x}) = 1$, on $\partial B_{\sqrt{1+R^2}}$; $0 < \psi(\mathbf{x}) < 1$ in D.

A direct calculation yields

$$\begin{aligned}
L\psi(\mathbf{x}) &= (\frac{1}{(1+R^2)^p}-1)^{-1}\{4p(p+1)\frac{a_{ij}x_ix_j}{|\mathbf{x}|^{2p+4}}-2p\frac{a_{ii}}{|\mathbf{x}|^{2p+2}} \\
&\qquad\qquad -2p\frac{b_ix_i}{|\mathbf{x}|^{2p+2}}+c(\frac{1}{|\mathbf{x}|^{2p}}-1)\} \\
&\le (\frac{1}{(1+R^2)^p}-1)^{-1}\{4p(p+1)\frac{a_{ij}x_ix_j}{|\mathbf{x}|^{2p+4}}-2p\frac{a_{ii}}{|\mathbf{x}|^{2p+2}}-2p\frac{b_ix_i}{|\mathbf{x}|^{2p+2}}\} \\
&\le (\frac{1}{(1+R^2)^p}-1)^{-1}\frac{2p}{|\mathbf{x}|^{2p+4}}\{2(p+1)\lambda|\mathbf{x}|^2-n\Lambda|\mathbf{x}|^2 \\
&\qquad\qquad -\|b\|_{L^\infty(C_{(0,2R)})}|\mathbf{x}|^3\},
\end{aligned}$$

where we used $c(\mathbf{x}) \le 0$ in $C_{(0,2R)} \cap D$, $b(\mathbf{x}) \in L^\infty(C_{(0,2R)}) \cap D$ and the ellipticity. If we choose $p > 0$ large enough, then we have

$$2(p+1)\lambda|\mathbf{x}|^2-n\Lambda|\mathbf{x}|^2-\|b\|_{L^\infty(C_{(0,2R)})}|\mathbf{x}|^3 \ge 0, \qquad \mathbf{x} \in C_{(0,2R)} \cap D.$$

Hence, we have

$$L\psi(\mathbf{x}) \le 0, \qquad \mathbf{x} \in C_{(0,2R)} \cap D.$$

We also can check that $v(\mathbf{x}) \le \psi(\mathbf{x})$ on $\partial(C_{(0,2R)} \cap D)$. Therefore, we have

$$\begin{cases} Lv(\mathbf{x}) & \ge & L\psi(\mathbf{x}), & \qquad \mathbf{x} \in \mathcal{C}_{(0,2R)} \cap D, \\ v(\mathbf{x}) & \le & \psi(\mathbf{x}), & \qquad \mathbf{x} \in \partial(\mathcal{C}_{(0,2R)} \cap D). \end{cases}$$

By maximum principle, we have

$$v(\mathbf{x}) \le \psi(\mathbf{x}), \quad \mathbf{x} \in \mathcal{C}_{(0,2R)} \cap D. \tag{8}$$

In particular, there exists a constant $\theta \in (0,1)$ such that

$$v(\mathbf{x}', R) \le \psi(\mathbf{x}', R) \le 1-\theta < 1, \quad \mathbf{x}' \in \mathcal{S}.$$

Noting that $u = v + w$, and combining (7) with (8), we have

$$u(\mathbf{x}) \le v(\mathbf{x}) + w(\mathbf{x}) \le \psi(\mathbf{x}) + C\|f\|_{L^n(\mathcal{C}_{(0,2R)})}, \quad \mathbf{x} \in \mathcal{C}_{(0,2R)} \cap D.$$

Taking $\varepsilon_0 \le \frac{\theta}{2C}$, and setting $\delta = \frac{\theta}{2}$, we have

$$u(\mathbf{x}', R) \le 1-\theta+C\varepsilon_0 \le 1-\frac{\theta}{2} \le 1-\delta, \quad \mathbf{x}' \in \mathcal{S}.$$

What's more, from the above, we also know, for any $\mathbf{x} \in \mathcal{C}_{(2kR,\ 2(k+1)R)}, k \in \mathbb{Z}$,

$$u(\mathbf{x}', (2k+1)R) \le 1-\delta, \quad \mathbf{x}' \in \mathcal{S}.$$

Remark 2.3. If we take the result in Remark 2.2 instead of the result in Lemma 2.1 to prove the corresponding theorems, we will use step size $2R$ with $R \ge (3 + \mathrm{diam}(\mathcal{S}))$ rather than step size 2.

Now we prove the Aleksandrov maximum principle in the unbounded cylinder $\mathcal{C}$, which is of interest in its own right. We remark that, to our best knowledge, such kind theorem depends on the diameter or the measure of the set as set forth in [4] and many other references. Our version of maximum principle is for an unbounded cylinder due to the zero boundary condition in our situation.

Proof of Theorem 1.1. The maximum principle has been proved with $f = 0$ by Lemma 2.2 in [1], here suppose $f \neq 0$.

We can assume $\sup\limits_{\mathbf{x}\in\partial\mathcal{C}} u^+(x) = 0$. Otherwise, we can consider $v(\mathbf{x}) = u(\mathbf{x}) - \sup\limits_{\mathbf{x}\in\partial\mathcal{C}} u^+(\mathbf{x})$. Therefore, we only need to prove

$$u^+(\mathbf{x}) \le C\|f\|_{L^n_*(\mathcal{C})}, \qquad \forall\ \mathbf{x} \in \mathcal{C}.$$

Suppose $u(\mathbf{x}) \le M$, since u has upper bound and M could be very large for now. Also let $\|f\|_{L^n_*(\mathcal{C})} = F$. Set $\mathcal{C}_{(k-1,k+1)} = \{\mathbf{x} = (\mathbf{x}', y) \in \mathbb{R}^n \mid \mathbf{x}' \in \mathcal{S},\ k-1 < y < k+1\}$, $k \in \mathbb{Z}$. In order to apply Lemma 2.1, we consider the function

$$\tilde{u}(\mathbf{x}) = \frac{\varepsilon_0 u(\mathbf{x})}{\varepsilon_0 \max\{\hat{u}(k-1), \hat{u}(k+1)\} + \|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}}, \qquad \mathbf{x} \in \mathcal{C}.$$

Thus, we obtain $\|\tilde{f}\|_{L^n(\mathcal{C}_{(k-1,k+1)})} \le \varepsilon_0$, where

$$\tilde{f}(\mathbf{x}) = \frac{\varepsilon_0 f(\mathbf{x})}{\varepsilon_0 \max\{\hat{u}(k-1), \hat{u}(k+1)\} + \|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}}, \qquad \mathbf{x} \in \mathcal{C}.$$

Obviously, $\tilde{u}(\mathbf{x})$ satisfies

$$\begin{cases} L\tilde{u}(\mathbf{x}) \ \ge\ \tilde{f}(\mathbf{x}), & \mathbf{x} \in \mathcal{C}_{(k-1,k+1)}, \\ \tilde{u}(\mathbf{x}) \ \le\ 0, & \mathbf{x} \in \partial_b\mathcal{C}_{(k-1,k+1)}, \\ \tilde{u}(\mathbf{x}) \ \le\ 1, & \mathbf{x} \in \partial\mathcal{C}_{(k-1,k+1)}. \end{cases}$$

Now we can apply Lemma 2.1 to $\tilde{u}(\mathbf{x})$ in $\mathcal{C}_{(k-1,k+1)}$, there exists a constant $\delta \in (0,1)$, such that $\tilde{u}(\mathbf{x}', k) \le (1-\delta)$, $\mathbf{x}' \in \mathcal{S}$, that is,

$$\begin{aligned} u(\mathbf{x}', k) &\le \frac{(1-\delta)}{\varepsilon_0}\{\varepsilon_0 \max\{\hat{u}(k-1), \hat{u}(k+1)\} + \|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}\} \\ &= (1-\delta)\{\max\{\hat{u}(k-1), \hat{u}(k+1)\} + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}\} \\ &\le (1-\delta)M + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}. \end{aligned}$$

Hence, we have

$$\begin{aligned} u^+(\mathbf{x}', k) &\le (1-\delta)\{\max\{\hat{u}(k-1), \hat{u}(k+1)\} + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}\} \\ &\le (1-\delta)M + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}. \end{aligned}$$

By the definition of $\hat{u}(y)$, we have

$$\begin{aligned}\hat{u}(k) &\le (1-\delta)\{\max\{\hat{u}(k-1),\hat{u}(k+1)\} + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}\}\\ &\le (1-\delta)M + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}.\end{aligned}$$

Then, for any $k \in \mathbb{Z}$,

$$\begin{aligned}\hat{u}(k) &\le (1-\delta)\max\{\hat{u}(k-1),\hat{u}(k+1)\} + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}\}\\ &\le (1-\delta)\max\{(1-\delta)\max\{\hat{u}(k-2),\hat{u}(k)\} + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-2,k)})},\\ &\quad (1-\delta)\max\{\hat{u}(k),\hat{u}(k+2)\} + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k,k+2)})}\}\\ &\quad + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}\\ &\le (1-\delta)^2\max\{\max\{\hat{u}(k-2),\hat{u}(k)\},\max\{\hat{u}(k),\hat{u}(k+2)\}\}\\ &\quad + \frac{(1-\delta)^2}{\varepsilon_0}F + \frac{(1-\delta)}{\varepsilon_0}F\\ &\ \ \vdots\\ &\le (1-\delta)^m M + \sum_{i=1}^{m}\frac{(1-\delta)^i}{\varepsilon_0}F\\ &= (1-\delta)^m M + \frac{F}{\varepsilon_0}\sum_{i=1}^{m}(1-\delta)^i.\end{aligned}$$

By applying Aleksandrov maximum principle (Theorem 9.1) in [6], for any $y \in [k-1,k+1]$, $k \in \mathbb{Z}$,

$$\begin{aligned}\hat{u}(y) &\le (1-\delta)^m M + \frac{F}{\varepsilon_0}\sum_{i=1}^{m}(1-\delta)^i + C\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}\\ &\le (1-\delta)^m M + \frac{F}{\varepsilon_0}\cdot\frac{1-\delta}{\delta} + CF,\end{aligned}$$

where C only depends on n,γ, $\mathrm{diam}(\mathcal{S})$. Letting $m \to \infty$, we have $\hat{u}(y) \le (\frac{1-\delta}{\varepsilon_0\delta} + C)F$, $\forall\, y \in \mathbb{R}$. Therefore, we obtain our result

$$u(\mathbf{x}) \le (\frac{1-\delta}{\varepsilon_0\delta} + C)\|f\|_{L^n_*(\mathcal{C})}, \qquad \forall\, \mathbf{x} \in \mathcal{C},$$

where $(\frac{1-\delta}{\varepsilon_0\delta} + C)$ depends on n,γ, $\mathrm{diam}(\mathcal{S})$. □

Similarly, we can obtain the Aleksandrov maximum principle in half cylinder $\mathcal{C}^+$ (or $\mathcal{C}^-$) by using the same method. Therefore, the proof of Corollary 1.2 is omitted here.

3. Boundary Harnack inequality

In this section, we mainly prove the boundary Harnack inequality in a bounded Lipschitz cylinder. The boundary Harnack inequality and comparison theorem are crucial for our consideration. Let $\psi(\mathbf{x}')$ be a Lipschitz function in $\mathbb{R}^{n-1}(n \geq 2)$ with a Lipschitz constant $\tilde{K}$, such that

$$|\psi(\mathbf{x}') - \psi(\mathbf{y}')| \leq \tilde{K}|\mathbf{x}' - \mathbf{y}'|, \qquad \forall\ \mathbf{x}', \mathbf{y}' \in \mathbb{R}^{n-1}. \tag{9}$$

For $r > 0$, denote

$$Q_r := \{\mathbf{x} = (\mathbf{x}', y) \in \mathbb{R}^n : |\mathbf{x}'| < r,\ 0 < y - \psi(\mathbf{x}') < r\},$$
$$\Gamma_r := \{\mathbf{x} = (\mathbf{x}', y) \in \mathbb{R}^n : |\mathbf{x}'| \leq r,\ y = \psi(\mathbf{x}')\} \subset \partial Q_r.$$

Lemma 3.1 (Boundary Harnack Inequality)**.** *Let $\psi(\mathbf{x}')$ be a function in $\mathbb{R}^{n-1}$ $(n \geq 2)$ satisfying the Lipschitz condition (9), $\psi(0) = 0$, and let $u(\mathbf{x})$ be a function in $W^{2,n}_{loc}(Q_{2r}) \bigcap C(\bar{Q}_{2r}), r > 0$, such that*

$$\begin{cases} Lu(\mathbf{x}) = 0, & \mathbf{x} \in Q_{2r}, \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \Gamma_{2r}, \\ u(\mathbf{x}) > 0, & \mathbf{x} \in Q_{2r}. \end{cases}$$

Then we have

$$\sup_{Q_r} u \leq C u(\mathbf{0}', r),$$

where C only depends on $n, \gamma, \tilde{K}$.

Proof. For convenience, we write

$$Lu(\mathbf{x}) = a_{ij}(\mathbf{x}) D_{ij} u(\mathbf{x}) + b_i(\mathbf{x}) D_i u(\mathbf{x}) + c(\mathbf{x}) u(\mathbf{x}),\ \mathbf{x} \in \mathcal{C}_{(k,k+2)}$$

as

$$Lu(\mathbf{x}) = a_{ij}(\mathbf{x}) u_{x_i x_j}(\mathbf{x}) + b_i(\mathbf{x}) u_{x_i}(\mathbf{x}) + c(\mathbf{x}) u(\mathbf{x}),\ \mathbf{x} \in \mathcal{C}_{(k,k+2)}.$$

We use a trick for adding new variables, clearly,

$$\begin{aligned} Lu(\mathbf{x}) = &\frac{a_{ij}(\mathbf{x})}{(1 + x_{n+1})}(1 + x_{n+1}) u_{x_i x_j} + b_i(\mathbf{x})((1 + x_{n+1}) u_{x_i})_{x_{n+1}} \\ &+ c(\mathbf{x})((1 + x_{n+1}) u)_{x_{n+1}} + d((1 + x_{n+1}) u)_{x_{n+1} x_{n+1}}, \end{aligned}$$

where $i, j = 1, 2, \ldots, n$, $|x_{n+1}| \leq \frac{1}{2}$, and d is a sufficiently large constant. Therefore, we can write

$$\begin{aligned} Lu(\mathbf{x}) = &\frac{a_{ij}(\mathbf{x})}{(1 + x_{n+1})}((1 + x_{n+1}) u)_{x_i x_j} + b_i(\mathbf{x})((1 + x_{n+1}) u)_{x_i x_{n+1}} \\ &+ c(\mathbf{x})((1 + x_{n+1}) u)_{x_{n+1}} + d((1 + x_{n+1}) u)_{x_{n+1} x_{n+1}}. \end{aligned}$$

Set $v(\mathbf{x}, x_{n+1}) = (1 + x_{n+1}) u(\mathbf{x})$, then

$$Lu(\mathbf{x}) = \frac{a_{ij}(\mathbf{x})}{(1 + x_{n+1})} v_{x_i x_j} + b_i(\mathbf{x}) v_{x_i x_{n+1}} + c(\mathbf{x}) v_{x_{n+1}} + d v_{x_{n+1} x_{n+1}}.$$

Denote

$$\tilde{a}_{kl}(\mathbf{x}, x_{n+1})v_{x_kx_l} = \frac{a_{ij}(\mathbf{x})}{(1+x_{n+1})}v_{x_ix_j} + b_i(\mathbf{x})v_{x_ix_{n+1}} + dv_{x_{n+1}x_{n+1}},$$

where $k, l = 1, 2, \ldots, n+1$.

Next, we prove $\tilde{a}_{kl}(\mathbf{x}, x_{n+1})$ satisfy uniformly elliptic condition, which is equivalent to prove $(\tilde{a}_{kl})$ is an $(n+1)\times(n+1)$ positive definite matrix.

We see, (a_{ij}) is an $n \times n$ positive definite matrix, and $x_{n+1} \in [-\frac{1}{2}, \frac{1}{2}]$, therefore, $(\frac{a_{ij}}{1+x_{n+1}})$ is an $n\times n$ positive definite matrix. Set $A = (\frac{a_{ij}}{1+x_{n+1}})$, $b = (b_1, b_2, \ldots, b_n)'$, then, we have

$$(\tilde{a}_{kl}) = \begin{pmatrix} A & b \\ b' & d \end{pmatrix}.$$

We know, $|A| > 0$. Since d sufficiently large, $|(\tilde{a}_{kl})| = d|A| - |b|^2 > 0$, therefore, $(\tilde{a}_{kl})$ is an $(n+1)\times(n+1)$ positive definite matrix.

As above, we can write $Lu(\mathbf{x}) = \tilde{a}_{kl}(\mathbf{x}, x_{n+1})v_{x_kx_l} + c(\mathbf{x})v_{x_{n+1}}$. Similarly, we add a variable again, then this equation can be written as

$$\begin{aligned} Lu(\mathbf{x}) = {} & \frac{\tilde{a}_{kl}(\mathbf{x}, x_{n+1})}{(1+x_{n+2})}((1+x_{n+2})v)_{x_kx_l} + c(\mathbf{x})((1+x_{n+2})v)_{x_{n+1}x_{n+2}} \\ & + (e(1+x_{n+2})v)_{x_{n+2}x_{n+2}}, \end{aligned}$$

where $k, l = 1, 2, \ldots, n+1, |x_{n+2}| \le \frac{1}{2}$, and e is a sufficiently large constant. Set $w(\mathbf{x}, x_{n+1}, x_{n+2}) = (1+x_{n+2})v(\mathbf{x}, x_{n+1})$, then the equation can be simplified as

$$Lu(\mathbf{x}) = \frac{\tilde{a}_{kl}(\mathbf{x}, x_{n+1})}{(1+x_{n+2})}w_{x_kx_l} + c(\mathbf{x})w_{x_{n+1}x_{n+2}} + ew_{x_{n+2}x_{n+2}}.$$

Denote

$$\hat{a}_{st}(\mathbf{x}, x_{n+1}, x_{n+2})w_{x_sx_t} = \frac{\tilde{a}_{kl}(\mathbf{x}, x_{n+1})}{(1+x_{n+2})}w_{x_kx_l} + c(\mathbf{x})w_{x_{n+1}x_{n+2}} + ew_{x_{n+2}x_{n+2}},$$

which is to say,

$$Lu(\mathbf{x}) = \hat{a}_{st}(\mathbf{x}, x_{n+1}, x_{n+2})w_{x_sx_t},$$

where $s, t = 1, 2, \ldots, n+2$, $w(\mathbf{x}, x_{n+1}, x_{n+2}) = (1+x_{n+1})(1+x_{n+2})u(\mathbf{x})$.

Similarly, we can show $(\hat{a}_{st})$ is an $(n+2)\times(n+2)$ positive definite matrix, provided e sufficiently large, then $\hat{a}_{st}(\mathbf{x}, x_{n+1}, x_{n+2})$ satisfy uniformly elliptic condition.

Denote $L_0w = \hat{a}_{st}(\tilde{\mathbf{x}})w_{x_sx_t}$, $\tilde{\mathbf{x}} = (\mathbf{x}, x_{n+1}, x_{n+2}) \in \mathbb{R}^{n+2}$, and $\tilde{Q}_{2r} = \{\tilde{\mathbf{x}} = (\mathbf{x}', y, x_{n+1}, x_{n+2}) \in \mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} : |\mathbf{x}'| < 2r,\ 0 < y - \psi(\mathbf{x}') < 2r,\ |x_{n+1}| < \frac{1}{2},\ |x_{n+2}| < \frac{1}{2}\}$. Then $w(\tilde{\mathbf{x}})$ satisfies the uniformly elliptic equations

$$\begin{cases} L_0w(\tilde{\mathbf{x}}) = 0, & \tilde{\mathbf{x}} \in \tilde{Q}_{2r}, \\ w(\tilde{\mathbf{x}}) = 0, & \tilde{\mathbf{x}} \in \tilde{\Gamma}_{2r}, \\ w(\tilde{\mathbf{x}}) > 0, & \tilde{\mathbf{x}} \in \tilde{Q}_{2r}, \end{cases}$$

where $\tilde{\Gamma}_{2r} = \{\tilde{\mathbf{x}} = (\mathbf{x}', y, x_{n+1}, x_{n+2}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : |\mathbf{x}'| \leq 2r,\ y = \psi(\mathbf{x}'),\ |x_{n+1}| \leq \frac{1}{2},\ |x_{n+2}| \leq \frac{1}{2}\}$. For this problem, we have a boundary Harnack inequality (Theorem 3.4) in [9]

$$\sup_{\tilde{Q}_r} w \leq C_1 w(\mathbf{0}', r, 0, 0),$$

where $\tilde{Q}_r = \{\tilde{\mathbf{x}} = (\mathbf{x}', y, x_{n+1}, x_{n+2}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : |\mathbf{x}'| < r,\ 0 < y - \psi(\mathbf{x}') < r,\ |x_{n+1}| < \frac{1}{4},\ |x_{n+2}| < \frac{1}{4}\}$ and C_1 only depends on $n, \gamma, \tilde{K}$. Since $w = (1 + x_{n+1})(1 + x_{n+2})u$, we replace w by u, then we can derive our result

$$\sup_{Q_r} u \leq Cu(\mathbf{0}', r),$$

where C only depends on $n, \gamma, \tilde{K}$. □

Remark 3.2. When L is in divergence form in Lemma 3.1, a similar proof can be found in Lemma 3.1. Here, we just point out some key steps. For convenience, we write

$$Lu(\mathbf{x}) = D_i(a_{ij}(\mathbf{x})D_j u(\mathbf{x})) + b_i(\mathbf{x})D_i u(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}),\ \mathbf{x} \in \mathcal{C}_{(k,k+2)}$$

as

$$Lu(\mathbf{x}) = \partial_i(a_{ij}(\mathbf{x})\partial_j u(\mathbf{x})) + b_i(\mathbf{x})\partial_i u(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}),\ \mathbf{x} \in \mathcal{C}_{(k,k+2)}.$$

We use a trick for adding new variables, set $u(\mathbf{x}, x_{n+1}) = u(\mathbf{x})$, then

$$\begin{aligned} Lu(\mathbf{x}) = {} & \partial_i(a_{ij}(\mathbf{x})\partial_j u) + \partial_{n+1}(b_i(\mathbf{x})x_{n+1}\partial_i u) + \partial_i(b_i(\mathbf{x})x_{n+1}\partial_{n+1}u) \\ & + d\partial_{n+1,n+1}u + \partial_{n+1}(c(\mathbf{x})x_{n+1}u), \end{aligned}$$

where $i, j = 1, 2, \ldots, n, |x_{n+1}| \leq \frac{1}{2}$, and d is a sufficiently large constant. Therefore, we can denote

$$Lu(\mathbf{x}) = \partial_k(\tilde{a}_{kl}(\mathbf{x}, x_{n+1})\partial_l u) + \partial_{n+1}(c(\mathbf{x})x_{n+1}u),$$

where $k, l = 1, 2, \ldots, n+1$. Like the proof in Lemma 3.1, we know $\tilde{a}_{kl}(\mathbf{x}, x_{n+1})$ satisfy uniformly elliptic condition by taking d large enough.

Similarly, we add a variable again, then this equation can be written as

$$\begin{aligned} Lu(\mathbf{x}) = {} & \partial_k[\frac{\tilde{a}_{kl}(\mathbf{x}, x_{n+1})}{(1 + x_{n+2})}\partial_l((1 + x_{n+2})u)] + \partial_{n+1}[c(\mathbf{x})x_{n+1}\partial_{n+2}((1 + x_{n+2})u)] \\ & + \partial_{n+2}[c(\mathbf{x})x_{n+1}\partial_{n+1}(1 + x_{n+2})u)] + e\partial_{n+2,n+2}[(1 + x_{n+2})u], \end{aligned}$$

where $k, l = 1, 2, \ldots, n+1,\ |x_{n+2}| \leq \frac{1}{2}$, and e is a sufficiently large constant. Set $v(\mathbf{x}, x_{n+1}, x_{n+2}) = (1 + x_{n+2})u(\mathbf{x}, x_{n+1})$, then we can rewrite as follows

$$\begin{aligned} Lu(\mathbf{x}) = {} & \partial_k(\frac{\tilde{a}_{kl}(\mathbf{x}, x_{n+1})}{(1 + x_{n+2})}\partial_l v) + \partial_{n+1}(c(\mathbf{x})x_{n+1}\partial_{n+2}v) + \partial_{n+2}(c(\mathbf{x})x_{n+1}\partial_{n+1}v) \\ & + e\partial_{n+2,n+2}v, \end{aligned}$$

and further, we can denote as

$$Lu(\mathbf{x}) = \partial_s(\hat{a}_{st}(\mathbf{x}, x_{n+1}, x_{n+2})\partial_t v),$$

where $s, t = 1, 2, \ldots, n+2$. If we take e large enough, then $\hat{a}_{st}(\mathbf{x}, x_{n+1}, x_{n+2})$ satisfy uniformly elliptic condition. The following proof is similar as that corresponds to the proof in Lemma 3.1, and finally we obtain the desired result.

From Lemma 3.1, we have the following corollary, which is the key to proving the asymptotic behaviors of positive solutions.

Corollary 3.3. *Let $u \in U$. There exists a universal constant C depending only on $n, \gamma, \mathcal{S}$, such that for any $y \in \mathbb{R}$, we have*

$$u(\mathbf{x}) \leq Cu(\mathbf{0}', y), \qquad \mathbf{x} \in \mathcal{C}_{(y-2,y+2)}.$$

Lemma 3.4 (Comparison Theorem)**.** *Let $\psi(\mathbf{x}')$ be a function in $\mathbb{R}^{n-1}(n \geq 2)$ satisfying the Lipschitz condition* (9)*, $\psi(0) = 0$, and let $u_1(\mathbf{x}), u_2(\mathbf{x})$ be functions in $W^{2,n}_{loc}(Q_{2r}) \bigcap C(\bar{Q}_{2r}), r > 0$, such that*

$$\begin{cases} Lu_{1,2}(\mathbf{x}) &=& 0, & \mathbf{x} \in Q_{3r}, \\ u_{1,2}(\mathbf{x}) &=& 0, & \mathbf{x} \in \Gamma_{3r}, \\ u_{1,2}(\mathbf{x}) &>& 0, & \mathbf{x} \in Q_{3r}. \end{cases}$$

Then we have

$$\sup_{Q_r} \frac{u_1}{u_2} \leq \tilde{C}^2 \inf_{Q_r} \frac{u_1}{u_2},$$

where $\tilde{C}$ only depends on $n, \gamma, \tilde{K}$.

Proof. Similarly, we can use the same method in Lemma 3.1 and combine with Corollary 3.7 in [9] to obtain our conclusion. Here, we omit detail proofs. $\square$

Likewise, we have the following corollary in some bounded cylinder.

Corollary 3.5. *Let $u_1, u_2 \in U$, and $u_1(\mathbf{0}', 0) = u_2(\mathbf{0}', 0)$. Then there exists a universal constant K depending only on $n, \gamma, \mathcal{S}$, such that for some $k \in \mathbb{Z}^+$*

$$\frac{1}{K} u_2(\mathbf{x}) \leq u_1(\mathbf{x}) \leq K u_2(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{C}_{[-k,k]}.$$

Remark 3.6. If the condition $u_1(\mathbf{0}', 0) = u_2(\mathbf{0}', 0)$ is replaced by $u_1(\mathbf{0}', 0) \leq u_2(\mathbf{0}', 0)$, the result also holds.

Now from Corollary 3.5, we obtain a lemma to compare the solutions.

Lemma 3.7. *Let $u_1, u_2 \in U$. If there exists a point $\mathbf{x}_0 = (\mathbf{x}'_0, y_0) \in \mathcal{C}$ such that $u_1(\mathbf{x}_0) = u_2(\mathbf{x}_0)$, then there exists a universal constant τ such that*

$$\tau u_2(\mathbf{x}', y_0) \leq u_1(\mathbf{x}', y_0) \leq \frac{1}{\tau} u_2(\mathbf{x}', y_0), \qquad \mathbf{x}' \in \mathcal{S}.$$

Proof. Set $\tilde{u}_1(\tilde{\mathbf{x}}', y) = u_1(\mathbf{x}'_0 - \tilde{\mathbf{x}}', y_0 - y)$, $\tilde{u}_2(\tilde{\mathbf{x}}', y) = u_2(\mathbf{x}'_0 - \tilde{\mathbf{x}}', y_0 - y)$. Then, we get $\tilde{u}_1(\mathbf{0}', 0) = \tilde{u}_2(\mathbf{0}', 0)$, from Corollary 3.5, there exists $K > 0$, such that $\frac{1}{K}\tilde{u}_2(\tilde{\mathbf{x}}) \leq \tilde{u}_1(\tilde{\mathbf{x}}) \leq K\tilde{u}_2(\tilde{\mathbf{x}})$, $\forall\ \tilde{\mathbf{x}} \in \mathcal{C}_{[-k,k]}$. Taking $y = 0$, we have $\frac{1}{K}\tilde{u}_2(\tilde{\mathbf{x}}', 0) \leq \tilde{u}_1(\tilde{\mathbf{x}}', 0) \leq K\tilde{u}_2(\tilde{\mathbf{x}}', 0)$, that is,

$$\frac{1}{K} u_2(\mathbf{x}'_0 - \tilde{\mathbf{x}}', y_0) \leq u_1(\mathbf{x}'_0 - \tilde{\mathbf{x}}', y_0) \leq K u_2(\mathbf{x}'_0 - \tilde{\mathbf{x}}', y_0).$$

Taking $\mathbf{x}' = \mathbf{x}'_0 - \tilde{\mathbf{x}}' \in \mathcal{S},\ \tau = \frac{1}{K}$, then we get

$$\tau u_2(\mathbf{x}', y_0) \le u_1(\mathbf{x}', y_0) \le \frac{1}{\tau} u_2(\mathbf{x}', y_0), \qquad \mathbf{x}' \in \mathcal{S}. \qquad \square$$

Remark 3.8. If the condition $u_1(\mathbf{x}_0) = u_2(\mathbf{x}_0)$ is replaced by $u_1(\mathbf{x}_0) \le u_2(\mathbf{x}_0)$, the result also holds.

4. The exponential decay of bounded solutions

In this section, we first demonstrate the existence and uniqueness of bounded solution in the unbounded cylinder, which plays an important role in proving the structure theorem with inhomogeneous term f. Then we show the exponential decay of bounded solutions.

Proof of Theorem 1.3. Firstly, we consider the following equations in the bounded cylinder

$$\begin{cases} Lu(\mathbf{x}) &= f(\mathbf{x}), & \mathbf{x} \in \mathcal{C}_{(-N,N)}, \\ u(\mathbf{x}) &= 0, & \mathbf{x} \in \partial\mathcal{C}_{(-N,N)}, \end{cases}$$

where $N \in \mathbb{Z}^+$. By the elliptic theory of strong solutions (Theorem 9.30) in [6], for this problem, there exists a unique solution $u_N(\mathbf{x}) \in W^{2,n}_{loc}(\mathcal{C}_{(-N,N)}) \bigcap C(\bar{\mathcal{C}}_{(-N,N)})$.

By Aleksandrov maximum principle (Theorem 9.1) in [6], we have

$$\|u_N\|_{L^\infty(\mathcal{C}_{(-N,N)})} \le C_N \|f\|_{L^n(\mathcal{C}_{(-N,N)})},$$

where C_N depends only on n, γ, N.

We will prove there exists a constant $C_0 > 0$ not depending on N, such that

$$\|u_N\|_{L^\infty(\mathcal{C}_{(-N,N)})} \le C_0 \|f\|_{L^n(\mathcal{C}_{(-N,N)})}.$$

For convenience, we denote $M = \|u_N\|_{L^\infty(\mathcal{C}_{(-N,N)})}$.

For any $\xi :\ -N+1 \le \xi \le N-1$, clearly, $\mathcal{C}_{(\xi-1,\xi+1)} \subset \mathcal{C}_{(-N,N)}$. Like the proof of Theorem 1.1 in $\mathcal{C}_{(\xi-1,\xi+1)}$, we have

$$u_N(\mathbf{x}', \xi) \le (1-\delta)M + \frac{1-\delta}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(-N,N)})},\ \mathbf{x}' \in \mathcal{S},$$

then we have

$$\sup_{\xi \in (-N+1,N-1)} \hat{u}_N(\xi) \le (1-\delta)M + \frac{1-\delta}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(-N,N)})},$$

that is,

$$\sup_{\mathbf{x} \in \mathcal{C}_{(-N+1,N-1)}} u_N(\mathbf{x}) \le (1-\delta)M + \frac{1-\delta}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(-N,N)})}.$$

We have further that

$$\sup_{\mathbf{x} \in \mathcal{C}_{(-N+1,N-1)}} |u_N(\mathbf{x})| \le (1-\delta)M + \frac{1-\delta}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(-N,N)})}.$$

For any $\mathbf{x} \in \mathcal{C}_{(-N,-N+1)}$, by Aleksandrov maximum principle (Theorem 9.1) in [6] again, we have

$$\begin{aligned}
\|u_N\|_{L^\infty(\mathcal{C}_{(-N,-N+1)})} &\leq \|u_N\|_{L^\infty(\partial\mathcal{C}_{(-N,-N+1)})} + C_1\|f\|_{L^n(\mathcal{C}_{(-N,-N+1)})} \\
&\leq \sup_{\mathbf{x}\in\mathcal{C}_{(-N+1,N-1)}} |u_N(\mathbf{x})| + C_1\|f\|_{L^n(\mathcal{C}_{(-N,-N+1)})} \\
&\leq (1-\delta)M + \frac{1-\delta}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(-N,N)})} + C_1\|f\|_{L^n(\mathcal{C}_{(-N,N)})} \\
&= (1-\delta)M + C_2\|f\|_{L^n(\mathcal{C}_{(-N,N)})},
\end{aligned}$$

where $C_2 = \frac{1-\delta}{\varepsilon_0} + C_1$ depends only on $n, \gamma, \operatorname{diam}(\mathcal{S})$.

For any $\mathbf{x} \in \mathcal{C}_{(N-1,N)}$, similarly, we have

$$\|u_N\|_{L^\infty(\mathcal{C}_{(N-1,N)})} \leq (1-\delta)M + C_2'\|f\|_{L^n(\mathcal{C}_{(-N,N)})},$$

where $C_2' = \frac{1-\delta}{\varepsilon_0} + C_1'$ depends only on $n, \gamma, \operatorname{diam}(\mathcal{S})$.

Therefore, for any $\mathbf{x} \in \mathcal{C}_{(-N,N)}$, we have

$$\|u_N\|_{L^\infty(\mathcal{C}_{(-N,N)})} \leq (1-\delta)M + 2C_3\|f\|_{L^n(\mathcal{C}_{(-N,N)})},$$

where $C_3 = \max\{C_2, C_2'\}$ depends only on $n, \gamma, \operatorname{diam}(\mathcal{S})$. That is, from the definition of M, we have $M \leq (1-\delta)M + 2C_3\|f\|_{L^n(\mathcal{C}_{(-N,N)})}$. Thus we obtain

$$\|u_N\|_{L^\infty(\mathcal{C}_{(-N,N)})} = M \leq \frac{2}{\delta}C_3\|f\|_{L^n(\mathcal{C}_{(-N,N)})} = C_0\|f\|_{L^n(\mathcal{C}_{(-N,N)})},$$

where $C_0 = \frac{2}{\delta}C_3$ depends only on $n, \gamma, \operatorname{diam}(\mathcal{S})$.

Then, with the boundary Hölder estimate (Corollary 9.29) in [6], there exists a constant $C_* > 0$ depending only on $n, \gamma, \mathcal{S}$, such that

$$[u_N(\mathbf{x})]_{C^\alpha(\mathcal{C}_{[-N,N]})} \leq C_*, \ \alpha \in (0,1).$$

Thus, for any bounded domain $\mathcal{C}_{[-l,l]}$ with $l > 0$, by Arzela-Ascoli theorem, there exists a subsequence of $\{u_N(\mathbf{x})\}$ which uniformly converges in $\mathcal{C}_{[-l,l]}$. Without loss of generality, we can assume that there exists a function $u(\mathbf{x})$ such that $u_N(\mathbf{x})$ uniformly converges to $u(\mathbf{x})$ in $W^{2,n}_{loc}(\mathcal{C}) \bigcap C(\bar{\mathcal{C}})$. Therefore, $u(\mathbf{x})$ is bounded in $\mathcal{C}$ and satisfies (6). By Aleksandrov maximum principle (Theorem 1.1), we know u is the desired unique bounded solution. □

Next, we give the proof of the exponential decay of bounded solutions (Theorem 1.4) with inhomogeneous term f.

Proof of Theorem 1.4. Assume $\|f\|_{L^n_*(\mathcal{C}^+)} = F$. Since u is bounded from above, we can apply Corollary 1.2,

$$\hat{u}(y) \leq \hat{u}(0) + C\|f\|_{L^n_*(\mathcal{C}^+)} = \hat{u}(0) + CF, \qquad \forall\ y \in (0, +\infty).$$

By applying Lemma 2.1, there exists a constant $\delta \in (0,1)$, such that

$$\hat{u}(1) \leq \frac{(1-\delta)}{\varepsilon_0}\{\varepsilon_0 \max\{\hat{u}(0), \hat{u}(2)\} + \|f\|_{L^n(\mathcal{C}_{(0,2)})}\}$$

$$
\begin{aligned}
&\le (1-\delta)\max\{\hat{u}(0), \hat{u}(0) + C\|f\|_{L^n_*(\mathcal{C}^+)}\} + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(0,2)})}\\
&\le (1-\delta)(\hat{u}(0) + C\|f\|_{L^n_*(\mathcal{C}^+)}) + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(0,2)})}\\
&\le (1-\delta)(\hat{u}(0) + CF) + \frac{(1-\delta)}{\varepsilon_0}F.
\end{aligned}
$$

$$
\begin{aligned}
\hat{u}(2) &\le \frac{(1-\delta)}{\varepsilon_0}\{\varepsilon_0 \max\{\hat{u}(1), \hat{u}(3)\} + \|f\|_{L^n(\mathcal{C}_{(1,3)})}\}\\
&\le (1-\delta)\max\{(1-\delta)\max\{\hat{u}(0), \hat{u}(2)\} + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(0,2)})},\\
&\quad (1-\delta)\max\{\hat{u}(2), \hat{u}(4)\} + \frac{(1-\delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(2,4)})}\} + \frac{1-\delta}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(1,3)})}\\
&\le (1-\delta)^2\max\{\hat{u}(0) + CF, \hat{u}(0) + CF\} + \frac{(1-\delta)^2}{\varepsilon_0}F + \frac{1-\delta}{\varepsilon_0}F\\
&= (1-\delta)^2(\hat{u}(0) + CF) + \frac{(1-\delta)^2}{\varepsilon_0}F + \frac{1-\delta}{\varepsilon_0}F.
\end{aligned}
$$

Do the operation repeatedly, we obtain

$$
\begin{aligned}
\hat{u}(k) &\le (1-\delta)^k(\hat{u}(0) + CF) + \sum_{i=1}^{k}\frac{(1-\delta)^i}{\varepsilon_0}F\\
&\le (1-\delta)^k(\hat{u}(0) + CF) + \frac{F}{\varepsilon_0}\cdot\frac{1-\delta}{\delta}.
\end{aligned}
$$

Therefore, we have the following estimate, for $\mathbf{x} = (\mathbf{x}', y) \in \mathcal{C}^+$,

$$
\begin{aligned}
u(\mathbf{x}) &\le \max\{\hat{u}([y]), \hat{u}([y]+1)\} + CF\\
&\le (1-\delta)^{[y]}(\hat{u}(0) + CF) + \frac{F}{\varepsilon_0}\cdot\frac{1-\delta}{\delta} + CF\\
&\le (1-\delta)^{y-1}(\hat{u}(0) + CF) + \frac{F}{\varepsilon_0}\cdot\frac{1-\delta}{\delta} + CF\\
&= \frac{(\hat{u}(0) + CF)}{(1-\delta)}e^{y\ln(1-\delta)} + \frac{F}{\varepsilon_0}\cdot\frac{1-\delta}{\delta} + CF\\
&= \frac{(\hat{u}(0) + CF)}{(1-\delta)}e^{-\alpha y} + \frac{F}{\varepsilon_0}\cdot\frac{1-\delta}{\delta} + CF,
\end{aligned}
$$

where $\alpha = -\ln(1-\delta) > 0$. Since $F = \|f\|_{L^n_*(\mathcal{C}^+)}$, we have

$$
\begin{aligned}
u(\mathbf{x}) &\le C_0(\hat{u}(0) + F)e^{-\alpha y} + (\frac{1-\delta}{\varepsilon_0\delta} + C)F\\
&\le C_0\hat{u}(0)e^{-\alpha y} + (\frac{1-\delta}{\varepsilon_0\delta} + C + C_0)\|f\|_{L^n_*(\mathcal{C}^+)}, \quad \mathbf{x} \in \mathcal{C}^+. \qquad \square
\end{aligned}
$$

The proof of Corollary 1.5 is similar as that Theorem 1.4. Here we omit the proof.

5. The structure theorems of solutions

In this section, we show the structure theorems of solutions with inhomogeneous term f. In the case of $f = 0$, the structure theorem is stated as Theorem 1.6. The asymptotic behaviors of the solutions are stated as Theorem 1.7 and Theorem 1.8. The details of proofs for these theorems are covered in [1], and are omitted here.

For the general f, we give the proof of Theorem 1.9.

Proof of Theorem 1.9. Applying Theorem 1.3, we take v such that v is the unique bounded solution of the following problem

$$\left\{ \begin{array}{rcll} Lv(\mathbf{x}) & = & f(\mathbf{x}) & \mathbf{x} \in \mathcal{C}, \\ v(\mathbf{x}) & = & 0 & \mathbf{x} \in \partial\mathcal{C}, \end{array} \right. \tag{10}$$

then we know that there exists a constant $C > 0$, such that $u - v > -C$, and $u - v$ satisfies

$$\left\{ \begin{array}{rcll} L(u-v)(\mathbf{x}) & = & 0 & \mathbf{x} \in \mathcal{C}, \\ (u-v)(\mathbf{x}) & = & 0 & \mathbf{x} \in \partial\mathcal{C}, \\ (u-v)(\mathbf{x}) & > & -C & \mathbf{x} \in \mathcal{C}. \end{array} \right. \tag{11}$$

Since $u-v$ is bounded below, by applying Theorem 1.1 (Aleksandrov maximum principle), we obtain $u-v \geq 0$. Thus, either $u \equiv v$, or $u-v > 0$. If $u = v$, then our conclusion clearly holds (taking $p = q = 0$); If $u - v > 0$, by applying the Theorem 1.6, we derive, there exist $w \in U^+,\ z \in U^-$, such that $u-v = pw+qz$, that is, $u = v+pw+qz$, where v is the bounded solution. Therefore, we obtain our conclusion

$$\tilde{U} = U_0 + U^+ + U^-,$$

where $U_0 = \{v\}$ is the unique bounded solution of the problem (10). $\square$

It's easy to obtain Corollary 1.10 from Theorem 1.9. Therefore, we omit the proof here.

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