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A GRADED MINIMAL FREE RESOLUTION OF THE m-TH ORDER SYMBOLIC POWER OF A STAR CONFIGURATION IN $\mathbb{P}^n$

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Abstract. In [30] the author finds a graded minimal free resolution of the 2-nd order symbolic power of a star configuration in $\mathbb{P}^n$ of any codimension r. In this paper, we find that of any m-th order symbolic power of a star configuration in $\mathbb{P}^n$ of codimension 2, which generalizes the result of Galetto, Geramita, Shin, and Van Tuyl in [15, Theorem 5.3]. Furthermore, we extend it to the m-th order symbolic power of a star configuration in $\mathbb{P}^n$ of any codimension r for $m = 3, 4$, which also generalizes the result of Biermann et al. in [1, Corollaries 4.6 and 5.7]. We also suggest how to find a graded minimal free resolution of the m-th order symbolic power of a star configuration in $\mathbb{P}^n$ of any codimension r for $m \geq 5$.

1. Introduction

Let $R = \Bbbk[x_0, x_1, \ldots, x_n]$ be an $(n+1)$-variable polynomial ring over an infinite field $\Bbbk$ of any characteristic and let I be a homogeneous ideal of R (or the ideal $I_{\mathbb{X}}$ of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$). Note that R/I has a *graded minimal free resolution* $\mathbb{F}$, as an R-module, of the form:

$$\mathbb{F}: \quad 0 \to \mathbb{F}_m \to \cdots \to \mathbb{F}_i \xrightarrow{\varphi_i} \mathbb{F}_{i-1} \to \cdots \to \mathbb{F}_1 \to R \to R/I \to 0,$$

where the $\mathbb{F}_i$ are free graded R-modules and the image of each homomorphism φ_i of free modules in the resolution lies in $(x_0, x_1, \ldots, x_n)\mathbb{F}_{i-1}$. In fact,

$$\mathbb{F}_i := \bigoplus_{t=0}^{r_i} R(-(i+1+t))^{\beta_{i,i+1+t}}.$$

The numbers $\{\beta_{i,j}\}$ (resp. $\{i+j\}$) for $0 \leq i \leq m$ are called the i^{th} *graded Betti numbers* (resp. *shifts*) of the ideal I.

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For positive integers r and s with $1 \leq r \leq \min\{n, s\}$, suppose $F_1, \ldots, F_s$ are general forms in R of degrees $d_1, \ldots, d_s$, respectively. We call the variety $\mathbb{X}$ defined by the ideal

$$\bigcap_{1 \leq i_1 < \cdots < i_r \leq s} (F_{i_1}, \ldots, F_{i_r})$$

a star configuration in $\mathbb{P}^n$ of type (r, s) (or simply codimension r). In particular, if $F_1, \ldots, F_s$ are general linear forms in R, then we call the variety $\mathbb{X}$ *a linear star configuration in* $\mathbb{P}^n$ *of type* (r, s).

Configurations of this type have been a very interesting family of points, curves, hypersurfaces, and so on. For example, their defining ideals are easy to describe algebraically ([27]). In addition, Bocci and Harbourne [3] have shown that these sets of points exhibit some nice extremal properties (see also [19]), and Catalisano, Geramita, Gimigliano, and Shin [8] have studied star-configurations in $\mathbb{P}^2$ to calculate the dimensions of the secant varieties to the varieties of reducible curves (see also [7, 28]). There have been many other papers which have further explored the properties of star-configurations in $\mathbb{P}^n$ (see [3, 5, 6, 8, 10, 17, 28, 29]).

In recent years, motivated by works of Ein-Lazarsfeld-Smith [13] and Hochster-Huneke [24], comparisons between the symbolic and the regular powers of I have attracted attention among the researchers in Commutative Algebra and Algebraic Geometry (see [14,20,21,26]). For example, there is a question called *the ideal containment problem on finding* all the pairs (m, r) of integers such that $I^{(m)} \subset I^r$, where $I^{(m)} = \bigcap_{\wp \in \mathrm{Ass}(I)} (I^m R_\wp \cap R)$ and I^r are the symbolic and the regular powers of I, respectively. In particular, if I is a radical ideal, then Zariski-Nagata theorem shows that the symbolic power $I^{(m)}$ defines a homogeneous scheme $\mathbb{Z}$ of fat points $I_\mathbb{Z} = I_{\wp_1}^m \cap \cdots \cap I_{\wp_s}^m$, or equivalently it consists of polynomials vanishing to order m along the projective variety defined by I. There is also a series of results on the ideal containment problem, the resurgence, and the Waldschmidt constant ([2, 4, 11, 12, 15, 16, 18, 25, 26]). In particular, if $\mathbb{X}$ is a star configuration in $\mathbb{P}^n$ of type (r, s) defined by general forms $F_1, \ldots, F_s$, then the m-th order symbolic power of the ideal of a star configuration $\mathbb{X}$ (the m-th order symbolic power of $\mathbb{X}$ for short) is

$$I_\mathbb{X}^{(m)} = \bigcap_{1 \leq i_1 < \cdots < i_r \leq s} (F_{i_1}, \ldots, F_{i_r})^m,$$

because each ideal $(F_{i_1}, \ldots, F_{i_r})$ is a minimal prime of $I_\mathbb{X}$ and it is generated by a regular sequence ([31, Appendix 6, Lemma 5]). The m-th order symbolic power $I_\mathbb{X}^{(m)}$ has been studied from the point of view of algebra, algebraic geometry, and combinatorics ([9, 15–18, 30]).

In [18], the authors find a graded minimal free resolution of the 2-nd order symbolic power of a linear star configuration $\mathbb{X}$ in $\mathbb{P}^n$ of any codimension r, and in [30], the author extends that of a star configuration $\mathbb{X}$ in $\mathbb{P}^n$ of any codimension r. In [1], the authors find a graded minimal free resolution of the m-th order symbolic power of a linear star configuration in $\mathbb{P}^n$ of any codimension r ([1, Corollary 5.6]), though the formula is not simple.

In this paper, we find a graded minimal free resolution of any m-th order symbolic power of a star configuration $\mathbb{X}$ in $\mathbb{P}^n$ of codimension 2 (see Theorems 3.1 and 3.3), which generalizes the result in [15, Theorem 5.3]. Moreover, we also find a graded minimal free resolution of the m-th order symbolic power of a star configuration in $\mathbb{P}^n$ of any codimension r for $m = 3, 4$ with $r \geq m$ (see Theorems 4.5 and 5.6), which generalizes the result in [1, Corollaries 4.6 and Corollary 5.7].

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2. Preliminaries

We recall basic concepts for simplicial complexes. Define $[s] = \{1, 2, \ldots, s\}$. A *matroid* Δ on a vertax set $[s]$ is a nonempty collection of subsets of $[s]$ that is closed under inclusion and satisfies the following property. If A, B are in Δ and $|A| > |B|$, then there is some $i \in A$ such that $B \cup \{i\} \in \Delta$. We will consider Δ as a simplicial complex. Let $T = \Bbbk[y_1, \ldots, y_s]$. For a subset $A \subseteq [s]$, we write y_A for the square free monomial $\prod_{i \in A} y_i$. The *Stanley-Reisner* ideal of Δ is $I_\Delta = \langle y_A \mid A \subseteq [s], A \notin \Delta \rangle$ and the corresponding *Stanley-Reisner* ring is $\Bbbk[\Delta] = T/I_\Delta$.

Theorem 2.1 ([18, Theorem 3.3])**.** *Assume $F_1, \ldots, F_s \in R = \Bbbk[x_0, x_1, \ldots, x_n]$ are homogeneous polynomials such that any subset of at most $r + 1$ of them forms an R-regular sequence. Let Δ be a matroid on $[s]$ of dimension $s - r - 1$ with $2 \leq r \leq \min\{s, n\}$. Consider the ring homomorphism*

$$\varphi : T = \Bbbk[y_1, \ldots, y_s] \to R, \ y_i \mapsto F_i.$$

Let I be an ideal of T. We write $\varphi_*(I)$ to denote the ideal in R generated by $\varphi(I)$. If $\mathbb{F}_{\Bbbk[\Delta]}$ is a graded minimal free resolution of $\Bbbk[\Delta]$ over T, then $\mathbb{F}_{\Bbbk[\Delta]} \otimes_T R$ is a graded minimal free resolution of $R/\varphi_*(I_\Delta)$ over R.

The ideal $\varphi_*(I_\Delta)$ is said to be obtained by *specialization* from the matroid ideal I_Δ. The subscheme of $\mathbb{P}^n$ defined by $\varphi_*(I_\Delta)$ is called a *matroid configuration* [18].

Note that if we look at a graded minimal free T-resolution of T/I_Δ, then the entries in all the maps are monomials in the y_i. Moreover, replacing each y_i by F_i and each T by R gives a graded minimal free resolution of $R/\varphi_*(I_\Delta)$. So the formula $\mathbb{F} \otimes_T R$ implies the following two meanings.

(a) The variable y_i in $T = \Bbbk[y_1, \ldots, y_s]$ moves to a form F_i in $R = \Bbbk[x_0, x_1, \ldots, x_n]$, and
(b) A T free module $\mathbb{F}_\ell$ changes to an R free module $\mathbb{E}_\ell := \mathbb{F}_\ell \otimes_T R$ for $\ell \geq 1$.

Since a linear star configuration in $\mathbb{P}^n$ is one of the matroid configurations, we shall use [18, Theorem 3.3] for the proof of this theorem.

A monomial ideal $I \subset S$ is said to have *linear quotients* if there is an order $u_1, \dots, u_s$ of monomials in $G(I)$ (see Definition 2.4 for this notation) such that the colon ideal

$$(u_1, \dots, u_{k-1}) : u_k$$

is generated by variables for all $k = 2, 3, \dots, s$. In [1], the authors proved that the m-th order symbolic power of a linear star configuration $I_{(r,s)}^{(m)}$ is a symmetric strongly shifted ideal (see [1, Theorem 4.3]), and so it has linear quotients (see [1, Theorem 3.2]). A structure of the minimal free resolution of an ideal having linear quotients is well known [23, Theorem 1.12]. This implies that $I = I_{(r,s)}^{(m)}$ is a componentwise linear ideal, and the ideal $\langle I_j \rangle$ generated by all homogeneous polynomials of degree j belonging to I has a linear resolution ([22]). Based on theses facts, the authors in [1, Theorem 3.2] showed that Betti numbers $\beta_{i,i+j}(I_{(r,s)}^{(m)})$ are determined by the set $C(u)$ (with notations in Notation 4.1) where

$$C(u) = \{y_{\sigma(1)}, \dots, y_{\sigma(p)}\} \cup \{y_{\sigma(k)} : p + 1 \leq k \leq n - r, \sigma(k) < \max(u)\}.$$

In this paper, we often use this formula in the proof of our main results since it is easier to calculate the Betti numbers and the shifts in the free modules than to use either linear quotients or the differential maps between free modules.

We are now ready to find the Betti numbers and the shifts of a graded minimal free resolution of the m-th order symbolic power of a star configuration in $\mathbb{P}^n$ of codimension 2.

Remark 2.2. (1) Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^n$ of type (r, s). By Theorem 2.1, the study of a graded minimal free resolution of $I_{\mathbb{X}}$ can be reduced to the monomial case. Let $I_{(r,s)}$ be the monomial ideal of $S = \Bbbk[y_1, \dots, y_s]$ defined by

$$I_{(r,s)} := \bigcap_{1 \leq i_1 < \cdots < i_r \leq s} (y_{i_1}, \dots, y_{i_r}).$$

By Theorem 2.1, a graded minimal free resolution of $I_{\mathbb{X}}^{(m)}$ is completely determined by $I_{(r,s)}^{(m)}$. Moreover, the ideal

$$I_{(r,s)}^{(m)} = \bigcap_{1 \leq i_1 < \cdots < i_r \leq s} (y_{i_1}, \dots, y_{i_r})^m$$

is $\mathfrak{S}_s$-fixed, where $\mathfrak{S}_s$ is a symmetric group on s letters.

(2) Note that every symbolic power of a linear star configuration is aCM (see [17, Theorem 3.1]), and so any symbolic power of a star configuration (matroid configuration) in $\mathbb{P}^n$ is also aCM (see [18, Theorem 3.3]).

Theorem 2.3 ([27, Theorem 2.3]). *Let $F_1, \ldots, F_s$ be general forms in $R = \Bbbk[x_0, x_1, \ldots, x_n]$ with $s \geq 2$ and $n \geq 2$. Then*

$$\bigcap_{1 \leq j_1 < \cdots < j_r \leq s} (F_{j_1}, \ldots, F_{j_r}) = \sum_{1 \leq i_1 < \cdots < i_{r-1} \leq s} \left(\frac{\prod_{\ell=1}^{s} F_\ell}{F_{i_1} \cdots F_{i_{r-1}}} \right)$$

for $1 \leq r \leq \min\{n, s\}$.

Definition 2.4. A sequence $\lambda = (\lambda_1, \ldots, \lambda_s)$ of non-negative integers is *a partition of d of length s* if $\lambda_1 \leq \cdots \leq \lambda_s$ and $|\lambda| = \lambda_1 + \cdots + \lambda_s = d$. Let

$$P_s = \{(\lambda_1, \ldots, \lambda_s) \mid 0 \leq \lambda_1 \leq \cdots \leq \lambda_s\}$$

be the set of partitions of length s. For a monomial $u = y_1^{\lambda_1} \cdots y_s^{\lambda_s}$ of degree d and $\lambda = (\lambda_1, \ldots, \lambda_s)$, we denote by $u = y^\lambda$. For a monomial ideal I, the set

$$P(I) = \{\lambda \in P_s \mid y^\lambda \in I\}.$$

We write $G(I)$ for the unique set of minimal monomial generators of I. When I is $\mathfrak{S}_s$-fixed, we define

$$\Lambda(I) = \{\lambda \in P(I) \mid y^\lambda \in G(I)\}.$$

Proposition 2.5 ([1, Proposition 4.1]). *For every integer $m \geq 1$, the ideal $I_{(r,s)}^{(m)}$ is $\mathfrak{S}_s$-fixed, and*

$$\begin{aligned} P(I_{(r,s)}^{(m)}) &= \{\lambda \in P_s \mid |\lambda_{\leq r}| \geq m\}, \\ \Lambda(I_{(r,s)}^{(m)}) &= \{\lambda \in P_s \mid |\lambda_{\leq r}| = m, \forall i > r, \lambda_i = \lambda_r\}, \end{aligned}$$

where $|\lambda_{\leq r}| = \lambda_1 + \cdots + \lambda_r$ for $\lambda = (\lambda_1, \ldots, \lambda_s)$ with $r \leq s$.

Corollary 2.6 ([1, Corollary 4.4]). (1) *The Castelnuovo-Mumford regularity of $I_{(r,s)}^{(m)}$ is $m(s - r + 1)$. Moreover, if $m \geq 2$, then the bottom row in the Betti table of $I_{(r,s)}^{(m)}$ is given by the following formula.*

$$\beta_{i,i+m(s-r+1)}(I_{(r,s)}^{(m)}) = \binom{s}{r-1}\binom{r-1}{i} \quad \textit{for all} \quad i \geq 0.$$

(2) *If $m \leq r$, then*

$$\beta_{i,i+m+(s-r)}(I_{(r,s)}^{(m)}) = \binom{s}{r-m-i}\binom{s-r+m+i-1}{i} \quad \textit{for all} \quad i \geq 0.$$

Corollary 2.7 ([1, Corollary 4.6]). *If $r \geq 3$, then*

$$\beta_{i,i+j}(I_{(r,s)}^{(3)}) = \begin{cases} \dbinom{s}{r-i-3}\dbinom{s-r+i+2}{i}, & j = s-r+3, \\ \dbinom{s}{r-2}\left[\dbinom{r-2}{i} + (s-r+1)\dbinom{r-1}{i}\right], & j = 2(s-r+1)+1, \\ \dbinom{s}{r-1}\dbinom{r-1}{i}, & j = 3(s-r+1). \end{cases}$$

Example 2.8. (1) Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^n$ of type $(2, s)$ and let $m = 4$. By Proposition 2.5, there are one generator of degree $2s$, and s-minimal generators of degrees $3s-2$ and $4s-4$. By Hilbert-Burch theorem, a graded minimal free resolution of $R/I_{\mathbb{X}}^{(4)}$ is

$$0 \to \begin{array}{c} R^s(-(3s-1)) \\ \oplus \\ R^s(-(4s-3)) \end{array} \to \begin{array}{c} R(-2s) \\ \oplus \\ R^s(-(3s-2)) \\ \oplus \\ R^s(-(4s-4)) \end{array} \to R \to R/I_{\mathbb{X}}^{(4)} \to 0.$$

(2) Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^n$ of type $(2, s)$ and let $m = 5$. By Proposition 2.5, there are s-minimal generators of degrees $3s-1$, $4s-3$, and $5s-5$. By Hilbert-Burch theorem, a graded minimal free resolution of $R/I_{\mathbb{X}}^{(5)}$ is

$$0 \to \begin{array}{c} R^{s-1}(-3s) \\ \oplus \\ R^s(-(4s-2)) \\ \oplus \\ R^s(-(5s-4)) \end{array} \to \begin{array}{c} R^s(-(3s-1)) \\ \oplus \\ R^s(-(4s-3)) \\ \oplus \\ R^s(-(5s-5)) \end{array} \to R \to R/I_{\mathbb{X}}^{(5)} \to 0.$$

The following two propositions are motivated by Example 2.8.

Proposition 2.9. *Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^n$ of type $(2, s)$ and let m be a positive even integer. Then a graded minimal free resolution of $R/I_{\mathbb{X}}^{(m)}$ is*

$$\begin{array}{rl} 0 & \to \bigoplus_{\ell=0}^{\frac{m}{2}-1} R^s(-((s-1)(m-\ell)+(\ell+1))) \\ & \to R(-\frac{sm}{2}) \oplus \bigoplus_{\ell=0}^{\frac{m}{2}-1} R^s(-((s-1)(m-\ell)+\ell)) \\ & \to R \to R/I_{\mathbb{X}}^{(m)} \to 0. \end{array}$$

Table 1. the number of minimal generators of each degree

degree	the number of minimal generators
$(s-1)\cdot m$	s
$1+(s-1)(m-1)$	s
$2+(s-1)(m-2)$	s
$\vdots$	$\vdots$
$(\frac{m}{2}-2)+(s-1)(\frac{m}{2}+2)$	s
$(\frac{m}{2}-1)+(s-1)(\frac{m}{2}+1)$	s
$s\cdot\frac{m}{2}$	1

Proof. By Proposition 2.5, there are s minimal generators of each degree

$$(s-1)m, 1+(s-1)(m-1), 2+(s-1)(m-2), \ldots,$$

$$\Big(\frac{m}{2}-2\Big)+(s-1)\Big(\frac{m}{2}+2\Big),\Big(\frac{m}{2}-1\Big)+(s-1)\Big(\frac{m}{2}+1\Big),$$

and one more minimal generator of degree $s\cdot\frac{m}{2}$ (see Table 1).

Therefore, by Hilbert-Burch theorem, we obtain a graded minimal free resolution of $R/I_{\mathbb{X}}^{(m)}$ as above. $\square$

Using the same idea as in the proof of Proposition 2.9, one can easily prove the following proposition, and so we omit the proof here.

Proposition 2.10. *Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^n$ of type $(2,s)$ and let m be a positive odd integer with $m\geq 3$. Then a graded minimal free resolution of $R/I_{\mathbb{X}}^{(m)}$ is*

$$\begin{array}{lll} 0 & \to & R^{s-1}\Big(-s\cdot\Big(\dfrac{m+1}{2}\Big)\Big)\oplus\Big[\displaystyle\bigoplus_{\ell=0}^{\frac{m-3}{2}}R^s(-((s-1)(m-\ell)+(\ell+1)))\Big] \\ & \to & \displaystyle\bigoplus_{\ell=0}^{\frac{m-1}{2}}R^s(-((s-1)(m-\ell)+\ell)) \\ & \to & R\to R/I_{\mathbb{X}}^{(m)}\to 0. \end{array}$$

3. A graded minimal free resolution of the m-th order symbolic power of a star configuration in $\mathbb{P}^n$ of codimension 2

In this section, we shall find a graded minimal free resolution of the m-th order symbolic power of a star configuration in $\mathbb{P}^n$ of codimension 2.

Theorem 3.1. *Let $\mathbb{X}$ be a star configuration in $\mathbb{P}^n$ of type $(2,s)$ defined by s general forms $F_1,\dots,F_s$ in R of degrees $d_1,\dots,d_s$, respectively, and let m be a positive even integer. Define $d:=d_1+\cdots+d_s$. Then a graded minimal free resolution of $R/I_{\mathbb{X}}^{(m)}$ is*

$$\begin{array}{lll} 0 & \to & \displaystyle\bigoplus_{\ell=0}^{\frac{m}{2}-1}\Big[\bigoplus_{1\leq i_1<\cdots<i_{s-1}\leq s}R(-((d_{i_1}+\cdots+d_{i_{s-1}})\cdot(m-\ell)+(\ell+1)\cdot d_{i_s}))\Big] \\ & \to & R(-\frac{dm}{2})\oplus\displaystyle\bigoplus_{\ell=0}^{\frac{m}{2}-1}\Big[\bigoplus_{\substack{1\leq i_1<\cdots<i_{s-1}\leq s\\ i_s\in\{1,2,\dots,s\}-\{i_1,\dots,i_{s-1}\}}}R(-((d_{i_1}+\cdots+d_{i_{s-1}})\cdot(m-\ell)+\ell\cdot d_{i_s}))\Big] \\ & \to & R\to R/I_{\mathbb{X}}^{(m)}\to 0. \end{array}$$

Proof. Let $T=\Bbbk[y_1,\dots,y_s]$ and consider the ideal of T

$$I_{(2,s)}=\bigcap_{1\leq i_1<i_2\leq s}(y_{i_1},y_{i_2}),$$

generated by all products of $s-1$ distinct variables in $\{y_1,\dots,y_s\}$ (see Theorem 2.3). It is the Stanley-Reisner ideal of a uniform matroid on $[s]$. Recall the map

$$\varphi:T\to R,\quad y_i\mapsto F_i. \tag{3.1}$$

Then

$$I_{\mathbb{X}}^{(m)} = \varphi_*(I_{(2,s)}^{(m)}).$$

By Theorem 2.1, for $i = 1, 2$, the i-th free module of a graded minimal free resolution of $R/I_{\mathbb{X}}^{(m)}$ is

$$\mathbb{E}_i := \mathbb{F}_i \otimes_T R,$$

where

$$\mathbb{F}_2 = \bigoplus_{\ell=0}^{\frac{m}{2}-1} T^s(-((s-1)(m-\ell)+(\ell+1))), \quad \text{and}$$

$$\mathbb{F}_1 = T\Big(-\frac{sm}{2}\Big) \oplus \bigoplus_{\ell=0}^{\frac{m}{2}-1} T^s(-((s-1)(m-\ell)+\ell)).$$

As mentioned before, the entries in all the maps in the graded minimal free resolution of T/I_Δ are monomials in the y_i, and replacing each y_i by F_i and each T by R gives a graded minimal free resolution of $R/\varphi_*(I_\Delta)$. Hence one can conclude that

$$s \overset{\varphi_*}{\mapsto} d \quad \text{and} \quad 1 \overset{\varphi_*}{\mapsto} d_i. \tag{3.2}$$

By Proposition 2.5 and equation (3.2), for $\ell = 0, 1, \ldots, \frac{m}{2}$, we have

$$(x_{i_1} \cdots x_{i_{s-1}})^{m-\ell} \cdot x_{i_s}^{\ell} \quad \overset{\varphi_*}{\mapsto} \quad (F_{i_1} \cdots F_{i_{s-1}})^{m-\ell} \cdot (F_{i_s})^{\ell} \ ,$$

i.e.,

$$\begin{aligned} &((\overset{i_1\text{-st}}{m-\ell}) \cdot 1 + \cdots + (\overset{i_{s-1}\text{-st}}{m-\ell}) \cdot 1) + \overset{i_s\text{-th}}{\ell} \cdot 1 \\ \overset{\varphi_*}{\mapsto}\ &((\overset{i_1\text{-st}}{m-\ell}) \cdot d_{i_1} + \cdots + (\overset{i_{s-1}\text{-st}}{m-\ell}) \cdot d_{i_{s-1}}) + \overset{i_s\text{-th}}{\ell} \cdot d_{i_s}, \end{aligned}$$

where $1 \le i_1 < \cdots < i_{s-1} \le s$ and $i_s \in \{1, \ldots, s\} - \{i_1, \ldots, i_{s-1}\}$. Therefore, we obtain a graded minimal free resolution of $R/I_{\mathbb{X}}^{(m)}$ from Hilbert-Burch theorem as above. This completes the proof. $\square$

The following corollary is immediate from Theorem 3.1.

Corollary 3.2 ([15, Theorem 5.3]). *Let $\mathbb{X}$ be a star configuration in $\mathbb{P}^n$ of type $(2, s)$ defined by s general forms in $R = \Bbbk[x_0, x_1, \ldots, x_n]$ of degrees $d_1, \ldots, d_s$ with $s \ge 2$, and let $d = d_1 + \cdots + d_s$. Then a graded minimal free resolution of $R/I_{\mathbb{X}}^{(2)}$ is*

$$0 \to \bigoplus_{1 \le i \le s} R(-(2d-d_i)) \to R(-d) \oplus \Big[\bigoplus_{1 \le i \le s} R(-(2(d-d_i)))\Big] \to R \to R/I_{\mathbb{X}}^{(2)} \to 0.$$

Using the same idea as in the proof of Theorem 3.1, we can obtain the following theorem, and so we omit the proof.

Theorem 3.3. *Let $\mathbb{X}$ be a star configuration in $\mathbb{P}^n$ of type $(2,s)$ and let m be a positive odd integer with $m \geq 3$. Define $d := d_1 + \cdots + d_s$. Then a graded minimal free resolution of $R/I_{\mathbb{X}}^{(m)}$ is*

$$\begin{array}{rl} & R^{s-1}\Big(- d \cdot \Big(\frac{m+1}{2}\Big)\Big) \\ & \oplus \\ 0 \to & \bigoplus_{\ell=0}^{\frac{m-3}{2}} \Bigg[\bigoplus_{\substack{1\le i_1<\cdots<i_{s-1}\le s \\ i_s\in\{1,2,\dots,s\}-\{i_1,\dots,i_{s-1}\}}} R(-((d_{i_1}+\cdots+d_{i_{s-1}})\cdot(m-\ell)+(\ell+1)\cdot d_{i_s})) \Bigg] \\ \to & \bigoplus_{\ell=0}^{\frac{m-1}{2}} \Bigg[\bigoplus_{\substack{1\le i_1<\cdots<i_{s-1}\le s \\ i_s\in\{1,2,\dots,s\}-\{i_1,\dots,i_{s-1}\}}} R(-((d_{i_1}+\cdots+d_{i_{s-1}})\cdot(m-\ell)+\ell\cdot d_{i_s})) \Bigg] \\ \to & R \to R/I_{\mathbb{X}}^{(m)} \to 0. \end{array}$$

4. A graded minimal free resolution of the 3-rd order symbolic power of a star configuration in $\mathbb{P}^n$ of codimension r

In this section, we shall find a graded minimal free resolution of the 3-rd order symbolic power of a star configuration in $\mathbb{P}^n$ of type (r,s) with $\min\{n,r\} \leq s$ and $3 \leq r$. We first introduce some notations in [1].

Notation 4.1 ([1])**.** Let $\lambda = (\lambda_1, \dots, \lambda_s)$ with $0 \leq \lambda_1 \leq \cdots \leq \lambda_s$ be a partition and $y^\lambda = y_1^{\lambda_1} \cdots y_s^{\lambda_s}$ be a monomial in $T = \Bbbk[y_1, \dots, y_s]$, and let $u = \sigma(y^\lambda)$ for $\sigma \in \mathfrak{S}_s$.

(1) $p := p(\lambda) = |\{k \mid \lambda_k < \lambda_s - 1\}|$.
(2) $q := q(\lambda) = |\{k \mid \lambda_k = \lambda_s\}|$.
(3) $\max(u) = \max\{\sigma(k) \mid \lambda_k = \lambda_s\}$.
(4) $C(u) = \{y_{\sigma(1)}, \dots, y_{\sigma(p)}\} \cup \{y_{\sigma(k)} \mid p+1 \leq k \leq s-q, \sigma(k) < \max(u)\}$.

Lemma 4.2. *For $3 \leq m \leq r \leq n$, let $\mathbb{X}$ be a star configuration in $\mathbb{P}^n$ of type (r,s) defined by forms of degrees $d_1, \dots, d_s$ with $r \leq s$. Let*

$$0 \to \mathbb{E}_r \to \cdots \to \mathbb{E}_1 \to R \to R/I_{\mathbb{X}}^{(m)} \to 0$$

be a graded minimal free resolution of $R/I_{\mathbb{X}}^{(m)}$, where $\mathbb{E}_{\ell+1} := \mathbb{F}_{\ell+1} \otimes_T R$ and $\mathbb{F}_{\ell+1}$ is the $(\ell+1)$-st free module of a graded minimal free resolution of $T/I_{(r,s)}^{(m)}$ for $0 \leq \ell \leq r-1$. Then the free submodule of the $(\ell+1)$-st free module $\mathbb{E}_{\ell+1} = \mathbb{F}_{\ell+1} \otimes_T R$, which is from the minimal generators of degree $(s-r+m)$ in $I_{(r,s)}^{(m)}$, is

$$\begin{aligned} & T^{\binom{s}{r-m-\ell}\binom{s-r+m+\ell-1}{\ell}}(-(s-(r-m)+\ell)) \otimes_T R \\ = & \bigoplus_{1\le i_1<\cdots<i_{r-m-\ell}\le s} R^{\binom{s-r+m+\ell-1}{\ell}}(-(d-(d_{i_1}+\cdots+d_{i_{r-m-\ell}}))) \end{aligned}$$

for $0 \leq \ell \leq r-1$.

Proof. Let $\lambda = (\underbrace{0,\dots,0}_{(r-m)}, \underbrace{1,\dots,1}_{(s-(r-m))})$. For $\sigma \in \mathfrak{S}_s$ and $u = \sigma(y^\lambda)$, we have

$$C(u) = \{y_{\sigma(k)} \mid 1 \le k \le r-m, \sigma(k) < \max(u)\}.$$

By the symmetry of scripts $1, 2, \dots, s$, we may assume that $\max(u) = s$ and

$$C(u) = \{\sigma(1), \dots, \sigma(r-m)\} = \{1, 2, \dots, r-m\}, \quad \text{and} \quad |C(u)| = r-m.$$

Note that the degree of (square free) minimal generators in this free module is $(s-r+m+\ell)$. This means that the number $(s-r+m+\ell)$ is the same as the number of subscripts of the monomial having degree 1, say $r-m-\ell+1, \dots, s$. From this step with $\max(u) = s$, one can choose ℓ-subscripts among subscripts $r-m-\ell+1, \dots, s$ except for $\max(u) = s$.

Hence we have the (square free) monomial generator of the form

$$(y_1^0 \cdots y_{r-m-\ell}^0)(y_{r-m-\ell+1}^1 \cdots y_{j_1}^1 \cdots y_{j_\ell}^1 \cdots y_s^1), \tag{4.1}$$

where $\{j_1, \dots, j_\ell\} \subset [s] - (\{1, \dots, r-m-\ell\} \cup \{s\})$.

If we summarize our arguments with the symmetry of subscripts $1, 2, \dots, s$, we obtain that the number of minimal generators of degree $s-(r-m-\ell)$ in this free module is

$$\binom{s}{s-(r-m-\ell)} = \binom{s}{r-m-\ell},$$

and for each monomial of degree $s-(r-m-\ell)$, the monomial repeats

$$\binom{s-(r-m-\ell)-1}{\ell}\text{-times.}$$

Therefore, the following shift repeats $\binom{s-r+m+\ell-1}{\ell}$-times, i.e.,

$$\begin{aligned} & s-(r-m-\ell) \\ &= (\overset{(r-m-\ell+1)\text{-st}}{1} + \cdots + \overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1} + \cdots + \overset{s\text{-th}}{1}), \quad \tbinom{s-r+m+\ell-1}{\ell}\text{-times,} \end{aligned}$$

where $\{j_1, \dots, j_\ell\} \subset [s] - (\{1, 2, \dots, r-m-\ell\} \cup \{s\})$. This implies

$$s-(r-m-\ell) = s - (\overset{1\text{-st}}{1} + \cdots + \overset{(r-m-\ell)\text{-th}}{1}), \quad \tbinom{s-r+m+\ell-1}{\ell}\text{-times.}$$

Applying the matroid configuration map φ_* with the symmetry of subscripts $1, 2, \dots, s$, we get that

$$s-(r-m-\ell) \overset{\varphi_*}{\mapsto} d - (d_{i_1} + \cdots + d_{i_{r-m-\ell}}), \quad \tbinom{s-r+m+\ell-1}{\ell}\text{-times,}$$

where $1 \le i_1 < \cdots < i_{r-m-\ell} \le s$.

Therefore, for such ℓ, we have the free submodule as follows

$$\begin{aligned} & T^{\binom{s}{r-m-\ell}\binom{s-r+m+\ell-1}{\ell}}(-(s-(r-m)+\ell)) \otimes_T R \\ &= \bigoplus_{1 \le i_1 < \cdots < i_{r-m-\ell} \le s} R^{\binom{s-r+m+\ell-1}{\ell}}(-(d-(d_{i_1} + \cdots + d_{i_{r-m-\ell}}))) \end{aligned}$$

as we wished. $\square$

Lemma 4.3. *With notations as in Lemma 4.2, for $3 = m \le r$, and $0 \le \ell \le r-1$, the free submodule of the $(\ell+1)$-st free module $\mathbb{E}_{\ell+1}$, which is from the minimal generators of degree $2(s-(r-1))+1$ in $I_{\mathbb{X}}^{(3)}$, is*

$$
\begin{aligned}
&T^{\binom{s}{r-2}\binom{r-2}{\ell}+(s-r+1)\binom{s}{r-2}\binom{r-1}{\ell}}(-(2(s-(r-1))+1+\ell))\otimes_T R\\
&= \left[\begin{array}{l}
\displaystyle\bigoplus_{\substack{1\le i_1<\cdots<i_{r-2}\le s\\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-2}\}\\ k=\max([s]-\{i_1,\dots,i_{r-2}\})}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-2}}))-d_k+(d_{j_1}+\cdots+d_{j_\ell})))\\
\displaystyle\oplus \bigoplus_{\substack{1\le i_1<\cdots<i_{r-2}\le s\\ k<\max([s]-\{i_1,\dots,i_{r-2}\})\\ k\notin\{i_1,\dots,i_{r-2}\}\\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-2},k\}}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-2}}))-d_k+(d_{j_1}+\cdots+d_{j_\ell})))
\end{array}\right].
\end{aligned}
$$

Proof. Let

$$\lambda = (\underbrace{0,\dots,0}_{(r-2)\text{-times}}, 1, \underbrace{2,\dots,2}_{(s-(r-1))\text{-times}}).$$

For $\sigma \in \mathfrak{S}_s$, and $u = \sigma(y^\lambda)$, we have

$$C(u) = \{\sigma(1),\dots,\sigma(r-2)\}, \quad \text{or} \quad \{\sigma(1),\dots,\sigma(r-2),\sigma(r-1)\}.$$

By the symmetry of subscripts $1,2,\dots,s$, we may assume that

$$\sigma(1) = 1,\dots,\sigma(r-2) = r-2, \text{ and } \sigma(r-1) = k,$$

where $r-1 \le k \le s$.

(1) Let $C(u) = \{\sigma(1),\sigma(2),\dots,\sigma(r-2)\} = \{1,2,\dots,r-2\}$. Then $\sigma(r-1) = k > \max\{\sigma(r),\dots,\sigma(s)\}$, i.e., $\sigma(r-1) = k = s$ (see the proof of Corollary 2.7). Note that the Betti number for this case is

$$\binom{s}{r-2}\binom{r-2}{\ell}.$$

Hence the minimal generators in this free module are of the form

$$
\begin{array}{rcl}
 & & \left\{\begin{array}{l} y_1^0 y_2^0 \cdots y_{r-2}^0 y_s^1 (y_{r-1}\cdots y_{s-1})^2 y_{j_1}^1 \cdots y_{j_\ell}^1,\\ \text{where } \{j_1,\dots,j_\ell\}\subset\{1,\dots,r-2\}\end{array}\right.\\
 & \mapsto & \overset{s\text{-th}}{1} + 2(\overset{(r-1)\text{-st}}{1} + \cdots + \overset{(s-1)\text{-st}}{1}) + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1})\\
(4.2) & = & 2(\overset{(r-1)\text{-st}}{1} + \cdots + \overset{(s-1)\text{-st}}{1} + \overset{s\text{-th}}{1}) - \overset{s\text{-th}}{1} + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1})\\
 & = & 2(s - (\overset{(1\text{-st}}{1} + \cdots + \overset{(r-2)\text{-nd}}{1})) - \overset{s\text{-th}}{1} + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1})\\
 & \overset{\varphi_*}{\mapsto} & 2(d - (d_1 + d_2 + \cdots + d_{r-2})) - d_s + (d_{j_1} + \cdots + d_{j_\ell}).
\end{array}
$$

From this observation, we obtain that

$$
\begin{aligned}
& T^{\binom{s}{r-2}\binom{r-2}{\ell}}(-(2(s-(r-1))+1+\ell)) \otimes_T R \\
= & \bigoplus_{\substack{1\le i_1<\cdots<i_{r-2}\le s\\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-2}\}\\ k=\max([s]-\{i_1,\dots,i_{r-2}\})}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-2}}))-d_k+(d_{j_1}+\cdots+d_{j_\ell}))).
\end{aligned}
$$

(2) Let $C(u)=\{\sigma(1),\sigma(2),\dots,\sigma(r-1)\}=\{1,2,\dots,r-2,k\}$. Since

$$\sigma(r-1)=k<\max(u)=\max\{\sigma(r),\dots,\sigma(s)\},$$

we get $r-1\le k\le s-1$ (see the proof of Corollary 2.7). Note that the Betti number for this case is

$$(s-(r-1))\binom{s}{r-2}\binom{r-1}{\ell}.$$

For $r-1\le k\le s-1$, we have

$$
\begin{aligned}
& \begin{cases} y_1^0y_2^0\cdots y_{r-2}^0y_k^1(y_{r-1}\cdots\hat{y}_k\cdots y_s)^2y_{j_1}^1\cdots y_{j_\ell}^1,\\ \text{where } j_1,\dots,j_\ell\in\{1,\dots,r-2\}\cup\{k\},\end{cases}\\
\mapsto\ & \overset{k\text{-th}}{1}+2(\overset{(r-1)\text{-st}}{1}+\cdots+\widehat{\overset{k\text{-th}}{1}}+\cdots+\overset{s\text{-th}}{1})+(\overset{j_1\text{-st}}{1}+\cdots+\overset{j_\ell\text{-th}}{1})\\
\mapsto\ & 2(\overset{(r-1)\text{-st}}{1}+\cdots+\overset{k\text{-th}}{1}+\cdots+\overset{s\text{-th}}{1})-\overset{k\text{-th}}{1}+(\overset{j_1\text{-st}}{1}+\cdots+\overset{j_\ell\text{-th}}{1})\\
=\ & 2(s-(\overset{1\text{-st}}{1}+\cdots+\overset{(r-2)\text{-nd}}{1}))-\overset{k\text{-th}}{1}+(\overset{j_1\text{-st}}{1}+\cdots+\overset{j_\ell\text{-th}}{1})\\
\overset{\varphi_*}{\mapsto}\ & 2(d-(d_1+\cdots+d_{r-2}))-d_k+(d_{j_1}+\cdots+d_{j_\ell}),
\end{aligned}
$$

where $\hat{*}$ means that $*$ is omitted. From this observation, we get that

$$
\begin{aligned}
& T^{(s-r+1)\binom{s}{r-2}\binom{r-1}{\ell}}(-(2(s-(r-1))+1+\ell)) \otimes_T R \\
= & \bigoplus_{\substack{1\le i_1<\cdots<i_{r-2}\le s\\ k<\max([s]-\{i_1,\dots,i_{r-2}\})\\ k\notin\{i_1,\dots,i_{r-2}\}\\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-2},k\}}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-2}}))-d_k+(d_{j_1}+\cdots+d_{j_\ell}))).
\end{aligned}
$$

From (1) and (2), we get the free module

$$
\begin{aligned}
& T^{\binom{s}{r-2}\binom{r-2}{\ell}+(s-r+1)\binom{s}{r-2}\binom{r-1}{\ell}}(-(2(s-(r-1))+1+\ell)) \otimes_T R \\
= & \left[\begin{array}{c}
\displaystyle\bigoplus_{\substack{1\le i_1<\cdots<i_{r-2}\le s\\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-2}\}\\ k=\max([s]-\{i_1,\dots,i_{r-2}\})}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-2}}))-d_k+(d_{j_1}+\cdots+d_{j_\ell})))\\
\oplus\\
\displaystyle\bigoplus_{\substack{1\le i_1<\cdots<i_{r-2}\le s\\ k<\max([s]-\{i_1,\dots,i_{r-2}\})\\ k\notin\{i_1,\dots,i_{r-2}\}\\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-2},k\}}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-2}}))-d_k+(d_{j_1}+\cdots+d_{j_\ell})))
\end{array}\right]
\end{aligned}
$$

as we wished. □

Lemma 4.4. *Let $\mathbb{X}$ be as in Lemma 4.2. For $3 \leq r$, $0 \leq \ell \leq r-1$, and $3 \leq m$, the free submodule of the $(\ell+1)$-st free module $\mathbb{E}_{\ell+1}$ of a graded minimal free resolution of $R/I_{\mathbb{X}}^{(m)}$, which is from the minimal generators of degree $m(s-(r-1))$ in $I_{(r,s)}^{(m)}$, is*

$$\begin{aligned} & T^{\binom{s}{r-1}\binom{r-1}{\ell}}(-(m(s-(r-1))+\ell)) \otimes_T R \\ = & \bigoplus_{\substack{1\leq i_1<\cdots<i_{r-1}\leq s \\ \{j_1,\ldots,j_\ell\}\subset\{i_1,\ldots,i_{r-1}\}}} R(-(m(d-(d_{i_1}+\cdots+d_{i_{r-1}}))+(d_{j_1}+\cdots+d_{j_\ell}))). \end{aligned}$$

Proof. The proof is immediate from Corollary 2.6(1). □

We are now ready to find a graded minimal free resolution of $R/I_{\mathbb{X}}^{(3)}$ when $\mathbb{X}$ is a star configuration in $\mathbb{P}^n$ for $n \geq 3$. Combining Lemmas 4.2, 4.3, and 4.4, one can obtain the following theorem.

Theorem 4.5. *Let $\mathbb{X}$ be as in Lemma 4.4. For $3 \leq r$ and $1 \leq \ell \leq r-1$, a graded minimal free resolution of $R/I_{\mathbb{X}}^{(3)}$ is*

$$0 \to \mathbb{E}_r \to \cdots \to \mathbb{E}_1 \to R \to R/I_{\mathbb{X}}^{(3)} \to 0,$$

where

$$\mathbb{E}_{\ell+1} = \left[\begin{array}{c} \displaystyle\bigoplus_{1\leq i_1<\cdots<i_{r-3-\ell}\leq s} R^{\binom{s-r+3+\ell-1}{\ell}}(-(d-(d_{i_1}+\cdots+d_{i_{r-3-\ell}}))) \\ \oplus \\ \displaystyle\bigoplus_{\substack{1\leq i_1<\cdots<i_{r-2}\leq s \\ \{j_1,\ldots,j_\ell\}\subset\{i_1,\ldots,i_{r-2}\} \\ k=\max([s]-\{i_1,\ldots,i_{r-2}\})}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-2}}))-d_k+(d_{j_1}+\cdots+d_{j_\ell}))) \\ \oplus \\ \displaystyle\bigoplus_{\substack{1\leq i_1<\cdots<i_{r-2}\leq s \\ k<\max([s]-\{i_1,\ldots,i_{r-2}\}) \\ k\notin\{i_1,\ldots,i_{r-2}\} \\ \{j_1,\ldots,j_\ell\}\subset\{i_1,\ldots,i_{r-2},k\}}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-2}}))-d_k+(d_{j_1}+\cdots+d_{j_\ell}))) \\ \oplus \\ \displaystyle\bigoplus_{\substack{1\leq i_1<\cdots<i_{r-1}\leq s \\ \{j_1,\ldots,j_\ell\}\subset\{i_1,\ldots,i_{r-1}\}}} R(-(3(d-(d_{i_1}+\cdots+d_{i_{r-1}}))+(d_{j_1}+\cdots+d_{j_\ell}))) \end{array}\right]$$

for $0 \leq \ell \leq r-1$.

Example 4.6. Let $\mathbb{X}$ be a star configuration in $\mathbb{P}^n$ of type $(3,4)$ defined by forms of degrees $1,2,3,5$ with $n \geq 3$ and $m = 3$. By Theorem 4.5, we have

$$\mathbb{E}_1 = R(-d) \oplus \bigoplus_{1\leq i\neq j\leq s} R(-(2(d-d_i)-d_j))) \oplus \bigoplus_{1\leq i<j\leq s} R(-(3(d-(d_i+d_j))).$$

First let $d = d_1 + d_2 + d_3 + d_4 = 11$. The following two tables summarize the shifts in $\mathbb{E}_1$.

$2(d-d_1)-d_2$	18	$2(d-d_1)-d_3$	17	$2(d-d_1)-d_4$	15
$2(d-d_2)-d_1$	17	$2(d-d_2)-d_3$	15	$2(d-d_2)-d_4$	13
$2(d-d_3)-d_1$	15	$2(d-d_3)-d_2$	14	$2(d-d_3)-d_4$	11
$2(d-d_4)-d_1$	11	$2(d-d_4)-d_2$	10	$2(d-d_4)-d_3$	9

$3(d-(d_1+d_2))$	24	$3(d-(d_1+d_3))$	21	$3(d-(d_1+d_4))$	15
$3(d-(d_2+d_3))$	18	$3(d-(d_2+d_4))$	12	$3(d-(d_3+d_4))$	9

Thus we have

$$\begin{aligned}\mathbb{E}_1 = {} & R(-9)^2 \oplus R(-10) \oplus R(-11)^3 \oplus R(-12) \oplus R(-13)\\ & \oplus R(-14) \oplus R(-15)^4 \oplus R(-17)^2 \oplus R(-18)^2 \oplus R(-21) \oplus R(-24).\end{aligned}$$

Similarly, from Theorem 4.5, one can find the free modules $\mathbb{E}_2$ and $\mathbb{E}_3$ as well.

Therefore a graded minimal free resolution of $R/I_{\mathbb{X}}^{(3)}$ is

$$\begin{aligned} 0 \to {} & R(-17)^3 \oplus R(-19)^3 \oplus R(-20)^2 \oplus R(-21)^3 \oplus R(-23)\\ & \oplus R(-25) \oplus R(-27)\\ \to {} & R(-12)^3 \oplus R(-14)^4 \oplus R(-15)^2 \oplus R(-16)^5 \oplus R(-17)^3\\ & \oplus R(-18)^4 \oplus R(-19)^2 \oplus R(-20)^4 \oplus R(-21) \oplus R(-22)\\ & \oplus R(-24) \oplus R(-25) \oplus R(-26)\\ \to {} & R(-9)^2 \oplus R(-10) \oplus R(-11)^3 \oplus R(-12) \oplus R(-13)\\ & \oplus R(-14) \oplus R(-15)^4 \oplus R(-17)^2 \oplus R(-18)^2 \oplus R(-21) \oplus R(-24)\\ \to {} & R \ \to \ R/I_{\mathbb{X}}^{(3)} \ \to \ 0.\end{aligned}$$

5. A graded minimal free resolution of the 4-th order symbolic power of a star configuration in $\mathbb{P}^n$ of codimension r

In this section, we shall find a graded minimal free resolution of the 4-th order symbolic power of a star configuration in $\mathbb{P}^n$ of codimension r with $n \geq r \geq 4$. By analogous ideas as in Section 4, we can find the Betti numbers and shifts of most cases for $m = 4$, and so we shall omit proofs in detail for those cases, except for a few of specific cases. Indeed, we know the graded Betti numbers and the degrees of the minimal generators in the following proposition using [1, Corollary 5.7] and Proposition 2.5, respectively. Here we introduce an outline of the proof.

Proposition 5.1. *With notations as above, for $s \geq r \geq 4$,*

$$\beta_{i,i+j}(I^{(4)}_{(r,s)}) = \begin{cases} \binom{s}{r-4-i}\binom{s-r+3+i}{i}, & j = s-(r-4), \\ \binom{s}{r-3}\left[\binom{r-3}{i} + (s-r+1)\binom{r-2}{i} + \binom{s-r+2}{2}\binom{r-1}{i}\right], & \\ & j = 2(s-(r-1))+2, \\ \binom{s}{r-2}\binom{r-2}{i}, & j = 2(s-(r-2)), \\ (s-r+2)\binom{s}{r-2}\binom{r-1}{i}, & j = 3(s-(r-1))+1, \\ \binom{s}{r-1}\binom{r-1}{i}, & j = 4(s-(r-1)). \end{cases}$$

Sketch of Proof. By Proposition 2.5, the degrees of the monomial minimal generators in $I^{(4)}_{(r,s)}$ corresponding to each partition λ are summarized in the following table.

	degree	partition
(1)	$s-(r-4)$	$\lambda = (\underbrace{0,\dots,0}_{(r-4)\text{-times}}, \underbrace{1,\dots,1}_{(s-(r-4))\text{-times}})$
(2)	$2(s-(r-1))+2$	$\lambda = (\underbrace{0,\dots,0}_{(r-3)\text{-times}}, 1, 1, \underbrace{2,\dots,2}_{(s-(r-1))\text{-times}})$
(3)	$2(s-(r-2))$	$\lambda = (\underbrace{0,\dots,0}_{(r-2)\text{-times}}, \underbrace{2,\dots,2}_{(s-(r-2))\text{-times}})$
(4)	$3(s-(r-1))+1$	$\lambda = (\underbrace{0,\dots,0}_{(r-2)\text{-times}}, 1, \underbrace{3,\dots,3}_{(s-(r-1))\text{-times}})$
(5)	$4(s-(r-1))$	$\lambda = (\underbrace{0,\dots,0}_{(r-1)\text{-times}}, \underbrace{4,\dots,4}_{(s-(r-1))\text{-times}})$

The top and the bottom rows of the Betti table are immediate from Corollary 2.6(1) and (2), respectively. The third and forth rows of the Betti table are also easy to find. So we shall prove the second row only of the Betti table for the monomial minimal generators in $I^{(4)}_{(r,s)}$ of degree $2(s-(r-1))+2$. Recall

$$\lambda = (\underbrace{0,\dots,0}_{(r-3)\text{-times}}, 1, 1, \underbrace{2,2,\dots,2}_{(s-r+1)\text{-times}}).$$

For $\sigma \in \mathfrak{S}_s$, let $u = \sigma(y^\lambda)$. Then $|C(u)| = r-3, r-2$, or $r-1$. Using the symmetry of $1,2,\dots,s$, we assume that $\sigma(1) = 1,\dots,\sigma(r-3) = r-3$.

(1) Let $|C(u)| = r-3$, i.e., $\max(u) \leq \sigma(r-2), \sigma(r-1)$. Then $\sigma(r-2) = s-1$ and $\sigma(r-1) = s$. Hence by [1, Corollary 3.4], the Betti number is $\binom{s}{r-3}\binom{r-3}{i}$.

(2) Let $|C(u)| = r-2$. Then we have $\sigma(r-2) < \max(u) \le \sigma(r-1)$. I.e., $\sigma(r-2) = k$ and $\sigma(r-1) = s$, where $r-2 \le k \le s-2$. So, by [1, Corollary 3.4], the Betti number is $(s-r+1)\binom{s}{r-3}\binom{r-2}{i}$.

(3) Let $|C(u)| = r-1$. Then $\sigma(r-2), \sigma(r-1) \le \max(u)$. Hence $r-2 \le \sigma(r-2) < \sigma(r-1) \le s-1$. Thus, by [1, Corollary 3.4], the Betti number is $\binom{s-r+2}{2}\binom{s}{r-3}\binom{r-1}{i}$.

Therefore, the second row of the Betti table is

$$\beta_{i,i+j}(I^{(4)}_{(r,s)}) = \binom{s}{r-3}\left[\binom{r-3}{i} + (s-r+1)\binom{r-2}{i} + \binom{s-r+2}{2}\binom{r-1}{i}\right],$$

as we wished. $\square$

Example 5.2. Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^n$ of type $(4,6)$ with $n \ge 4$ and $m = 4$. By Proposition 2.5, a graded minimal free resolution of $R/I^{(4)}_{\mathbb{X}}$ is

$$\begin{aligned} 0 &\to R(-11)^{36} \oplus R(-13)^{60} \oplus R(-15)^{20} \\ &\to R(-10)^{141} \oplus R(-12)^{180} \oplus R(-14)^{60} \\ &\to R(-9)^{180} \oplus R(-11)^{180} \oplus R(-13)^{60} \\ &\to R(-6) \oplus R(-8)^{75} \oplus R(-10)^{60} \oplus R(-12)^{20} \\ &\to R \ \to \ R/I^{(4)}_{\mathbb{X}} \ \to \ 0. \end{aligned}$$

Now we shall find a graded minimal free resolution of the 4-th order symbolic power of a star configuration $\mathbb{X}$ in $\mathbb{P}^n$ of type (r,s) with $4 \le r \le n$. As we mentioned before, we shall skip the proofs in detail, as it is not hard to prove.

Lemma 5.3. *With notations as in Proposition 5.1, the free submodule of the $(\ell+1)$-st free module $\mathbb{E}_{\ell+1}$, which is from the minimal generators of degree $2(s-(r-1))+2$ in $I^{(4)}_{(r,s)}$, is*

$$T^{\binom{s}{r-3}\left[\binom{r-3}{\ell}+(s-r+1)\binom{r-2}{\ell}+\binom{s-r+2}{2}\binom{r-1}{\ell}\right]}(-(2(s-(r-1))+2+\ell)) \otimes_T R$$

$$= \left[\begin{array}{c} \displaystyle\bigoplus_{\substack{1\le i_1<\cdots<i_{r-3}\le s \\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-3}\} \\ k_1=\max([s]-\{i_1,\dots,i_{r-3}\}) \\ k_2=\max([s]-\{i_1,\dots,i_{r-3},k_1\})}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-3}}))-(d_{k_1}+d_{k_2})+(d_{j_1}+\cdots+d_{j_\ell}))) \\ \oplus \\ \displaystyle\bigoplus_{\substack{1\le i_1<\cdots<i_{r-3}\le s \\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-3},k_2\} \\ k_1=\max([s]-\{i_1,\dots,i_{r-3}\}) \\ k_2<\max([s]-\{i_1,\dots,i_{r-3},k_3\}) \\ k_2\notin\{i_1,\dots,i_{r-3},k_3\} \\ k_3=\max([s]-\{i_1,\dots,i_{r-3},k_1\})}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-3}}))-(d_{k_1}+d_{k_2})+(d_{j_1}+\cdots+d_{j_\ell}))) \\ \oplus \\ \displaystyle\bigoplus_{\substack{1\le i_1<\cdots<i_{r-3}\le s \\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-3},k_1,k_2\} \\ \{k_1,k_2\}\subset([s]-\{i_1,\dots,i_{r-3},k_3\}) \\ k_3=\max([s]-\{i_1,\dots,i_{r-3}\})}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-3}}))-(d_{k_1}+d_{k_2})+(d_{j_1}+\cdots+d_{j_\ell}))) \end{array}\right].$$

Proof. Let

$$\lambda = (\underbrace{0,\dots,0}_{(r-3)\text{-times}}, 1, 1, \underbrace{2,\dots,2}_{(s-(r-1))\text{-times}})$$

For $\sigma \in \mathfrak{S}_s$ and $u = \sigma(y^\lambda)$, recall that $|C(u)| = r-3, r-2$, or $r-1$ (see the proof of Proposition 5.1), i.e.,

$$C(u) = \{\sigma(1),\dots,\sigma(r-3)\}, \quad \{\sigma(1),\dots,\sigma(r-2)\}, \quad \text{or} \\ \{\sigma(1),\dots,\sigma(r-2),\sigma(r-1)\}.$$

(1) Let $C(u) = \{\sigma(1),\dots,\sigma(r-3)\}$, i.e., $\sigma(r-2), \sigma(r-1) \geq \max(u)$ (see the proof of Proposition 5.1). Using the symmetry of subscripts $1,2,\dots,s$, we assume that

$$C(u) = \{\sigma(1) = 1,\dots,\sigma(r-3) = r-3\}, \quad \sigma(r-2) = s-1, \quad \sigma(r-1) = s.$$

Then one of the minimal generators of this case in the free module is of the following form, and thus we get that

$$\begin{aligned}
&\begin{cases} y_1^0 y_2^0 \cdots y_{r-3}^0 (y_{s-1}^1 y_s^1)(y_{r-2}^2 \cdots y_{s-2}^2)(y_{j_1}^1 \cdots y_{j_\ell}^1), \\ \text{where } j_1,\dots,j_\ell \in \{1,\dots,r-3\} \end{cases} \\
\mapsto\ & (\overset{(s-1)\text{-st}}{1} + \overset{s\text{-th}}{1}) + 2(\overset{(r-2)\text{-nd}}{1} + \cdots + \overset{(s-2)\text{-nd}}{1}) + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\
=\ & 2(\overset{(r-2)\text{-nd}}{1} + \cdots + \overset{(s-2)\text{-nd}}{1} + \overset{(s-1)\text{-st}}{1} + \overset{s\text{-th}}{1}) - (\overset{(s-1)\text{-st}}{1} + \overset{s\text{-th}}{1}) \\
& + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\
=\ & 2(s - (\overset{1\text{-st}}{1} + \cdots + \overset{(r-3)\text{-rd}}{1})) - (\overset{(s-1)\text{-st}}{1} + \overset{s\text{-th}}{1}) + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\
\overset{\varphi_*}{\mapsto}\ & 2(d - (d_1 + d_2 + \cdots + d_{r-3})) - (d_{s-1} + d_s) + (d_{j_1} + \cdots + d_{j_\ell}).
\end{aligned}$$

From this information with the symmetry of the subscripts $1,2,\dots,s$, we get that

$$\begin{aligned}
& T^{\binom{s}{r-3}\binom{r-3}{\ell}}(-(2(s-(r-1)) + 2 + \ell)) \otimes_T R \\
=\ & \bigoplus_{\substack{1 \leq i_1 < \cdots < i_{r-3} \leq s \\ \{j_1,\dots,j_\ell\} \subset \{i_1,\dots,i_{r-3}\} \\ k_1 = \max([s] - \{i_1,\dots,i_{r-3}\}) \\ k_2 = \max([s] - \{i_1,\dots,i_{r-3},k_1\})}} R(-(2(d - (d_{i_1} + \cdots + d_{i_{r-3}})) - (d_{k_1} + d_{k_2}) + (d_{j_1} + \cdots + d_{j_\ell}))).
\end{aligned}$$

(2) Let $|C(u)| = r-2$. Then $\sigma(r-2) < \max(u) \leq \sigma(r-1)$. Using the same idea as in (1), assume

$$C(u) = \{\sigma(1) = 1, \sigma(2) = 2, \dots, \sigma(r-3) = r-3, \sigma(r-2) = k\}, \quad \sigma(r-1) = s.$$

Since

$$k_2 := k = \sigma(r-2) < \max(u) = \max\{\sigma(r),\dots,\sigma(s)\} \leq \sigma(r-1) = s := k_1,$$

we have that $r-2 \leq k \leq s-2$. Recall that the Betti number for this case is

$$(s-r+1)\binom{s}{r-3}\binom{r-2}{\ell}.$$

For such k, we have

$$
\begin{aligned}
&\begin{cases} y_1^0 y_2^0 \cdots y_{r-3}^0 y_k^1 y_s^1 (y_{r-2} \cdots \hat{y}_k \cdots y_{s-1} \hat{y}_s)^2 y_{j_1}^1 \cdots y_{j_\ell}^1, \\ \text{where } j_1, \dots, j_\ell \in \{1, \dots, r-3, k\} \text{ and } r-2 \le k \le s-2 \end{cases} \\
\mapsto\ & (\overset{k\text{-th}}{1} + \overset{s\text{-th}}{1}) + 2(\overset{(r-2)\text{-nd}}{1} + \cdots + \overset{\widehat{k\text{-th}}}{1} + \cdots + \overset{(s-1)\text{-st}}{1} + \overset{\widehat{s\text{-th}}}{1}) \\
& +(\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\
=\ & 2(\overset{(r-2)\text{-nd}}{1} + \cdots + \overset{k\text{-th}}{1} + \cdots + \overset{s\text{-th}}{1}) - (\overset{k\text{-th}}{1} + \overset{s\text{-th}}{1}) + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\
=\ & 2(s - (\overset{1\text{-st}}{1} + \cdots + \cdots + \overset{(r-3)\text{-rd}}{1})) - (\overset{k\text{-th}}{1} + \overset{s\text{-th}}{1}) + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\
\overset{\varphi_*}{\mapsto}\ & 2(d - (d_1 + \cdots + d_{r-3})) - (d_k + d_s) + (d_{j_1} + \cdots + d_{j_\ell}).
\end{aligned}
$$

From this observation with the symmetry of subscripts $1, 2, \dots, s$, we have

$$
\begin{aligned}
& T^{\binom{s}{r-3}(s-r+1)\binom{r-2}{\ell}}(-(2(s-(r-1)) + 2 + \ell)) \otimes_T R \\
=\ & \bigoplus_{\substack{1 \le i_1 < \cdots < i_{r-3} \le s \\ \{j_1, \dots, j_\ell\} \subset \{i_1, \dots, i_{r-3}, k_2\} \\ k_1 = \max([s] - \{i_1, \dots, i_{r-3}\}) \\ k_2 < \max([s] - \{i_1, \dots, i_{r-3}, k_3\}) \\ k_2 \notin \{i_1, \dots, i_{r-3}, k_3\} \\ k_3 = \max([s] - \{i_1, \dots, i_{r-3, k_1}\})}} R(-(2(d - (d_{i_1} + \cdots + d_{i_{r-3}})) - (d_{k_1} + d_{k_2}) + (d_{j_1} + \cdots + d_{j_\ell}))).
\end{aligned}
$$

(3) Let $|C(u)| = r-1$. Then $\sigma(r-2), \sigma(r-1) < \max(u)$. Using the same ideas as above, assume

$$C(u) = \{\sigma(1) = 1, \sigma(2) = 2, \dots, \sigma(r-3) = r-3, \sigma(r-2) = k, \sigma(r-1) = l\}.$$

Since

$$k_1 := k = \sigma(r-2), k_2 := l = \sigma(r-1) < \max(u) = \max\{\sigma(r), \dots, \sigma(s)\} = s,$$

we get that $r-2 \le k, l \le s-1$ (see the proof of Proposition 5.1). Recall that the Betti number for this case is

$$\binom{s}{r-3}\binom{s-(r-2)}{2}\binom{r-1}{\ell}.$$

For $r-2 \le k, l \le s-1$, we have

$$
\begin{aligned}
&\begin{cases} y_1^0 y_2^0 \cdots y_{r-3}^0 y_k^1 y_l^1 (y_{r-2} \cdots \hat{y}_k \cdots \hat{y}_l \cdots y_s)^2 y_{j_1}^1 \cdots y_{j_\ell}^1, \\ \text{where } j_1, \dots, j_\ell \in \{1, \dots, r-3, k, l\}, \end{cases} \\
\mapsto\ & (\overset{k\text{-th}}{1} + \overset{l\text{-th}}{1}) + 2(\overset{(r-2)\text{-nd}}{1} + \cdots + \overset{\widehat{k\text{-th}}}{1} + \cdots + \overset{\widehat{l\text{-th}}}{1} + \cdots + \overset{s\text{-th}}{1}) \\
& +(\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\
=\ & 2(\overset{(r-2)\text{-nd}}{1} + \cdots + \overset{k\text{-th}}{1} + \cdots + \overset{l\text{-th}}{1} + \cdots + \overset{s\text{-th}}{1}) - (\overset{k\text{-th}}{1} + \overset{l\text{-th}}{1}) \\
& +(\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\
=\ & 2(s - (\overset{1\text{-st}}{1} + \cdots + \overset{(r-3)\text{-rd}}{1})) - (\overset{k\text{-th}}{1} + \overset{l\text{-th}}{1}) + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\
\overset{\varphi_*}{\mapsto}\ & 2(d - (d_1 + \cdots + d_{r-3})) - (d_k + d_l) + (d_{j_1} + \cdots + d_{j_\ell}).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
&T^{\binom{s}{r-3}\binom{s-r+2}{2}\binom{r-1}{\ell}}(-(2(s-(r-1))+2+\ell)) \otimes_T R \\
= &\bigoplus_{\substack{1\le i_1<\cdots<i_{r-3}\le s \\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-3},k_1,k_2\} \\ \{k_1,k_2\}\subset([s]-\{i_1,\dots,i_{r-3},k_3\}) \\ k_3=\max([s]-\{i_1,\dots,i_{r-3}\})}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-3}}))-(d_{k_1}+d_{k_2})+(d_{j_1}+\cdots+d_{j_\ell}))).
\end{aligned}
$$

This completes the proof. □

The following lemma is also easy to prove, and we omit the proof.

Lemma 5.4. *With notations as in Lemma 5.3, the free submodule of the* $(\ell+1)$*-st free module* $\mathbb{E}_{\ell+1}$*, which is from the minimal generators of degree* $2(s-(r-2))$ *in* $I^{(4)}_{(r,s)}$*, is*

$$
\begin{aligned}
&T^{\binom{s}{r-2}\binom{r-2}{\ell}}(-(2(s-(r-2))+\ell)) \otimes_T R \\
= &\bigoplus_{\substack{1\le i_1<\cdots<i_{r-2}\le s \\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-2}\}}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{(r-2)}}))+(d_{j_1}+\cdots+d_{j_\ell}))).
\end{aligned}
$$

Lemma 5.5. *With notations as in Lemma 5.3, the free submodule of the* $(\ell+1)$*-st free module* $\mathbb{E}_{\ell+1}$*, which is from the minimal generators of degree* $3(s-(r-1))+1$ *in* $I^{(4)}_{(r,s)}$*, is*

$$
\begin{aligned}
&T^{(s-r+2)\binom{s}{r-2}\binom{r-1}{\ell}}(-(3(s-(r-1))+1+\ell)) \otimes_T R \\
= &\bigoplus_{\substack{1\le i_1<\cdots<i_{r-2}\le s \\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-2},k\} \\ k\in\{1,2,\dots,s\}-\{i_1,\dots,i_{r-2}\}}} R(-(3(d-(d_{i_1}+\cdots+d_{i_{(r-2)}}))-2\cdot d_k+(d_{j_1}+\cdots+d_{j_\ell})))
\end{aligned}
$$

Proof. Let $\lambda = (\underbrace{0,\dots,0}_{(r-2)\text{-times}}, 1, \underbrace{3,\dots,3}_{(s-(r-1))\text{-times}})$. For $\sigma \in \mathfrak{S}_s$, let $u = \sigma(y^\lambda)$.

Then $C(u) = \{\sigma(1),\dots,\sigma(r-1)\}$, i.e., $|C(u)| = r-1$. To produce monomial u with $|C(u)| = r-1$, we can first choose whose variables have degree 0. Among the remaining variables $(s-r+2)$, any single one has degree 1. Using the symmetry of subscripts $1,2,\dots,s$, we may assume that

$$\sigma(1)=1,\dots,\sigma(r-2)=r-2, \quad \text{and} \quad \sigma(r-1)=k,$$

where $r-1\le k\le s$. Recall that the Betti number for this case is

$$\binom{s}{r-2}\binom{r-1}{\ell}.$$

For $r-1 \le k \le s$, we have

$$\begin{array}{cl} & \left\{ \begin{array}{l} y_1^0 y_2^0 \cdots y_{r-2}^0 y_k^1 (y_{r-1} \cdots \hat{y}_k \cdots y_s)^3 y_{j_1}^1 \cdots y_{j_\ell}^1, \\ \text{where } j_1, \ldots, j_\ell \in \{1, \underset{\hat{}}{\ldots}, r-2, k\} \end{array} \right. \\ \mapsto & \overset{k\text{-th}}{1} + 3(\overset{(r-1)\text{-st}}{1} + \cdots + \overset{\hat{k\text{-th}}}{1} + \cdots + \overset{s\text{-th}}{1}) + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\ = & 3(\overset{(r-1)\text{-st}}{1} + \cdots + \overset{k\text{-th}}{1} + \cdots + \overset{s\text{-th}}{1}) - 2 \cdot \overset{k\text{-th}}{1} + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\ = & 3(s - (\overset{1\text{-st}}{1} + \cdots + \overset{(r-2)\text{-nd}}{1})) - 2 \cdot \overset{k\text{-th}}{1} + (\overset{j_1\text{-st}}{1} + \cdots + \overset{j_\ell\text{-th}}{1}) \\ \overset{\varphi_*}{\mapsto} & 3(d - (d_1 + \cdots + d_{r-2})) - 2 \cdot d_k + (d_{j_1} + \cdots + d_{j_\ell}). \end{array}$$

Therefore, by the symmetry of subscripts $1, 2, \ldots, s$ with this argument, we get that

$$\begin{aligned} & T^{(s-r+2)\binom{s}{r-2}\binom{r-1}{\ell}}(-(3(s-(r-1))+1+\ell)) \otimes_T R \\ = & \bigoplus_{\substack{1 \le i_1 < \cdots < i_{r-2} \le s \\ \{j_1, \ldots, j_\ell\} \subset \{i_1, \ldots, i_{r-2}, k\} \\ k \in \{1,2,\ldots,s\} - \{i_1, \ldots, i_{r-2}\}}} R(-(3(d - (d_{i_1} + \cdots + d_{i_{(r-2)}})) - 2 \cdot d_k + (d_{j_1} + \cdots + d_{j_\ell}))). \end{aligned}$$

This completes the proof. $\square$

From Lemmas 4.2, 5.3, 5.4, 5.5, and 4.4, we obtain the following theorem.

Theorem 5.6. *With notations as in Lemma 5.5, for $4 \le r \le n$, the $(\ell+1)$-st free module $\mathbb{E}_{\ell+1}$ of a graded minimal free resolution of $R/I_{\mathbb{X}}^{(4)}$ is*

$$\begin{aligned} & \mathbb{E}_{\ell+1} \\ = & \left[\bigoplus_{1 \le i_1 < \cdots < i_{r-4-\ell} \le s} R^{\binom{s-r+3+\ell}{\ell}}(-(d - (d_{i_1} + \cdots + d_{i_{(r-4-\ell)}}))) \right] \\ & \oplus \\ & \left[\begin{array}{c} \displaystyle\bigoplus_{\substack{1 \le i_1 < \cdots < i_{r-3} \le s \\ \{j_1, \ldots, j_\ell\} \subset \{i_1, \ldots, i_{r-3}\} \\ k_1 = \max([s] - \{i_1, \ldots, i_{r-3}\}) \\ k_2 = \max([s] - \{i_1, \ldots, i_{r-3}, k_1\})}} R(-(2(d - (d_{i_1} + \cdots + d_{i_{r-3}})) - (d_{k_1} + d_{k_2}) + (d_{j_1} + \cdots + d_{j_\ell}))) \\ \oplus \\ \displaystyle\bigoplus_{\substack{1 \le i_1 < \cdots < i_{r-3} \le s \\ \{j_1, \ldots, j_\ell\} \subset \{i_1, \ldots, i_{r-3}, k_2\} \\ k_1 = \max([s] - \{i_1, \ldots, i_{r-3}\}) \\ k_2 < \max([s] - \{i_1, \ldots, i_{r-3}, k_3\}) \\ k_2 \notin \{i_1, \ldots, i_{r-3}, k_3\} \\ k_3 = \max([s] - \{i_1, \ldots, i_{r-3}, k_1\})}} R(-(2(d - (d_{i_1} + \cdots + d_{i_{r-3}})) - (d_{k_1} + d_{k_2}) + (d_{j_1} + \cdots + d_{j_\ell}))) \\ \oplus \\ \displaystyle\bigoplus_{\substack{1 \le i_1 < \cdots < i_{r-3} \le s \\ \{j_1, \ldots, j_\ell\} \subset \{i_1, \ldots, i_{r-3}, k_1, k_2\} \\ \{k_1, k_2\} \subset ([s] - \{i_1, \ldots, i_{r-3}, k_3\}) \\ k_3 = \max([s] - \{i_1, \ldots, i_{r-3}\})}} R(-(2(d - (d_{i_1} + \cdots + d_{i_{r-3}})) - (d_{k_1} + d_{k_2}) + (d_{j_1} + \cdots + d_{j_\ell}))) \end{array} \right] \\ & \oplus \end{aligned}$$

$$\left[\bigoplus_{\substack{1\le i_1<\cdots<i_{r-2}\le s\\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-2}\}}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{(r-2)}}))+(d_{j_1}+\cdots+d_{j_\ell})))\right]$$

$$\oplus$$

$$\left[\bigoplus_{\substack{1\le i_1<\cdots<i_{r-2}\le s\\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-2},k\}\\ k\in\{1,2,\dots,s\}-\{i_1,\dots,i_{r-2}\}}} R(-(3(d-(d_{i_1}+\cdots+d_{i_{(r-2)}}))-2\cdot d_k+(d_{j_1}+\cdots+d_{j_\ell})))\right]$$

$$\oplus$$

$$\left[\bigoplus_{\substack{1\le i_1<\cdots<i_{r-1}\le s\\ \{j_1,\dots,j_\ell\}\subset\{i_1,\dots,i_{r-1}\}}} R(-(4(d-(d_{i_1}+\cdots+d_{i_{(r-1)}}))+(d_{j_1}+\cdots+d_{j_\ell})))\right]$$

for $0\le \ell\le r-1$.

Example 5.7. Let $\mathbb{X}$ be a star configuration in $\mathbb{P}^n$ of type $(4,5)$ defined by forms of degrees $1,1,1,1,5$ with $n\ge 4$. Let $d=d_1+\cdots+d_5=9$ and let

$$0\to\mathbb{E}_4\to\mathbb{E}_3\to\mathbb{E}_2\to\mathbb{E}_1\to R\to R/I_{\mathbb{X}}^{(4)}\to 0$$

be a graded minimal free resolution of $R/I_{\mathbb{X}}^{(4)}$. By Theorem 5.6, the first free module $\mathbb{E}_1$ is

$$\begin{aligned}\mathbb{E}_1=R(-d)&\oplus\left[\bigoplus_{\substack{1\le i\le s\\ k<l,\{k,l\}\subset\{1,2,\dots,s\}-\{i\}}} R(-(2(d-d_i)-(d_k+d_l)))\right]\\ &\oplus\left[\bigoplus_{1\le i<j\le s} R(-2(d-(d_i+d_j)))\right]\\ &\oplus\left[\bigoplus_{\substack{1\le i<j\le s\\ k\in\{1,2,\dots,s\}-\{i,j\}}} R(-(3(d-(d_i+d_j))-2d_k))\right]\\ &\oplus\left[\bigoplus_{1\le i<j<k\le s} R(-4(d-(d_i+d_j+d_k)))\right].\end{aligned}$$

From the following table on the shifts in $\mathbb{E}_1$ with $d_1=d_2=d_3=d_4=1$,

$2(d-d_1)-(d_2+d_3)$	14	$2(d-d_1)-(d_2+d_4)$	14	$2(d-d_1)-(d_2+d_5)$	10
$2(d-d_1)-(d_3+d_4)$	14	$2(d-d_1)-(d_3+d_5)$	10	$2(d-d_1)-(d_4+d_5)$	10

we get the shifts

$$14^{\otimes 12},\ 10^{\otimes 12},$$

where $*^{\otimes a}$ means that we have the shift $*$ a-times.

Since $(d-d_5)-(d_i+d_j)=6$ for $i,j\neq 5$ with $i\neq j$, we also have the shift

$$6^{\otimes 6}.$$

Based on the following table on the shifts in $\mathbb{E}_1$,

$2(d-(d_1+d_2))$	14	$2(d-(d_1+d_3))$	14
$2(d-(d_1+d_4))$	14	$2(d-(d_1+d_5))$	6

we have the shifts

$$14^{\otimes 6},\ 6^{\otimes 4}.$$

The following two tables shows the shifts in $\mathbb{E}_1$.

$3(d-(d_1+d_2))-2d_3$	19	$3(d-(d_1+d_2))-2d_4$	19	$3(d-(d_1+d_2))-2d_5$	11
$3(d-(d_1+d_5))-2d_2$	7	$3(d-(d_1+d_5))-2d_3$	7	$3(d-(d_1+d_5))-2d_4$	7

$4(d-(d_1+d_2+d_3))$	24	$4(d-(d_1+d_2+d_5))$	8

Thus we have the shifts

$$19^{\otimes 12},\ 11^{\otimes 6},\ 7^{\otimes 12},\quad \text{and}\quad 24^{\otimes 4},\ 8^{\otimes 6}.$$

So the first free module $\mathbb{E}_1$ is

$$\begin{aligned}\mathbb{E}_1 = {} & R(-6)^{10}\oplus R(-7)^{12}\oplus R(-8)^{6}\oplus R(-9)\oplus R(-10)^{12}\oplus R(-11)^{6}\\ & \oplus R(-14)^{18}\oplus R(-19)^{12}\oplus R(-24)^{4}.\end{aligned}$$

Similarly, by Theorem 5.6, one can find $\mathbb{E}_2, \mathbb{E}_3$, and $\mathbb{E}_4$ as well. Therefore, a graded minimal free resolution of $R/I_{\mathbb{X}}^{(4)}$ is

$$\begin{aligned}0 & \to R(-13)^{3}\oplus R(-14)^{12}\oplus R(-15)^{6}\oplus R(-17)^{12}\oplus R(-18)^{6}\\ & \quad \oplus R(-22)^{12}\oplus R(-27)^{4}\\ & \to R(-8)^{3}\oplus R(-9)^{12}\oplus R(-10)^{6}\oplus R(-12)^{20}\oplus R(-13)^{30}\oplus R(-14)^{12}\\ & \quad \oplus R(-16)^{42}\oplus R(-17)^{12}\oplus R(-21)^{36}\oplus R(-26)^{12}\\ & \to R(-7)^{12}\oplus R(-8)^{24}\oplus R(-9)^{12}\oplus R(-11)^{30}\oplus R(-12)^{24}\oplus R(-13)^{6}\\ & \quad \oplus R(-15)^{48}\oplus R(-16)^{6}\oplus R(-20)^{36}\oplus R(-25)^{12}\\ & \to R(-6)^{10}\oplus R(-7)^{12}\oplus R(-8)^{6}\oplus R(-9)\oplus R(-10)^{12}\oplus R(-11)^{6}\\ & \quad \oplus R(-14)^{18}\oplus R(-19)^{12}\oplus R(-24)^{4}\\ & \to R \ \to\ R/I_{\mathbb{X}}^{(4)} \ \to\ 0.\end{aligned}$$

6. Additional comments

In Section 3, we find a graded minimal free resolution for any m-th symbolic power of a star configuration in $\mathbb{P}^n$ of codimension 2. However, that of a star configuration in $\mathbb{P}^n$ of codimension $r>2$ (fixed, i.e., m is possibly $>r$) is not explored, but it is expected to be feasible. For example, if we take $r = 3$, $m = 4$, and $s \geq 3$, then we have four different partitions as follows (see Proposition 2.5).

1	1	2	2	$\cdots$	2
0	2	2	2	$\cdots$	2
0	1	3	3	$\cdots$	3
0	0	4	4	$\cdots$	4

Using these 4 partitions, we can find a graded minimal free resolution of the 4-th symbolic power of a linear star configuration by [1, Corollary 5.6], and a star configuration in $\mathbb{P}^n$ of codimension 3 by Theorem 2.1 using analogous ideas in Sections 3, 4, and 5. It appears that we can find a closed formula for a graded minimal free resolution of any m-th symbolic power for a star configuration in $\mathbb{P}^n$ of codimension $r > 2$, though the formula is not simple. It may take time, but feasible.

In Sections 4 and 5, we find a graded minimal free resolution in $\mathbb{P}^n$ of the m-th symbolic power for a star configuration in $\mathbb{P}^n$ of any codimension r for $m = 3, 4$ with $r \geq m$. If we consider the case of $s \geq r \geq 5$ and $m = 5$, then we obtain the graded Betti numbers and shifts of the 5-th symbolic power for a linear star configuration in $\mathbb{P}^n$ of codimension r, using Proposition 2.5 and [1, Corollary 4.6] (see also [1, Corollary 5.7]) as follows.

$$\beta_{\ell,\ell+d}(I^{(5)}_{(r,s)}) = \begin{cases} \binom{s}{r-5-\ell}\binom{s-r+4+\ell}{\ell} & \text{for } d = s-(r-5), \\ \binom{s}{r-x-3}\binom{s-r+x+3}{s-r+x}\binom{s-r+x-1}{0}\binom{r-x-3}{\ell} & \\ +\binom{s}{r-x-3}\binom{s-r+x+3}{s-r+x+1}\binom{s-r+x}{1}\binom{r-x-3}{\ell-1} & \\ +\binom{s}{r-x-3}\binom{s-r+x+3}{s-r+x+2}\binom{s-r+x+1}{2}\binom{r-x-3}{\ell-2} & \\ +\binom{s}{r-x-3}\binom{s-r+x+3}{s-r+x+3}\binom{s-r+x+2}{3}\binom{r-x-3}{\ell-3} & \\ & \text{for } d = 2(s-(r-4))-3, \\ \binom{s}{r-3}\binom{s-r+3}{s-r+2}\binom{r-3}{\ell}+\binom{s}{r-3}\binom{s-r+2}{1}\binom{r-3}{\ell-1} & \\ & \text{for } d = 2(s-(r-3))-1, \\ \binom{r-1}{2}\binom{s}{r-1}\binom{r-1}{\ell} & \text{for } d = 3(s-(r-1))+2, \\ \binom{r-1}{1}\binom{s}{r-1}\binom{r-1}{\ell} & \text{for } d = 4(s-(r-1))+1, \\ \binom{s}{r-1}\binom{r-1}{\ell} & \text{for } d = 5(s-(r-1)). \end{cases}$$

We then use the same ideas in Sections 4 and 5 to find a graded minimal free resolution of the 5-th symbolic power for a star configuration in $\mathbb{P}^n$ of codimension r with $s \geq r \geq 5$. With this approach, we can continue to find that of a star configuration in $\mathbb{P}^n$ of codimension $r \geq m \geq 5$. As seen before, it seems that no recursive formula is available for a graded minimal free resolution of the m-th symbolic power for a linear or star configuration in $\mathbb{P}^n$. However, we have introduced some ideas on how to find that of any m-th symbolic power for a star configuration in $\mathbb{P}^n$ of any codimension r in Sections 3, 4, and 5, hoping the ideas are useful to further extensions.

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