STATIC AND RELATED CRITICAL SPACES WITH HARMONIC CURVATURE AND THREE RICCI EIGENVALUES

Jongsu Kim

Abstract. In this article we make a local classification of n-dimensional Riemannian manifolds (M, g) with harmonic curvature and less than four Ricci eigenvalues which admit a smooth non constant solution f to the following equation

(1) $$\nabla df = f(r - \frac{R}{n-1}g) + x \cdot r + y(R)g,$$

where ∇ is the Levi-Civita connection of g, r is the Ricci tensor of g, x is a constant and $y(R)$ a function of the scalar curvature R. Indeed, we showed that, in a neighborhood V of each point in some open dense subset of M, either (i) or (ii) below holds;

(i) $(V, g, f + x)$ is a static space and isometric to a domain in the Riemannian product of an Einstein manifold N and a static space $(W, g_W, f + x)$, where g_W is a warped product metric of an interval and an Einstein manifold.

(ii) (V, g) is isometric to a domain in the warped product of an interval and an Einstein manifold.

For the proof we use eigenvalue analysis based on the Codazzi tensor properties of the Ricci tensor.

1. Introduction

A number of geometric spaces can be defined as the solutions of tensor field equations which involve Hessian of a function. Gradient Ricci solitons or static spaces are examples of such spaces.

Here we are interested in geometric spaces as follows: any Riemannian manifold (M, g) of constant scalar curvature R with a smooth non constant function

f satisfying the following equation with Hessian

$$(2) \qquad \nabla df = f(r - \frac{R}{n-1}g) + x \cdot r + y(R)g,$$

where ∇ is the Levi-Civita connection of g, r is the Ricci curvature of g, x is a constant and $y(R)$ a function of R. As explained in [12], the collection of solutions to (2) includes static spaces, Miao-Tam critical metrics, V-static spaces and critical point metrics. There are lots of recent interests and developments for these spaces from various sources [1, 3–5, 7, 8, 14, 15, 17, 18], but to understand them in general is nontrivial.

In this article, we shall confine ourselves to study (2) among Riemannian manifolds *with harmonic curvature*. This class of Riemannain manifolds has already been well studied when the manifolds are compact, for critical point metrics in [17] and Miao-Tam critical metrics in [2].

Harmonic curvature condition is interesting for its own sake. We recall that Riemannian manifolds with harmonic curvature are characterized when it admits at most two Ricci eigenvalues [6, Chap. 16] and [9]. So, we are interested in the spaces satisfying (2) with harmonic curvature and more than two Ricci eigenvalues.

Four dimensional spaces satisfying (2) with harmonic curvature have already been characterized in [12]. The approach therein depends on eigenvalue analysis based on the so-called Codazzi tensors. Although it was effective enough to yield explicit local and global description of the spaces, the computation becomes harder as the dimension of the manifold increases, leaving an interesting challenge.

Here we search for what can be done in such approach for higher dimension. Indeed, we managed to find a complete characterization of spaces with less than four Ricci eigenvalues;

Theorem 1. *Let (M^n, g, f) be a (not necessarily complete) n-dimensional Riemannian manifold satisfying (2) with harmonic curvature and less than four distinct Ricci eigenvalues at each point. Then for each point in some open dense subset of M, there exists a neighborhood V such that one of the following three assertions holds:*

(i) there are Einstein manifolds $(N^{k-1}, \tilde{g}_1)$ and $(U^{n-k}, \tilde{g}_2)$ with the Ricci tensor relation $r^{\tilde{g}_1} = (k-2)k_2\tilde{g}_1$ and $r^{\tilde{g}_2} = (n-k-1)k_n\tilde{g}_2$ for constants k_2, k_n such that

(V, g) is isometric to a domain in the Riemannian product of $(N^{k-1}, p^2\tilde{g}_1)$ for a constant p and a static space $(W^{n-k+1} := I \times U^{n-k}, \ ds^2 + h(s)^2\tilde{g}_2, \ f+x)$ which is a warped product over an open interval I; the function h on I satisfies (30), $f + x = c \cdot h'(s)$ for a constant c, and p satisfies (32). (V, g) has exactly three distinct Ricci eigenvalues at each point. It holds that $x\frac{R}{n-1} + y = 0$ and $(V, g, f+x)$ is a static space.

(ii) (V, g) is isometric to a domain in the warped product $(I \times N^{n-1}, ds^2 + h(s)^2 g_N)$ where g_N is an Einstein metric with the Ricci tensor $r^{g_N} = (n-2)k g_N$

for a constant k. The function h satisfies $(h')^2 + \frac{R}{n(n-1)}h^2 + 2\frac{a}{(n-2)h^{n-2}} = k$ for a constant a and $h'f' - fh'' = x(h'' + \frac{R}{n-1}h) + y(R)h$. (V, g) has exactly two distinct Ricci eigenvalues at each point.

(iii) (V, g) is Einstein and a warped product over an interval; $g = ds^2 + (f'(s))^2\tilde{g}$, where s is a function such that $\nabla s = \frac{\nabla f}{|\nabla f|}$ and $\tilde{g}$ is Einstein. And f satisfies $f'' = -\frac{R}{n(n-1)}f + x\frac{R}{n} + y(R)$.

Conversely, the Riemannian product of an Einstein metric and a warped-product static space as described in (i) *is a static space. Moreover, the warped product metric and the function f as in* (ii) *or* (iii) *satisfy* (2). *All these have harmonic curvature with less than four distinct Ricci eigenvalues at each point.*

To prove this theorem we analyze various cases about Ricci eigenvalues, based on the framework of [12]. On a space satisfying (2), the gradient ∇f of the potential function f is known to be an eigenvector for the Ricci tensor. So, we let $E_1 = \frac{\nabla f}{|\nabla f|}$ and form a Ricci-eigen orthonormal frame field E_i, $i = 1, \ldots, n$ with corresponding eigenvalues λ_i. As λ_1 plays a unique role, we divide the proof into two cases; the first is when the space has $\lambda_1 = \lambda_i$ for some $i > 1$, and the second is when $\lambda_1 \neq \lambda_i$ for any $i > 1$. We note that the argument of resolving the first case, done in Section 3, may provide a way to analyze the case of many eigenvalues.

In Section 2, we prepare for the framework of argument. In Section 3, we prove the case when the space has three eigenvalues with $\lambda_1 = \lambda_i$ for some $i > 1$. In Section 4, we prove the case of three eigenvalues with $\lambda_1 \neq \lambda_i$ for any $i > 1$. In the last Section 5, we treat the case of one or two eigenvalues and finish the proof of Theorem 1.

2. Preliminaries

In this section we recall some results from [12] with additional explanation. A Riemannian metric has harmonic curvature if and only if the Ricci tensor is a Codazzi tensor, written in local coordinates as $\nabla_k r_{ij} = \nabla_i r_{kj}$, [6, Chap. 16]. A Riemannian manifold with harmonic curvature is real analytic in harmonic coordinates [10]. Below we shall denote the Ricci tensor as r or $R(\cdot, \cdot)$.

Lemma 1. *For an n-dimensional manifold (M^n, g, f) with harmonic curvature satisfying* (2), *it holds that*

$$-R(X, Y, Z, \nabla f) = -R(X, Z)g(\nabla f, Y) + R(Y, Z)g(\nabla f, X)$$
$$- \frac{R}{n-1}\{g(\nabla f, X)g(Y, Z) - g(\nabla f, Y)g(X, Z)\}.$$

Proof. The proof is the same as that of Lemma 2.2 of [12] except the difference of dimensions. □

Lemma 2 (Lemma 2.3, [12])**.** *Let (M^n, g, f) have harmonic curvature, satisfying* (2). *Let c be a regular value of f and $\Sigma_c = \{x \mid f(x) = c\}$ be the level surface of f. Then the following assertions hold:*

(i)   *Where $\nabla f \neq 0$, $E_1 := \frac{\nabla f}{|\nabla f|}$ is an eigenvector field of r.*

(ii) *$|\nabla f|$ is constant on a connected component of Σ_c.*

(iii) *There is a function s locally defined with $s(x) = \int \frac{df}{|df|}$, so that $ds = \frac{df}{|df|}$ and $E_1 = \nabla s$.*

(iv) *$R(E_1, E_1)$ is constant on a connected component of Σ_c.*

(v)   *Near a point in Σ_c, the metric g can be written as*

$$g = ds^2 + \sum_{i,j>1} g_{ij}(s, x_2, \ldots, x_n)dx_i \otimes dx_j,$$

where $x_2, \ldots, x_n$ is a local coordinates system on Σ_c.

(vi) *$\nabla_{E_1} E_1 = 0$.*

For a point x in M, let $E_r(x)$ be the number of distinct eigenvalues of the Ricci tensor r_x, and set $M_r = \{x \in M \mid E_r$ is constant in a neighborhoodof $x\}$, following Derdziński [9], so that M_r is an open dense subset of M. Then we have:

Lemma 3 (Lemma 2.4, [12])**.** *For a Riemannian metric g of dimension $n \geq 4$ with harmonic curvature, consider orthonormal vector fields E_i, $i = 1, \ldots, n$ such that $R(E_i, \cdot) = \lambda_i g(E_i, \cdot)$. Then the followings hold in each connected component of M_r;*

(i) $(\lambda_j - \lambda_k)\langle \nabla_{E_i} E_j, E_k \rangle + E_i\{R(E_j, E_k)\}$

       $= (\lambda_i - \lambda_k)\langle \nabla_{E_j} E_i, E_k \rangle + E_j\{R(E_k, E_i)$ *for any $i, j, k = 1, \ldots, n$.*

(ii)   *If $k \neq i$ and $k \neq j$, $(\lambda_j - \lambda_k)\langle \nabla_{E_i} E_j, E_k \rangle = (\lambda_i - \lambda_k)\langle \nabla_{E_j} E_i, E_k \rangle$.*

(iii) *Given distinct eigenfunctions λ and μ of the Ricci tensor r and local vector fields v and u such that $rv = \lambda v$, $ru = \mu u$ with $|u| = 1$, it holds $v(\mu) = (\mu - \lambda)\langle \nabla_u u, v \rangle$.*

(iv)   *For each eigenfunction λ of r, the λ-eigenspace distribution is integrable and its leaves are totally umbilic submanifolds of M.*

Lemma 2 implies that for any point p in the open dense subset $M_r \cap \{\nabla f \neq 0\}$ of M^n, there is a neighborhood U of p where there exists an orthonormal Ricci-eigenvector fields E_i, $i = 1, \ldots, n$ such that

(i) $E_1 = \frac{\nabla f}{|\nabla f|}$,

(ii) for $i > 1$, E_i is tangent to smooth level hypersurfaces of f.

These local orthonormal Ricci-eigenvector fields $\{E_i\}$ shall be called an *adapted frame field* of (M, g, f). We set $\zeta_i := -\langle \nabla_{E_i} E_i, E_1 \rangle = \langle E_i, \nabla_{E_i} E_1 \rangle$ for $i > 1$. By (2), $\nabla_{E_i} E_1 = \nabla_{E_i}(\frac{\nabla f}{|\nabla f|}) = \frac{fR(E_i, \cdot) - f\frac{R}{n-1}g(E_i, \cdot) + xR(E_i, \cdot) + y(R)g(E_i, \cdot)}{|\nabla f|}$. So we may write:

(3) $\nabla_{E_i} E_1 = \zeta_i E_i$ where $\zeta_i = \dfrac{(f + x)R(E_i, E_i) - \frac{R}{n-1}f + y(R)}{|\nabla f|}$.

Due to Lemma 2, in a neighborhood of a point $p \in M_r \cap \{\nabla f \neq 0\}$, f may be considered as functions of the variable s only, and $f' := \frac{df}{ds} = |\nabla f|$.

Lemma 4. *Let (M, g, f) be an n-dimensional space with harmonic curvature, satisfying (2). The Ricci eigenfunctions λ_i associated to an adapted frame field E_i are constant on a connected component of a regular level hypersurface Σ_c of f, and so depend on the local variable s only. Moreover, ζ_i, $i = 2, \ldots, n$, in (3) also depend on s only. In particular, we have $E_i(\lambda_j) = E_i(\zeta_k) = 0$ for $i, k > 1$ and any j.*

Proof. Lemma 3.1 of [12] gives the proof in the four dimensional case. Similar argument can be given in higher dimension. One can refer to Lemma 3 in [16]. $\qquad\square$

3. Three eigenvalues with $\lambda_1 = \lambda_i$ for some $i > 1$

In this section we shall prove that if an n-dimensional space (M^n, g, f) with harmonic curvature satisfying (2) has exactly three Ricci eigenvalues, then it is not possible to have $\lambda_1 = \lambda_i$ for some $i > 1$.

Assume that $\lambda_1 = \lambda_i$ for some $i > 1$. We may assume that $\lambda_1 = \lambda_i$ for $1 \le i \le k$, $\lambda_{k+1} = \cdots = \lambda_{k+m}$, $\lambda_{k+m+1} = \cdots = \lambda_n$ and $\lambda_1, \lambda_{k+1}, \lambda_n$ are pairwise distinct.

By (3) and Lemma 2, setting $p := f + x$, we take (E_i, E_i) to (2) and get

$$(4) \qquad R_{11} = \frac{p''}{p} + \frac{R}{n-1} - \frac{z}{p},$$

$$(5) \qquad R_{ii} = \zeta_i \frac{p'}{p} + \frac{R}{n-1} - \frac{z}{p} \quad \text{for } i > 1,$$

where $R_{ij} := R(E_i, E_j)$ and $z := x\frac{R}{n-1} + y(R)$.

From the harmonic curvature condition, we have $0 = \nabla_1 R_{ii} - \nabla_i R_{1i}$ for $i > 1$, which gives

$$0 = R'_{ii} + \zeta_i(R_{ii} - R_{11}) = (\zeta_i\frac{p'}{p} + \frac{R}{n-1} - \frac{z}{p})' + \zeta_i(\zeta_i\frac{p'}{p} - \frac{p''}{p})$$

$$= \zeta_i'\frac{p'}{p} + \frac{zp'}{p^2} + \zeta_i^2\frac{p'}{p} - \zeta_i(\frac{p'}{p})^2.$$

Note that p is not constant. Multiplying the above by p^2, we have

$$(6) \qquad (\zeta_i'p + z + \zeta_i^2 p - \zeta_i p')p' = 0.$$

From $\lambda_1 = \lambda_2$, by (4) and (5), we get $\zeta_2 = \frac{p''}{p'}$. Put this into (6) and get

$$(7) \qquad p'''p + zp' - p'p'' = 0.$$

Integrating, $pp'' - (p')^2 + zp + b_1 = 0$ for a constant b_1. Multiply by $2p^{-3}p'$ to get $2p^{-2}p'p'' - 2p^{-3}(p')^3 + 2zp^{-2}p' + 2b_1p^{-3}p' = 0$. Integrating, $p^{-2}(p')^2 - 2zp^{-1} - b_1p^{-2} + b_2 = 0$ for a constant b_2. We have

$$(8) \qquad (p')^2 = 2zp + b_1 - b_2p^2,$$

$$(9) \qquad p'' = z - b_2p.$$

From (4) and (9), $R_{ii} = R_{11} = -b_2 + \frac{R}{n-1}$ for $2 \leq i \leq k$.

Now we shall exploit the equation $-R_{1ii1} = R_{ii} - \frac{R}{n-1}$, $i \geq k+1$ from Lemma 1. From (3) and Lemma 2(vi), we compute $R_{1ii1} = -\zeta_i' - \zeta_i^2$. So, we get $p(\zeta_i' + \zeta_i^2) - p'\zeta_i + z = 0$ from (5). Set $\zeta_i = \frac{u_i'}{u_i}$ and we obtain

$$(10) \qquad pu_i'' - p'u_i' + zu_i = 0.$$

Notice that (10) is a linear second order differential equation for u_i with one solution being p'; see (7). To find the second solution by reduction of order, we put $u_i = p'v$ into (10) and get a first order linear ODE for v': $pp'v'' + (2pp'' - (p')^2)v' = 0$. With (8) and (9), this gives $pp'v'' = (b_2p^2 + b_1)v'$. Now, we have

$$\frac{v''}{v'} = \frac{(b_2p^2 + b_1)}{pp'} = \frac{(b_2p^2 + b_1)p'}{pp'p'}$$

$$= \frac{(b_2p^2 + b_1)}{p(2zp + b_1 - b_2p^2)}p' = \{\frac{1}{p} + \frac{(2b_2p - 2z)}{(2zp + b_1 - b_2p^2)}\}p'.$$

By integration, for some constant C,

$$(11) \qquad v' = \frac{Cp}{2zp + b_1 - b_2p^2} = \frac{Cp}{(p')^2}.$$

Assume $b_2 > 0$. From (8) and (9) $p = c_0 \sin(\sqrt{b_2}(s-s_0)) + \frac{z}{b_2}$ for some numbers $c_0 \neq 0$ and s_0. By (11) $v' = \frac{C}{c_0^2 b_2} \cdot \frac{c_0 \sin(\sqrt{b_2}(s-s_0)) + \frac{z}{b_2}}{\cos^2(\sqrt{b_2}(s-s_0))}$. Integrating, for a constant $\tilde{c}$, we achieve

$$(12) \qquad v = \frac{C}{c_0^2 b_2}\{\frac{c_0}{\sqrt{b_2}} \sec(\sqrt{b_2}(s-s_0)) + \frac{z}{b_2}\frac{1}{\sqrt{b_2}}\tan(\sqrt{b_2}(s-s_0)) + \tilde{c}\}.$$

Any solution u_i of (10) can be written as $u_i = d_1p' + d_2p'v = p'(d_1 + d_2v)$ for constants d_1 and d_2. Then $\zeta_i = \frac{u_i'}{u_i} = \frac{p''(d_1+d_2v)+p'd_2v'}{p'(d_1+d_2v)} = \frac{p''}{p'} + \frac{d_2v'}{d_1+d_2v}$. So we may write $\zeta_i = \frac{p''}{p'} + \frac{v'}{c_i+v}$ for a constant c_i, $i \geq k+1$.

On the other hand, we have $R = R_{11} + \sum_{i=2}^n R_{ii} = k(-b_2 + \frac{R}{n-1}) + \sum_{i=k+1}^n \{\zeta_i \frac{p'}{p} + \frac{R}{n-1} - \frac{z}{p}\}$. Use (9), (11) and $\zeta_i = \frac{p''}{p'} + \frac{v'}{c_i+v}$, to get

$$(13) \qquad ap' + m\frac{1}{c_{k+1}+v} + (n-k-m)\frac{1}{c_n+v} = 0,$$

where $a := \frac{1}{C}(\frac{R}{n-1} - nb_2)$. Set $x := \cos(\sqrt{b_2}(s-s_0))$ and $y := \sin(\sqrt{b_2}(s-s_0))$.

Now (12) becomes $v = \frac{C}{c_0^2 b_2 \sqrt{b_2} x}\{c_0 + \frac{z}{b_2}y + \sqrt{b_2}\tilde{c}x\}$. We can write $c_i + v = \frac{C}{c_0^2 b_2 \sqrt{b_2} x}\{c_0 + \frac{z}{b_2}y + \hat{c}_i x\}$ for a constant $\hat{c}_i$.

From (13), $ac_0\sqrt{b_2}x(c_{k+1}+v)(c_n+v)+m(c_n+v)+(n-k-m)(c_{k+1}+v) = 0$, which yields

$$aC\{c_0 + \frac{z}{b_2}y + \hat{c_{k+1}}x\}\{c_0 + \frac{z}{b_2}y + \hat{c_n}x\} + mc_0b_2\{c_0 + \frac{z}{b_2}y + \hat{c_n}x\}$$

$$+ (n-k-m)c_0b_2\{c_0 + \frac{z}{b_2}y + \hat{c_{k+1}}x\} = 0.$$

By expanding, the above equation has the form of

$$(14) \qquad a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6 = 0,$$

where

$$x^2 + y^2 = 1, \ a_1 = aC c_{\hat{k}+1} \hat{c}_n, \ a_2 = aC(c_{\hat{k}+1} + \hat{c}_n)\frac{z}{b_2},$$

$$a_3 = aC\frac{z^2}{b_2^2}, \ a_4 = aCc_0(c_{\hat{k}+1} + \hat{c}_n) + mc_0 b_2 \hat{c}_n + (n-k-m)c_0 b_2 c_{\hat{k}+1},$$

$$a_5 = 2aCc_0\frac{z}{b_2} + (n-k)c_0^2 b_2\frac{z}{b_2} \text{ and } a_6 = aCc_0^2 + (n-k)c_0^2 b_2.$$

From (14), $(a_1 - a_3)x^2 + a_3 + a_4 x + a_6 = -(a_2 x + a_5)y$. By taking squares, we obtain $\{(a_1 - a_3)x^2 + a_3 + a_4 x + a_6\}^2 = (a_2 x + a_5)^2(1 - x^2)$. We can easily see that $a_1 = a_3$, $a_2 = 0$, $a_4 = a_5 = 0$ and $a_1 + a_6 = 0$. So, $a_2 = aC(c_{\hat{k}+1} + \hat{c}_n)\frac{z}{b_2} = 0$. Note that $C \neq 0$.

If $az \neq 0$, then $c_{\hat{k}+1} + \hat{c}_n = 0$. And $a_1 = a_3$ gives $0 \leq \frac{z^2}{b_2^2} = c_{\hat{k}+1}\hat{c}_n \leq 0$. Then we get $c_{\hat{k}+1} = \hat{c}_n = 0$. Then $c_{k+1} = c_n$, so $\zeta_{k+1} = \zeta_n$, a contradiction.

If $a = 0$, then $a_6 = (n-k)c_0 b_2$ and $a_1 = 0$. As $a_1 + a_6 = 0$, we get $b_2 = 0$, a contradiction to the assumption $b_2 > 0$.

If $z = 0$ and $a \neq 0$, then $a_3 = 0$. So, $a_1 = aCc_{\hat{k}+1}\hat{c}_n = 0$. We may assume $c_{\hat{k}+1} = 0$. Then $a_6 = aCc_0^2 + (n-k)c_0^2 b_2 = 0$, so $a_4 = aCc_0\hat{c}_n + mc_0 b_2\hat{c}_n = (k - n + m)c_0 b_2 \hat{c}_n$. We get $\hat{c}_n = 0$, a contradiction to $c_{k+1} \neq c_n$.

We have gotten only contradictions, so we cannot have $\lambda_1 = \lambda_i$ for some $i > 1$.

The other cases of $b_2 = 0$ and $b_2 < 0$ can be proved similarly and we omit them. We have proved:

Lemma 5. *Suppose that an n-dimensional manifold (M^n, g, f) with harmonic curvature satisfies (2) and has exactly three Ricci-eigenvectors. Then it is not possible to have $\lambda_1 = \lambda_i$ for some $i > 1$.*

We remark that the argument for Lemma 5 may be extended to any number of eigenvalues.

4. Three eigenvalues with $\lambda_1 \neq \lambda_i$ for any $i > 1$

In this section we treat the case (M, g) has exactly three Ricci-eigenvectors but $\lambda_1 \neq \lambda_i$ for any $i > 1$. We may assume that $\lambda_2 = \cdots = \lambda_k \neq \lambda_{k+1} = \cdots = \lambda_n$.

Lemma 6. *Let (M, g, f) be an n-dimensional Riemannian manifold with harmonic curvature satisfying (2). Suppose that for an adapted frame fields E_j, $j = 1, \ldots, n$, in an open subset W of $M_r \cap \{\nabla f \neq 0\}$, the eigenvalue λ_1 is distinct from any other λ_i and $\lambda_2 = \cdots = \lambda_k \neq \lambda_{k+1} = \cdots = \lambda_n$. Then there exist coordinates $(x_1 := s, x_2, \ldots, z_n)$ in a neighborhood of each point in W*

such that $\nabla s = \frac{\nabla f}{|\nabla f|}$ and g can be written as

(15) $$g = ds^2 + p(s)^2 \tilde{g}_1 + h(s)^2 \tilde{g}_2,$$

where $p := p(s)$ and $h := h(s)$ are smooth functions and $\tilde{g}_i$, $i = 1, 2$, is a pull-back of an Einstein metric on a $(k-1)$-dimensional domain N^{k-1} with $x_2, \ldots, x_k$ coordinates, and on an $(n-k)$-dimensional domain U^{n-k} with $x_{k+1}, \ldots, x_n$, respectively.

We have $E_1 = \frac{\partial}{\partial s}$, $E_i = \frac{1}{p} e_i$, $i = 2, \ldots, k$ and $E_j = \frac{1}{p} e_j$, $j = k+1, \ldots, n$ where $\{e_i\}$ and $\{e_j\}$ are orthonormal frame fields on U_1 and U_2, respectively.

Proof. For $i \in \{2, \ldots, k\}$ and $j \in \{k+1, \ldots, n\}$, from Lemma 3(ii), we have

$$(\lambda_i - \lambda_j)\langle \nabla_{E_1} E_i, E_j \rangle = (\lambda_1 - \lambda_j)\langle \nabla_{E_i} E_1, E_j \rangle.$$

As $\langle \nabla_{E_i} E_1, E_j \rangle = 0$ by (3), $\langle \nabla_{E_1} E_i, E_j \rangle = 0$. By Lemma 2(vi) we get, $\nabla_{E_1} E_i = \sum_{l \in \{2,\ldots,k\}, l \neq i} \Gamma^l_{1i} E_l$. And $[E_1, E_i]$ belongs to the span of $E_2, \ldots, E_k$. As the span of $E_2, \ldots, E_k$ is integrable by Lemma 3, so the span D_1 of $E_1, E_2, \ldots, E_k$ is integrable. The span D_2 of $E_{k+1}, \ldots, E_n$ is also integrable. By a higher dimensional version of Lemma 4.2 of [11], there exist local coordinates y_i in which $\frac{\partial}{\partial y_i}$, $i = 1, \ldots, k$, span D_1 and $\frac{\partial}{\partial y_i}$, $i = k+1, \ldots, n$, span D_2 and $g = \sum_{i,j=1}^{k} \tilde{g}_{ij} dy_i \odot dy_j + \sum_{i,j=k+1}^{n} \tilde{g}_{ij} dy_i \odot dy_j$, where $\odot$ is the symmetric tensor product and $\tilde{g}_{ij}$ are functions of y_i, and $\sum_{i=1}^{k} E_i^* \odot E_i^* = \sum_{i,j=1}^{k} \tilde{g}_{ij} dy_i \odot dy_j$ and $\sum_{i=k+1}^{n} E_i^* \odot E_i^* = \sum_{i,j=k+1}^{n} \tilde{g}_{ij} dy_i \odot dy_j$, where E_i^* is the dual of E_i with respect to g.

By the symmetrical argument to the above, the span D_3 of $E_1, E_{k+1}, \ldots, E_n$ and the span D_4 of $E_2, \ldots, E_k$ is integrable and so there exist local coordinates z_i in which $\frac{\partial}{\partial z_i}$, $i = 1$ and $i = k+1, \ldots, n$, span D_3, and $\frac{\partial}{\partial z_i}$, $i = 2, \ldots, k$, span D_4 so that $g = \hat{g}_{11} dz_1^2 + \sum_{i=k+1}^{n} \hat{g}_{1i} dz_1 \odot dz_i + \sum_{i,j=k+1}^{n} \hat{g}_{ij} dz_i \odot dz_j + \sum_{i,j=2}^{k} \hat{g}_{ij} dz_i \odot dz_j$, where $\hat{g}_{ij}$ are functions of z_i, where $E_1^* \odot E_1^* + \sum_{i=k+1}^{n} E_i^* \odot E_i^* = \hat{g}_{11} dz_1^2 + \sum_{i=k+1}^{n} \hat{g}_{1i} dz_1 \odot dz_i + \sum_{i,j=k+1}^{n} \hat{g}_{ij} dz_i \odot dz_j$ and $\sum_{i=2}^{k} E_i^* \odot E_i^* = \sum_{i,j=2}^{k} \hat{g}_{ij} dz_i \odot dz_j$.

Now recall the metric expression in Lemma 2(v). The functions $s, z_2, \ldots, z_k$, $y_{k+1}, \ldots, y_n$ form local coordinates near a point and $g = ds^2 + \sum_{i=2}^{n} E_i^* \odot E_i^* = ds^2 + \sum_{i,j=2}^{k} \hat{g}_{ij} dz_i \odot dz_j + \sum_{i,j=k+1}^{n} \tilde{g}_{ij} dy_i \odot dy_j$.

Denoting by $(x_1 := s, x_2, \ldots, x_n)$ the coordinates $s, z_2, \ldots, z_k, y_{k+1}, \ldots, y_n$, the metric g can be written as $g = ds^2 + \sum_{i,j=2}^{k} g_{ij} dx_i dx_j + \sum_{i,j=k+1}^{n} g_{ij} dx_i dx_j$.

We write $\partial_1 = \frac{\partial}{\partial s}$, $\partial_i = \frac{\partial}{\partial x_i}$. From Lemma 3(ii), (iii) and Lemma 4, we have $\langle \nabla_{E_i} E_j, E_a \rangle = 0$ for $i, j \in \{2, \ldots, k\}$ and $a = k+1, \ldots, n$. So, we should have $\langle \nabla_{\partial_i} \partial_j, \partial_a \rangle = 0$. Computing the Christoffel symbol in local coordinates, we get

(16) $$0 = \langle \nabla_{\partial_i} \partial_j, \partial_a \rangle = -\frac{1}{2} \partial_a g_{ij}.$$

By (3), $\langle \nabla_{E_i} E_i, E_1 \rangle = -\zeta_2$ for $i \in \{2, \ldots, k\}$. So we get $\langle \nabla_{\partial_i} \partial_j, \frac{\partial}{\partial s} \rangle = -\zeta_2 g_{ij}$ for $i, j \in \{2, \ldots, k\}$. Computing $\nabla_{\partial_i} \partial_j$, we obtain

$$\text{(17)} \qquad \zeta_2 g_{ij} = \frac{1}{2} \frac{\partial}{\partial s} g_{ij}.$$

From (16) and (17), for $i, j \in \{2, \ldots, k\}$, we get $g_{ij} = e^{C_{ij}} p(s)^2$, where the function $p(s) > 0$ is independent of i, j and each function C_{ij} depends only on $x_2, \ldots, x_k$.

Similarly, we can get $g_{ij} = e^{\tilde{C}_{ij}} h(s)^2$ for $i, j \in \{k+1, \ldots, n\}$. So, g can be written as $g = ds^2 + p(s)^2 \tilde{g}_1 + h(s)^2 \tilde{g}_2$, where $\tilde{g}_i$, $i = 1, 2$, is a Riemannian metric on a $(k-1)$-dimensional domain U_1 with $x_2, \ldots, x_k$ coordinates, and on an $(n-k)$-dimensional domain U_2 with $x_{k+1}, \ldots, x_n$, respectively.

In local coordinates $(x_1 := s, x_2, \ldots, x_n)$, we shall write some Christofel symbols Γ_{ij}^k and Ricci curvature of g. In this proof, for any $(0, 2)$-tensor P, $P(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ shall be denoted by P_{ij}. We let $\tilde{\nabla}$, $\tilde{\Gamma}_{ij}^k$ and $R_{ij}^{\tilde{g}_1}$ be the Levi-Civita connection, Christoffel symbols and Ricci curvature of $\tilde{g}_1$, respectively. For $i, j, l \in \{2, \ldots, k\}$, we get;

$$\text{(18)} \qquad \Gamma_{ij}^l = \tilde{\Gamma}_{ij}^l,$$

$$R_{ij} = -\tilde{g}_{1\,ij} \left\{ pp'' + (n-k)\frac{h'}{h} pp' + (k-2)p'^2 \right\} + R_{ij}^{\tilde{g}_1}.$$

By the assumption $\lambda_2 = \cdots = \lambda_k$, we have $R_{ij} = \lambda_2 g_{ij} = \lambda_2 p^2 \tilde{g}_{1\,ij}$. So, $R_{ij}^{\tilde{g}_1} = \lambda_2 p^2 \tilde{g}_{1\,ij} + \tilde{g}_{1\,ij} \{ pp'' + (n-k)\frac{h'}{h} pp' + (k-2)p'^2 \}$. So, if $k > 3$, $\tilde{g}_1$ is Einstein.

Assume $k = 3$. From (18), for $i, j, l \in \{2, \ldots, k\}$, we have $\nabla_l \tilde{g}_{1\,ij} = \tilde{\nabla}_l \tilde{g}_{1\,ij} = 0$ and $\nabla_l R_{ij}^{\tilde{g}_1} = \tilde{\nabla}_l R_{ij}^{\tilde{g}_1}$ so that $\nabla_l R_{ij} = \tilde{\nabla}_l R_{ij}^{\tilde{g}_1}$. The harmonic curvature condition gives $\nabla_l R_{ij} = \nabla_j R_{il}$ so that $\tilde{\nabla}_l R_{ij}^{\tilde{g}_1} = \tilde{\nabla}_j R_{il}^{\tilde{g}_1}$. By the contracted second Bianchi identity the 2-dimensional metric $\tilde{g}_1$ then has constant curvature.

If $k = 2$, $\tilde{g}_1$ is one-dimensional metric.

We can similarly prove that $\tilde{g}_2$ is Einstein and this proves the lemma. $\square$

Lemma 7. *For the local metric $g = ds^2 + p(s)^2 \tilde{g}_1 + h(s)^2 \tilde{g}_2$ of (15) with the frame E_i, if we write the Ricci tensors of $\tilde{g}_i$ as $R^{\tilde{g}_1} = (k-2)k_2 \tilde{g}_1$ and $R^{\tilde{g}_2} = (n-k-1)k_n \tilde{g}_2$ for numbers k_2 and k_n, then the following assertions hold*:

For $i \geq 2$, $\langle \nabla_{E_i} E_i, E_1 \rangle = -\zeta_i$ with $\zeta_2 = \cdots = \zeta_k = \frac{p'}{p}$ and $\zeta_{k+1} = \cdots = \zeta_n = \frac{h'}{h}$.

For the Ricci tensor components $R_{ij} = R(E_i, E_j)$ of g,

$$R_{11} = -(k-1)(\zeta_2' + \zeta_2^2) - (n-k)(\zeta_n' + \zeta_n^2),$$

$$R_{ii} = -\zeta_2' - (k-1)\zeta_2^2 - (n-k)\zeta_2 \zeta_n + \frac{(k-2)}{p^2} k_2 \quad \text{for } i \in \{2, \ldots, k\},$$

$$R_{jj} = -\zeta_n' - (n-k)\zeta_n^2 - (k-1)\zeta_2\zeta_n + \frac{(n-k-1)}{h^2}k_n \quad \text{for } j \in \{k+1,\ldots,n\}.$$

Moreover, $R_{1ii1} := R(E_1, E_i, E_i, E_1) = -\zeta_i' - \zeta_i^2$.

Proof. One may verify all the formulas by direct computation or using the Gauss equation for submanifolds. □

Next, we can prove:

Lemma 8. *Under the hypothesis of Lemma 6, it holds that* $x\frac{R}{n-1} + y = 0$ *and* $\zeta_2\zeta_n = 0$.

Proof. Recall, for $j > 1$,

$$(19) \qquad\qquad\qquad -R_{1jj1} = R_{jj} - \frac{R}{n-1}.$$

We put $j = 2$ and $j = n$ into (19), and from Lemma 7 get

$$(20) \qquad -2\zeta_2' - k\zeta_2^2 - (n-k)\zeta_2\zeta_n + \frac{(k-2)}{p^2}k_2 - \frac{R}{n-1} = 0,$$

$$(21) \qquad -2\zeta_n' - (n-k+1)\zeta_n^2 - (k-1)\zeta_2\zeta_n + \frac{(n-k-1)}{h^2}k_n - \frac{R}{n-1} = 0.$$

Differentiating (20),

$$(22) \qquad -2\zeta_2'' - 2k\zeta_2\zeta_2' - (n-k)\zeta_2'\zeta_n - (n-k)\zeta_2\zeta_n' - 2\frac{(k-2)p'}{p^3}k_2 = 0.$$

The harmonic curvature condition gives $\nabla_1 R_{ii} - \nabla_i R_{1i} = (R_{ii})' + \zeta_i R_{ii} - \zeta_i R_{11} = 0$. Put $i = 2$ and get

$$-\zeta_2'' - k\zeta_2\zeta_2' - (n-k)\zeta_2'\zeta_n - \frac{(k-2)p'}{p^3}k_2 - (n-k)\zeta_2^2\zeta_n + (n-k)\zeta_2\zeta_n^2 = 0.$$

Comparing this with (22) gives

$$(23) \qquad\qquad \zeta_2'\zeta_n - \zeta_2\zeta_n' + 2\zeta_2^2\zeta_n - 2\zeta_2\zeta_n^2 = 0.$$

From (5) and (19), $\zeta_2' + \zeta_2^2 = \zeta_2\frac{f'}{f+x} - \frac{z}{f+x}$ and $\zeta_n' + \zeta_n^2 = \zeta_n\frac{f'}{f+x} - \frac{z}{f+x}$. Then,

$$(24) \qquad\qquad (\zeta_2' + \zeta_2^2)\zeta_n - (\zeta_n' + \zeta_n^2)\zeta_2 = \frac{z}{f+x}(\zeta_2 - \zeta_n).$$

The above (23) and (24) yield $-\zeta_2^2\zeta_n + \zeta_2\zeta_n^2 = \frac{z}{f+x}(\zeta_2 - \zeta_n)$, so

$$(25) \qquad\qquad\qquad \zeta_2\zeta_n = -\frac{z}{f+x}.$$

Differentiating (25), and using (25) again,

$$(26) \qquad\qquad -(\zeta_2'\zeta_n + \zeta_2\zeta_n') = -\frac{zf'}{(f+x)^2} = \zeta_2 f' \cdot \frac{\zeta_n}{f+x}.$$

Using (5) and (19), $\zeta_2 f' = (f+x)(R_{22} - \frac{R}{n-1}) + z = f(R_{22} - \frac{R}{n-1}) + x(R_{22} - \frac{R}{n-1}) + x\frac{R}{n-1} + y = f(\zeta_2' + \zeta_2^2) + xR_{22} + y$. Together with (26), $-(\zeta_2'\zeta_n + \zeta_2\zeta_n')(1 + \frac{x}{f}) = \zeta_2 f' \cdot \frac{\zeta_n}{f} = \zeta_n(\zeta_2' + \zeta_2^2) + x\frac{\zeta_n}{f}\{-\zeta_2' - (k-1)\zeta_2^2 - (n-k)\zeta_2\zeta_n + \frac{(k-2)}{p^2}k_2\} + y\frac{\zeta_n}{f}$, which is rearranged as

$$\frac{x}{f}\{-\zeta_2\zeta_n' + (k-1)\zeta_2^2\zeta_n + (n-k)\zeta_2\zeta_n^2 - \frac{(k-2)}{p^2}k_2\zeta_n\}$$

$$(27) \qquad = 2\zeta_2'\zeta_n + \zeta_2\zeta_n' + \zeta_2^2\zeta_n + \frac{y}{f}\zeta_n.$$

We shall remove $\zeta_2{}'$ and $\zeta_n{}'$ in (27). By (20) and (23) we get

$$\zeta_2'\zeta_n = \frac{1}{2}\{-k\zeta_2^2\zeta_n - (n-k)\zeta_2\zeta_n^2 + \alpha\zeta_n\}, \text{ where } \alpha := \frac{(k-2)}{p^2}k_2 - \frac{R}{n-1},$$

$$\zeta_2\zeta_n' = \zeta_2'\zeta_n + 2\zeta_2^2\zeta_n - 2\zeta_2\zeta_n^2 = (\frac{4-k}{2})\zeta_2^2\zeta_n - \frac{(4+n-k)}{2}\zeta_2\zeta_n^2 + \frac{1}{2}\alpha\zeta_n.$$

With these, and setting $\beta := (k-2)\zeta_2^2 + (n-k+2)\zeta_2\zeta_n - \frac{(k-2)}{p^2}k_2$, the left hand side (LHS) of (27) equals $\frac{x}{2f}\{3\beta\zeta_n - 2\zeta_2\zeta_n^2 + \frac{R}{n-1}\zeta_n\}$, while the RHS equals

$$\frac{1}{2}\{-3\beta\zeta_n + 2\zeta_2\zeta_n^2 - 3\frac{R}{n-1}\zeta_n + 2\frac{y}{f}\zeta_n\}.$$

The equality of LHS=RHS gives

$$3(1 + \frac{x}{f})\beta\zeta_n = \frac{x}{f}\{2\zeta_2\zeta_n^2 - \frac{R}{n-1}\zeta_n\} + 2\zeta_2\zeta_n^2 - 3\frac{R}{n-1}\zeta_n + 2\frac{y}{f}\zeta_n.$$

From (25), we get $(1 + \frac{x}{f})\zeta_2\zeta_n = -\frac{x\frac{R}{n-1} + y}{f}$. So,

$$3(1 + \frac{x}{f})\beta\zeta_n = -3(1 + \frac{x}{f})\frac{R}{n-1}\zeta_n + 2\frac{x}{f}\zeta_2\zeta_n^2 + 2\zeta_2\zeta_n^2 + 2\frac{x}{f}\frac{R}{n-1}\zeta_n + 2\frac{y}{f}\zeta_n$$

$$= -3(1 + \frac{x}{f})\frac{R}{n-1}\zeta_n.$$

We have obtained $3(1 + \frac{x}{f})(\beta + \frac{R}{n-1})\zeta_n = 0$.

As f is not constant, we have either $\zeta_n = 0$ or $\beta + \frac{R}{n-1} = (k-2)\zeta_2^2 + (n-k+2)\zeta_2\zeta_n - \frac{(k-2)}{p^2}k_2 + \frac{R}{n-1} = 0$.

If $\zeta_n = 0$, then $z = 0$ from (25). So, Lemma holds.

If $\zeta_n \neq 0$, i.e., $h' \neq 0$, and $\beta + \frac{R}{n-1} = 0$, then (20) gives

$$-2\zeta_2' - k\zeta_2^2 + \frac{(n-k)}{(n-k+2)}\{(k-2)\zeta_2^2 - \frac{(k-2)}{p^2}k_2 + \frac{R}{n-1}\} + \frac{(k-2)}{p^2}k_2 - \frac{R}{n-1} = 0,$$

which reduces to

$$-(n-k+2)\frac{p''}{p} - (k-2)\frac{(p')^2}{p^2} + \frac{(k-2)}{p^2}k_2 - \frac{R}{n-1} = 0$$

as we put $\zeta_2 = \frac{p'}{p}$. Comparing with $\beta + \frac{R}{n-1} = 0$, we get $\frac{p''}{p} = \zeta_2 \zeta_n$. So, $\frac{p''}{p} = \frac{p'h'}{ph}$.

Assume that $p' \neq 0$. Then $\frac{p''}{p'} = \frac{h'}{h}$ and integration gives $p' = c_1 h$ for a constant $c_1 \neq 0$. While we are assuming $h' \neq 0$ and $p' \neq 0$, by a symmetrical argument between p and h we can also get $h' = c_2 p$ for constant $c_2 \neq 0$. (25) again gives $\frac{p'h'}{ph} = c_1 c_2 = -\frac{z}{f+x}$. So, f is a constant, a contradiction. So, $p'h' = 0$, i.e., $\zeta_2 \zeta_n = 0$. By (25), $z = 0$. This proves Lemma. $\qquad\square$

Now with the above lemma given, we may assume $\zeta_2 = \frac{p'}{p} = 0$ and $\zeta_n \neq 0$. The other case that $\zeta_2 \neq 0 = \zeta_n$ will be symmetrical. Now p is a constant, and (21) becomes

$$(28) \qquad -2\frac{h''}{h} - (n-k-1)\left(\frac{h'}{h}\right)^2 + \frac{(n-k-1)}{h^2}k_n - \frac{R}{n-1} = 0.$$

Multiply (28) by h^2, differentiate and then multiply by $-\frac{1}{2}h^{n-k-1}$ to get $h^{n-k}h''' + (n-k)h^{n-k-1}h'h'' + \frac{R}{n-1}h^{n-k}h' = \{h''h^{n-k} + \frac{R}{(n-1)(n-k+1)}h^{n-k+1}\}'$ $= 0$. Integration gives, for a constant c_2,

$$(29) \qquad \frac{h''}{h} + \frac{R}{(n-1)(n-k+1)} = \frac{c_2}{h^{n-k+1}}.$$

Put (29) into (28) for $n - k \geq 2$, or just integrate (28) when $n - k = 1$ and get

$$(30) \qquad (h')^2 + \frac{Rh^2}{(n-1)(n-k+1)} + \frac{2c_2 h^{-n+k+1}}{(n-k-1)} = k_n \quad \text{for} \quad n-k \geq 2,$$

$$(h')^2 + \frac{R}{2(n-1)}h^2 = c_3 \quad \text{for a constant } c_3, \quad \text{for } n-k = 1.$$

Recall $\zeta_n' + \zeta_n^2 = \zeta_n \frac{f'}{f+x} - \frac{z}{f+x}$ from (5) and (19). As $z = 0$, we then have $\frac{h''}{h'} = \frac{f'}{f+x}$. Integration gives, for a constant $c_1 \neq 0$,

$$(31) \qquad\qquad\qquad f + x = c_1 h'.$$

And $\zeta_2 = 0$ in (20) gives

$$(32) \qquad\qquad\qquad \frac{(k-2)}{p^2}k_2 = \frac{R}{n-1}.$$

One can check that the metric g with the above p, h and f in (30)$\sim$(32) satisfy the equation (2) and the harmonic curvature condition. As p is constant, $g = ds^2 + p(s)^2\tilde{g}_1 + h(s)^2\tilde{g}_2$ is the Riemannian product of an Einstein metric $(N^{k-1}, p^2\tilde{g}_1)$ and $(W^{n-k+1}, ds^2 + h(s)^2\tilde{g}_2)$. Summarizing all the discussion in this section, we state:

Proposition 1. *Let (M, g, f) be an n-dimensional Riemannian manifold with harmonic curvature satisfying (2). Suppose that for an adapted frame fields E_j, $j = 1, \ldots, n$, in an open subset W of $M_r \cap \{\nabla f \neq 0\}$, the eigenvalue λ_1 is distinct from any other λ_i and $\lambda_2 = \cdots = \lambda_k \neq \lambda_{k+1} = \cdots = \lambda_n$. Then*

there exist coordinates $(x_1 := s, x_2, \ldots, z_n)$ in a neighborhood of each point in W such that $\nabla s = \frac{\nabla f}{|\nabla f|}$ and g can be written as

$$(33) \qquad\qquad g = ds^2 + p^2 \tilde{g}_1 + h(s)^2 \tilde{g}_2,$$

where p is a constant and $h := h(s)$ a smooth function satisfying (30) and (32) and $\tilde{g}_i$, $i = 1, 2$, is an Einstein metric with Ricci tensor $R^{\tilde{g}_1} = (k-2)k_2\tilde{g}_1$ and $R^{\tilde{g}_2} = (n-k-1)k_n\tilde{g}_2$ for numbers k_2 and k_n. We also get $f + x = c_1 h'$ and $z := x\frac{R}{n-1} + y = 0$.

Conversely, any Riemannian metric as in (33) satisfying (30) and (32) is a solution of (2) with $f = c_1 h' - x$.

5. The proof of Theorem 1

Now we are ready to prove the main theorem.

Proof of Theorem 1. Lemma 5 and Proposition 1 resolve the case of exactly three Ricci eigenvalues. When $\lambda_1 = \lambda_i$ for some $i > 1$, Lemma 5 shows no existence of any space (M, g, f). When $\lambda_1 \neq \lambda_i$ for any $i > 1$, we have $x\frac{R}{n-1} + y = 0$ by Proposition 1. Then (2) becomes $\nabla df = (f + x)(Rc - \frac{R}{n-1}g)$, which means $(M, g, f+x)$ is a static space. One can see that the $(n-k+1)$-dimensional space $(W^{n-k+1} := I \times U^{n-k}, ds^2 + h(s)^2\tilde{g}_2, f + x)$ is also a static space where the equation (30) corresponds to (2.2) in [13]. It is easy to see that the metric $ds^2 + h(s)^2\tilde{g}_2$ itself has harmonic curvature.

If there are exactly two distinct Ricci eigenvalues in an open subset of $M_r \cap \{\nabla f \neq 0\}$, setting μ_i to be the dimension of λ_i-eigenspace, the multiplicity (μ_a, μ_b) of Ricci eigenvalues can be either $(1, n - 1)$ or $(p, n - p)$ with $1 < p < n - 1$.

In the $(p, n - p)$ case, we observe that the proof of Lemma 5 contains the proof for our case. Indeed, as $\lambda_1 = \lambda_i$ for some $i > 1$, following through the proof of Lemma 5, we see that as there are only two eigenvalues, we still have $c_i + v = \frac{C}{c_0^2 b_2 \sqrt{b_2}x}\{c_0 + \frac{z}{b_2}y + \hat{c}_i x\}$ for a constant $\hat{c}_i$, but the equation (13) has only two terms, not three. So, the argument can be done more simply to show no existence.

In the $(1, n - 1)$ case and if $\lambda_1 = \lambda_i$ for some $i > 1$, then this does not occur by the above paragraph.

In the $(1, n - 1)$ case and if $\lambda_1 \neq \lambda_i$ for any $i > 1$, locally the metric g is known to be a warped product metric of an interval with an Einstein metric; see the proof of 16.38 Theorem in [6]. To see precise description, we refer to Proposition 7.1 of [12], where it is analyzed in dimension four, but the argument still works in higher dimension, yielding (ii).

If there is exactly one Ricci eigenvalue in an open subset of $M_r \cap \{\nabla f \neq 0\}$, i.e., when g is Einstein, this case is discussed as Example 1 of [12]. Though it is written for four dimension, the argument works for higher dimension. According to that Example 1, in some neighborhood of a point we can write

$g = ds^2 + (f'(s))^2\tilde{g}$, where s is a function such that $\nabla s = \frac{\nabla f}{|\nabla f|}$ and $\tilde{g}$ is considered as a Riemannian metric on a level surface of f. The metric $\tilde{g}$ is Einstein and f satisfies $f'' = -\frac{R}{n(n-1)}f + x\frac{R}{n} + y(R)$. $\square$

Theorem 1 indicates that there may be fewer solutions of other geometric equations such as Miao-Tam metrics or critical point metrics than static spaces.

The converse part of Theorem 1 provides all the examples of Riemannian manifolds with harmonic curvature and less than four Ricci eigenvalues, satisfying (2). In particular we can get a compact static space $(N^{k-1}, \tilde{g}_1) \times \mathbb{S}^1 \times_h (\mathbb{Y}^{n-k}, \tilde{g}_2)$, where $\tilde{g}_1$ and $\tilde{g}_2$ are some positive Einstein metrics and $\mathbb{S}^1 \times_h (\mathbb{Y}^{n-k}, \tilde{g}_2)$ means a warped product metric with warping function h.

We avoid making the long list of all the examples, but refer to the list of four dimensional spaces in [12].

Remark 1. In this work we have studied static and related spaces with less than four Ricci eigenvalues. It would be interesting to study four eigenvalues.

It is also interesting to understand in our terms other geometric equations, such as gradient Ricci solitons or warped product Einstein metrics. In these mentioned cases the scalar curvature is not constant, which makes the problem somewhat more difficult.

References

[1] L. Ambrozio, *On static three-manifolds with positive scalar curvature*, J. Differential Geom. **107** (2017), no. 1, 1–45. https://doi.org/10.4310/jdg/1505268028

[2] H. Baltazar, R. Batista, and K. Bezerra, *On the volume functional of compact manifolds with boundary with harmonic Weyl tensor*, arXiv:1710.06247v1 [math.DG].

[3] H. Baltazar and E. Ribeiro, Jr., *Remarks on critical metrics of the scalar curvature and volume functionals on compact manifolds with boundary*, Pacific J. Math. **297** (2018), no. 1, 29–45. https://doi.org/10.2140/pjm.2018.297.29

[4] A. Barros and E. Ribeiro, Jr., *Critical point equation on four-dimensional compact manifolds*, Math. Nachr. **287** (2014), no. 14-15, 1618–1623. https://doi.org/10.1002/mana.201300149

[5] R. Batista, R. Diógenes, M. Ranieri, and E. Ribeiro Jr., *Critical metrics of the volume functional on compact three-manifolds with smooth boundary*, J. Geom. Anal. **27** (2017), no. 2, 1530–1547. https://doi.org/10.1007/s12220-016-9730-y

[6] A. L. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **10**, Springer-Verlag, Berlin, 1987. https://doi.org/10.1007/978-3-540-74311-8

[7] J. Corvino, *Scalar curvature deformation and a gluing construction for the Einstein constraint equations*, Comm. Math. Phys. **214** (2000), no. 1, 137–189. https://doi.org/10.1007/PL00005533

[8] J. Corvino, M. Eichmair, and P. Miao, *Deformation of scalar curvature and volume*, Math. Ann. **357** (2013), no. 2, 551–584. https://doi.org/10.1007/s00208-013-0903-8

[9] A. Derdziński, *Classification of certain compact Riemannian manifolds with harmonic curvature and nonparallel Ricci tensor*, Math. Z. **172** (1980), no. 3, 273–280. https://doi.org/10.1007/BF01215090

[10] D. DeTurck and H. Goldschmidt, *Regularity theorems in Riemannian geometry. II. Harmonic curvature and the Weyl tensor*, Forum Math. **1** (1989), no. 4, 377–394. https://doi.org/10.1515/form.1989.1.377

[11] J. Kim, *On a classification of 4-d gradient Ricci solitons with harmonic Weyl curvature*, J. Geom. Anal. **27** (2017), no. 2, 986–1012. `https://doi.org/10.1007/s12220-016-9707-x`

[12] J. Kim and J. Shin, *Four-dimensional static and related critical spaces with harmonic curvature*, Pacific J. Math. **295** (2018), no. 2, 429–462. `https://doi.org/10.2140/pjm.2018.295.429`

[13] O. Kobayashi, *A differential equation arising from scalar curvature function*, J. Math. Soc. Japan **34** (1982), no. 4, 665–675. `https://doi.org/10.2969/jmsj/03440665`

[14] P. Miao and L.-F. Tam, *Einstein and conformally flat critical metrics of the volume functional*, Trans. Amer. Math. Soc. **363** (2011), no. 6, 2907–2937. `https://doi.org/10.1090/S0002-9947-2011-05195-0`

[15] J. Qing and W. Yuan, *On scalar curvature rigidity of vacuum static spaces*, Math. Ann. **365** (2016), no. 3-4, 1257–1277. `https://doi.org/10.1007/s00208-015-1302-0`

[16] J. Shin, *On the classification of 4-dimensional (m, ρ)-quasi-Einstein manifolds with harmonic Weyl curvature*, Ann. Global Anal. Geom. **51** (2017), no. 4, 379–399. `https://doi.org/10.1007/s10455-017-9542-8`

[17] G. Yun, J. Chang, and S. Hwang, *Total scalar curvature and harmonic curvature*, Taiwanese J. Math. **18** (2014), no. 5, 1439–1458. `https://doi.org/10.11650/tjm.18.2014.1489`

[18] G. Yun and S. Hwang, *Rigidity of generalized Bach-flat vacuum static spaces*, J. Geom. Phys. **121** (2017), 195–205. `https://doi.org/10.1016/j.geomphys.2017.07.016`

Jongsu Kim
Department of Mathematics
Sogang University
Seoul 04107, Korea
Email address: `jskim@sogang.ac.kr`