

EXAMPLES OF SIMPLY REDUCIBLE GROUPS

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ABSTRACT. Simply reducible groups are important in physics and chemistry, which contain some of the important groups in condensed matter physics and crystal symmetry. By studying the group structures and irreducible representations, we find some new examples of simply reducible groups, namely, dihedral groups, some point groups, some dicyclic groups, generalized quaternion groups, Heisenberg groups over prime field of characteristic 2, some Clifford groups, and some Coxeter groups. We give the precise decompositions of product of irreducible characters of dihedral groups, Heisenberg groups, and some Coxeter groups, giving the Clebsch-Gordan coefficients for these groups. To verify some of our results, we use the computer algebra systems GAP and SAGE to construct and get the character tables of some examples.

1. Introduction

Wigner introduced the concept of ‘simply reducible group’ in his study of group representations and quantum mechanics [79]. This concept is quite useful because many of the symmetry groups (in particular, the point groups) we have in atomic and molecular systems are simply reducible, ‘and algebraic manipulations of tensor operators become much easier for those groups’ [71, page 146]. The following paragraph is excerpted from [15, page 45].

Wigner wrote: “The groups of most eigenvalue problems occurring in quantum theory are S.R.” (where “S.R.” stands for “simply reducible”) having in mind the study of “small perturbations” of the “united system” of two eigenvalue problems invariant under some group G of symmetries. Then simple reducibility guarantees that the characteristic functions of the eigenvalues into which the united system splits can be determined in “first approximation” by the invariance of the eigenvalue problem under G . This is the case, for instance, for the *angular momentum* in quantum mechanics. We mention that the

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multiplicity-freeness of the representations in the definition of simply reducible groups is the condition for the validity of the well-known *Eckart-Wigner theorem* in quantum mechanics.

Now we give the definition of simply reducible group.

Definition 1.1 ([47, Definition 1], [79, page 57]). A group G is called a *simply reducible group* if

- **ambivalence**: every element of G is conjugate to its inverse,
- **multiplicity-free**: tensor product of any two irreducible representations of G decomposes into a direct sum of irreducible representations of G with multiplicities 0 or 1.

Remark 1.2 ([36, page 152], [78, page 529]).

- (1) Here the representation vector space is over \mathbb{C} .
- (2) Condition (i) means that all classes of G are ambivalent, that all characters' values are real numbers, that all irreducible representations of G are integral (i.e., *real*) or half-integral (i.e., *quaternionic*).
- (3) Condition (ii) is important for physical applications. It implies that the 'correct linear combinations' of products of basis functions are determined to within a phase factor, and that the solution of the physical problem is uniquely determined from symmetry arguments.
- (4) If one drops the property of ambivalence no essential new difficulties arise in the definition and in the symmetry relations of Wigner coefficients. However, if the multiplicity free condition is dropped a multiplicity index enters the Wigner coefficients.

Lemma 1.3 ([9, Page 426, Exercise 1.8]). *Given two finite simply reducible groups G and H , then the direct product group $G \times H$ is also a simply reducible group.*

If we drop the 'ambivalence' requirement in Definition 1.1, the group we get is called an ASR-group. Here 'ASR' is the abbreviation for 'almost simply reducible'.

Definition 1.4 ([47, Definition 2]). A group G is called an *ASR-group* if the tensor product of any two irreducible representations of G is multiplicity-free.

Remark 1.5 ([48, page 931]). Any simply reducible group is an ASR-group. The converse, generally speaking, is false. For example, every finite abelian group is an ASR-group (because all the irreducible representations of abelian groups are one-dimensional and tensor product of two one-dimensional representations is still one-dimensional) whereas the abelian simply reducible groups are elementary abelian 2-groups.

There are some famous examples of simply reducible groups in chemistry, mathematics and physics papers.

Example 1.6 ([79], [15, pages 45, 46], [71, Appendix 3A]). Symmetric groups S_3 , S_4 , quaternion group \mathbb{H} , and the rotation groups $O(3)$, $SO(3)$, or $SU(2)$.

Furthermore, most of the *molecular symmetry groups* such as (using Schönflies notation) $D_{\infty h}$, $C_{\infty v}$, C_{2v} , C_{3v} , C_{2h} , D_{3h} , D_{6h} , T_d and O_h are simply reducible. These groups are also called *point groups* (cf. Definition 2.47, [16, Section 8.2]). Character tables of these point groups can also be found in [28, 29, 32], Chapter 4 and Appendix 4 of [75]. Molecular examples for these point groups are (cf. [32] and [81]):

Point group	$D_{\infty h}$	$C_{\infty v}$	C_{2v}	C_{3v}	C_{2h}	D_{3h}	D_{6h}	T_d	O_h
Molecule	H_2	HCN	H_2O	NH_3	$C_{16}H_{10}$	C_3H_6	C_6H_6	CH_4	SF_6

The molecular symbols in the above table are *hydrogen*, *hydrogen cyanide*, *water*, *ammonia*, *dibenzopentalene*, *cyclopropane*, *benzene*, *methane*, *sulfur hexafluoride*, respectively.

There is an open problem for simply reducible groups: give the classification of all simply reducible groups. This question is interesting to physicists. It is Problem 11.94 in *Unsolved Problems in Group Theory The Kourovka Notebook* (cf. [49, Problem 11.94]). To classify the simply reducible groups, we need to find as many groups as possible that meet the requirement. That's why we find some new examples of simply reducible groups.

Our new examples are as followings.

Group	Result
Dihedral group D_n	Theorem 2.42
Some crystallographic point groups	Theorem 2.52
Dicyclic group $\Gamma(2, 2, n)$ with n even	Theorem 2.54
Generalized quaternionic group $\Gamma(2, 2, 2^{k-1})$	Corollary 2.56
Heisenberg group $\mathcal{H}_n(\mathbb{F}_2)$	Theorem 2.69
Clifford group $\Gamma(n)$ with $n \not\equiv 1 \pmod{4}$	Theorem 2.81
Coxeter groups $\bar{W}(\mathcal{A}_1), \bar{W}(\mathcal{A}_2), \bar{W}(\mathcal{A}_3), \bar{W}(\mathcal{B}_2),$ $\bar{W}(\mathcal{B}_3), \bar{W}(\mathcal{G}_2), \bar{W}(I_2(n))$ with $n = 5$ or $n \geq 7$	Theorem 3.32

To verify some of the finite groups in the above table, we use the computer algebra systems GAP (cf. [26]) and SAGE (cf. [69]) to get the character tables and some other groups' information.

The following theorem is a helpful tool for classification, which helps us to exclude some impossible groups.

Theorem 1.7 (Kazarin-Chankov, [47, page 656]).

- (1) Let G be a finite ASR-group. Then G is solvable.
- (2) Let G be a finite simply reducible group. Then G is solvable.

All the nonabelian groups in the *classification of finite simple groups* [5, Theorem 0.1.1] are not solvable, so by the Kazarin-Chankov Theorem 1.7, we get:

Corollary 1.8. *All the nonabelian groups in the classification of finite simple groups are not ASR-groups or simply reducible groups.*

By the Feit-Thompson odd order theorem (i.e., every finite group of odd order is solvable, cf. [67, page 107, Exercise 5.23]), we know all the nonabelian groups in the classification of finite simple groups are of even order. In fact, if the order of a finite group is an odd number, then the group identity element is the only element that is conjugate to its inverse (cf. [20, Corollary 23.4 (Burnside)]); this is the ‘most’ non-ambivalent group. Therefore, we have the following result.

Theorem 1.9.

- (1) *The orders of all the nontrivial finite simply reducible groups are even.*
- (2) *Given any even number, we can always find a simply reducible group of the given order. For example, the dihedral group (cf. Theorem 2.42).*

2. Finite groups

2.1. Review of representation theory

We follow [14] to review some basic facts of representation theory of finite groups.

For any set X , we use $\sharp X$ to denote the number of elements in X . Let G be a finite group and V be a finite dimensional vector space over \mathbb{C} . We denote by $GL(V)$ the *linear group* of V consisting of all invertible linear maps $A : V \rightarrow V$.

Definition 2.1 ([14, Definition 3.2.1]). A *representation* of G on V is a group homomorphism

$$\rho : G \longrightarrow GL(V).$$

We denote this representation by the pair (ρ, V) .

Assume the dimension of V is equal to n , then $GL(V)$ is isomorphic to the general linear group $GL(n, \mathbb{C})$. We may regard a representation of G as a group homomorphism

$$\rho : G \longrightarrow GL(n, \mathbb{C}).$$

We call this n the *dimension* or *degree* of ρ .

Definition 2.2 ([14, page 83]). A representation (ρ, V) of a group G is *irreducible* if the only G -invariant subspaces are $\{0\}$ and V .

Definition 2.3 ([14, page 83]). Let (ρ, V) and (σ, W) be two representations of the group G . If there exists a linear isomorphism of vector spaces

$$A : V \longrightarrow W$$

such that, for all $g \in G$, $\sigma(g)A = A\rho(g)$, then we say the two representations are *equivalent* (or *isomorphic*).

Suppose the vector space V is endowed with an inner product $\langle \cdot, \cdot \rangle$. Then we can define the notion of a *unitary representation*.

Definition 2.4 ([14, page 84]). A representation (ρ, V) is *unitary* if it preserves the inner product, i.e.,

$$\langle \rho(g)u, \rho(g)v \rangle = \langle u, v \rangle$$

for all $g \in G$ and $u, v \in V$.

Given an arbitrary representation (ρ, V) of G , it is always possible to endow V with an inner product: for any chosen basis $\{v_1, v_2, \dots, v_n\}$ of V , for any $x = \sum_{j=1}^n \xi_j v_j$ and $y = \sum_{j=1}^n \eta_j v_j$ in V , define

$$(2.5) \quad \langle x, y \rangle := \sum_{j=1}^n \overline{\xi_j} \eta_j.$$

Then $\langle \cdot, \cdot \rangle$ defined in the equation (2.5) is an inner product of V .

Given an arbitrary inner product $\langle \cdot, \cdot \rangle$ for V , define

$$(2.6) \quad (v, w) = \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle$$

for all v and w in V (cf. [14, page 84]).

Proposition 2.7 ([14, Proposition 3.3.1]). *The representation $(\rho, V_{\langle \cdot, \cdot \rangle})$ is unitary and equivalent to $(\rho, V_{\langle \cdot, \cdot \rangle})$. In particular, every representation of G is equivalent to a unitary representation.*

Definition 2.8 ([14, page 84]). Let V and W be two vector spaces endowed with scalar products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, respectively.

- (1) A linear operator

$$U : V \longrightarrow W$$

is *unitary* if

$$\langle U(v), U(v') \rangle_W = \langle v, v' \rangle_V$$

for every $v, v' \in V$.

- (2) Two representations $(\rho, V_{\langle \cdot, \cdot \rangle_V})$ and $(\sigma, W_{\langle \cdot, \cdot \rangle_W})$ are *unitarily equivalent* if there exists a unitary linear operator $U : V \rightarrow W$ such that $\sigma(g)U = U\rho(g)$ for all $g \in G$.

Proposition 2.9 ([14, Lemma 3.3.2, Lemma 3.3.3]).

- (1) *Suppose that (ρ, V) and (σ, W) are unitary representations of a finite group G . If they are equivalent, then they are also unitarily equivalent.*
- (2) *Every representation of G is the direct sum of a finite number of irreducible representations.*

Definition 2.10 ([14, Definition 3.3.4]). Let G be a finite group. We denote by \widehat{G} , the *dual* of G , a complete set of all irreducible pairwise nonequivalent (unitary) representations of G (in other words, \widehat{G} contains exactly one element

belonging to each equivalence class of irreducible representations). Sometimes, we call the equivalence class as the isomorphism class.

Theorem 2.11 ([4, Theorem 10.4.6(b)]). *Let G be a finite group. There are finitely many isomorphism classes of irreducible representations of G , the same number as the number of conjugacy classes in the group G .*

Definition 2.12 ([4, pages 298–299]). Let G be a finite group and (ρ, V) be a representation of G . The *character* of the representation ρ is the complex-valued function whose domain is the group G , defined by

$$\chi_\rho(g) = \text{trace}(\rho(g)).$$

For simplicity, we sometimes use $\chi(g)$ for the character.

Proposition 2.13 ([4, Proposition 10.4.2(f)]). *Isomorphic representations have the same character.*

Definition 2.14 ([14, page 271]). Given two representations (ρ, V) and (σ, W) of G , we define the *tensor product* of ρ and σ by

$$\rho \otimes \sigma : G \longrightarrow GL(V \otimes W),$$

the action is

$$(\rho \otimes \sigma)(g)(v \otimes w) = (\rho(g)v) \otimes (\sigma(g)w)$$

for any $g \in G$ and $v \otimes w \in V \otimes W$.

By Proposition 2.9(2), we know the tensor product representation can be decomposed into direct sum of irreducible representations of G . By the following theorem, we see the decomposition of representation is precisely the decomposition of its corresponding character.

Theorem 2.15 ([4, Corollary 10.4.8]). *Let ρ_1, \dots, ρ_r represent the isomorphism classes of all irreducible representations of a finite group G , and let ρ be any representation of G . Let χ_j and χ be the characters of ρ_j and ρ , respectively, and let*

$$\begin{aligned} n_j &= \langle \chi, \chi_j \rangle \\ &= \frac{1}{\#G} \sum_{g \in G} \overline{\chi(g)} \chi_j(g). \end{aligned}$$

Then

- (1) $\chi = n_1\chi_1 + n_2\chi_2 + \dots + n_r\chi_r$, with each $n_j \in \mathbb{Z}^{\geq 0}$,
- (2) ρ is isomorphic to $n_1\rho_1 \oplus n_2\rho_2 \oplus \dots \oplus n_r\rho_r$.
- (3) Two representations ρ and σ of a finite group G are isomorphic if and only if their characters are equal.

Therefore, to study the decomposition of representations of a finite group G , we can study the decomposition of their corresponding characters. We use Theorem 2.15 to calculate the multiplicities of irreducible representations for the decomposition of tensor product.

Definition 2.16 ([60, page 3264]). For a finite group G and its dual \widehat{G} . The *Kronecker multiplicity* $g(\rho, \phi, \psi)$, where $\rho, \phi, \psi \in \widehat{G}$, is defined by the equation

$$(2.17) \quad \chi_\phi \cdot \chi_\psi = \sum_{\rho \in \widehat{G}} g(\rho, \phi, \psi) \chi_\rho,$$

here $\chi_\phi \cdot \chi_\psi$ is the usual (Kronecker) product of characters:

$$(\chi_\phi \cdot \chi_\psi)(x) = \chi_\phi(x) \cdot \chi_\psi(x).$$

Remark 2.18 ([43, Proposition 19.6], [30, Abstract]).

- (1) $\chi_{\rho \otimes \sigma} = \chi_\rho \cdot \chi_\sigma$. Therefore, to study the decomposition of tensor product of irreducible representations, we just need to study the Kronecker multiplicity of the product of characters.
- (2) The Kronecker multiplicity is also called the *Clebsch-Gordan coefficient* in physics.

Let (ρ, V) be a representation of G and $V^* = \text{Hom}(V, \mathbb{C})$ be the dual vector space with the natural pairing $\langle \cdot, \cdot \rangle_{\text{dual}}$. We define the *conjugate representation* (or the *contragredient representation*) (ρ^*, V^*) by

$$\langle \rho^*(g)v^*, v \rangle_{\text{dual}} := \langle v^*, \rho(g^{-1})v \rangle_{\text{dual}}.$$

Definition 2.19 ([14, Definition 9.5.1]). The representation ρ is *selfconjugate* (or *selfadjoint*) if ρ and ρ^* are equivalent.

Note 2.20 ([14, page 294]).

- (1) The representation ρ^* is irreducible if and only if ρ is irreducible.
- (2) For an irreducible representation (ρ, V) of G , ρ is selfconjugate if and only if $\chi_\rho(g) \in \mathbb{R}$ for any $g \in G$.

Lemma 2.21 ([38, Page 10, Proposition]). Let χ_ρ be the character of a representation (ρ, V) of a finite group G . Then

$$\chi_{\rho^*}(g) = \chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$$

for all $g \in G$.

Definition 2.22 ([14, Definition 9.7.1, Definition 9.7.2]). Let $(\rho, V_{\langle \cdot, \cdot \rangle})$ be an irreducible representation of a finite group G .

- (1) The representation ρ is *complex* when it is not selfconjugate. Equivalently, ρ is complex if and only if ρ and its conjugate representation ρ^* are not unitarily equivalent, i.e., $\chi_\rho \neq \chi_{\rho^*}$.
- (2) If ρ is selfconjugate and suppose that there exists an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ in V such that the corresponding matrix coefficients

$$\begin{aligned} u_{s,t} : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \langle \rho(g)v_t, v_s \rangle \end{aligned}$$

are real valued for any $g \in G$ and $s, t = 1, 2, \dots, n$. Then we say that ρ is *real*. Otherwise, we say that ρ is *quaternionic*.

For any $\rho \in \widehat{G}$, define (cf. [14, equation (9.34)]):

$$(2.23) \quad c(\rho) = \begin{cases} 1 & \text{if } \rho \text{ is real,} \\ 0 & \text{if } \rho \text{ is complex,} \\ -1 & \text{if } \rho \text{ is quaternionic.} \end{cases}$$

The following theorem is a helpful tool to know whether a representation is real, complex or quaternionic, by working on the corresponding character.

Theorem 2.24 (Frobenius-Schur Indicator, [14, Theorem 9.7.7]). *Let χ be the character of an irreducible representation (ρ, V) of G . Then*

$$(2.25) \quad \frac{1}{\#G} \sum_{g \in G} \chi(g^2) = c(\rho).$$

For any $h \in G$, Wigner defined a function on G as following

$$(2.26) \quad \xi(h) := \#\{g \in G : g^2 = h\}.$$

Corollary 2.27 ([14, Corollary 9.7.8]).

$$\xi(h) = \sum_{\rho \in \widehat{G}} c(\rho) \chi_{\rho}(h).$$

In particular,

$$\xi(1_G) = \sum_{\substack{\rho \in \widehat{G} \\ \rho \text{ real}}} d_{\rho} - \sum_{\substack{\rho \in \widehat{G} \\ \rho \text{ quaternionic}}} d_{\rho},$$

here d_{ρ} means the dimension of representation ρ .

Theorem 2.28 (Wigner, [14, Theorem 9.7.10]). *Let ψ be the number of selfconjugate representations of G . Then*

- (1) $\psi = \frac{1}{\#G} \sum_{g \in G} \xi(g)^2$;
- (2) ψ is equal to the number of ambivalent classes of G .

For $g \in G$, denote by $\nu(g) = \#\{h \in G : hg = gh\}$ for the cardinality of centralizer. Wigner gave a criterion for simply reducibility as the following theorem.

Theorem 2.29 ([79, Theorem 2]). *For a finite group G , the equality*

$$(2.30) \quad \sum_{g \in G} \xi(g)^3 = \sum_{g \in G} \nu(g)^2$$

holds if and only if G is simply reducible.

2.2. Symmetric groups

Symmetric group S_3 and S_4 are simply reducible groups. These two examples are listed in [79]. In fact, we can settle down all the symmetric groups. Symmetric groups S_1 and S_2 are cyclic groups of order 1 and 2, respectively. Character tables for S_1 and S_2 are

order of the class	1
representative element	1
χ_1	1

and

order of the class	1	1
representative element	1	(1 2)
χ_1	1	1
χ_2	1	-1

respectively. We can see from the above two character tables that S_1 and S_2 are simply reducible groups.

Since symmetric group S_n is not solvable for $n \geq 5$, by Theorem 1.7, we know S_n is not an ASR-group or a simply reducible group for $n \geq 5$.

Therefore, we have the following result:

Theorem 2.31.

- (1) *Symmetric group S_n is a simply reducible group for $n \in \{1, 2, 3, 4\}$.*
- (2) *Symmetric group S_n is not a simply reducible group or an ASR-group for $n \geq 5$.*

2.3. Boolean group

By Remark 1.5, we know that abelian simply reducible groups are elementary abelian 2-groups. In this subsection, we make a review of basic facts about the elementary abelian groups, then we can find some simply reducible subgroups in the ambivalent group.

Definition 2.32 ([66, page 88], [31, page 6]). An abelian group all of whose non-identity elements have order p , for some fixed prime number p , is called an *elementary abelian group* (or sometimes an *elementary abelian p -group*).

When $p = 2$, the elementary abelian 2-group is sometimes called a *Boolean group*.

Proposition 2.33 ([66, pages 27, 88, 161]).

- (1) *Every elementary abelian p -group is a vector space over the prime field \mathbb{F}_p with p many elements; and conversely every such vector space is an elementary abelian group.*
- (2) *Every finite elementary abelian group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$ with $n \in \mathbb{Z}^{\geq 0}$.*
- (3) *Every elementary abelian group has a fairly simple finite presentation:*

$$(\mathbb{Z}/p\mathbb{Z})^n \simeq \langle e_1, \dots, e_n : e_j^p = 1, e_j e_k = e_k e_j \rangle.$$

We denote the Boolean group $(\mathbb{Z}/2\mathbb{Z})^n$ by $Bln(n)$, its presentation is

$$(2.34) \quad (\mathbb{Z}/2\mathbb{Z})^n \simeq \langle e_1, \dots, e_n : e_j^2 = 1, e_j e_k = e_k e_j \rangle.$$

By the relations in the equation (2.34), we know $Bln(n)$ is ambivalent; since $Bln(n)$ is abelian, we know that all the irreducible representations of $Bln(n)$ are 1-dimensional. Since tensor product of any two 1-dimensional representations is still 1-dimensional, this tensor product must be one of the irreducible representations. Therefore, we know the tensor product is multiplicity-free.

Theorem 2.35. *Boolean group $Bln(n)$ is a simply reducible group for $n \in \mathbb{Z}^{\geq 0}$.*

Remark 2.36 ([31, page 22]). Boolean group $Bln(2)$ is the Klein four-group.

For a finite ambivalent group, some of its subgroups are elementary abelian 2-groups.

Proposition 2.37 ([3, Propositions 2 and 3]). *Let G be a finite ambivalent group, $Z(G)$ be the center of G , and G' be the derived subgroup. Then:*

- (1) $Z(G)$ and G/G' are elementary abelian 2-groups.
- (2) If G is abelian, then it is an elementary abelian 2-group.

By Proposition 2.37, we get the following result.

Corollary 2.38. *Let G be a finite ambivalent group. Then both the center and abelianization of G are simply reducible groups. Furthermore, if G is abelian, then G is a simply reducible group.*

2.4. Dihedral group

Leonardo da Vinci proved that a finite group of planar isometries (i.e., planar symmetries) is either a cyclic group C_n or a dihedral group D_n (cf. [54, Theorem 8.8], [17, Page 142]). These groups are also called the rosette groups. By the Remark 1.5, we know that each cyclic group C_n is an ASR-group. For a cyclic group C_n , an element $x \in C_n$ is conjugate to its inverse only when $x^2 = 1$, therefore, only C_1 and C_2 are simply reducible groups. As for the dihedral group D_n , we discuss the simply reducibility in this subsection.

Let D_n be the *dihedral group* of order $2n$, generated by two elements x and y such that

$$x^n = 1, y^2 = 1, \text{ and } yx = x^{-1}y.$$

The presentation of D_n is

$$(2.39) \quad \langle x, y : x^n = y^2 = 1, yx = x^{-1}y \rangle$$

for positive integer n . For $n = 1$, we have

$$D_1 = S_2,$$

hence it is a simply reducible group. For $n \geq 2$, we can write down the conjugacy classes and irreducible representations of D_n precisely. We use the conjugacy classes and character tables listed in [43] in the following discussions.

We give the conjugacy classes, character tables and decompositions of tensor product representations of D_n with respect to the parity of n .

The dimensions of all the irreducible representations of D_n are equal to one or two (cf. [43, Section 18.3]). Write $\epsilon = e^{\frac{2\pi i}{n}}$. To make the notations concise, we make the following adjustments for the two dimensional irreducible character ψ_j , with $1 \leq j \leq \frac{n-1}{2}$. The character ψ_j is given in the following character tables.

- For $\frac{n+1}{2} \leq j \leq n-1$, we have $1 \leq n-j \leq \frac{n-1}{2}$ and

$$\epsilon^{jr} + \epsilon^{-jr} = \epsilon^{(n-j)r} + \epsilon^{-(n-j)r},$$

thus we know $\psi_j = \psi_{n-j}$ and ψ_j for $j > \frac{n-1}{2}$ is still making sense.

- For ψ_{-j} with $1 \leq j \leq \frac{n-1}{2}$, we have

$$\epsilon^{(-j)r} + \epsilon^{-(-j)r} = \epsilon^{jr} + \epsilon^{-jr},$$

hence we have $\psi_{-j} = \psi_j$.

Case 1: n is an odd number

There are precisely $\frac{1}{2}(n+3)$ conjugacy classes (cf. [43, page 108, equation (12.11)]) of D_n :

$$\{1\}, \{x, x^{-1}\}, \dots, \{x^{\frac{n-1}{2}}, x^{-\frac{n-1}{2}}\}, \{y, xy, \dots, x^{n-1}y\}.$$

And the character table (cf. [43, page 182]) is

order of the class	1	2	n
representative element	1	x^r ($1 \leq r \leq \frac{n-1}{2}$)	y
χ_1	1	1	1
χ_2	1	1	-1
ψ_j ($1 \leq j \leq \frac{n-1}{2}$)	2	$\epsilon^{jr} + \epsilon^{-jr}$	0

The decompositions of product of irreducible characters are

$$\begin{aligned}
 \chi_u \cdot \chi_u &= \chi_1 \quad u \in \{1, 2\}, \\
 \chi_1 \cdot \chi_2 &= \chi_2, \\
 \chi_u \cdot \psi_j &= \psi_j \quad u \in \{1, 2\} \text{ and } 1 \leq j \leq \frac{n-1}{2}, \\
 \psi_j \cdot \psi_j &= \psi_{2j} + \chi_1 + \chi_2 \quad 1 \leq j \leq \frac{n-1}{2}, \\
 \psi_j \cdot \psi_k &= \psi_{j+k} + \psi_{j-k} \quad j \neq k \text{ and } 1 \leq j, k \leq \frac{n-1}{2}.
 \end{aligned}
 \tag{2.40}$$

Case 2: n is an even number

Write $n = 2m$. There are precisely $m+3$ conjugacy classes (cf. [43, page 108, equation (12.12)]) of D_n :

$$\begin{aligned}
 &\{1\}, \{x^m\}, \{x, x^{-1}\}, \dots, \{x^{m-1}, x^{-(m-1)}\}, \\
 &\{x^{2j}y : 0 \leq j \leq m-1\}, \{x^{2j+1}y : 0 \leq j \leq m-1\}.
 \end{aligned}$$

And the character table (cf. [43, page 183]) is

order of the class	1	1	2	m	m
representative element	1	x^m	x^r ($1 \leq r \leq m-1$)	y	xy
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^m$	$(-1)^r$	1	-1
χ_4	1	$(-1)^m$	$(-1)^r$	-1	1
ψ_j ($1 \leq j \leq m-1$)	2	$2(-1)^j$	$\epsilon^{jr} + \epsilon^{-jr}$	0	0

The decompositions of product of irreducible characters are

$$\begin{aligned}
 \chi_u \cdot \chi_u &= \chi_1 \quad u \in \{1, 2, 3, 4\}, \\
 \chi_1 \cdot \chi_u &= \chi_u \quad u \in \{1, 2, 3, 4\}, \\
 \chi_2 \cdot \chi_3 &= \chi_4, \\
 \chi_2 \cdot \chi_4 &= \chi_3, \\
 \chi_3 \cdot \chi_4 &= \chi_2, \\
 (2.41) \quad \chi_u \cdot \psi_j &= \begin{cases} \psi_j & u \in \{1, 2\} \text{ and } 1 \leq j \leq m-1, \\ \psi_{m-j} & u \in \{3, 4\} \text{ and } 1 \leq j \leq m-1, \end{cases} \\
 \psi_j \cdot \psi_j &= \begin{cases} \psi_{2j} + \chi_1 + \chi_2 & 1 \leq j \leq m-1 \text{ and } m \text{ is odd,} \\ \chi_1 + \chi_2 + \chi_3 + \chi_4 & 1 \leq j \leq m-1 \text{ and } m \text{ is even,} \end{cases} \\
 \psi_j \cdot \psi_k &= \begin{cases} \psi_{j-k} + \chi_3 + \chi_4 & j \neq k, j+k=m \text{ and } 1 \leq j, k \leq m-1, \\ \psi_{j+k} + \psi_{j-k} & j \neq k, j+k \neq m \text{ and } 1 \leq j, k \leq m-1. \end{cases}
 \end{aligned}$$

Now we have the following result about dihedral groups.

Theorem 2.42. *Dihedral group D_n is a simply reducible group for all positive integers n .*

The following result offers us one method to verify the multiplicity-free condition.

Theorem 2.43 (Mackey, [53, Theorem 7]). *For a finite group G , if G has a commutative normal subgroup N such that G/N is of order 2, then the tensor product of any two irreducible representations of G is multiplicity-free.*

Now we come to the proof of Theorem 2.42.

Proof. By the above calculations about the conjugacy classes of D_n , we can see that every element of D_n is conjugate to its inverse. We can also write the proof via direct calculations: for any x^j and $yx^j \in D_n$, we have

$$\begin{aligned}
 yx^jy^{-1} &= yxx \cdots xy^{-1} \\
 &= yxy^{-1}yxy^{-1}yxy^{-1}y \cdots xy^{-1}yxy^{-1} \\
 &= x^{-1}x^{-1}x^{-1} \cdots x^{-1}x^{-1} = (x^{-1})^j \\
 &= x^{-j},
 \end{aligned}$$

hence we know x^j and x^{-j} are conjugate for any $j \in \{0, 1, 2, \dots, n-1\}$. We also have

$$\begin{aligned} (yx^j)^2 &= yx^j yx^j \\ &= yxy^{-1}yx \cdots y^{-1}yxy^{-1}x^j \\ &= x^{-1}x^{-1} \cdots x^{-1}x^j = (x^{-1})^j x^j \\ &= 1, \end{aligned}$$

thus

$$(yx^j)^{-1} = yx^j,$$

we get yx^j and $(yx^j)^{-1}$ are conjugate for any $j \in \{0, 1, 2, \dots, n-1\}$. Therefore, every element in D_n is conjugate to their inverses.

There are two method to show the ‘multiplicity-free’ requirement.

Method 1. By the decompositions in (2.40) and (2.41), we know the decompositions are multiplicity-free.

Method 2. The subgroup $\langle x \rangle$ generated by x is a commutative normal subgroup of D_n with index 2, by Mackey Theorem 2.43, we know the decomposition is multiplicity-free.

Therefore, we know D_n is a simply reducible group for all the positive integer n . \square

Remark 2.44.

- (1) The Klein four group is isomorphic to D_2 , by Theorem 2.42, we know Klein four group is also a simply reducible group.
- (2) A finite group F is called a *Frobenius group* if it contains a subgroup H such that

$$H \cap gHg^{-1} = \{1\} \quad \text{for every } g \in F - H.$$

Dihedral group D_n is a Frobenius group for odd number $n \geq 3$, by taking $H = \langle y \rangle$ (cf. [73, page 80]). However, not every Frobenius group is a simply reducible group. For example, the Frobenius group F_{20} of order 20 is given by the following presentation

$$\langle x, y : x^5 = y^4 = yxy^{-1}x^{-2} = 1 \rangle.$$

We calculate its character table as follows:

order of the class	1	4	5	5	5
representative element	1	x	y	y^2	y^3
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	i	-1	$-i$
χ_4	1	1	$-i$	-1	i
χ_5	4	-1	0	0	0

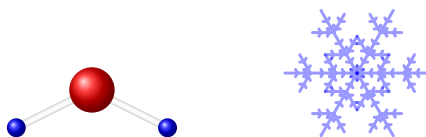
We see

$$\chi_5 \cdot \chi_5 = 3\chi_5 + \chi_1 + \chi_2 + \chi_3 + \chi_4,$$

hence it is not multiplicity-free. Therefore F_{20} is not a simply reducible group.

In solid state physics and molecular symmetry theory, the dihedral group D_n is a point group. We have proved that D_n is a simply reducible group (Theorem 2.42), we want to know whether there are more simply reducible groups in the family of point groups. Firstly, we make a review about the point groups (cf. [16, 40, 51, 58, 70]).

Many microscopic and macroscopic assemblages (or objects, or figures, or systems, or bodys, etc. in different literatures) exhibit some form of symmetry. For example, water molecule and snowflake:



Definition 2.45 ([70, page 14], [51, page 64]). An assemblage is said to *have symmetry* if some movement (or operation) of the assemblage leaves the assemblage in a situation indistinguishable from its original (or initial) situation. A *symmetry operation* on an assemblage is a movement that moves the assemblage to a situation which is indistinguishable from its initial situation.

Remark 2.46 ([51, page 5, 64]).

- (1) Symmetry is a spatial property of an assemblage, by which the parts of the assemblage can be moved from an initial situation to another indistinguishable situation by the movement called the symmetry operation.
- (2) The term “assemblage” can be used to describe the distribution of faces on a crystal, of bonds radiating from a central atom and of diffractions spectra from crystalline materials. That’s the reason we pick this term to give Definition 2.45.
- (3) The symmetry operation reveals the symmetry property inherent in the assemblage according to the nature of the operation.

Definition 2.47 ([51, page 65]). A *point group* is the group of symmetry operations on an assemblage, all of these operations pass through a single fixed point.

Example 2.48 ([58, page 63], [81, Table]). The point group for a molecule (or a finite cluster of atoms or some similar assemblage) is finite if the molecule consists of a finite number of atoms and is mapped onto itself by a finite number of isometries. The point group for the triphenylphosphine molecule is the cyclic group C_3 . However, the group is infinite for linear molecules like the hydrogen and the carbon dioxide:



Because the molecular axis is infinite order, the resulting point groups are infinite. In fact, they share the same point group $D_{\infty h}$ (cf. Example 1.6).

Remark 2.49 ([51, page 65], [58, page 64], [40, page 169]).

- (1) The symmetry operations of a point group must leave at least one point unmoved: in some cases, it is a line or a plane that is unmoved under the action of the point group. For a finite molecule, all its symmetry operations leave its center of gravity unchanged.
- (2) Point groups describe the microscopic symmetry of molecules and the macroscopic symmetry of crystals. Therefore, they are frequently used in studying electronic states and vibrations of molecules as well as the symmetry of the macroscopic properties of crystals.

Definition 2.50 ([40, pages 169–170], [16, pages 389–393]). We denote the geometrical symmetry operations by the following symbols:

E : Identity operation.

\mathcal{C}_l : Rotation through an angle $\frac{2\pi}{l}$. The rotation axis is called an l -fold axis. If we need to write down the axis in precise, we use $\mathcal{C}_l^{\mathbf{n}}$ to denote the rotation through angle $\frac{2\pi}{l}$ about the axis \mathbf{n} , this \mathbf{n} is called a *principal axis*. One usually chooses the principal axis as the z -axis.

\mathcal{I} : Space inversion. It takes \mathbf{r} into $-\mathbf{r}$.

σ : Mirror reflection. It carries three kinds of suffixes according to the property of the mirror plane.

σ_h : Mirror reflection in the horizontal plane. This σ_h is equal to $\mathcal{I}\mathcal{C}_2^z$, where the reflection plane's normal vector is in the z direction.

σ_v : Mirror reflection in the vertical plane. This is a reflection across a plane containing the principal axis.

σ_d : Mirror reflection in the vertical diagonal plane.

$\mathcal{S}\mathcal{C}_l$: Rotatory inversion. Rotation through the angle $\frac{2\pi}{l}$ followed by inversion. In general, a rotatory inversion may also be understood as a rotatory reflection (rotation followed by reflection).

For the rotation \mathcal{C}_l , the number l can be any positive integer. However, we need to consider the translational symmetry when we work on the crystals. To make the rotational symmetry be compatible with the translational symmetry, the value l must be 1, 2, 3, 4, and 6. Point groups composed of rotations and inversion under this restriction are called *crystallographic point groups*. Using the Schönflies symbols, all the 32 crystallographic point groups are listed as followings (cf. [40, pages 171–172] [4, Theorem 6.12.1]).

Group C_n : This group has only an n -fold rotation axis. It is a cyclic group of order n consisting of $E, \mathcal{C}_n, \mathcal{C}_n^2, \dots, \mathcal{C}_n^{n-1}$ ($n = 1, 2, 3, 4, 6$).

Group C_i : This group is composed of the space inversion \mathcal{I} and the identity E .

Group C_{nv} : This group has n vertical mirror planes and an n -fold axis ($n = 2, 3, 4, 6$).

Group C_{nh} : This group has a horizontal mirror plane and an n -fold axis ($n = 1, 2, 3, 4, 6$). It contains the inversion \mathcal{I} for $n = 2, 4, 6$. In some literatures, C_{1h} is also denoted as C_s .

Group \mathfrak{S}_n : This group has only an n -fold rotatory reflection axis ($n = 4, 6$). For $n = 2$ and 3 , other symbols, C_i and C_{3h} , are commonly used in place of \mathfrak{S}_2 and \mathfrak{S}_3 .

Group D_n : This group has n two-fold axes perpendicular to the n -fold rotation axis ($n = 2, 3, 4, 6$).

Group D_{nd} : Addition of n diagonal mirror planes to the group D_n results in this group ($n = 2, 3$). The mirror planes bisect the angles between the two-fold axes.

Group D_{nh} : Addition of a horizontal mirror plane to D_n results in this group ($n = 2, 3, 4, 6$). D_{nh} contains the inversion for $n = 2, 4$, and 6 .

Group O : The *octahedral group* of 24 rotational symmetries of a cube or an octahedron. It is isomorphic to the symmetric group S_4 .

Group O_h : This group is the combine of O and the inversion: $O_h = O \times C_i$.

Group T : The *tetrahedral group* of 12 rotational symmetries of a tetrahedron. It is isomorphic to the alternating group A_4 .

Group T_h : This group is the combine of T and the inversion: $T_h = T \times C_i$.

Group T_d : The full symmetry group of a tetrahedron. It is obtained by adding $6\mathcal{I}\mathcal{C}_4$ and $6\sigma_d$ operations to T .

Remark 2.51 ([16, page 399, Ex.8.1], [40, pages 172–173]).

- (1) The group \mathfrak{S}_n is not the symmetric group S_n . In fact, $\mathfrak{S}_{2m} = C_m \times C_i$ for odd m , and $\mathfrak{S}_{2m} \simeq C_{2m}$ for even m .
- (2) The point groups O, O_h, T, T_h , and T_d are called the *cubic point groups*. Of these five groups, O_h and T_d often appear in physical applications. They are related by $O_h = T_d \times C_i$.
- (3) In addition to the above 32 crystallographic point groups, the following two groups describe the symmetry of linear molecules:

Group $C_{\infty v}$: This group consists of rotations of arbitrary angles about the molecular axis and vertical mirror reflections.

Group $D_{\infty h}$: Homonuclear diatomic molecules have this symmetry.

Addition of the horizontal mirror plane to $C_{\infty v}$ leads to this group. According to Example 1.6, we know both $C_{\infty v}$ and $D_{\infty h}$ are simply reducible groups.

By the already known facts and Lemma 1.3, we can settle down all the 32 crystallographic point groups in accordance with the simply reducibility.

Theorem 2.52. *Point groups $C_1, C_2, C_i, C_{2v}, C_{3v}, C_{4v}, C_{6v}, C_{1h}, C_{2h}, D_2, D_3, D_4, D_6, D_{2d}, D_{3d}, D_{2h}, D_{3h}, D_{4h}, D_{6h}, O, O_h$ and T_d are simply reducible*

groups. The remaining crystallographic point groups, C_3 , C_4 , C_6 , C_{3h} , C_{4h} , C_{6h} , \mathfrak{S}_4 , \mathfrak{S}_6 , T and T_h are not simply reducible groups.

Proof. By Remark 1.5, we know C_1 , C_2 and C_i are simply reducible groups, while C_3 , C_4 and C_6 are not simply reducible groups.

By Example 1.6, we know C_{2v} and C_{3v} are simply reducible groups. In fact, C_{nv} is isomorphic to the dihedral group D_n [16, page 394]. Therefore, we know C_{4v} and C_{6v} are also simply reducible groups.

For the group C_{nh} , since C_{1h} is a group of order 2, it is isomorphic to C_2 . Hence we know C_{1h} is a simply reducible group. The simply reducibility for C_{2h} is already known in Example 1.6. In fact, C_{nh} is isomorphic to C_{2n} (respectively $C_n \times C_i$) for odd (respectively even) n [16, page 399, Ex. 8.2]. Hence, we have the following isomorphisms:

$$\begin{aligned} C_{3h} &\simeq C_6, \\ C_{4h} &\simeq C_4 \times C_i, \\ C_{6h} &\simeq C_6 \times C_i, \end{aligned}$$

easily we know these three groups are not simply reducible groups.

For the group \mathfrak{S}_n , we have the following isomorphisms by Remark 2.51(1):

$$\begin{aligned} \mathfrak{S}_4 &\simeq C_4, \\ \mathfrak{S}_6 &= C_3 \times C_i. \end{aligned}$$

Hence, we know both of them are not simply reducible groups.

Dihedral group D_n ($n = 2, 3, 4, 6$) are simply groups, this is proved in Theorem 2.42.

For the group D_{2d} and D_{3d} . By [16, page 399, Ex. 8.4], we know $D_{nd} = D_n \times C_i$ for odd n and $D_{nd} \simeq D_{2n}$ for even n . Hence we know D_{2d} and D_{3d} are simply reducible groups.

For the group D_{nh} , because $D_{nh} \simeq D_{2n}$ for odd n and $D_{nh} = D_n \times C_i$ for even n [16, page 399, Ex. 8.5], we know D_{2h} , D_{3h} , D_{4h} and D_{6h} are simply reducible groups.

For the remaining five cubic point groups, they are easy to settle down by the already know facts. \square

2.5. Dicyclic group

Consider the *dicyclic group* $\Gamma(2, 2, n)$ of order $4n$ (cf. [65, pages 347–348]), its presentation is

$$\langle x, y : x^{2n} = 1, y^2 = x^n, y^{-1}xy = x^{-1} \rangle.$$

According to the following Theorem 2.53 and proofs in [65, page 348], we know the order of x is equal to $2n$.

Theorem 2.53 ([65, Theorem 12.29(2)]). *The presentation*

$$\langle Y : S \rangle = \langle x, y : x^n = 1, y^m = 1, yx = x^s y \rangle$$

defines the group

$$H = \{x^i y^j : 0 \leq i < n \text{ and } 0 \leq j < m\},$$

where $\text{order}(H) = mn$, $\text{order}(x) = n$, $\text{order}(y) = m$ and

$$(x^i y^k)(x^j y^l) = x^{i+j} y^{k+l}.$$

Moreover, any group of order mn that is generated by Y and satisfies the relations S is defined by $\langle Y : S \rangle$.

Now we come to our result about the simply reducibility:

Theorem 2.54. *Dicyclic group $\Gamma(2, 2, n)$ is an ASR-group for $n \in \mathbb{Z}^{\geq 1}$. In particular, $\Gamma(2, 2, n)$ is a simply reducible group when n is an even number.*

Proof. Consider the subgroup $\langle x \rangle$ generated by x , since

$$\frac{\sharp \Gamma(2, 2, n)}{\sharp \langle x \rangle} = \frac{4n}{2n} = 2,$$

we know $\langle x \rangle$ is a normal subgroup of $\Gamma(2, 2, n)$. $\langle x \rangle$ is also a commutative group. By Theorem 2.43, we know tensor product of any two irreducible representations of $\Gamma(2, 2, n)$ is multiplicity-free. Hence we know $\Gamma(2, 2, n)$ is an ASR-group.

When n is an even number, say $n = 2r$. Then x is conjugate to x^{-1} is given by the definition of $\Gamma(2, 2, n)$, hence we also have x^j is conjugate to x^{-j} for any $j \in \{1, 2, 3, \dots, 2n\}$. For the generator y , we have

$$\begin{aligned} x^r y x^{-r} &= x^r y x^{3r} \\ &= y x^{-r} x^{3r} \quad (\text{by } y^{-1} x y = x^{-1}, \text{ i.e., } x y = y x^{-1}) \\ &= y x^{2r} = y x^n = y^3 = y^{-1}, \end{aligned}$$

thus we know y is conjugate to y^{-1} . Because $y^2 = x^n$, we know the order of y is equal to 4. Hence we have y^k is conjugate to y^{-k} for any $k \in \{1, 2, 3, 4\}$.

Any element in the dicyclic group $\Gamma(2, 2, n)$ can be written as $x^j y^k$ for $j \in \{1, 2, 3, \dots, 2n\}$ and $k \in \{1, 2, 3, 4\}$. When $k = 2$ or 4 , we have

$$\begin{aligned} x^j y^2 &= x^j x^n = x^{j+n}, \\ x^j y^4 &= x^j \cdot 1 = x^j. \end{aligned}$$

Hence we know they are conjugate to their inverses. When $k = 3$, we have

$$x^j y^3 = x^j \cdot y^2 \cdot y = x^j \cdot x^n \cdot y = x^{j+n} y,$$

hence we only need to prove that $x^j y$ is conjugate to its inverse for any $j \in \{1, 2, 3, \dots, 2n\}$. If $j > r$, then we have

$$\begin{aligned} x^{j-r} y \cdot x^j y \cdot (x^{j-r} y)^{-1} &= x^{j-r} y \cdot x^j y \cdot y^{-1} x^{r-j} \\ &= x^{j-r} y \cdot x^r = y y^{-1} x y y^{-1} x y y^{-1} \cdots x y y^{-1} x y \cdot x^r \\ &= y \cdot (x^{-1})^{j-r} \cdot x^r \quad (\text{because } y^{-1} x y = x^{-1}) \end{aligned}$$

$$\begin{aligned}
 &= yx^{-j+2r} = y \cdot x^{2r} \cdot x^{-j} = y \cdot y^2 \cdot x^{-j} = y^{-1} \cdot x^{-j} \\
 &= (x^j y)^{-1},
 \end{aligned}$$

thus we know $x^j y$ is conjugate to its inverse when $j > r$. If $j = r$, then

$$y \cdot x^r y \cdot y^{-1} = yx^r,$$

since

$$x^r y \cdot yx^r = x^r y^2 x^r = x^r x^n x^r = x^{2n} = 1,$$

we know $(x^r y)^{-1} = yx^r$ and $x^r y$ is conjugate to its inverse. If $j < r$, then

$$\begin{aligned}
 x^{j+3r} y \cdot x^j y \cdot (x^{j+3r} y)^{-1} &= x^{j+3r} yx^{-3r} = yy^{-1}xyy^{-1}x \cdots yy^{-1}xy \cdot x^{3r} \\
 &= yx^{-j-6r} = yx^{-j+2r} \quad (\text{because } x^{4r} = x^{2n} = 1) \\
 &= y \cdot x^n \cdot x^{-j} = y^3 \cdot x^{-j} = y^{-1}x^{-j} \\
 &= (x^j y)^{-1},
 \end{aligned}$$

thus we know $x^j y$ is conjugate to its inverse when $j < r$.

By the above discussion, we know that every element of $\Gamma(2, 2, n)$ is conjugate to its inverse, and $\Gamma(2, 2, n)$ is a simply reducible group when n is an even number.

When n is an odd number, say $n = 2r + 1$. For any $x^b y^c \in \Gamma(2, 2, n)$, since $x^b y^c y(x^b y^c)^{-1} = x^b yx^{-b}$, we only need to consider the existence of x^b such that

$$x^b yx^{-b} = y^{-1}.$$

By the relation $xy = yx^{-1}$, we have

$$x^b yx^{-b} = yx^{-b}x^{-b} = yx^{-2b}.$$

Since $y^2 = x^{2r+1}$ and $2b$ is an even number, there is no number b such that $x^{-2b} = y^2$. Thus we know y is not conjugate to y^{-1} , and $\Gamma(2, 2, n)$ is not a simply reducible group when n is an odd number. \square

As a special example, the *generalized quaternion group* of order 2^{k+1} is

$$(2.55) \quad \langle x, y : x^{2^k} = 1, y^2 = x^{2^{k-1}}, yxy^{-1} = x^{-1} \rangle.$$

Hence we can denote the generalized quaternion group by using symbol of dicyclic group: $\Gamma(2, 2, 2^{k-1})$, here $k \in \mathbb{Z}^{\geq 2}$. Since 2^{k-1} is an even number, by Theorem 2.54, we get:

Corollary 2.56. *The generalized quaternion group $\Gamma(2, 2, 2^{k-1})$ is simply reducible for $k \in \mathbb{Z}^{\geq 2}$.*

Remark 2.57 ([18, page 8]). The smallest dicyclic group is $\Gamma(2, 2, 2)$, it is the quaternion group

$$(2.58) \quad \mathbb{H} = \left\{ \pm 1, \pm i, \pm j \pm k : \begin{array}{l} i \cdot j = -j \cdot i = k \\ j \cdot k = -k \cdot j = i \\ k \cdot i = -i \cdot k = j \\ i^2 = j^2 = k^2 = -1 \end{array} \right\}.$$

This is the case when $k = 2$. Therefore \mathbb{H} is a simply reducible group.

2.6. Metacyclic group and binary polyhedral group

Definition 2.59 ([44, page 88]). A group G is called *metacyclic* if it has a normal subgroup H such that both H and G/H are cyclic.

Remark 2.60. Both dihedral group and dicyclic group are special examples of finite metacyclic group.

The following theorem gives a description of any finite metacyclic group in terms of the four parameters m, n, r, s , note that it is not a classification theorem.

Theorem 2.61 ([44, Proposition 7.1]). *Consider the group*

$$(2.62) \quad G = \langle x, y : x^m = 1, y^{-1}xy = x^r, y^n = x^s \rangle,$$

where $m, n, r, s \in \mathbb{Z}^{>0}$, $r, s \leq m$, and

$$r^n \equiv 1 \pmod{m}, rs \equiv s \pmod{m}.$$

Then $H = \langle x \rangle$ is a normal subgroup of G such that

$$H \simeq C_m, G/H \simeq C_n.$$

Thus, G is a finite metacyclic group, and moreover, every finite metacyclic group has a presentation of this form.

By the above theorem, we can easily get the following result.

Corollary 2.63. *Finite metacyclic group G is an ASR-group when $n = 2$.*

Proof. When $n = 2$, then H is a commutative normal subgroup of G and the order of the quotient group G/H is 2. By the Mackey Theorem 2.43, we know G is multiplicity-free, and G is an ASR-group. \square

More generally, consider the *binary polyhedral group* (or von Dyck group) $\Gamma(p, q, n)$ with parameters (p, q, n) (cf. [19, page 68], [45, page 276])

$$\langle x_1, x_2, x_3 : x_1^p = x_2^q = x_3^n = x_1x_2x_3 \rangle.$$

$\Gamma(p, q, n)$ is finite only when $(p, q, n) = (2, 3, 3)$ or $(2, 3, 4)$ or $(2, 3, 5)$ or $(2, 2, n)$ (cf. [19, Section 6.4]). In these situations, the group has an order 2 center, which is generated by $x_1x_2x_3$. In fact, we have

$\Gamma(2, 3, 3)$ = tetrahedral group of order 12, i.e., the alternating group A_4 ,

$\Gamma(2, 3, 4)$ = octahedral group of order 24, i.e., the symmetric group S_4 ,

$\Gamma(2, 3, 5)$ = icosahedral group of order 60, i.e., the alternating group A_5 ,

$\Gamma(2, 2, n)$ = dicyclic group.

Only $\Gamma(2, 3, 4)$ and $\Gamma(2, 2, n)$ with even number n (cf. Theorem 2.54) are simply reducible. For other values (p, q, n) , the center may be infinite. We can summarize as the following result.

Theorem 2.64. *Finite binary polyhedral group $\Gamma(p, q, n)$ is simply reducible when $(p, q, r) = (2, 3, 4)$ or $(p, q, r) = (2, 2, n)$ with n being an even number.*

2.7. Heisenberg group

Heisenberg group over the real numbers \mathbb{R} is important to quantum mechanics, “the state space of a quantum particle, either free or moving in a potential, will be a unitary representation of this group, with the group of spatial translations a subgroup.” (cf. [83]) As for the Heisenberg group over a finite field, it is useful in time-frequency analysis [63], Gabor analysis [23], phase space methods in quantum information theory [33, 77], mobile communication and radar applications [37]. Therefore, we want to know whether the Heisenberg group is simply reducible.

The *Heisenberg group* $\mathcal{H}_n(K)$ is defined for any commutative ring K with $n \in \mathbb{Z}^{\geq 1}$. Firstly, we assume $K = \mathbb{Z}/p\mathbb{Z}$ is a prime field, i.e., p is a prime number. Then $\mathcal{H}_n(K)$ is defined as following:

$$(2.65) \quad \mathcal{H}_n(K) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \in GL(n+2, K) : \begin{array}{l} a \text{ is a row vector of length } n \\ b \text{ is a column vector of length } n \\ I_n \text{ is the identity matrix of size } n \\ c \text{ is a number in } K \end{array} \right\}.$$

By direct calculation, we see the order of $\mathcal{H}_n(K)$ is p^{2n+1} .

The field $K = \mathbb{Z}/p\mathbb{Z}$ has the property that there is an embedding ω of K as an additive group into the circle group \mathbf{S}^1 :

$$\begin{aligned} \omega : K &\longrightarrow \mathbf{S}^1 \\ j &\longmapsto e^{\frac{2\pi i}{p}j}. \end{aligned}$$

Lemma 2.66. *Every element of the Heisenberg group $\mathcal{H}_n(K)$ is conjugate to its inverse only when the characteristic of K is equal to 2.*

Proof. By the equation (2.65), we know that matrix in $\mathcal{H}_n(K)$ is determined by vectors a , b , and scalar c . Thus we use $M(a, b, c)$ to denote a matrix in $\mathcal{H}_n(K)$.

For any $M(a, b, c) \in \mathcal{H}_n(K)$, its inverse is

$$M(a, b, c)^{-1} = M(-a, -b, \sum_{j=1}^n a_j b_j - c).$$

We want to find some matrix $M(x, z, y) \in \mathcal{H}_n(K)$ such that

$$M(x, z, y)M(a, b, c)M(x, z, y)^{-1} = M(a, b, c)^{-1}.$$

By direct calculation, we have

$$M(x, z, y)M(a, b, c)M(x, z, y)^{-1} = M(a, b, c + \sum_{j=1}^n x_j b_j - \sum_{j=1}^n a_j z_j).$$

The equality

$$M(a, b, c + \sum_{j=1}^n x_j b_j - \sum_{j=1}^n a_j z_j) = M(-a, -b, \sum_{j=1}^n a_j b_j - c)$$

holds if $a = -a$, $b = -b$ and $c + \sum_{j=1}^n x_j b_j - \sum_{j=1}^n a_j z_j = \sum_{j=1}^n a_j b_j - c$, the only possibility is when $p = 2$. Then let $x = a$, $z = 0$ and for any $y \in K$, we have

$$M(a, 0, y)M(a, b, c)M(a, 0, y)^{-1} = M(a, b, c)^{-1}.$$

Therefore, we know when $p = 2$, every element in $\mathcal{H}_n(K)$ is conjugate to its inverse. When the characteristic of K is not equal to 2, not every elements in $\mathcal{H}_n(K)$ are conjugate to their inverses. \square

Lemma 2.67. *For the Heisenberg group $\mathcal{H}_n(K)$, denote its commutator subgroup and center by $\mathcal{H}_n(K)'$ and $Z_{\mathcal{H}_n(K)}$, respectively. Then we have*

$$\mathcal{H}_n(K)' = Z_{\mathcal{H}_n(K)} \simeq K.$$

Proof. For any $M(x, z, y)$, $M(a, b, c) \in \mathcal{H}_n(K)$, we calculate the elements of commutator subgroup as followings.

$$\begin{aligned} & M(x, z, y)M(a, b, c)M(x, z, y)^{-1}M(a, b, c)^{-1} \\ &= M(0, 0, \sum_{j=1}^n (x_j b_j - a_j z_j)) \\ &= \begin{pmatrix} 1 & 0 & \sum_{j=1}^n (x_j b_j - a_j z_j) \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

here we have

$$\sum_{j=1}^n (x_j b_j - a_j z_j) = \sum_{j=1}^n \det \begin{pmatrix} x_j & a_j \\ z_j & b_j \end{pmatrix}$$

it can take any value of K , thus we get the commutator subgroup

$$\mathcal{H}_n(K)' = \{M(0, 0, c) \in \mathcal{H}_n(K) : \forall c \in K\}.$$

The center of $\mathcal{H}_n(K)$ can be calculated similarly, it is

$$Z_{\mathcal{H}_n(K)} = \{M(0, 0, y) \in GL(n, K) : y \in K\} = \mathcal{H}_n(K)' \simeq K. \quad \square$$

By Lemma 2.67, we know the number of non-equivalent one-dimensional irreducible representations of $\mathcal{H}_n(K)$ is equal to

$$\sharp(\mathcal{H}_n(K)/\mathcal{H}_n(K)') = \sharp(\mathcal{H}_n(K))/\sharp K = p^{2n}.$$

Since $\mathcal{H}_n(K)/Z_{\mathcal{H}_n(K)} \simeq K^{2n}$ is abelian, the irreducible characters with trivial central character ω are one-dimensional, the p^{2n} characters of the additive group K^{2n} . All the p^{2n} one-dimensional irreducible characters of $\mathcal{H}_n(K)$ are induced from the p^{2n} one-dimensional irreducible characters of K^{2n} by letting the trivial action on the center $Z_{\mathcal{H}_n(K)}$ (cf. [57, page 164, Lemma 5]).

For any nonzero h in K , define the representation ρ_h on the finite-dimensional inner product space $\ell^2(K^n)$ by

$$(\rho_h(M(a, b, c))f)(x) = \omega(b \cdot x + hc)f(x + ha)$$

for any $M(a, b, c) \in \mathcal{H}_n(K)$ and $f \in \ell^2(K^n)$. The corresponding Stone-von Neumann theorem for Heisenberg group (cf. [59, Chapter 1], [80], [35, Exercises 14.5]):

- (i) ρ_h is an irreducible representation of $\mathcal{H}_n(K)$,
- (ii) ρ_h and ρ_k are pairwise non-equivalent, for all $h \neq k$,

and all irreducible representations of $\mathcal{H}_n(K)$ on which the center acts nontrivially arise in this way.

We also have

$$p^{2n+1} = 1^2 \times p^{2n} + (p^n)^2 \times (p-1),$$

here $p^{2n+1} = \#\mathcal{H}_n(K)$, p^{2n} is the number of non-equivalent one-dimensional irreducible representations of $\mathcal{H}_n(K)$, $p-1$ is the number of non-equivalent p^n -dimensional irreducible representations ρ_h of $\mathcal{H}_n(K)$.

By direct calculation, we see the character χ of ρ_h is given by (cf. [80] [35, Exercises 14.5])

$$(2.68) \quad \chi(M(a, b, c)) = \begin{cases} p^n \omega(hc) & \text{if } a = b = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.69. *Finite Heisenberg group $\mathcal{H}_n(K)$ over prime field $K = \mathbb{Z}/p\mathbb{Z}$ is a simply reducible group when $p = 2$; $\mathcal{H}_n(K)$ is not a simply reducible group over prime field K when the prime number $p \geq 3$. Meanwhile, Heisenberg group over a field whose characteristic is not equal to 2 is not simply reducible.*

Proof. By Lemma 2.66, we know $\mathcal{H}_n(K)$ is ambivalent only if $p = 2$. For other field whose characteristic is not equal to 2, we know the Heisenberg group cannot be ambivalent.

Now we verify the multiplicity-free condition for $p = 2$. There are 2^{2n} one-dimensional irreducible representations of $\mathcal{H}_n(\mathbb{Z}/2\mathbb{Z})$, and only one 2^n -dimensional irreducible representation, we denote these representations by $\eta_1, \eta_2, \dots, \eta_{2^{2n}}, \pi_{2^n}$. All the η_j 's are induced from the irreducible characters of $(\mathbb{Z}/2\mathbb{Z})^{2n}$ by letting the trivial action on the center $Z_{\mathcal{H}_n(\mathbb{Z}/2\mathbb{Z})}$. By the equation (2.68), we know the character of this 2^n -dimensional irreducible representation is

$$\chi_{\pi_{2^n}}(M(a, b, c)) = \begin{cases} 2^n \omega(c) & \text{if } a = b = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Tensor product of any two one-dimensional irreducible representations is still a one-dimensional representation. Tensor product of any one-dimensional representation with π_{2^n} is still π_{2^n} , because $\chi_{\pi_{2^n}}$ has nonzero value only if $a = 0$ and $b = 0$, this is just the character values of the center $Z_{\mathcal{H}_n(\mathbb{Z}/2\mathbb{Z})}$, while the one-dimensional representation's character values at center $Z_{\mathcal{H}_n(\mathbb{Z}/2\mathbb{Z})}$ are all equal to 1, hence the tensor product is still π_{2^n} .

Character of tensor product of π_{2^n} with itself is

$$\chi_{\pi_{2^n} \otimes \pi_{2^n}}(M(a, b, c)) = \begin{cases} 2^{2n} & \text{if } a = b = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 2.15, we can calculate the inner product of $\chi_{\pi_{2^n} \otimes \pi_{2^n}}$ with all the irreducible representations, we get

$$\chi_{\pi_{2^n} \otimes \pi_{2^n}} = \chi_1 + \chi_2 + \cdots + \chi_{2^{2n}},$$

here χ_j is the character of the one-dimensional irreducible representation of $\mathcal{H}_n(\mathbb{Z}/2\mathbb{Z})$, $1 \leq j \leq 2^{2n}$. Thus we know the tensor product is multiplicity-free, and $\mathcal{H}_n(\mathbb{Z}/2\mathbb{Z})$ is a simply reducible group. \square

By Theorem 2.69, we know any field (finite or infinite) whose characteristic is not 2 is not simply reducible. For the finite field \mathbb{F}_{2^r} of characteristic 2, we have proved that the Heisenberg group is simply reducible when $r = 1$. When $r > 1$, we have the following result.

Theorem 2.70. *Heisenberg group $\mathcal{H}_n(\mathbb{F}_{2^r})$ is not simply reducible when $r > 1$.*

Proof. We denote by $q = 2^r$. Let us denote the set of irreducible characters of \mathbb{F}_{2^r} by $\widehat{\mathbb{F}_{2^r}}$, these characters are \mathbf{S}^1 -valued functions on \mathbb{F}_{2^r}

$$\omega : \mathbb{F}_{2^r} \longrightarrow \mathbf{S}^1.$$

Characters of all irreducible representations of $\mathcal{H}_n(\mathbb{F}_{2^r})$ on which the center acts nontrivially is still given by

$$\chi_h(M(a, b, c)) = \begin{cases} q^n \omega(hc) & \text{if } a = b = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here h is some (any) nonzero element in \mathbb{F}_{2^r} . There are nonzero h_1 and h_2 in \mathbb{F}_{2^r} such that $h_1 + h_2 \neq 0$ because $r > 1$. In this case, we have

$$\begin{aligned} \chi_{h_1} \cdot \chi_{h_2}(M(0, 0, c)) &= q^{2n} \omega(h_1 c) \omega(h_2 c) \\ &= q^{2n} \omega((h_1 + h_2)c) \\ &= q^n \chi_{h_1 + h_2}(M(0, 0, c)), \end{aligned}$$

here the last equation is calculated directly by using Theorem 2.15. $\chi_{h_1 + h_2}$ is another irreducible character because $h_1 + h_2 \neq 0$. Thus we know the decomposition is not multiplicity-free. Therefore, we know $\mathcal{H}_n(\mathbb{F}_{2^r})$ is not simply reducible for $r > 1$. \square

Theorem 2.71. *For the Heisenberg group $\mathcal{H}_n(K)$ over the prime field $K = \mathbb{Z}/p\mathbb{Z}$.*

- (1) *When $p = 2$ and $n \geq 1$, all the irreducible representations of $\mathcal{H}_n(\mathbb{Z}/2\mathbb{Z})$ are real.*
- (2) *When $p \geq 3$ and $n \geq 1$, the trivial representation is real, all the non-trivial irreducible representations are complex.*

Proof. (1) When $p = 2$ and $n \geq 1$, all the one-dimensional irreducible representations are real because the matrices corresponding to these representations are 1×1 matrices, and the matrices' entries are just the character values, which are ± 1 .

For the representation $\pi_{2^{2n}}$, denote $G = \mathcal{H}_n(\mathbb{Z}/2\mathbb{Z})$, we calculate

$$\begin{aligned} \frac{1}{\#G} \sum_{g \in G} \chi_{\pi_{2^{2n}}}(g^2) &= \frac{1}{\#G} \sum_{g \in G} \chi_{\pi_{2^{2n}}}(M(a, b, c)^2) \\ &= \frac{1}{\#G} \sum_{g \in G} \chi_{\pi_{2^{2n}}}(M(0, 0, \sum_{j=1}^n a_j b_j)) \\ &= \frac{1}{\#G} \sum_{g \in G} 2^n \omega\left(\sum_{j=1}^n a_j b_j\right). \end{aligned}$$

Here, $\sum_{j=1}^n a_j b_j = 0$ or 1 , we need to settle down how many 0's and 1's appear. Let

$$\begin{aligned} \alpha_n &= \#\{(a_j, b_j) : \sum_{j=1}^n a_j b_j = 0\}, \\ \beta_n &= \#\{(a_j, b_j) : \sum_{j=1}^n a_j b_j = 1\}. \end{aligned}$$

Then we have

$$\sum_{j=1}^{n-1} a_j b_j = 0 \text{ or } 1; \quad \sum_{j=1}^n a_j b_j = 0 \text{ or } 1,$$

and

$$(2.72) \quad \begin{aligned} \alpha_n &= 3\alpha_{n-1} + \beta_{n-1}, \\ \beta_n &= \alpha_{n-1} + 3\beta_{n-1}, \end{aligned}$$

for $n \geq 2$, with $\alpha_1 = 3$ and $\beta_1 = 1$. Write the equations (2.72) inductively, we get

$$\begin{aligned} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} &= \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{n-1} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(4^n + 2^n) \\ \frac{1}{2}(4^n - 2^n) \end{pmatrix}. \end{aligned}$$

Now we can continue our calculation

$$\begin{aligned} \frac{1}{\#G} \sum_{g \in G} \chi_{\pi_{2^{2n}}}(g^2) &= \frac{1}{\#G} \left(2^n \omega(0) \frac{1}{2}(4^n + 2^n) + 2^n \omega(1) \frac{1}{2}(4^n - 2^n) \right) \cdot 2 \\ &= \frac{1}{\#G} \left(2^n \frac{1}{2}(4^n + 2^n) - 2^n \frac{1}{2}(4^n - 2^n) \right) \cdot 2 \\ &= \frac{1}{2^{2n+1}} 2^{2n+1} = 1, \end{aligned}$$

hence by the Frobenius-Schur indicator Theorem 2.24, we know the representation π_{2^n} is *real*.

Therefore, all the irreducible representations of $\mathcal{H}_n(\mathbb{Z}/2\mathbb{Z})$ are real when $n \geq 1$.

(2) When $p \geq 3$ and $n \geq 1$. Since elements in $\mathcal{H}_n(K)$ cannot always be conjugate to their inverses when $p \geq 3$, we know all the nontrivial irreducible representations are not selfconjugate, therefore, all the nontrivial irreducible representations are complex.

The trivial representation of $\mathcal{H}_n(K)$ is *real*. \square

2.8. Clifford group

Definition 2.73 ([55, 56]). *Miller group* on n generators is defined by

$$M_n := \langle x_1, x_2, \dots, x_n : (x_k)^{m_k} = 1, x_j^{-1} x_k x_j = x_k^{-1} \rangle,$$

such that $n > 1$, $m_k > 2$ for $1 \leq k \leq n$, and for any $j \neq k$.

Theorem 2.74 ([8, Corollary 2.1]). *The Miller group M_n is ambivalent if and only if $n \not\equiv 1 \pmod{4}$.*

Definition 2.75 ([52, pages 36–37]). *Clifford group* is a finite group generated by an orthonormal basis e_1, e_2, \dots, e_n of \mathbb{R}^n . It can be presented by the abstract elements $e_1, e_2, \dots, e_n, -1$ subject to the relations that -1 is central and that $(-1)^2 = 1$, $e_j^2 = -1$ and $e_j e_k = -e_k e_j$ for all $k \neq j$.

We denote the Clifford group by Γ_n , by direct calculation, we see the order of Γ_n is 2^{n+1} . By the defining relations for the Clifford group, we know the Clifford group is a Miller group. We denote its center by $Z(\Gamma_n)$ whose elements are given by the following lemma.

Lemma 2.76 ([74, Proposition IV.3.2.(b)]). *For the Clifford group Γ_n and its center $Z(\Gamma_n)$, we have*

$$(2.77) \quad Z(\Gamma_n) = \begin{cases} \{\pm 1\} & n \text{ is even,} \\ \{\pm 1, \pm e_1 e_2 \cdots e_n\} & n \text{ is odd.} \end{cases}$$

For the convenience, we give the detailed proof of this lemma by the method of [74]:

Proof. For any element $\pm e_{m_1} e_{m_2} \cdots e_{m_k} \in \Gamma_n$, here

$$\{m_1, m_2, \dots, m_k\} \subset \{1, 2, \dots, n\}$$

and m_j 's are distinct with each other. If k is even and for any $m_j \in \{m_1, m_2, \dots, m_k\}$, we have

$$(2.78) \quad e_{m_j} (e_{m_1} e_{m_2} \cdots e_{m_k}) = -(e_{m_1} e_{m_2} \cdots e_{m_k}) e_{m_j} \neq (e_{m_1} e_{m_2} \cdots e_{m_k}) e_{m_j},$$

so $e_{m_1} e_{m_2} \cdots e_{m_k} \notin Z(\Gamma_n)$. If k is odd and $m_j \notin \{m_1, m_2, \dots, m_k\}$, then the equation (2.78) holds, and $e_{m_1} e_{m_2} \cdots e_{m_k} \notin Z(\Gamma_n)$ as long as $\{m_1, \dots, m_k\} \neq$

$\{1, \dots, n\}$. This means the center is not bigger than those in the equation (2.77).

When n is even, by the equation (2.78) with $k = n$, we know $\pm e_1 \cdots e_n$ is not in the center; hence $Z(\Gamma_n) = \{\pm 1\}$.

When n is odd, we need to show

$$(2.79) \quad (e_{m_1} e_{m_2} \cdots e_{m_k})(e_1 e_2 \cdots e_n) = (e_1 e_2 \cdots e_n)(e_{m_1} e_{m_2} \cdots e_{m_k}).$$

In fact, we just need to show

$$e_{m_j}(e_1 e_2 \cdots e_n) = (e_1 e_2 \cdots e_n)e_{m_j},$$

because the equation (2.79) can be proved step by step by the above equation. m_j is a number in $\{1, 2, \dots, n\}$, to explain the movement clearly, we denote the e_{m_j} in $e_1 e_2 \cdots e_{m_j} \cdots e_n$ by $\widetilde{e_{m_j}}$: $e_1 e_2 \cdots \widetilde{e_{m_j}} \cdots e_n$. We have

$$(2.80) \quad \begin{aligned} e_{m_j}(e_1 e_2 \cdots e_n) &= e_{m_j} e_1 e_2 \cdots e_{m_j} \cdots e_n \\ &= e_{m_j} e_1 e_2 \cdots \widetilde{e_{m_j}} \cdots e_n \\ &= e_1 e_2 \cdots e_{m_j} \cdots e_n \widetilde{e_{m_j}} \\ &= (e_1 e_2 \cdots e_{m_j} \cdots e_n)e_{m_j}. \end{aligned}$$

The number of elements on the left and right sides of $\widetilde{e_{m_j}}$ share the same parity, this results in the movement of the equation (2.80). \square

By using the Wigner Theorem 2.29, we find the following result.

Theorem 2.81. *Clifford group Γ_n is a simply reducible group when $n \not\equiv 1 \pmod{4}$.*

To calculate the formula in the Wigner Theorem 2.29, we need the following lemmas about binomial coefficients and Orbit-Stabilizer.

Lemma 2.82 ([22, Page 16, Exercise 55]).

$$(2.83) \quad \begin{aligned} \binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots &= \frac{1}{2}(2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4}), \\ \binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \cdots &= \frac{1}{2}(2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4}), \\ \binom{n}{2} + \binom{n}{6} + \binom{n}{10} + \cdots &= \frac{1}{2}(2^{n-1} - 2^{\frac{n}{2}} \cos \frac{n\pi}{4}), \\ \binom{n}{3} + \binom{n}{7} + \binom{n}{11} + \cdots &= \frac{1}{2}(2^{n-1} - 2^{\frac{n}{2}} \sin \frac{n\pi}{4}). \end{aligned}$$

Lemma 2.84 (Orbit-Stabilizer, [41, Corollary 4.11]). *Let G be a finite group. For any $g \in G$, denote by $C(g)$ for the conjugacy class containing g . Then*

$$\sharp(C(g)) = [G : Z_G(g)],$$

where $[G : Z_G(g)]$ is the index in G of $Z_G(g)$.

Now we can proof Theorem 2.81.

Proof. By Theorem 2.74, we know Γ_n is ambivalent if and only if $n \not\equiv 1 \pmod{4}$. Firstly we calculate the left side of the equation in the Wigner Theorem 2.29:

$$\begin{aligned}
 & \sum_{g \in \Gamma_n} \xi(g)^3 \\
 (2.85) \quad &= \xi(1)^3 + \xi(-1)^3 \\
 (2.86) \quad &= \left(2 \cdot \left(\binom{n}{0} + \binom{n}{3} + \binom{n}{4} + \binom{n}{7} + \binom{n}{8} + \cdots \right) \right)^3 \\
 & \quad + \left(2 \cdot \left(\binom{n}{1} + \binom{n}{2} + \binom{n}{5} + \binom{n}{6} + \binom{n}{9} + \binom{n}{10} + \cdots \right) \right)^3 \\
 &= \left(2^n + 2^{\frac{n}{2}} \left(\cos \frac{n\pi}{4} - \sin \frac{n\pi}{4} \right) \right)^3 \\
 & \quad + \left(2^n - 2^{\frac{n}{2}} \left(\cos \frac{n\pi}{4} - \sin \frac{n\pi}{4} \right) \right)^3 \quad (\text{by Lemma 2.82}) \\
 &= 2^{3n+1} + 3 \cdot 2^{2n+1} \left(1 - \sin \frac{n\pi}{2} \right) \\
 (2.87) \quad &= \begin{cases} 2^{3n+1} + 3 \cdot 2^{2n+1} & n \equiv 0 \text{ or } 2 \pmod{4}, \\ 2^{3n+1} + 3 \cdot 2^{2n+2} & n \equiv 3 \pmod{4}, \\ 2^{3n+1} & n \equiv 1 \pmod{4}. \end{cases}
 \end{aligned}$$

Now we explain the reasons of the above equations. For any element $\pm e_{m_1} e_{m_2} \cdots e_{m_k} \in \Gamma_n$, here

$$\{m_1, m_2, \dots, m_k\} \subset \{1, 2, \dots, n\}$$

and m_j 's are distinct with each other, we have

$$\begin{aligned}
 (\pm e_{m_1} e_{m_2} \cdots e_{m_k})^2 &= (e_{m_1} e_{m_2} \cdots e_{m_k})^2 \\
 &= \pm e_{m_1} e_{m_1} e_{m_2} e_{m_2} \cdots e_{m_k} e_{m_k} \\
 &= \pm (-1) \cdot (-1) \cdots (-1) = \pm (-1)^k \\
 (2.88) \quad &= \pm 1,
 \end{aligned}$$

thus we know only ± 1 in Γ_n can have “square root” in the sense of the equation (2.26), hence we have the equation (2.85). In fact, we can settle down precisely the value in the equation (2.88):

$$\begin{aligned}
 & (e_{m_1} e_{m_2} \cdots e_{m_k})^2 \\
 &= e_{m_1} (-1)^{k-1} e_{m_1} e_{m_2} e_{m_3} \cdots e_{m_k} e_{m_2} e_{m_3} \cdots e_{m_k} \\
 &= (e_{m_1} (-1)^{k-1} e_{m_1}) (e_{m_2} (-1)^{k-2} e_{m_2}) (e_{m_3} (-1)^{k-3} e_{m_3}) \cdots \\
 & \quad (e_{m_{k-1}} (-1)^{k-(k-1)} e_{m_{k-1}}) (e_{m_k} (-1)^{k-k} e_{m_k}) \\
 &= (-1)^{(k-1)+(k-2)+(k-3)+\dots+(k-(k-1))+(k-k)} e_{m_1} e_{m_1} e_{m_2} e_{m_2} \cdots e_{m_k} e_{m_k} \\
 &= (-1)^{\frac{k(k-1)}{2}} (-1)^k \\
 &= (-1)^{\frac{k(k+1)}{2}},
 \end{aligned}$$

hence we get

$$(e_{m_1}e_{m_2}\cdots e_{m_k})^2 = \begin{cases} 1 & k \equiv 0, 3 \pmod{4}, \\ -1 & k \equiv 1, 2 \pmod{4}. \end{cases}$$

Both $e_{m_1}e_{m_2}\cdots e_{m_k}$ and $-e_{m_1}e_{m_2}\cdots e_{m_k}$ have the same square result, therefore, we have the equation (2.86).

Secondly we calculate the right side of the equation in the Wigner Theorem 2.29. By Lemma 2.84, we know

$$\nu(g) = \sharp Z_G(g) = \frac{\sharp G}{\sharp C(g)},$$

here $C(g)$ is the conjugacy class of g .

For any $e_{m_1}e_{m_2}\cdots e_{m_k}$ and $e_{p_1}e_{p_2}\cdots e_{p_j} \in \Gamma_n$, we have

$$\begin{aligned} & (e_{p_1}e_{p_2}\cdots e_{p_j})(e_{m_1}e_{m_2}\cdots e_{m_k})(e_{p_1}e_{p_2}\cdots e_{p_j})^{-1} \\ &= (\pm e_{m_1}e_{m_2}\cdots e_{m_k})(e_{p_1}e_{p_2}\cdots e_{p_j})(e_{p_1}e_{p_2}\cdots e_{p_j})^{-1} \\ &= \pm e_{m_1}e_{m_2}\cdots e_{m_k}, \end{aligned}$$

so each conjugacy class is either $\{e_{m_1}e_{m_2}\cdots e_{m_k}\}$ or $\{\pm e_{m_1}e_{m_2}\cdots e_{m_k}\}$. For any element g in the center $Z(\Gamma_n)$ of Γ_n , its conjugacy class $C(g)$ contains only one element, namely itself. For any element $g \in \Gamma_n - Z(\Gamma_n)$, its conjugacy class $C(g)$ contains two elements, namely g and g^{-1} . Thus we have

$$\nu(g) = \begin{cases} 2^{n+1} & g \in Z(\Gamma_n), \\ 2^n & g \in \Gamma_n - Z(\Gamma_n). \end{cases}$$

When n is even, its center is (cf. Lemma 2.76)

$$Z(\Gamma_n) = \{\pm 1\},$$

and we have $\sharp(\Gamma_n - Z(\Gamma_n)) = 2 \cdot (2^n - 1)$. Then we have

$$\begin{aligned} \sum_{g \in \Gamma_n} \nu(g)^2 &= \nu(1)^2 + \nu(-1)^2 + \sum_{g \in \Gamma_n - Z(\Gamma_n)} \nu(g)^2 \\ &= 2 \cdot ((2^{n+1})^2 + (2^n)^2 \cdot (2^n - 1)) \\ (2.89) \quad &= 2^{3n+1} + 3 \cdot 2^{2n+1}. \end{aligned}$$

When n is odd, its center is (cf. Lemma 2.76)

$$Z(\Gamma_n) = \{\pm 1, \pm e_1e_2\cdots e_k\},$$

and we have $\sharp(\Gamma_n - Z(\Gamma_n)) = 2 \cdot (2^n - 2)$. Then we have

$$\begin{aligned} \sum_{g \in \Gamma_n} \nu(g)^2 &= \nu(1)^2 + \nu(-1)^2 + \nu\left(\prod_{k=1}^n e_k\right)^2 + \nu\left(-\prod_{k=1}^n e_k\right)^2 + \sum_{g \in \Gamma_n - Z(\Gamma_n)} \nu(g)^2 \\ &= 2 \cdot ((2^{n+1})^2 \cdot 2 + (2^n)^2 \cdot (2^n - 2)) \\ (2.90) \quad &= 2^{3n+1} + 3 \cdot 2^{2n+2}. \end{aligned}$$

Therefore, when $n \not\equiv 1 \pmod{4}$, the equation (2.87) is consistent with equations (2.89) and (2.90). By the Wigner Theorem 2.29, we know Γ_n is simply reducible when $n \not\equiv 1 \pmod{4}$. \square

Remark 2.91. Thanks to Professor Alexander Hulpke answering my question in *Mathematics Stack Exchange* (cf. [1]), I know how to use the computer algebra system GAP to construct a Clifford group and get its character table. For example, we can construct the Clifford group Γ_3 by the following codes in GAP:

```
gap> f:=FreeGroup("e1","e2","e3","m");
<free group on the generators [ e1, e2, e3, m ]>
gap> AssignGeneratorVariables(f);
#I Assigned the global variables [ e1, e2, e3, m ]
gap> rels:=[Comm(e1,m),Comm(e2,m),Comm(e3,m),e1^2/m,e2^2/m,
e3^2/m, e1*e2/(m*e2*e1),e1*e3/(m*e3*e1),e2*e3/(m*e3*e2)];;
gap> g:=f/rels;
<fp group on the generators [ e1, e2, e3, m ]>
```

Then we can get the conjugacy classes and irreducible character tables of Γ_3 by the following codes:

```
gap> ConjugacyClasses(g);
gap> Irr(CharacterTable(g));
```

We can construct and get the character table of other Clifford group Γ_n in GAP, just by adding more generators and modifying the relations. We can get all the irreducible characters of Clifford group via GAP for $1 \leq n \leq 12$. For $n \geq 13$, the GAP can not return the needed character tables because of the pre-set memory limit.

3. Coxeter groups

Since dihedral groups are Coxeter groups, and some Coxeter groups are related to the quasicrystallographic structures in condensed matter physics (cf. [50] and [61]), we want to find out whether the other finite Coxeter groups are simply reducible.

3.1. Definition and properties

Firstly we make a brief review about Coxeter groups. The notations and results are taken from [11] and [39].

Definition 3.1 ([39, page 3]). Let V be a real Euclidean space endowed with a positive definite symmetric bilinear form (λ, μ) . A *reflection* is a linear operator s on V which sends some nonzero vector α to its negative while fixing pointwise the hyperplane H_α orthogonal to α .

We may write $s = s_\alpha$ and we have a formula

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha.$$

A finite group generated by reflections is called a *finite reflection group*. It is a finite subgroup of the orthogonal group $O(V)$.

Definition 3.2 ([39, pages 6,8]). Consider a finite reflection group W which is generated by all reflections s_α , $\alpha \in \Phi$. Call Φ a *root system* with associated reflection group W . The elements of Φ are called *roots*. Call a subset Δ of Φ a *simple system* (and call its elements *simple roots*) if Δ is a vector space basis for the \mathbb{R} -span of Φ in V and if moreover each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign (all nonnegative or all nonpositive).

Definition 3.3 ([39, page 105], [46, page 66]).

- (1) A *Coxeter system* is a pair (W, S) consisting of a group W and a set of generators $S \subset W$, subject only to relations of the form

$$(ss')^{m(s,s')} = 1,$$

where $m(s, s) = 1$, $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$ in S . In case no relation occurs for a pair s, s' , we make the convention that $m(s, s') = \infty$. We shall be interested in *finite Coxeter systems*, namely those for which W is finite.

- (2) Formally, W is the quotient F/N , where F is a free group on the set S and N is the normal subgroup generated by all elements

$$(ss')^{m(s,s')}.$$

Call $\sharp S$ the *rank* of (W, S) , and refer to W as a *Coxeter group*.

Theorem 3.4 ([39, page 133]). *W is a Coxeter group. The following are equivalent:*

- (1) *W is finite.*
- (2) *W is a finite reflection group.*

Theorem 3.5 ([62, Theorem 3]). *All finite reflection groups are ambivalent.*

By Theorem 3.4 and Theorem 3.5, we know all the finite Coxeter groups are ambivalent.

Definition 3.6 ([11, Definition IV.1.9.4]). Let Ξ be a set. A *Coxeter matrix* of type Ξ is a symmetric square matrix $M = (m_{jk})_{j,k \in \Xi}$ whose entries are integers or $+\infty$ satisfying the relations

$$\begin{aligned} m_{jj} &= 1 && \text{for all } j \in \Xi, \\ m_{jk} &\geq 2 && \text{for } j, k \in \Xi \text{ with } j \neq k. \end{aligned}$$

A *Coxeter graph* of type Ξ is (by abuse of language) a pair consisting of a graph Ψ having Ξ as its set of vertices and a map f from the set of edges of

this graph to the set consisting of $+\infty$ and the set of integers ≥ 3 . Ψ is called the *underlying graph* of the Coxeter graph (Ψ, f) .

Definition 3.7 ([11, page 14]). If (W, S) is a Coxeter system, the matrix $M = (m(s, s'))_{s, s' \in S}$, where $m(s, s')$ is the order of ss' , is a Coxeter matrix of type S which is called the *Coxeter matrix* of (W, S) . The Coxeter graph (Ψ, f) associated to M is called the *Coxeter graph* of (W, S) .

We remark that two vertices s and s' of Ψ are joined if and only if s and s' do not commute.

Definition 3.8 ([11, page 14]). A Coxeter system (W, S) is said to be *irreducible* if the underlying graph of its Coxeter graph is connected and non-empty.

Theorem 3.9 ([11, Theorem VI.4.1.1]). *The graph of any irreducible finite Coxeter system (W, S) is isomorphic to one of the following:*

$$\begin{array}{ll}
 \mathcal{A}_n & \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad (n \geq 1 \text{ vertices}) \\
 \mathcal{B}_n & \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet^4 \quad (n \geq 2 \text{ vertices}) \\
 \mathcal{D}_n & \text{---} \bullet \text{---} \bullet \text{---} \bullet \begin{array}{l} \nearrow \bullet \\ \searrow \bullet \end{array} \quad (n \geq 4 \text{ vertices}) \\
 \mathcal{E}_6 & \begin{array}{c} \bullet \\ | \\ \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \\
 \mathcal{E}_7 & \begin{array}{c} \bullet \\ | \\ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \\
 \mathcal{E}_8 & \begin{array}{c} \bullet \\ | \\ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \\
 \mathcal{F}_4 & \text{---} \bullet \text{---} \bullet^4 \text{---} \bullet \\
 \mathcal{G}_2 & \text{---} \bullet^6 \text{---} \bullet \\
 \mathcal{H}_3 & \text{---} \bullet^5 \text{---} \bullet \\
 \mathcal{H}_4 & \text{---} \bullet^5 \text{---} \bullet \text{---} \bullet \\
 \mathcal{I}_2(n) & \text{---} \bullet^n \quad (n = 5 \text{ or } n \geq 7).
 \end{array}
 \tag{3.10}$$

No two of these graphs are isomorphic.

Theorem 3.11 ([11, Theorem VI.4.1.2]). *The Coxeter groups defined by the Coxeter graph $\mathcal{A}_n, \mathcal{B}_n, \dots, \mathcal{I}_2(n)$ of Theorem 3.9 are finite.*

3.2. Representations of $W(\mathcal{A}_n)$, $W(\mathcal{B}_n)$ and $W(\mathcal{D}_n)$

For the finite Coxeter groups corresponding to the Coxeter graphs \mathcal{A}_n , \mathcal{B}_n and \mathcal{D}_n in Theorem 3.9, we denote these groups by $W(\mathcal{A}_n)$, $W(\mathcal{B}_n)$ and $W(\mathcal{D}_n)$, respectively. Finite Coxeter groups $W(\mathcal{A}_n)$, $W(\mathcal{B}_n)$ and $W(\mathcal{D}_n)$ are isomorphic to some already known finite groups that are easier to work with. To write down the results, we first review the definition of wreath product of groups.

Definition 3.12 ([67, page 55]). Let G be a group and X be a set, we call X a G -set if there is a function

$$\alpha : G \times X \rightarrow X$$

(called an *action*), denoted by

$$\alpha : (g, x) \mapsto gx ,$$

such that:

- (i) $1x = x$ for all $x \in X$,
- (ii) $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

One also says that G acts on X . If $\#X = n$, then n is called the *degree* of the G -set X .

Definition 3.13 ([67, page 172]). Let D and Q be groups, let Ω be a finite Q -set, and let $K = \prod_{\omega \in \Omega} D_{\omega}$, where $D_{\omega} \simeq D$ for all $\omega \in \Omega$. Then the *wreath product* of D by Q , denoted by $D \wr Q$, is the semidirect product of K by Q , where Q acts on K by

$$q \cdot (d_{\omega}) = (d_{q\omega})$$

for $q \in Q$ and $(d_{\omega}) \in \prod_{\omega \in \Omega} D_{\omega}$.

Theorem 3.14 ([39, pages 41–42], [7, page 3172], [76, page 384], [13, pages 374–376]). For the Coxeter groups $W(\mathfrak{A}_n)$, $W(\mathfrak{B}_n)$ and $W(\mathfrak{D}_n)$, we have the following isomorphisms:

$$\begin{aligned} W(\mathfrak{A}_n) &\simeq S_{n+1}, \\ W(\mathfrak{B}_n) &\simeq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \simeq S_2 \wr S_n, \\ W(\mathfrak{D}_n) &\simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n, \end{aligned}$$

here S_n is the symmetric group. Therefore, we get the orders of these groups:

$$\begin{aligned} \#W(\mathfrak{A}_n) &= (n+1)!, \\ \#W(\mathfrak{B}_n) &= 2^n \cdot (n!), \\ \#W(\mathfrak{D}_n) &= 2^{n-1} \cdot (n!). \end{aligned}$$

Definition 3.15 ([68, page 2]). For a nonnegative integer n , a *partition* of n is a sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l),$$

where the nonnegative integers $\{\lambda_j\}$ are weakly decreasing and

$$\begin{aligned} |\lambda| &:= \sum_{j=1}^n \lambda_j \\ &= n. \end{aligned}$$

We write $\lambda \vdash n$ to denote that λ is a partition of n , and we use $\mathcal{P}(n)$ to denote the set of all partitions of n , and we set

$$\mathcal{P}(0) := \{\emptyset\}.$$

For a symmetric group S_n and its dual (cf. Definition 2.10) $\widehat{S_n}$, there is a one-to-one correspondence between the irreducible representation of S_n and a partition of n (cf. [76, page 382]):

$$\begin{aligned}\widehat{S_n} &\longrightarrow \mathcal{P}(n) \\ (\rho_\lambda, V_\lambda) &\longmapsto \lambda.\end{aligned}$$

For the Coxeter group $W(\mathfrak{A}_n)$, we know

$$W(\mathfrak{A}_n) \simeq S_{n+1}$$

according to Theorem 3.14. Therefore, the dual $\widehat{W(\mathfrak{A}_n)}$ is determined by $\mathcal{P}(n+1)$.

For the Coxeter group $W(\mathfrak{B}_n)$, we know

$$W(\mathfrak{B}_n) \simeq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \simeq S_2 \wr S_n$$

according to Theorem 3.14. We still use S_n and $(\mathbb{Z}/2\mathbb{Z})^n$ to denote their subgroups

$$\begin{aligned}S_n &= \{(1, \sigma) : \sigma \in S_n\} \subset (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n, \\ (\mathbb{Z}/2\mathbb{Z})^n &= \{(\sigma, 1) : \sigma \in (\mathbb{Z}/2\mathbb{Z})^n\} \subset (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n.\end{aligned}$$

Then $W(\mathfrak{B}_n)$ is generated by S_n and $\sigma_0 = (-1, 1, 1, \dots, 1) \in (\mathbb{Z}/2\mathbb{Z})^n$ ([76, page 384]).

Following notations in [76, Section 2], we review the representations of $W(\mathfrak{B}_n)$. Let

$$\begin{aligned}B\mathcal{P}(n) &:= \{(\lambda, \mu) : \lambda \text{ and } \mu \text{ are partitions such that } |\lambda| + |\mu| = n\} \\ &= \bigcup_{m=0}^n \mathcal{P}(m) \times \mathcal{P}(n-m).\end{aligned}$$

Let $(\rho_\lambda, V_\lambda)$ be the irreducible representation of the symmetric group S_n . We define a representation $\rho(\lambda, \emptyset)$ of $W(\mathfrak{B}_n)$ on V_λ by

$$\begin{aligned}\rho(\lambda, \emptyset)(\tau) &= \rho_\lambda(\tau) \quad \text{for any } \tau \in S_n, \\ \rho(\lambda, \emptyset)(\sigma_0) &= I_{V_\lambda} \quad \text{identity transformation.}\end{aligned}$$

Similarly, we define a representation $\rho(\emptyset, \lambda)$ of $W(\mathfrak{B}_n)$ on V_λ by

$$\begin{aligned}\rho(\emptyset, \lambda)(\tau) &= \rho_\lambda(\tau) \quad \text{for any } \tau \in S_n, \\ \rho(\emptyset, \lambda)(\sigma_0) &= -I_{V_\lambda}.\end{aligned}$$

Let λ and μ be partitions where $|\lambda| = n_1$, $|\mu| = n_2$, and $n_1 + n_2 = n$. We define $\rho(\lambda, \mu)$ by the following induced representation:

$$\rho(\lambda, \mu) = \rho(\lambda, \emptyset) \times \rho(\emptyset, \mu) \Big|_{W(\mathfrak{B}_{n_1}) \times W(\mathfrak{B}_{n_2})}^{W(\mathfrak{B}_n)}.$$

Theorem 3.16 ([76, page 385]). $\rho(\lambda, \mu)$ is an irreducible representation of $W(\mathfrak{B}_n)$ and

$$\{\rho(\lambda, \mu) : (\lambda, \mu) \in B\mathcal{P}(n)\}$$

is the dual of $W(\mathfrak{B}_n)$.

Since the number of irreducible representations of a finite group is equal to its number of conjugacy classes (cf. Theorem 2.11), we know $\# \widehat{W(\mathfrak{B}_n)}$ according to [7, page 3174]:

$$(3.17) \quad \# \widehat{W(\mathfrak{B}_n)} = \sum_{\substack{\eta_1, \eta_2, \dots, \eta_n \geq 0 \\ \sum_{j=1}^n j \cdot \eta_j = n}} \left(\prod_{j=1}^n (\eta_j + 1) \right).$$

For the Coxeter group $W(\mathfrak{D}_n)$, it is a subgroup of index 2 in $W(\mathfrak{B}_n)$ [76, page 388]. We can characterize its characters using those of $W(\mathfrak{B}_n)$. For the irreducible representation $\rho(\lambda, \mu)$ of $W(\mathfrak{B}_n)$, we denote its character by $\chi_{(\lambda, \mu)}$. Then we define a character $\chi_{[\lambda, \mu]}$ of $W(\mathfrak{D}_n)$ by restriction:

$$\chi_{[\lambda, \mu]} := \chi_{(\lambda, \mu)}|_{W(\mathfrak{D}_n)}.$$

Theorem 3.18 ([76, page 388]).

- (i) $\chi_{[\lambda, \mu]}$ is irreducible if and only if $\lambda \neq \mu$.
- (ii) $\chi_{[\lambda, \lambda]}$ is a direct sum of two non-equivalent irreducible characters.
- (iii) $\chi_{[\lambda, \mu]} = \chi_{[\mu, \lambda]}$.

The complete irreducible characters of $\widehat{W(\mathfrak{D}_n)}$ consist of the characters constructed as above in (i), (ii), and (iii).

Based on the calculation of $W(\mathfrak{B}_n)$, we know the number of irreducible characters of $W(\mathfrak{D}_n)$ is given by (cf. [72], [13, page 376])

$$(3.19) \quad \# \widehat{W(\mathfrak{D}_n)} = \begin{cases} \frac{\# \widehat{W(\mathfrak{B}_n)}}{2} & n \text{ is odd,} \\ \frac{\# \widehat{W(\mathfrak{B}_n)} + 3\# \mathcal{P}(\frac{n}{2})}{2} & n \text{ is even.} \end{cases}$$

3.3. Bounds of largest Kronecker multiplicity of $W(\mathfrak{B}_n)$ and $W(\mathfrak{D}_n)$

Definition 3.20 ([60, page 3264]). For a finite group G and its dual \widehat{G} . Define the largest Kronecker multiplicity of G by

$$\mathbf{K}(G) := \max_{\rho, \phi, \psi \in \widehat{G}} g(\rho, \phi, \psi),$$

here $g(\rho, \phi, \psi)$ is the Kronecker multiplicity (cf. Definition 2.16).

Proposition 3.21 ([60, Proposition 7.3]). For a finite group G and its dual \widehat{G} , let

$$k(G) := \# \widehat{G},$$

and $b(G)$ be the biggest dimension of all the dimensions of irreducible representations in \widehat{G} . Then we have

$$(3.22) \quad \frac{(\sharp G)^{\frac{1}{2}}}{(k(G))^{\frac{3}{2}}} \leq \mathbf{K}(G) \leq b(G).$$

By using Proposition 3.21, we calculate the lower bounds of largest Kronecker multiplicity for $W(\mathfrak{B}_n)$ and $W(\mathfrak{D}_n)$.

Theorem 3.23. *For the Coxeter group $W(\mathfrak{B}_n)$, the largest Kronecker multiplicity is greater than or equal to 2 when $n \geq 8$.*

Proof. We want to find the condition for $\mathbf{K}(W(\mathfrak{B}_n)) \geq 2$. By Proposition 3.21, Theorem 3.14 and the equation (3.17), we have

$$\begin{aligned} \mathbf{K}(W(\mathfrak{B}_n)) &\geq \frac{(\sharp W(\mathfrak{B}_n))^{\frac{1}{2}}}{(k(W(\mathfrak{B}_n)))^{\frac{3}{2}}} \\ &= \frac{(2^n \cdot (n!))^{\frac{1}{2}}}{\left(\sum_{\substack{\eta_1, \eta_2, \dots, \eta_n \geq 0 \\ \sum_{j=1}^n j \cdot \eta_j = n}} \left(\prod_{j=1}^n (\eta_j + 1) \right) \right)^{\frac{3}{2}}}. \end{aligned}$$

For the denominator part of the above expression, we have

$$\begin{aligned} \prod_{j=1}^n (\eta_j + 1) &\leq (\eta_1 + 1) \cdot (\eta_2 + 1)^2 \cdots (\eta_n + 1)^n \quad (\text{because } (\eta_j + 1) \geq 1) \\ (3.24) \quad &\leq \left(\frac{(\eta_1 + 1) + 2(\eta_2 + 1) + \cdots + n(\eta_n + 1)}{\frac{n(n+1)}{2}} \right)^{\frac{n(n+1)}{2}} \\ &= \left(\frac{(\eta_1 + 2\eta_2 + \cdots + n\eta_n) + 1 + 2 + \cdots + n}{\frac{n(n+1)}{2}} \right)^{\frac{n(n+1)}{2}} \\ &= \left(\frac{n + \frac{n(n+1)}{2}}{\frac{n(n+1)}{2}} \right)^{\frac{n(n+1)}{2}} = \left(\frac{n+3}{n+1} \right)^{\frac{n(n+1)}{2}} = \left(\left(1 + \frac{1}{\frac{n+1}{2}} \right)^{\frac{n+1}{2}} \right)^n \\ &< e^n, \end{aligned}$$

where the inequality in (3.24) is given by the inequality of arithmetic and geometric means. The number of elements of

$$\{(\eta_1, \dots, \eta_n) : \eta_1, \dots, \eta_n \geq 0, \sum_{j=1}^n j \cdot \eta_j = n\}$$

is equal to $\sharp\mathcal{P}(n)$ (cf. [82, page 6], [10]). By the Hardy-Ramanujan asymptotic formula (cf. [60, page 3268], [2, Theorem 6.3]), we know

$$\sharp\mathcal{P}(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \text{ as } n \rightarrow \infty.$$

In our calculation, we just use the relation $\sharp\mathcal{P}(n) < e^{\pi\sqrt{\frac{2n}{3}}}$ [6, page 197]. Hence we get

$$(3.25) \quad \sum_{\substack{\eta_1, \eta_2, \dots, \eta_n \geq 0 \\ \sum_{j=1}^n j \cdot \eta_j = n}} \left(\prod_{j=1}^n (\eta_j + 1) \right) < e^{\pi\sqrt{\frac{2n}{3}}} \cdot e^n = e^{(\sqrt{\frac{2}{3}}\pi\sqrt{n} + n)},$$

and

$$(3.26) \quad \begin{aligned} \mathbf{K}(W(\mathfrak{B}_n)) &\geq \sqrt{\frac{2^n \cdot (n!)}{\left(\sum_{\substack{\eta_1, \eta_2, \dots, \eta_n \geq 0 \\ \sum_{j=1}^n j \cdot \eta_j = n}} \left(\prod_{j=1}^n (\eta_j + 1) \right) \right)^3}} \\ &> \sqrt{\frac{2^n \cdot (n!)}{e^{(\sqrt{6}\pi\sqrt{n} + 3n)}}} \\ &\geq 1.09914 \quad \text{when } n \geq 67, \end{aligned}$$

this means $\mathbf{K}(W(\mathfrak{B}_n)) \geq 2$ when $n \geq 67$. The inequality (3.26) holds because of the following Stirling's formula (cf. [64, page 26]) for $n = 1, 2, \dots$

$$n! = \sqrt{2\pi n} n^{n+1/2} e^{-n} e^{r_n}$$

where r_n satisfies the double inequality

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}.$$

If we pick a weaker inequality

$$0 < r_n,$$

then we get the following lower bound of the Stirling's formula

$$(3.27) \quad n! > \sqrt{2\pi n} n^{n+1/2} e^{-n}.$$

Therefore, we have the following inequality in the equation (3.26):

$$(3.28) \quad \begin{aligned} \sqrt{\frac{2^n \cdot (n!)}{e^{(\sqrt{6}\pi\sqrt{n} + 3n)}}} &> \sqrt{\frac{2^n \cdot \sqrt{2\pi} \cdot n^{n+1/2}}{e^{(\sqrt{6}\pi\sqrt{n} + 4n)}}} = \sqrt{\frac{\sqrt{\pi} \cdot (2n)^{n+1/2}}{e^{(\sqrt{6}\pi\sqrt{n} + 4n)}}} \\ &\geq \sqrt{\frac{\sqrt{\pi} \cdot (2n)^{n+1/2}}{e^{(\sqrt{6}\pi + 4)n}}} > \sqrt{\frac{\sqrt{\pi} \cdot (n)^n}{e^{(\sqrt{6}\pi + 4)n}}}. \end{aligned}$$

We know

$$\frac{n^n}{e^{mn}} > 1 \text{ when } n > e^m, \ m, n \in \mathbb{Z}^{>0},$$

therefore, the right hand side formula in the equation (3.28) is strictly greater than 1 if we pick the number n greater than a big number. By our calculation, when $n \geq 67$, the equation (3.26) holds.

For $n < 67$, we can calculate directly by using the table of number of conjugacy classes in [7, page 3174]. Our calculation is listed in Table 1 of Subsection 3.5. \square

Theorem 3.29. *For the Coxeter group $W(\mathfrak{D}_n)$, the largest Kronecker multiplicity is greater than or equal to 2 when $n \geq 7$.*

Proof. We want to find the condition for $\mathbf{K}(W(\mathfrak{D}_n)) \geq 2$. By Proposition 3.21, Theorem 3.14 and the equation (3.19), we have

$$\mathbf{K}(W(\mathfrak{D}_n)) \geq \frac{(\#W(\mathfrak{D}_n))^{\frac{1}{2}}}{(k(W(\mathfrak{D}_n)))^{\frac{3}{2}}}.$$

When n is odd, we know

$$k(W(\mathfrak{D}_n)) = \frac{k(W(\mathfrak{B}_n))}{2},$$

hence we get

$$\begin{aligned} \mathbf{K}(W(\mathfrak{D}_n)) &\geq \sqrt{\frac{2^{n-1} \cdot (n!)}{\left(\sum_{\substack{\eta_1, \eta_2, \dots, \eta_n \geq 0 \\ \sum_{j=1}^n j \cdot \eta_j = n}} \left(\prod_{j=1}^n (\eta_j + 1)\right)\right)^3}} \\ &= \sqrt{\frac{2^{n+2} \cdot (n!)}{\left(\sum_{\substack{\eta_1, \eta_2, \dots, \eta_n \geq 0 \\ \sum_{j=1}^n j \cdot \eta_j = n}} \left(\prod_{j=1}^n (\eta_j + 1)\right)\right)^3}} \\ &> \sqrt{\frac{2^{n+2} \cdot (n!)}{e^{(\sqrt{6}\pi\sqrt{n}+3n)}}} \quad (\text{use the equation (3.25)}) \\ (3.30) \quad &\geq 2.19828 \quad \text{for odd } n \geq 67. \end{aligned}$$

When n is even, we have

$$\begin{aligned}
 k(W(\mathcal{D}_n)) &= \frac{\sum_{\substack{\eta_1, \eta_2, \dots, \eta_n \geq 0 \\ \sum_{j=1}^n j \cdot \eta_j = n}} \left(\prod_{j=1}^n (\eta_j + 1) \right) + 3 \cdot \sharp \mathcal{P}(\frac{n}{2})}{2} \\
 &< \frac{\sum_{\substack{\eta_1, \eta_2, \dots, \eta_n \geq 0 \\ \sum_{j=1}^n j \cdot \eta_j = n}} \left(\prod_{j=1}^n (\eta_j + 1) \right) + 3 \cdot e^{\sqrt{\frac{2}{3}}\pi\sqrt{\frac{n}{2}}}}{2} \\
 &< \frac{e^{(\sqrt{\frac{2}{3}}\pi\sqrt{n}+n)} + 3e^{\frac{\pi}{\sqrt{3}}\sqrt{n}}}{2},
 \end{aligned}$$

hence we get

$$\begin{aligned}
 \mathbf{K}(W(\mathcal{D}_n)) &> \sqrt{\frac{2^{n+2} \cdot (n!)}{\left(e^{(\sqrt{\frac{2}{3}}\pi\sqrt{n}+n)} + 3e^{\frac{\pi}{\sqrt{3}}\sqrt{n}}\right)^3}} \\
 (3.31) \quad &\geq 1.07753 \quad \text{for even } n \geq 66.
 \end{aligned}$$

The inequalities (3.30) and (3.31) hold because of the Stirling's lower bound formula (3.27).

For $n < 66$, we find that $\mathbf{K}(W(\mathcal{D}_n)) \geq 2$ when $n \geq 7$. Our calculation is listed in Table 2 of Subsection 3.5. \square

3.4. List of simply reducible groups

For the finite Coxeter groups with irreducible finite Coxeter systems, their Coxeter graphs are given in Theorem 3.9. We denote these groups by $W(\mathcal{A}_n)$, $W(\mathcal{B}_n)$, \dots , respectively. Now we classify them according to whether they are simply reducible group (abbr. SR-group).

Theorem 3.32. *Finite Coxeter groups with irreducible finite Coxeter systems are simply reducible groups under certain requirements about number n .*

Coxeter group	Order of the group	SR-group	not SR-group
$W(\mathcal{A}_n)$	$(n+1)!$	$W(\mathcal{A}_1), W(\mathcal{A}_2), W(\mathcal{A}_3)$	$W(\mathcal{A}_n), n \geq 4$
$W(\mathcal{B}_n)$	$2^n \cdot (n!)$	$W(\mathcal{B}_2), W(\mathcal{B}_3)$	$W(\mathcal{B}_n), n \geq 4$
$W(\mathcal{D}_n)$	$2^{n-1} \cdot (n!)$	none	$W(\mathcal{D}_n), n \geq 4$
$W(\mathcal{E}_6)$	51840	none	$W(\mathcal{E}_6)$
$W(\mathcal{E}_7)$	52903040	none	$W(\mathcal{E}_7)$
$W(\mathcal{E}_8)$	696729600	none	$W(\mathcal{E}_8)$
$W(\mathcal{F}_4)$	1152	none	$W(\mathcal{F}_4)$
$W(\mathcal{G}_2)$	12	$W(\mathcal{G}_2)$	none
$W(\mathcal{H}_3)$	120	none	$W(\mathcal{H}_3)$
$W(\mathcal{H}_4)$	14400	none	$W(\mathcal{H}_4)$
$W(\mathcal{I}_2(n))$	$2 \cdot n$	$W(\mathcal{I}_2(n)), n = 5, n \geq 7$	none

Proof. We use $\chi_{a,b}$ to denote the characters in the character table of Coxeter group, here a is the dimension of the character and b is the b -th appearance of this same dimension character appearing in the character table. If there is only one such character of dimension a , then we denote this character by χ_a . In the following decompositions of tensor product representations, the coefficients of the decompositions are calculated directly by Theorem 2.15.

For the Coxeter group $W(\mathcal{A}_n)$, by Theorem 3.14, we know it is isomorphic to the symmetric group S_{n+1} . By Theorem 2.31, we know $W(\mathcal{A}_1)$, $W(\mathcal{A}_2)$, $W(\mathcal{A}_3)$ are simply reducible groups and all the other groups are not simply reducible groups.

For the Coxeter group $W(\mathcal{B}_n)$, the largest Kronecker multiplicity is greater than or equal to 2 when $n \geq 8$ (cf. Theorem 3.23), this means $W(\mathcal{B}_n)$ is not multiplicity-free for $n \geq 8$, and we get that $W(\mathcal{B}_n)$ is not simply reducible for $n \geq 8$. For $n < 8$, we can verify them one by one. Since $W(\mathcal{B}_2)$ is isomorphic to the dihedral groups D_4 , we know it is a simply reducible group by Theorem 2.42. For $W(\mathcal{B}_3) \simeq S_2 \wr S_3$, its character table can be found in [42, page 443]:

Class order	1	3	3	6	6	1	6	6	8	8
$\chi_{1,1}$	1	1	1	1	1	1	1	1	1	1
$\chi_{2,1}$	2	2	2	0	0	2	0	0	-1	-1
$\chi_{1,2}$	1	1	1	-1	-1	1	-1	-1	1	1
$\chi_{1,3}$	1	-1	1	1	-1	-1	-1	1	1	-1
$\chi_{2,2}$	2	-2	2	0	0	-2	0	0	-1	1
$\chi_{1,4}$	1	-1	1	-1	1	-1	1	-1	1	-1
$\chi_{3,1}$	3	1	-1	1	1	-3	-1	-1	0	0
$\chi_{3,2}$	3	1	-1	-1	-1	-3	1	1	0	0
$\chi_{3,3}$	3	-1	-1	1	-1	3	1	-1	0	0
$\chi_{3,4}$	3	-1	-1	-1	1	3	-1	1	0	0

We can use GAP to get the above character table. The codes are as following.

```
gap> WBthree:=WreathProduct(SymmetricGroup(2),
                             SymmetricGroup(3));
gap> Irr(CharacterTable(WBthree));
```

We can also use SAGE to get the above character table. The codes are as following.

```
sage: WeylGroup(["B",3]).character_table()
```

Denote the set of all the irreducible characters of $W(\mathcal{B}_3)$ by

$$\widehat{W(\mathcal{B}_3)} = \{\chi_{1,1}, \chi_{2,1}, \chi_{1,2}, \chi_{1,3}, \chi_{2,2}, \chi_{1,4}, \chi_{3,1}, \chi_{3,2}, \chi_{3,3}, \chi_{3,4}\}.$$

By direct calculations, we have the following decompositions for product of any two irreducible characters in $\widehat{W(\mathcal{B}_3)}$, this also gives us the Clebsch-Gordan

coefficients for $W(\mathcal{B}_3)$:

$$\begin{aligned}
 & \chi_{1,c} \cdot \chi_{a,b} = \chi_{a,d}, \text{ here } \left\{ \begin{array}{l} \forall c \in \{1, 2, 3, 4\}, \forall \chi_{a,b} \in \widehat{W(\mathcal{B}_3)} \\ \text{some } d \text{ such that } \chi_{a,d} \in \widehat{W(\mathcal{B}_3)} \end{array} \right\}, \\
 & \chi_{2,1} \cdot \chi_{2,2} = \chi_{2,2} \cdot \chi_{2,2} = \chi_{1,1} + \chi_{1,2} + \chi_{2,1}, \\
 & \chi_{2,1} \cdot \chi_{2,2} = \chi_{1,3} + \chi_{1,4} + \chi_{2,2}, \\
 & \chi_{3,1} \cdot \chi_{3,1} = \chi_{3,2} \cdot \chi_{3,2} = \chi_{3,3} \cdot \chi_{3,3} = \chi_{3,4} \cdot \chi_{3,4} \\
 & \quad = \chi_{1,1} + \chi_{2,1} + \chi_{3,3} + \chi_{3,4}, \\
 & \chi_{3,1} \cdot \chi_{3,2} = \chi_{3,3} \cdot \chi_{3,4} = \chi_{1,2} + \chi_{2,1} + \chi_{3,3} + \chi_{3,4}, \\
 & \chi_{3,1} \cdot \chi_{3,3} = \chi_{3,2} \cdot \chi_{3,4} = \chi_{1,3} + \chi_{2,2} + \chi_{3,1} + \chi_{3,2}, \\
 & \chi_{3,1} \cdot \chi_{3,4} = \chi_{3,2} \cdot \chi_{3,3} = \chi_{1,4} + \chi_{2,2} + \chi_{3,1} + \chi_{3,2}.
 \end{aligned}
 \tag{3.33}$$

The above decomposition (3.33) is multiplicity-free. Hence we know $W(\mathcal{B}_3)$ is a simply reducible group. For $W(\mathcal{B}_4) = S_2 \wr S_4$, by the character table in [42, page 445], we have

$$\begin{aligned}
 \chi_{8,1} \cdot \chi_{8,1} &= (64, 16, 0, \dots, 0) \\
 &= 2\chi_{3,1} + \dots,
 \end{aligned}$$

here

$$\begin{aligned}
 \chi_{8,1} &= (8, 4, 0, 0, 0, -4, 0, 0, -1, -1, -8, 0, 0, 0, 0, 1, 1, 0, 0, 0), \\
 \chi_{3,1} &= (3, 3, 3, 1, 1, 3, 1, 1, 0, 0, 3, 1, -1, 1, -1, 0, 0, -1, -1, -1).
 \end{aligned}$$

The decomposition is not multiplicity-free. Hence $W(\mathcal{B}_4)$ is not a simply reducible group. We can use SAGE or GAP to get the character tables of the Coxeter groups $W(\mathcal{B}_5)$, $W(\mathcal{B}_6)$ and $W(\mathcal{B}_7)$. For $W(\mathcal{B}_5)$, we get its character table via the following SAGE codes

```
sage: WeylGroup(["B",5]).character_table()
```

or GAP codes

```
gap> Irr(CharacterTable(WreathProduct(SymmetricGroup(2),
SymmetricGroup(5))));
```

We get all the 36 irreducible characters of the Coxeter group $W(\mathcal{B}_5)$. Among all these characters, there are four 20-dimensional irreducible characters. Consider the last one,

$$\begin{aligned}
 \chi_{20,4} &= (20, -4, -4, 4, 4, -20, 2, -2, 2, -2, -2, 2, -2, 2, \\
 & \quad -1, -1, -1, 1, 1, 1, \underbrace{0, \dots, 0}_6, -1, -1, 1, 1, \underbrace{0, \dots, 0}_6).
 \end{aligned}$$

We have

$$\chi_{20,4} \cdot \chi_{20,4} = (400, \dots).$$

Since the sum of the dimensions of all the 36 irreducible representations is equal to 312, it is strictly less than 400, this means some of the coefficients in the

decomposition of $\chi_{20,4} \cdot \chi_{20,4}$ must be strictly larger than 1, hence we know the decomposition is not multiplicity-free and $W(\mathcal{B}_5)$ is not a simply reducible group.

For $W(\mathcal{B}_6)$, there are 65 irreducible characters, the sum of all these dimensions is equal to 1384. The biggest dimension of these irreducible character is 80. The dimension of the product of this biggest dimension character with itself is equal to 6400, which is strictly greater than 1384. Thus we know the decomposition of this tensor product must not be multiplicity-free.

For $W(\mathcal{B}_7)$, there are 110 irreducible characters, the sum of all these dimensions is equal to 6512. The biggest dimension of these irreducible character is 210. The dimension of the product of this biggest dimension character with itself is equal to 44100, which is strictly greater than 6512. Thus we know the decomposition of this tensor product must not be multiplicity-free.

For the Coxeter group $W(\mathcal{D}_n)$, the largest Kronecker multiplicity is greater than or equal to 2 when $n \geq 7$ (cf. Theorem 3.29), this means $W(\mathcal{D}_n)$ is not multiplicity-free for $n \geq 7$, and we get that $W(\mathcal{D}_n)$ is not simply reducible for $n \geq 7$. For $n < 7$, we can verify them one by one. We use SAGE to get the character tables of $W(\mathcal{D}_4)$, $W(\mathcal{D}_5)$, and $W(\mathcal{D}_6)$.

For $W(\mathcal{D}_4)$, the SAGE codes are:

```
sage: WeylGroup(["D",4]).character_table()
```

There is only one 8-dimensional irreducible representation, its character is

$$\chi_8 = (8, 0, 0, 1, 0, 0, 0, 0, 0, 0, -1, 0, -8).$$

The product is

$$\chi_8 \cdot \chi_8 = (64, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 64).$$

The sum of the dimensions of all the irreducible representations is equal to

$$44 = 1 + 1 + 2 + 3 + 3 + 3 + 3 + 3 + 3 + 4 + 4 + 6 + 8,$$

which is strictly less than 64, this means some of the coefficients in the decomposition of $\chi_8 \cdot \chi_8$ must be strictly larger than 1, hence we know the decomposition is not multiplicity-free and $W(\mathcal{D}_4)$ is not a simply reducible group.

For $W(\mathcal{D}_5)$, there are 18 irreducible characters, the sum of all these dimensions is equal to 156. The biggest dimension of these irreducible character is 20. The dimension of the product of this biggest dimension character with itself is equal to 400, which is strictly greater than 156. Thus we know the decomposition of this tensor product must not be multiplicity-free.

For $W(\mathcal{D}_6)$, there are 37 irreducible characters, the sum of all these dimensions is equal to 752. The biggest dimension of these irreducible character is 45. The dimension of the product of this biggest dimension character with itself is equal to 2025, which is strictly greater than 752. Thus we know the decomposition of this tensor product must not be multiplicity-free.

For the Coxeter group $W(\mathcal{E}_6)$, its character table can be found in [24, page 104]. We can also use SAGE to get its character table via the following codes:

sage: `WeylGroup(["E", 6]).character_table()`

The biggest dimension is equal to 90, we denote this character by χ_{90} , it is

$$\chi_{90} = (90, -6, -6, 2, 2, \underbrace{0, \dots, 0}_5, 9, -3, -1, \underbrace{0, \dots, 0}_{10}),$$

here we use the character value in [24, page 104]. The product of χ_{90} with itself is

$$\begin{aligned} \chi_{90} \cdot \chi_{90} &= (8100, 36, 36, 4, 4, \underbrace{0, \dots, 0}_5, 81, 9, 1, \underbrace{0, \dots, 0}_{10}) \\ &= 14\chi_{90} + \dots \end{aligned}$$

Hence we know the decomposition is not multiplicity-free and $W(\mathfrak{E}_6)$ is not a simply reducible group.

For the Coxeter group $W(\mathfrak{E}_7)$, its character table can be got from [24, Table III] or SAGE. The biggest dimension is equal to 512 and there are two of them, we denote one of these characters by $\chi_{512,1}$. It is

$$\begin{aligned} \chi_{512,1} &= (512, 512, \underbrace{0, \dots, 0}_8, -16, -16, \underbrace{0, \dots, 0}_4, -4, -4, 0, 0, 8, 8, \\ &\quad \underbrace{0, \dots, 0}_4, -1, -1, 2, 2, \underbrace{0, \dots, 0}_{20}, -1, -1, \underbrace{0, \dots, 0}_6, 1, 1). \end{aligned}$$

The product of $\chi_{512,1}$ with itself is

$$\chi_{512,1} \cdot \chi_{512,1} = (512^2, 512^2, \dots) = 91\chi_{512,1} + \dots,$$

hence we know the decomposition is not multiplicity-free and $W(\mathfrak{E}_7)$ is not a simply reducible group.

For the Coxeter group $W(\mathfrak{E}_8)$, its character table can be got from [25] or [27, page 415] or SAGE. The biggest dimension is equal to 7168, we denote this character by χ_{7168} . The product of χ_{7168} with itself is

$$\chi_{7168} \cdot \chi_{7168} = (51380224, 51380224, \dots).$$

Since the sum of the dimensions of all the 112 irreducible representations is equal to 199952, it is strictly less than 51380224, this means some of the coefficients in the decomposition of $\chi_{7168} \cdot \chi_{7168}$ must be strictly larger than 1, hence we know the decomposition is not multiplicity-free and $W(\mathfrak{E}_8)$ is not a simply reducible group.

For the Coxeter group $W(\mathfrak{F}_4)$, its character table can be got by combining tables in [13, page 413] and [12, page 49]. The biggest dimension is equal to 16, we denote this character by χ_{16} . The product of χ_{16} with itself is

$$\begin{aligned} \chi_{16} \cdot \chi_{16} &= (256, 256, 0, 4, 4, 0, 4, 4, 4, 4, \underbrace{0, \dots, 0}_{15}) \\ &= 5\chi_{12} + \dots, \end{aligned}$$

here

$$\begin{aligned}\chi_{12} &= (12, 12, -4, 0, 0, 4, 0, 0, -3, -3, \underbrace{1, 0, \dots, 0}_{14}), \\ \chi_{16} &= (16, -16, 0, -2, 2, 0, -2, 2, -2, 2, \underbrace{0, \dots, 0}_{15}).\end{aligned}$$

Hence we know the decomposition is not multiplicity-free and $W(\mathcal{F}_4)$ is not a simply reducible group.

For the Coxeter group $W(\mathcal{G}_2)$, it is isomorphic to the dihedral group D_6 , hence we know $W(\mathcal{G}_2)$ is a simply reducible group.

For the Coxeter group $W(\mathcal{H}_3)$, it is the full icosahedral group. Its character table can be got from [21, page 485]. The biggest dimension is equal to 5 and there are two of them, we denote one of these characters by $\chi_{5,1}$. The product of $\chi_{5,1}$ with itself is

$$\begin{aligned}\chi_{5,1} \cdot \chi_{5,1} &= (25, 0, 0, 1, 1, 25, 0, 0, 1, 1) \\ &= \chi_{1,1} + \chi_{3,1} + \chi_{3,2} + 2\chi_{4,1} + 2\chi_{5,1},\end{aligned}$$

hence we know the decomposition is not multiplicity-free and $W(\mathcal{H}_3)$ is not a simply reducible group.

For the Coxeter group $W(\mathcal{H}_4)$, its character table can be got from [34, page 167]. The biggest dimension of irreducible representations is 48. We denote this character by χ_{48} , and the product of χ_{48} with itself is

$$\begin{aligned}\chi_{48} \cdot \chi_{48} &= (48^2, 48^2, \dots) \\ &= (2304, 2304, 0, 0, 0, 4, 4, 4, 4, 0, 36, 36, \\ &\quad 4, 4, 4, 4, 0, 0, 0, 1, 1, 1, 1, 4, 4, \underbrace{0, \dots, 0}_9) \\ &= 13\chi_{40} + \dots,\end{aligned}$$

here

$$\begin{aligned}\chi_{40} &= (40, 40, 0, -2, -2, 0, 0, 0, 0, 4, 1, 1, -5, -5, -5, -5, \\ &\quad 1, -1, -1, 1, 1, 1, 1, \underbrace{0, \dots, 0}_{11}), \\ \chi_{48} &= (48, -48, 0, 0, 0, -2, 2, -2, 2, 0, -6, 6, 2, -2, -2, 2, 0, 0, 0, \\ &\quad 1, -1, -1, 1, 2, -2, \underbrace{0, \dots, 0}_9).\end{aligned}$$

Hence we know the decomposition is not multiplicity-free and $W(\mathcal{H}_4)$ is not a simply reducible group.

For the Coxeter group $W(I_2(n))$, this group is isomorphic to the dihedral group D_n . By Theorem 2.42 we know $W(I_2(n))$ is simply reducible for all $n \geq 5$. \square

3.5. Appendix: Tables of lower bounds

We calculate the lower bounds of the largest Kronecker multiplicity by the equation (3.22). The lower bound calculated from the equation (3.22) is less than 1 for $W(\mathfrak{B}_n)$ (respectively, $W(\mathfrak{D}_n)$) when $n \leq 7$ (respectively, $n \leq 6$), this is not useful for our verification of the multiplicity-free requirement. Hence we do not list those cases. We use the software MATHEMATICA (cf. [84]) to do those calculations.

TABLE 1. $\mathbf{K}(W(\mathfrak{B}_n)) \geq 2$ for $8 \leq n \leq 67$

n	$k(W(\mathfrak{B}_n))$	$\mathbf{K}(W(\mathfrak{B}_n)) \geq$	n	$k(W(\mathfrak{B}_n))$	$\mathbf{K}(W(\mathfrak{B}_n)) \geq$
8	185	2	38	5374390	9.62357×10^{17}
9	300	3	39	6978730	5.74398×10^{18}
10	481	6	40	9035539	3.48732×10^{19}
11	752	14	41	11664896	2.15282×10^{20}
12	1164	36	42	15018300	1.35063×10^{21}
13	1770	96	43	19283830	8.60852×10^{21}
14	2665	275	44	24697480	5.57161×10^{22}
15	3956	832	45	31551450	3.6606×10^{23}
16	5822	2636	46	40210481	2.44043×10^{24}
17	8470	8760	47	51124970	1.6504×10^{25}
18	12230	30291	48	64854575	1.13178×10^{26}
19	17490	109181	49	82088400	7.86801×10^{26}
20	24842	407927	50	103679156	5.54307×10^{27}
21	35002	1.58069×10^6	51	130673928	3.95643×10^{28}
22	49010	6.32829×10^6	52	164363280	2.8602×10^{29}
23	68150	2.61754×10^7	53	206327710	2.09373×10^{30}
24	94235	1.11531×10^8	54	258508230	1.55152×10^{31}
25	129512	4.89478×10^8	55	323275512	1.16361×10^{32}
26	177087	2.2076×10^9	56	403531208	8.82996×10^{32}
27	240840	1.02283×10^{10}	57	502810130	6.7783×10^{33}
28	326015	4.85997×10^{10}	58	625425005	5.26251×10^{34}
29	439190	2.36715×10^{11}	59	776616430	4.1313×10^{35}
30	589128	1.18023×10^{12}	60	962759294	3.27877×10^{36}
31	786 814	6.02099×10^{12}	61	1191580872	2.63016×10^{37}
32	1046705	3.13927×10^{13}	62	1472454540	2.13214×10^{38}
33	1386930	1.67208×10^{14}	63	1816715170	1.74636×10^{39}
34	1831065	9.08943×10^{14}	64	2238075315	1.44496×10^{40}
35	2408658	5.04056×10^{15}	65	2753078840	1.20757×10^{41}
36	3157789	2.84926×10^{16}	66	3381689157	1.01912×10^{42}
37	4126070	1.64104×10^{17}	67	4147937540	8.68422×10^{42}

TABLE 2. $\mathbf{K}(W(\mathcal{D}_n)) \geq 2$ for $7 \leq n \leq 66$

n	$k(W(\mathcal{D}_n))$	$\mathbf{K}(W(\mathcal{D}_n)) \geq$	n	$k(W(\mathcal{D}_n))$	$\mathbf{K}(W(\mathcal{D}_n)) \geq$
7	55	2	37	2063035	3.28207×10^{17}
8	100	3	38	2687930	1.92393×10^{18}
9	150	6	39	3489365	1.1488×10^{19}
10	251	11	40	4518710	6.97246×10^{19}
11	376	28	41	5832448	4.30564×10^{20}
12	599	68	42	7510338	2.70063×10^{21}
13	885	192	43	9641915	1.7217×10^{22}
14	1355	536	44	12350243	1.11412×10^{23}
15	1978	1664	45	15775725	7.32119×10^{23}
16	2944	5184	46	20107123	4.88017×10^{24}
17	4235	17519	47	25562485	3.3008×10^{25}
18	6160	59918	48	32429650	2.26332×10^{26}
19	8745	218362	49	41044200	1.5736×10^{27}
20	12484	809686	50	51842515	1.10852×10^{28}
21	17501	3.16139×10^6	51	65336964	7.91287×10^{28}
22	24589	1.25918×10^7	52	82185294	5.72002×10^{29}
23	34075	5.23508×10^7	53	103163855	4.18746×10^{30}
24	47233	2.22244×10^8	54	129258630	3.10288×10^{31}
25	64756	9.78955×10^8	55	161637756	2.32721×10^{32}
26	88695	4.40388×10^9	56	201771181	1.76592×10^{33}
27	120420	2.04566×10^{10}	57	251405065	1.35566×10^{34}
28	163210	9.70186×10^{10}	58	312719350	1.05247×10^{35}
29	219595	4.7343×10^{11}	59	388308215	8.2626×10^{35}
30	294828	2.35728×10^{12}	60	481388053	6.55736×10^{36}
31	393407	1.2042×10^{13}	61	595790436	5.26032×10^{37}
32	523699	6.27232×10^{13}	62	736237533	4.26419×10^{38}
33	693465	3.34415×10^{14}	63	908357585	3.49271×10^{39}
34	915978	1.81656×10^{15}	64	1119050181	2.88987×10^{40}
35	1204329	1.00811×10^{16}	65	1376539420	2.41514×10^{41}
36	1579472	5.6954×10^{16}	66	1690859793	2.03822×10^{42}

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