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PRESERVATION OF EXPANSIVITY IN HYPERSPACE DYNAMICAL SYSTEMS

Namjip Koo and Hyunhee Lee

Abstract. In this paper we study the preservation of various notions of expansivity in discrete dynamical systems and the induced map for n-fold symmetric products and hyperspaces. Then we give a characterization of a compact metric space admitting hyper N-expansive homeomorphisms via the topological dimension. More precisely, we show that C^0-generically, any homeomorphism on a compact manifold is not hyper N-expansive for any $N \in \mathbb{N}$. Also we give some examples to illustrate our results.

1. Introduction

Let $f : X \to X$ be a homeomorphism on a compact metric space (X, d). For given $\delta > 0$, the dynamical δ-ball centered at $x \in X$ is defined by

$$\Gamma^f_\delta(x) = \{y \in X \mid d(f^n(x), f^n(y)) \le \delta \text{ for all } n \in \mathbb{Z}\}.$$

We say that a homeomorphism f is *expansive* if there is $\delta > 0$ such that $\Gamma^f_\delta(x) = \{x\}$ for all $x \in X$. The idea of expansivity in dynamical systems was first conceived by Utz [18] in the middle of the twenty century and since then several generalizations have been considered. Some of them as the *N-expansivity, countable expansivity or continuum-wise expansive* homeomorphisms were summarized in levels of generalized expansivity (see [6,9,11,12,14]).

On the other hand, the study of collective dynamics is also an important topic in the area of dynamical systems. The most usual context for collective dynamics is that of the induced map on the hyperspace of all nonempty closed subsets of a compact metric space endowed with the Hausdorff metric. It is natural to combine both topics and study the relationship between initial dynamics and its collective dynamics. W. Bauer and K. Sigmund [4] investigated the relationship between homeomorphisms on a compact metric space and their induced actions on the space of probability measures and obtained the result that the induced homeomorphism on the probabilities is expansive if and only

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if the initial space is finite. S. Mazurkiewicz [13] proved that the hyperspace of a space with positive dimension has infinite dimension and the initial space admitting an induced expansive homeomorphism on the hyperspace has the zero dimension. Also, A. Artigue [2] introduced the strong concept of expansiveness on a hyperspace which is called the *hyper-expansiveness* and proved that a compact metric space X admitting a hyper-expansive homeomorphism on X is countable.

In this paper we investigate the preservation of various notions of expansivity in discrete dynamical systems and the corresponding induced maps, and show that C^0-generically there are no hyper N-expansive homeomorphisms on a compact manifold. Thus we give a characterization of a compact metric space admitting hyper N-expansive homeomorphisms via the topological dimension. Also we give some examples to illustrate our results.

2. Basic notions

Throughout this paper, X denotes a compact metric space with a metric d and $f : X \to X$ a homeomorphism on it. Let 2^X denote the collection of all nonempty closed subsets of X, i.e.,

$$2^X = \{A \subseteq X \mid A \text{ is nonempty and closed}\}$$

which will be referred to as a *hyperspace* of X.

For any $A \in 2^X$ and any $r > 0$, the *open d-ball* in X about A of radius r is given by

$$B_d(A, r) = \{x \in X \mid d(x, A) < r\}.$$

Then 2^X is a compact metric space with Hausdorff metric d_H given by

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq B_d(B, \varepsilon) \text{ and } B \subseteq B_d(A, \varepsilon)\}$$

for any $A, B \in 2^X$. For our main results of this paper, we need the following hyperspaces of X and the corresponding induced maps on hyperspaces by a homeomorphism $f : X \to X$:

- $\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}$ is the *n-fold symmetric product* of X;
- $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$ is the collection of all finite subsets of X;
- $C(X)$ is the subspace of connected compact subsets of X;
- $2^f : 2^X \to 2^X$ is given by $2^f(A) = f(A)$;
- $f_n : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ is given by $f_n = 2^f|_{\mathcal{F}_n(X)}$;
- $f^{<\omega} : \mathcal{F}(X) \to \mathcal{F}(X)$ is given by $f^{<\omega} = 2^f|_{\mathcal{F}(X)}$;
- $f_c : C(X) \to C(X)$ is given by $f_c = 2^f|_{C(X)}$.

Now we recall basic concepts of dynamical systems that will be necessary in our examples and results.

We say that a point $p \in X$ is *periodic* of f if there is $k \in \mathbb{N}$ such that $f^k(p) = p$. The smallest non-negative integer number satisfying this equality is called the *period* of p and will be denoted by $\pi(p)$. Periodic points of period

1 are fixed points. We say that a point $p \in X$ is *non-wandering* if for each neighborhood U of p in X there exists $n \in \mathbb{Z}$ such that $f^n(U) \cap U \neq \emptyset$. We say that a point $p \in X$ is *stable* if for all $\epsilon > 0$ there is $\delta > 0$ such that if $d(x,p) < \delta$, then $d(f^n(x), f^n(p)) < \epsilon$ for all $n \geq 0$. A point $p \in X$ is said to be *unstable* if it is stable for f^{-1}. We say that a point $p \in X$ is *asymptotically stable* if it is stable and there is $\gamma > 0$ such that if $d(x,p) < \gamma$, then $d(f^n(x), f^n(p)) \to 0$ as $n \to \infty$. If p is an asymptotically stable periodic point, then the orbit of p is said to be an *attractor*. A *repeller* is an attractor of f^{-1}. Next, we write $O_f(x)$ for the orbit of f at x, $\mathrm{Fix}(f)$ for the set of fixed points of f, $\Omega(f)$ the set of non-wandering points of f, $\mathrm{Per}(f)$ the set of all periodic points of f, $\mathrm{Per}_a(f)$ the set of attractor periodic points of f and $\mathrm{Per}_r(f)$ the set of repeller periodic points of f. We say that a point $x \in X$ is said to be *chain recurrent* if for any $\epsilon > 0$, there is an ϵ-chain from x to itself. The set of all chain recurrent points of f will be denoted by $CR(f)$.

We give the generalized notions of expansiveness for induced maps on a hyperspace concerning our main results.

Definition 1. Let $N \in \mathbb{N}$ be given. We say that a homeomorphism $f : X \to X$ is *hyper N-expansive* if its induced map $2^f : 2^X \to 2^X$ is N-expansive, that is, if there is $\delta > 0$ such that $\sharp\Gamma_\delta^{2^f}(A) \leq N$ for all $A \in 2^X$. Here $\sharp B$ stands for the cardinality of a set B as an element of 2^X.

We note that if $N = 1$ in Definition 1, then hyper N-expansiveness reduces to the notion of hyper-expansiveness introduced by A. Artigue [2].

Definition 2. Let $f : X \to X$ be a homeomorphism on a compact metric space X. We say that its induced homeomorphism $2^f : 2^X \to 2^X$ is

(1) *countably expansive* if there is $\delta > 0$ such that for all $A \in 2^X$ the set $\Gamma_\delta^{2^f}(A)$ is countable;
(2) *continuum-wise expansive* if there is $\delta > 0$ such that if $C \in 2^X$ is a connected set with $\mathrm{diam}(2^{f^l}(C)) < \delta$ for all $l \in \mathbb{Z}$, then C is a singleton. Here $\mathrm{diam}(\cdot)$ is the diameter operator.

3. Main results

In this section we describe implications between various expansive homeomorphisms on a compact metric space and the corresponding induced maps on a hyperspace and give some examples to illustrate their implications. Furthermore we show that C^0-generically, any homeomorphism on a compact manifold is not hyper N-expansive for any $N \in \mathbb{N}$.

A. Artigue [2] obtained the following result from the expansiveness of the induced maps on a hyperspace.

Lemma 3.1 ([2, Theorem 2.2]). *A homeomorphism $f : X \to X$ on a compact metric space is hyper-expansive if and only if f has a finite number of orbits and $\Omega(f) = \mathrm{Per}_r \cup \mathrm{Per}_a$.*

Lemma 3.2 ([8, Proposition 8.3]). *Let $2^f : 2^X \to 2^X$ be the induced map by $f : X \to X$. Then a point A in 2^X is an isolated fixed point of 2^f if and only if each point of A is an isolated fixed point of f.*

Recall that a compact metric space X has topological dimension less than $n \in \mathbb{N}_0$ if for all $r > 0$ there exists an open covering $\mathcal{U}$ of X with diameter less than r such that every point belongs to at most $n+1$ sets of $\mathcal{U}$ [7]. We obtain the following result which adapted from Theorem 5.2 in [9].

Lemma 3.3 ([17, Theorem 5]). *If X is a continuum with $\dim X \geq 3$, then the topological dimension of $C(X)$ is infinite, i.e., $\dim C(X) = \infty$.*

Remark 3.4. It follows from [9, Theorem 5.3] that if $f_c : C(X) \to C(X)$ is a continuum-wise expansive homeomorphism, then $\dim C(X)$ is finite. From Lemma 3.3, we have $\dim X \leq 2$.

Also, M. Levin and Y. Sternfeld [10, Theorem 2.1] showed that if X is a continuum with $\dim X = 2$, then $\dim C(X) = \infty$. From [10, Theorem 2.1] we can improve Remark 3.4 as follows.

Corollary 3.5. *If the induced homeomorphism $f_c : C(X) \to C(X)$ is continuum-wise expansive, then $\dim X \leq 1$.*

Since any homeomorphism of a 0-dimensional compact metric space is always continuum-wise expansive (see [9]), we obtain the following result for induced maps 2^f and f_c.

Proposition 3.6. *If the topological dimension of a compact metric space X is zero, then two induced maps 2^f and f_c are continuum-wise expansive.*

Proof. It follows from [8, Proposition 8.6] that if $\dim X = 0$, then $\dim 2^X = 0$. From the totally disconnectedness of 2^X with $\dim 2^X = 0$ (see [1, Theorem 2.2.36]), two hyperspaces 2^X and $C(X)$ also are totally disconnected. Thus two induced homeomorphisms $2^f : 2^X \to 2^X$ and $f_c : C(X) \to C(X)$ by a homeomorphism $f : X \to X$ are continuum-wise expansive. □

We give an example of the induced homeomorphisms that f_c is continuum-wise expansive but 2^f is not expansive.

Example 3.7. Let $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ be the compact metric space of 0 and 1 with the metric d given by

$$d(x,y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}},\ x = (x_i), y = (y_i) \in \Sigma_2.$$

The shift map $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(x)_i = x_{i+1}$. Note that $\dim \Sigma_2 = \dim C(\Sigma_2) = 0$ (see [8, Proposition 8.6]) and Σ_2 is totally disconnected. Also $C(\Sigma_2)$ is a 0-dimensional compact metric space with Hausdorff metric d_H given by

$$d_H(A,B) = d(a,b) \text{ for all } A = \{a\}, B = \{b\} \in C(\Sigma_2).$$

Thus an induced homeomorphism $\sigma_c : C(\Sigma_2) \to C(\Sigma_2)$ given by $\sigma_c(\{a\}) = 2^{\sigma}(\{a\}) = \{\sigma(a)\}$ for every $\{a\} \in C(\Sigma_2)$ is continuum-wise expansive but $2^{\sigma} : 2^{\Sigma_2} \to 2^{\Sigma_2}$ is not expansive.

B. Carvalho and W. Cordeiro [6] showed that for given $N \in \mathbb{N}$ there is an N-expansive homeomorphism with the shadowing property defined on a compact metric space and discussed the dynamics of such N-expansive homeomorphisms. From the proof of [6, Theorem A], we obtain the following result.

Lemma 3.8. *Let $g : M \to M$ be an expansive homeomorphism on a compact metric space with metric d_0 containing an infinite number of periodic points $\{p_k\}_{k\in\mathbb{N}}$. Given $N \in \mathbb{N}$, there exist a compact metric space X with metric d and an N-expansive homeomorphism $f : X \to X$ such that $X = M \cup E$ and f is not $(N-1)$-expansive, where E splits into an infinite number of periodic orbits of f.*

Example 3.9 ([6]). Let $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ be the compact metric space of all bisequences of 0 and 1 with the metric d_0 given by

$$d_0(x,y) = \begin{cases} \frac{1}{2^n} & \text{if } \ n = \max\{k \in \mathbb{N} \mid x_i = y_i \text{ for all } |i| < k\}, \\ 0 & \text{if } \ x = y, \end{cases}$$

where $x = (x_i)$, $y = (y_i) \in \Sigma_2$. Then the shift map σ is an expansive homeomorphism.

For each $n \in \mathbb{N}$, choose a periodic point $p_n \in \Sigma_2$ with period $\pi(p_n) = n$, and let $E = \bigcup_{n\in\mathbb{N}} O_\sigma(p_n) \subset \Sigma_2$. Take a copy F of E such that $\Sigma_2 \cap F = \emptyset$. Let X as the set $\Sigma_2 \cup F$. Then F can be enumerated by a bijection $Q : E \to F$ which assigns an element $\sigma^k(p_n)$ of E to an element $Q(\sigma^k(p_n)) = q(n,k)$ in F, where $n \in \mathbb{N}$ and $0 \le k < \pi(p_n)$. Define a metric d on X by

$$d(x,y) = \begin{cases} 0 & \text{if } \ x = y, \\ d_0(x,y) & \text{if } \ x, y \in \Sigma_2, \\ \frac{1}{n} + d_0(x, \sigma^k(p_n)) & \text{if } \ x \in \Sigma_2, y = q(n,k) \in F, \\ \frac{1}{n} + \frac{1}{m} + d_0(\sigma^k(p_n), \sigma^l(p_m)) & \text{if } \ x = q(n,k), y = q(m,l) \in F. \end{cases}$$

Then (X,d) is a compact metric space. Consider a homeomorphism $f : X \to X$ given by

$$f(x) = \begin{cases} \sigma(x) & \text{if } \ x \in \Sigma_2, \\ q(n,k+1) & \text{if } \ x = q(n,k) \in F, \ 0 \le k \le n-2, \\ q(n,0) & \text{if } \ x = q(n,n-1) \in F. \end{cases}$$

Then we can see that f is 2-expansive but not expansive. From $\dim \Sigma_2 = 0$ we see that X is totally disconnected and $\dim X = 0$. Thus f is continuum-wise expansive (see [9]). Similarly, we easily see that $f_c : C(X) \to C(X)$ is continuum-wise expansive but not expansive.

Next, we give examples of expansive homeomorphisms f of compact metric spaces which their induced maps f_c and 2^f are not expansive.

Example 3.10. Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be the hyperbolic toral automorphism on the 3-dimensional torus $\mathbb{T}^3$. It easily see that f is expansive (see [15, p. 108]). So, f is continuum-wise expansive. It follows from Lemma 3.3 and Proposition 3.6 that f_c is not continuum-wise expansive since $\dim X > 2$. Hence f_c is not expansive.

Example 3.11. Let f be the identity homeomorphism on the Cantor set. Note that the topological dimension of the Cantor set is zero. From Proposition 3.6, the induced identity map 2^f is continuum-wise expansive. We easily show that 2^f is not expansive. Suppose that 2^f is expansive. Then every fixed point is an attractor or a repeller and 2^f has a finite number of orbits. But the identity homeomorphism f of Cantor set has infinitely many orbits. This is absurd.

From well known some results on hyper-dynamics and our examples given in this paper, we describe implications between expansiveness and continuum-wise (abbrev. cw) expansiveness for homeomorphisms and their induced maps via the below table.

$(2^X, 2^f)$	expansive	$\Rightarrow$ $\not\Leftarrow$ Example 3.11	cw-expansive
	$\Downarrow$ $\not\Uparrow$ Example 3.7		$\Downarrow$ $\not\Uparrow$?
$(C(X), f_c)$	expansive	$\Rightarrow$ $\not\Leftarrow$ Example 3.9	cw-expansive
	$\Downarrow$ $\not\Uparrow$ Example 3.10		$\Downarrow$ $\not\Uparrow$ Example 3.10
(X, f)	expansive	$\Rightarrow$ $\not\Leftarrow$ Example 3.9	cw-expansive

We comment the following: Does there exists a homeomorphism $f : X \to X$ that f_c is continuum-wise expansive but $2^f : 2^X \to 2^X$ is not continuum-wise expansive?

Now, we will prove implications between n-expansiveness and countable expansiveness for homeomorphisms and their induced maps via the below table.

$f^{<\omega}$ is expansive	$\Rightarrow$ $\not\Leftarrow$ Example 3.13	f is countably expansive
$\Downarrow\not\Uparrow$		$\Downarrow\not\Uparrow$
f_n is expansive	$\Rightarrow$ $\not\Leftarrow$ Example 3.13	f is n-expansive

Theorem 3.12. *Let $n \in \mathbb{N}$ be given and $f : X \to X$ be a homeomorphism on a compact metric space X. If each corresponding induced homeomorphism $f_n : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ on the n-fold symmetric product of X is expansive, then $f : X \to X$ is n-expansive.*

Proof. Suppose by contradiction that f is not n-expansive. Let $\delta > 0$ be an expansive constant for f_n. Since f is not n-expansive, f is not expansive as well. Then there exists $x \in X$ such that $\{y, z\} \subseteq \Gamma^f_\delta(x)$ for $y, z \in X$ with $y \neq z$. Let $A = \{y\}$ and $B = \{z\}$. Then we have $d_H(f^l_n(A), f^l_n(B)) \leq \delta$ for all

$l \in \mathbb{Z}$. It means that $A \in \Gamma_\delta^{f_n}(B)$. This contradicts the assumption that each f_n is expansive. □

The following example shows that the converse of Theorem 3.12 is not true in general.

Example 3.13 ([3]). Let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group with n elements. For each $m \in \mathbb{Z}$ and given $n \geq 2$, let $A_m = \{a_{m,i} \mid i \in \mathbb{Z}_n\} \subset \mathbb{R}^+$ be such that $A_m \cap A_{m'} = \emptyset$ if $m \neq m'$, and A_m converges to $\{0\}$ as $m \to \infty$ under the Hausdorff metric. Let

$$X = \{\infty\} \cup (\mathbb{Z} \times \{0\}) \cup \bigcup_{m \in \mathbb{N}} \{-m, \dots, m\} \times A_m$$

be a subspace of the sphere $\mathbb{R}^2 \cup \{\infty\}$. We define a homeomorphism f on X by

$$f(x) = \begin{cases} \infty & \text{if } \; x = \infty, \\ (m+1, 0) & \text{if } \; x = (m, 0), \\ (j + 1.a_{m,i}) & \text{if } \; x = (j, a_{m,i}), \\ (-m, a_{m,i+1}) & \text{if } \; x = (m, a_{m,i}), \\ (-m, a_{m,0}) & \text{if } \; x = (m, a_{m,n-1}), \end{cases}$$

where $-m \leq j < m, i \in \mathbb{Z}_n$ and $m \in \mathbb{N}$. Then $f : X \to X$ is n-expansive but each $f_n : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ is not expansive.

Proof. Suppose by contradiction that f_n is expansive with expansive constant $\delta > 0$. Take

$$A = \{(0, a_{m,k}), (0, a_{m+1,k}), \dots, (0, a_{m+n-1,k})\}$$

and

$$B = \{(0, a_{m,k+1}), (0, a_{m+1,k+1}), \dots, (0, a_{m+n-1,k+1})\}$$

for some $k \in \mathbb{Z}_n$. Since

$$d(f^l((0, a_{m+i,k})), f^l((0, a_{m+i,k+1}))) < \delta$$

for every $i \in \mathbb{Z}_n$ and $l \in \mathbb{Z}$, we see that

$$f^l(A) \subset B_d(f^l(B), \delta)$$

for every $l \in \mathbb{Z}$. Similarly, we get $f^l(B) \subset B_d(f^l(A), \delta)$ for every $l \in \mathbb{Z}$. Hence, $d_H(f^l(A), f^l(B)) < \delta$ for every $l \in \mathbb{Z}$. This is a contradiction. □

Corollary 3.14. *Let $f : X \to X$ be a homeomorphism on a compact metric space X. If its induced homeomorphism $f^{<\omega} : \mathcal{F}(X) \to \mathcal{F}(X)$ is expansive, then $f : X \to X$ is countably expansive.*

We can easily check that the converse of Corollary 3.14 is not true as we see in the previous Example 3.13.

A. Artigue [2, Theorem 2.2.] gave a characterization of a compact metric space admitting a hyper-expansive homeomorphism. We extend A. Artigue's results about hyper-expansive homeomorphisms to hyper N-expansive homeomorphisms on a compact manifold. Recall that a compact f-invariant set $K \subset X$ is said to be *minimal* if for all $x \in K$ the orbit $O_f(x)$ is dense in K. If X is minimal, then a homeomorphism $f : X \to X$ is called *minimal*.

Lemma 3.15. *Assume that for given $N \in \mathbb{N}$, $f : X \to X$ is a hyper N-expansive homeomorphism on a compact metric space X. If $K \subset X$ is minimal, then K is a periodic orbit (i.e., finite).*

Proof. Since K is minimal, then for every $\epsilon > 0$ there is $n = n(\epsilon) \in \mathbb{N}$ such that for all $x \in K$ the set $O_n(x) := \{x, f(x), \dots, f^n(x)\}$ is ϵ-dense in K. That is, for any $y \in K$ there is $j \in \{0, 1, \dots, n\}$ such that $d(y, f^j(x)) < \epsilon$ for each $x \in K$.

By contradiction we assume that there exists $\epsilon_0 > 0$ such that for every $n = n(\epsilon_0) \in \mathbb{N}$ there is $x_n \in K$ satisfying the set $\{x_n, f(x_n), \dots, f^n(x_n)\}$ is not ϵ_0-dense in K. That is, for every $n \in \mathbb{N}$ there is a sequence (y_n) of K such that

$$B_d(y_n, \epsilon_0) \cap \{x_n, f(x_n), \dots, f^n(x_n)\} = \emptyset. \tag{3.1}$$

From the compactness of K, we assume that $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Since the sequence (y_n) of K converges to y, there exists $N_0 \in \mathbb{N}$ such that $B_d(y, \frac{\epsilon_0}{2}) \subset B_d(y_n, \epsilon_0)$ for each $n \geq N_0$. In view of (3.1), we have that $\{x_n, f(x_n), \dots, f^n(x_n)\} \cap B_d(y, \frac{\epsilon_0}{2}) = \emptyset$ for each $n \geq N_0$. Choose a large positive integer m with $m > n$ such that $d(f^i(x), f^i(x_m)) < \frac{\epsilon_0}{4}$ for all $i = 0, 1, \dots, n$. Then we have that

$$\begin{aligned} d(y, f^i(x)) &\geq d(y, f^i(x_m)) - d(f^i(x_m), f^i(x)) \\ &\geq \frac{\epsilon_0}{2} - \frac{\epsilon_0}{4} = \frac{\epsilon_0}{4}, \ i = 0, 1, \dots, n. \end{aligned}$$

Consequently, $\{x, f(x), \dots, f^n(x)\} \cap B_d(y, \frac{\epsilon_0}{4}) = \emptyset$ for any $n \geq N_0$. So we have $O_f(x) \cap B_d(y, \frac{\epsilon_0}{4}) = \emptyset$ for some $y \in K$. Hence this is a contradiction to minimality of K.

Since $f^j(O_n(x)) = O_n(f^j(x))$ for fixed $j \in \mathbb{N}$ and $x \in K$, then $f^j(O_n(x))$ is ϵ-dense in K for each $j \in \mathbb{N}$. If ϵ is an N-expansive constant for 2^f, then $d_H(f^i(O_n(x)), f^i(K)) \leq \epsilon$ for every $i \in \mathbb{Z}$ and $x \in K$. Since K is an f-invariant set and $n \in \mathbb{N}$ depends on ϵ and K, we see that

$$d_H(f^i(O_n(x)), f^i(K)) = d_H(f^i(O_n(x)), K) \leq \epsilon \ \text{ for all } i \in \mathbb{Z}.$$

Again, for fixed $j \in \mathbb{Z}$ and every $x \in K$, we see that $d_H(f^i(O_n(f^j(x))), f^i(K)) \leq \epsilon$ for all $i \in \mathbb{Z}$. By triangle inequality, we get $d_H(f^i(O_n(f^j(x))), f^i(O_n(x))) \leq 2\epsilon$ for every $i \in \mathbb{Z}$. This means that $O_n(f^j(x)) \in \Gamma^{2^f}_{2\epsilon}(O_n(x))$. Since 2^f is N-expansive, $\Gamma^{2^f}_{2\epsilon}(O_n(x))$ contains at most N-elements. Since j is arbitrary for

$O_n(f^j(x))$, there exist $k, l \in \mathbb{N}$ (say $k < l$) such that $O_n(f^k(x)) = O_n(f^l(x))$ for each $x \in K$. From $f^{l+n}(x) \in O_n(f^l(x)) = O_n(f^k(x))$ for each $x \in K$, we obtain $f^{l+n}(x) = f^{k+i}(x)$ for some $i \in \{0, 1, \ldots, n\}$. But we note that $l+n \neq k+i$ for every $i \in \{0, 1, \ldots, n\}$. Therefore each point x of K is periodic and $K = O_n(x)$ is finite for some $n = n(\epsilon) \in \mathbb{N}$. This completes the proof. □

Notice that there exists a metric d on a compact manifold M. Next we study hyper N-expansive homeomorphisms on a compact manifold M in the C^0-generic sense. We denote by Homeo(M) the collection of all homeomorphisms on M endowed with C^0-topology. A property is said to be *generic* if it contains a dense G_δ subset. For instance, it is known that the shadowing property is generic for compact manifold ([16, Theorem 1]) or a Cantor set ([5, Theorem 4.3]). Recall that a sequence $\{x_n\}_{n\in\mathbb{Z}}$ is a *δ-pseudo orbit* ($\delta > 0$) of $f \in$ Homeo(M) if $d(f(x_n), x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$. We say that $f \in$ Homeo(M) has the *shadowing property* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$ there is a point $x \in M$ which ϵ-traces the δ-pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$, that is, $d(f^n(x), x_n) < \epsilon$ for all $n \in \mathbb{Z}$.

Lemma 3.16. *Given $N \in \mathbb{N}$, if a homeomorphism $f : M \to M$ on a compact manifold is N-expansive and has the shadowing property, then $CR(f) = \overline{\mathrm{Per}(f)}$.*

Proof. It is enough to show that $CR(f) \subset \overline{\mathrm{Per}(f)}$. Let U be a neighborhood of $x \in CR(f)$ and c be an N-expansive constant of f. It is sufficient to show that for any $\epsilon > 0$, there is a periodic point y_ϵ of f such that $d(y_\epsilon, x) < \epsilon$. We fix $0 < \epsilon < \frac{c}{2}$. By the shadowing property of f, there exists $\delta > 0$ such that any δ-pseudo orbit in M can be ϵ-shadowed. Since $x \in CR(f)$, there is a finite δ-chain $\{x_i\}_{i=0}^n$ from x to itself. We extend it to be a periodic δ-pseudo orbit by $x_{kn+i} = x_i$ for all $k \in \mathbb{Z}$ and $i \in \{0, \ldots, n-1\}$. By the shadowing property of f, there is $y_\epsilon \in X$ such that $d(f^i(y_\epsilon), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. Then we have

$$\begin{aligned} d(f^{kn+i}(y_\epsilon), f^i(y_\epsilon)) &\leq d(f^{kn+i}(y_\epsilon), x_{kn+i}) + d(x_i, f^i(y_\epsilon)) \\ &< \epsilon + \epsilon < c \quad \text{for all } i \in \mathbb{Z}. \end{aligned}$$

Then we derive that $f^{kn}(y_\epsilon) \in \Gamma_c^f(y_\epsilon)$ for all $k \in \mathbb{Z}$. Since f is N-expansive, there are distinct integers k and l such that $f^{kn}(y_\epsilon) = f^{ln}(y_\epsilon)$ and so $f^{(k-l)n}(y_\epsilon)$ $= y_\epsilon$. Moreover, we see that $d(y_\epsilon, x) < \epsilon$ and so $y_\epsilon \in \mathrm{Per}(f) \cap U$. Then we have $x \in \overline{\mathrm{Per}(f)}$. Consequently we have $CR(f) = \overline{\mathrm{Per}(f)}$. □

Lemma 3.17. *If f is a hyper N-expansive homeomorphism on a compact manifold, then $\sharp\mathrm{Per}(f)$ is finite.*

Proof. By contradiction, we assume that $\sharp\mathrm{Per}(f)$ is infinite. For any $p \in \mathrm{Per}(f)$, $O_f(p)$ is a fixed point of 2^f. We see that $\sharp\mathrm{Fix}(2^f)$ is infinite, which is a contradiction. □

Theorem 3.18. *C^0-generically, any homeomorphism on a compact manifold is not hyper N-expansive.*

Proof. Let $\mathcal{R}$ be the residual subset of Homeo(M) such that $f \in \mathcal{R}$ has the shadowing property. Suppose by contradiction that the homeomorphism $f \in$ Homeo(M) on a compact manifold M is hyper N-expansive. Then for every $x \in M$, $\omega(x)$ contains a fixed point. By Lemma 3.16, we obtain $CR(f) = \Omega(f) = \overline{\mathrm{Per}(f)}$. Since f is hyper N-expansive, then by Lemma 3.17 we have $\mathrm{Per}(f) = \overline{\mathrm{Per}(f)} = \Omega(f)$. We can assume $\mathrm{Fix}(f) = \mathrm{Per}(f)$. Then there exist $p, q \in \mathrm{Fix}(f)$ such that there is a sequence $\{x_n\}_{n=1}^{\infty} \subset M$ satisfying $\omega(x_n) = p, \alpha(x_n) = q$ and $O_f(x_n) \neq O_f(x_m)$ for $n \neq m$. Since 2^M is compact, there is a subsequence $\{n_k\}$ such that $\overline{O_f(x_{n_k})}$ converges to $K \in 2^M$. We observe that K is f-invariant. Then we derive that

$$d_H(2^f(\overline{O_f(x_{n_k})}), 2^f(K)) = d_H(\overline{O_f(x_{n_k})}, K) \to 0 \quad \text{as} \quad k \to \infty.$$

Let c' be an N-expansive constant of 2^f. Then there is $k' > 0$ such that $d_H(\overline{O_f(x_{n_k})}, K) < c'$ for all $k \geq k'$. Then we see that $\overline{O_f(x_{n_k})} \in \Gamma_{c'}^{2^f}(K)$ for all $k \geq k'$, which is a contradiction. $\square$

We give an example of a hyper 2^n-expansive homeomorphism on a compact metric space that is not hyper $(2^n - 1)$-expansive.

Example 3.19. Let $X = \{p\} \cup \{a_i \,|\, a_i \to p \text{ as } i \to \pm\infty\}$ be a countable compact metric subspace of $\mathbb{R}^2$ and a homeomorphism $f : X \to X$ be given by

$$f(x) = \begin{cases} x, & \text{if } x = p, \\ a_{i+1}, & \text{if } x = a_i,\ i \in \mathbb{Z}. \end{cases}$$

Then f is hyper 2^n-expansive but not hyper $(2^n - 1)$-expansive.

Proof. For simplicity, we assume that $n = 1$. We easily see that $\Omega(f) = \{p\} = \mathrm{Fix}(f)$. Also, f has two orbits, i.e., $X = \{p\} \cup O_f(a_0)$ for some $a_0 \in X$. Since the fixed point p of f is neither an attractor nor a repeller, then f is not hyper-expansive by Lemma 3.1. Now we will show that f is hyper 2-expansive. Let $\delta > 0$ be such that $B_d(a_0, \delta) = \{a_0\}$. Suppose that there are two compact distinct subsets A and B of 2^X such that $A \in \Gamma_\delta^{2^f}(B)$. Without loss of generality, we assume that there exists $a \in A \setminus B$. Now we claim that $a = p$. Suppose by contradiction that $a \neq p$, and then there is $i \in \mathbb{Z}$ such that $B_d(f^i(a), \delta) = \{f^i(a)\}$. Then $d_H(f^i(A), f^i(B)) \geq \delta$ for some $i \in \mathbb{Z}$ and we have a contradiction from condition $A \in \Gamma_\delta^{2^f}(B)$. Next, suppose $b \in B \setminus A$. By the same proof of above claim, we see that $b = p$. But it is impossible since $a = p \in A \setminus B$. This means $B \subset A$. Consequently, $A = B \cup \{p\}$. In other words, if $A \in \Gamma_\delta^{2^f}(B)$ and $A \neq B$, then $A = B \cup \{p\}$. It implies that $\sharp\Gamma_\delta^{2^f}(B) \leq 2$. This completes the proof. $\square$

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Namjip Koo
Department of Mathematics
Chungnam National University
Daejeon 34134, Korea
Email address: njkoo@cnu.ac.kr

Hyunhee Lee
Department of Mathematics
Chungnam National University
Daejeon 34134, Korea
Email address: avechee@cnu.ac.kr