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COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF AANA RANDOM VARIABLES AND ITS APPLICATION IN NONPARAMETRIC REGRESSION MODELS

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Abstract. In this paper, we main study the strong law of large numbers and complete convergence for weighted sums of asymptotically almost negatively associated (AANA, in short) random variables, by using the Marcinkiewicz-Zygmund type moment inequality and Roenthal type moment inequality for AANA random variables. As an application, the complete consistency for the weighted linear estimator of nonparametric regression models based on AANA errors is obtained. Finally, some numerical simulations are carried out to verify the validity of our theoretical result.

1. Introduction

The following concept of asymptotically almost negatively associated random variables was introduced by Chandra and Ghosal [4].

Definition 1.1. A sequence $\{X_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated (AANA, in short) if there exists a nonnegative sequence $q(n) \to 0$ as $n \to \infty$ such that

$$\begin{aligned} &Cov(f(X_n), g(X_{n+1}, X_{n+2}, \ldots, X_{n+k})) \\ &\leq q(n) \left[Var(f(X_n)) Var(g(X_{n+1}, X_{n+2}, \ldots, X_{n+k}))\right]^{1/2} \end{aligned}$$

for all $n,\ k \geq 1$ and for all coordinatewise nondecreasing functions f and g whenever the variances exist.

It is easily seen that the family of AANA sequence contains independent sequence and NA sequence (with $q(n) = 0,\ n \geq 1$) as special cases. Since the

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concept of AANA sequence was introduced by Chandra and Ghosal [4], many authors were devoted to studying the limiting behavior of sums of AANA random variables. For example, Chandra and Ghosal [5] derived the Kolmogorov type inequality and the strong law of large number of Marcinkiewicz-Zygmund, Yuan and An [27] established the Marcinkiewicz-Zygmund type inequality and some Rosenthal type inequality for maximum partial sums of AANA random variables, Baek [1] studied the almost sure convergence for AANA random variable sequences, Wang et al. [21] gave the strong law of large numbers, strong growth rate, and the integrability of supremum for the partial sums of AANA sequence, and studied the complete convergence for weighted sums of AANA sequence, Cai and Guo [3] obtained the complete convergence and Marcinkiewicz-Zygmund type strong laws of large numbers for an AANA sequence of random variables, Shen et al. [15] investigated the limiting behavior of the maximum partial sums and Chung-type strong law of large numbers for arrays of rowwise AANA variables, Wang et al. [22] studied the complete convergence and complete moment convergence of weighted sums for an array of rowwise of AANA random variables, Shen and Shi [16] studied L^r convergence for AANA random variables under some suitable conditions, Zhang and Lan [28] proved some different types of Rosenthal's inequalities for sub-additive expectations under the upper expectation space of AANA random variables and studied a strong law of large numbers as the application of Rosenthal's inequalities, Huang et al. [10] studied the complete moment convergence for weighted sums of arrays of rowwise AANA random variable, Xi et al. [26] established the L^p convergence and complete convergence for weighted sums of AANA random variables, and so forth.

The aim of this work is to further study the complete convergence for weighted sum of AANA random variables under some suitable conditions.

Now let us recall the concept of complete convergence, which was introduced by Hsu and Robbins [9] as follows.

Definition 1.2. A sequence $\{X_n, n \geq 1\}$ of random variables converges completely to a constant C if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty. \tag{1.1}$$

The converse is true if the $\{X_n, n \geq 1\}$ are independent. By the Borel-Cantelli lemma, (1.1) implies that $X_n \to C$ a.s. Hence, complete convergence is a strong concept than a.s. convergence.

The concept of stochastic domination below will play a significant role in this work.

Definition 1.3. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

An array $\{X_{nk}, k \geq 1, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exits a positive constant C such that

$$P(|X_{nk}| > x) \leq CP(|X| > x)$$

for all $x \geq 0$, $k \geq 1$ and $n \geq 1$.

Definition 1.4. A double array $\{a_{nk}, k \geq 1, n \geq 1\}$ of real numbers is said to be a Toeplitz array if $\lim_{n\to\infty} a_{nk} = 0$ for each $k \geq 1$ and

$$\sum_{k=1}^{\infty} |a_{nk}| \leq C$$

for all $n \geq 1$, where C is a positive constant.

Recently, Qiu and Yang [13] studied the strong law of large numbers for weighted sums of negatively associated (NA, for short) random variables and obtained the following theorem.

Theorem 1.1. *Suppose* $1/r = 1/\alpha + 1/\beta$ *for* $1 < \alpha, \beta < \infty$ *and* $1 < r < 2$. *Let* $\{X_n, n \geq 1\}$ *be a sequence of NA random variables with identical distribution, and let* $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ *be an array of constants satisfying*

$$A_\alpha \doteq \limsup_{n\to\infty} A_{\alpha,n} < \infty, \ A^\alpha_{\alpha,n} \doteq \sum_{k=1}^{n} |a_{nk}|^\alpha / n. \tag{1.2}$$

If $E|X_1|^\beta < \infty$ *and* $EX_1 = 0$, *then*

$$n^{-1/r} \sum_{k=1}^{n} a_{nk} X_k \to 0 \ a.s.$$

Baek et al. [2] extended the strong law of large numbers for NA to complete convergence and obtained the following result.

Theorem 1.2. *Let* $\{X_{nk}, k \geq 1, n \geq 1\}$ *be an array of rowwise NA random variables, which is stochastically dominated by a random variable* X. *Let* $\{a_{nk}, k \geq 1, n \geq 1\}$ *be an array of constants satisfying*

$$\sup_{k\geq 1} |a_{nk}| = O(n^{-\gamma}) \text{ for some } \gamma > 0, \tag{1.3}$$

and

$$\sum_{k=1}^{\infty} |a_{nk}| = O(n^\alpha) \text{ for some } \alpha < \gamma. \tag{1.4}$$

Suppose that there exists $\delta > 0$ *such that* $1 + \alpha/\gamma < \delta \leq 2$. *Let* $\beta \geq -1$ *and* $\nu = \max\{1 + (1 + \alpha + \beta)/\gamma, \delta\}$. *If* $EX_{nk} = 0$ *for all* $k \geq 1$ *and* $n \geq 1$, *and*

$$\begin{cases} E|X|\log|X| < \infty & \text{for } 1 + \alpha + \beta = 0, \\ E|X|^\nu < \infty & \text{for } 1 + \alpha + \beta > 0, \end{cases}$$

then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{k=1}^{\infty} a_{nk} X_{nk}\right| > \varepsilon\right) < \infty.$$

The main purpose of this paper is to study the strong law of large numbers for sequences of AANA random variables and obtain the complete convergence for weighted sums of AANA random variables under suitable moment conditions. As an application, the complete consistency for weighted estimator of nonparametric regression model is achieved.

2. Main results

In this section, we will present our main results. The first one is the strong law of large numbers for AANA random variables.

Theorem 2.1. *Suppose* $1/r = 1/\alpha + 1/\beta$ *for* $1 < \alpha, \beta < \infty$ *and* $1 < r < 2$. *Let* $\{X_n, n \geq 1\}$ *be a sequence of zero mean AANA random variables with* $\sum_{n=1}^{\infty} q^2(n) < \infty$, *which is stochastically dominated by a random variable* X. *Assume that* $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ *is an array of real numbers satisfying* (1.2). *If* $E|X|^{\beta} < \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leq m \leq n} \left|\sum_{k=1}^{m} a_{nk} X_k\right| > \varepsilon n^{1/r}\right) < \infty. \tag{2.1}$$

If $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ is replaced by $\{a_n, n \geq 1\}$, then we can get the following complete convergence and Marcinkiewicz-Zygmund type strong law of large numbers for AANA random variables.

Corollary 2.1. *Suppose* $1/r = 1/\alpha + 1/\beta$ *for* $1 < \alpha, \beta < \infty$ *and* $1 < r < 2$. *Let* $\{X_n, n \geq 1\}$ *be a sequence of zero mean AANA random variables with* $\sum_{n=1}^{\infty} q^2(n) < \infty$, *which is stochastically dominated by a random variable* X. *Assume that* $\{a_k, k \geq 1\}$ *is a sequent of real numbers satisfying* $\limsup_{n\to\infty} \sum_{k=1}^{n} |a_k|^{\alpha}/n < \infty$. *If* $E|X|^{\beta} < \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leq m \leq n} \left|\sum_{k=1}^{m} a_k X_k\right| > \varepsilon n^{1/r}\right) < \infty, \tag{2.2}$$

and thus

$$\lim_{n\to\infty} \frac{1}{n^{1/r}} \sum_{k=1}^{n} a_k X_k = 0 \ \ a.s. \tag{2.3}$$

The next one is the complete convergence for weighted sums of AANA random variables.

Theorem 2.2. *Suppose that* $\beta \geq -1$. *Let* $\{X_{nk}, k \geq 1, n \geq 1\}$ *be an array of rowwise AANA random variables with the mixing coefficients* $\{q(n), n \geq 1\}$, *which is stochastically dominated by a random variable* X. *Let* $\{a_{nk}, k \geq 1, n \geq 1\}$ *be an array of constants satisfying* (1.3) *and* (1.4).

(a) $1 + \alpha + \beta < 0$.

If $\sum_{n=1}^{\infty} q^2(n) < \infty$ *and* $E|X| < \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\sup_{m\geq 1}\left|\sum_{k=1}^{m} a_{nk}X_{nk}\right| > \varepsilon\right) < \infty. \tag{2.4}$$

(b) $1 + \alpha + \beta > 0$, $\beta > -1$.

Let $\nu = 1+(1+\alpha+\beta)/\gamma$ *and* $\sum_{n=1}^{\infty} q^{q/p}(n) < \infty$ *for some* $p \in (3\cdot 2^{k-1}, 4\cdot 2^{k-1}]$ *and* $p > \max\{\nu, \frac{2+2\beta}{\gamma-\alpha}, 2\}$, *where integer numbers* $k \geq 1$ *and* $1/p + 1/q = 1$. *If* $EX_{nk} = 0$ *for all* $k \geq 1$ *and* $n \geq 1$, *and* $E|X|^{\nu} < \infty$, *then* (2.4) *also holds for all* $\varepsilon > 0$.

(c) $1 + \alpha + \beta = 0$.

If $\sum_{n=1}^{\infty} q^2(n) < \infty$ *and* $E|X|\log|X| < \infty$, *then* (2.4) *also holds for all* $\varepsilon > 0$.

Corollary 2.2. *Let* $\{X_{nk}, k \geq 1, n \geq 1\}$ *be an array of rowwise AANA random variables with the mixing coefficients* $\{q(n), n \geq 1\}$, *which is stochastically dominated by a random variable* X. *Let* $\{a_{nk}, k \geq 1, n \geq 1\}$ *be a Toeplitz array satisfying*

$$\sup_{k\geq 1}|a_{nk}| = O\left(n^{1/r-\delta}\right) \quad \text{for some} \quad r > 0,\ \delta > 1/r. \tag{2.5}$$

(a) *If* $0 < r < 1$, $\sum_{n=1}^{\infty} q^2(n) < \infty$ *and* $E|X| < \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\sup_{m\geq 1}\left|\sum_{k=1}^{m} a_{nk}X_{nk}\right| > \varepsilon n^{1/r}\right) < \infty. \tag{2.6}$$

(b) *If* $r > 1$, $t = 1 + (1 - 1/r)/\delta$, *and* $\sum_{n=1}^{\infty} q^{q/p}(n) < \infty$ *for some* $p \in (3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}]$ *and* $p > \max\{t, 2\}$, *where integer numbers* $k \geq 1$ *and* $1/p+1/q = 1$, *then, under* $E|X|^t < \infty$, (2.6) *also holds for all* $\varepsilon > 0$.

(c) *If* $r = 1$, $\sum_{n=1}^{\infty} q^2(n) < \infty$ *and* $E|X|\log|X| < \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\sup_{m\geq 1}\left|\sum_{k=1}^{m} a_{nk}X_{nk}\right| > \varepsilon n\right) < \infty. \tag{2.7}$$

3. Preliminary lemmas

In order to prove the main results of the paper, we need the following important lemmas. The first one is a basic property for AANA random variables which was obtained by Yuan and An [27].

Lemma 3.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of AANA random variables with mixing coefficients* $\{q(n), n \geq 1\}$, *and let* $f_1, f_2, \ldots$ *be all nondecreasing* (*or all nonincreasing*) *continuous functions. Then* $\{f_n(X_n), n \geq 1\}$ *is still a sequence of AANA random variable with mixing coefficients* $\{q(n), n \geq 1\}$.

The next one is about the Marcinkiewicz-Zygmund type moment inequality and Roenthal type moment inequality for AANA random variables, which can be found in Yuan and An [27].

Lemma 3.2. *Let $p > 1$ and $\{X_n, n \geq 1\}$ be a sequence of zero mean AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$. If $1 < p \leq 2$ and $\sum_{n=1}^{\infty} q^2(n) < \infty$, then there exists a positive constant C_p depending only on p such that for all $n \geq 1$,*

$$E\left(\max_{1\leq m\leq n}\left|\sum_{k=1}^{m} X_k\right|^p\right) \leq C_p \sum_{k=1}^{n} E|X_k|^p.$$

If $\sum_{n=1}^{\infty} q^{q/p}(n) < \infty$ for some $p \in (3 \cdot 2^{j-1}, 4 \cdot 2^{j-1}]$, where integer numbers $j \geq 1$ and $1/p + 1/q = 1$, then there exists a positive constant D_p depending only on p such that for all $n \geq 1$,

$$E\left(\max_{1\leq m\leq n}\left|\sum_{k=1}^{m} X_k\right|^p\right) \leq C_p \left\{\sum_{k=1}^{n} E|X_k|^p + \left(\sum_{k=1}^{n} EX_k^2\right)^{p/2}\right\}.$$

By the definition of stochastic domination and integration by parts, one can get the following important property for stochastic domination. For the details of the proof, one can refer to Wu [24] or Shen [14] for instance.

Lemma 3.3. *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of random variables which is stochastically dominated by a random variable X. For any $a > 0$ and $b > 0$, the following three statements hold:*

$$\begin{aligned}
E|X_{nk}|^a I(|X_{nk}| \leq b) &\leq C_1[E|X|^a I(|X| \leq b) + b^a P(|X| > b)],\\
E|X_{nk}|^a I(|X_n| > b) &\leq C_2 E|X|^a I(|X| > b),\\
E|X_{nk}|^a &\leq C_3 E|X|^a,
\end{aligned}$$

where C_1 C_2 and C_3 are positive constants.

Throughout this paper, the symbols $C, C_1, C_2, \ldots$ represent positive constants which may vary in different places. Let $I(A)$ be the indicator function of the set A and $\lfloor x \rfloor$ be the integer part of x. Denote $X^+ = XI(X > 0)$ and $X^- = -XI(X < 0)$.

4. The proofs of main results

In this section, we will give the proofs of our main results as follows.

Proof of Theorem 2.1. Since $a_{nk} = a_{nk}^+ - a_{nk}^-$, without loss of generality, we assume that $a_{nk} \geq 0$. For fixed $n \geq 1$, denote for $1 \leq k \leq n$ that

$$\begin{aligned}
Y_{nk} = &-n^{1/r} I(a_{nk}X_k < -n^{1/r}) + a_{nk}X_k I(a_{nk}|X_k| \leq n^{1/r})\\
&+ n^{1/r} I(a_{nk}X_k > n^{1/r}),
\end{aligned}$$

$$Z_{nk} = a_{nk}X_k - Y_{nk} = (a_{nk}X_k + n^{1/r})I(a_{nk}X_k < -n^{1/r}) + (a_{nk}X_k - n^{1/r})I(a_{nk}X_k > n^{1/r}).$$

It is easy to check that

$$P\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m} a_{nk}X_k\right| > \varepsilon n^{1/r}\right) \le \sum_{k=1}^{n} P(a_{nk}|X_k| > n^{1/r}) + P\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m} Y_{nk}\right| > \varepsilon n^{1/r}\right). \tag{4.1}$$

Form (1.2), we may assume that $\sum_{k=1}^{n}|a_{nk}|^{\alpha} \le n$. Then by the Hölder's inequality, we can obtain

$$\sum_{k=1}^{n}|a_{nk}|^{\gamma} \le \left(\sum_{k=1}^{n}|a_{nk}|^{\gamma\frac{\alpha}{\gamma}}\right)^{\frac{\gamma}{\alpha}}\left(\sum_{k=1}^{n}1\right)^{\frac{\alpha-\gamma}{\alpha}} \le n \tag{4.2}$$

for any $0 < \gamma < \alpha$ and

$$\sum_{k=1}^{n}|a_{nk}|^{\gamma} = \sum_{k=1}^{n}|a_{nk}|^{\alpha}|a_{nk}|^{\gamma-\alpha} \le \sum_{k=1}^{n}|a_{nk}|^{\alpha}\left(\sum_{k=1}^{n}|a_{nk}|^{\alpha}\right)^{\frac{\gamma-\alpha}{\alpha}} \le n^{\frac{\gamma}{\alpha}} \tag{4.3}$$

for any $\gamma \ge \alpha$. By (4.2) and (4.3), we get $\sum_{k=1}^{n} a_{nk}^{\beta} \le n^{\max\{1,\beta/\alpha\}}$.

Firstly, we will show that

$$n^{-1/r}\max_{1\le m\le n}\left|\sum_{k=1}^{m} EY_{nk}\right| \to 0 \quad\text{as}\quad n\to\infty. \tag{4.4}$$

Noting that $|Z_{nk}| \le a_{nk}|X_k|I(a_{nk}|X_k| > n^{1/r})$, by $EX_k = 0$, Lemma 3.3, $\beta > r$ and $-\beta/r + \beta/\alpha = -1$, we get

$$\begin{aligned} n^{-1/r}\max_{1\le m\le n}\left|\sum_{k=1}^{m} EY_{nk}\right| &= n^{-1/r}\max_{1\le m\le n}\left|\sum_{k=1}^{m} EZ_{nk}\right| \\ &\le n^{-1/r}\sum_{k=1}^{n} a_{nk}E|X_k|I(a_{nk}|X_k| > n^{1/r}) \\ &\le n^{-\beta/r}\sum_{k=1}^{n} a_{nk}^{\beta}E|X|^{\beta}I(a_{nk}|X| > n^{1/r}) \\ &\le Cn^{-\beta/r+\max\{1,\beta/\alpha\}}E|X|^{\beta} \to 0 \quad\text{as}\quad n\to\infty, \end{aligned}$$

which implies (4.4).

Hence, to prove (2.1), it suffices to show that

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n} P(a_{nk}|X_k| > n^{1/r}) < \infty,$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}Y_{nk}-EY_{nk}\right|>\varepsilon n^{1/r}/2\right)<\infty.$$

By Definition 1.3, $E|X|^{\beta}<\infty$, $\sum_{k=1}^{n}|a_{nk}|^{\alpha}\le n$ and $1/r=1/\alpha+1/\beta$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}P(a_{nk}|X_k|>n^{1/r})&\le C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}P(|X|^{\alpha}>n^{\alpha/r}a_{nk}^{-\alpha})\\&\le C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}P\left(|X|^{\alpha}>n^{\alpha/r}\left(\sum_{k=1}^{n}|a_{nk}|^{\alpha}\right)^{-1}\right)\\&\le C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}P(|X|>n^{1/r-1/\alpha})\\&=C\sum_{n=1}^{\infty}P(|X|>n^{1/\beta})\le CE|X|^{\beta}<\infty.\end{aligned}\tag{4.5}$$

By Lemma 3.1, we can see that $\{Y_{nk}-EY_{nk},1\le k\le n,n\ge 1\}$ is still an array of rowwise AANA random variables. Then by the Markov's inequality, Lemmas 3.2 and 3.3, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{-1}P\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(Y_{nk}-EY_{nk})\right|>\varepsilon n^{1/r}/2\right)\\&\le C\sum_{n=1}^{\infty}n^{-1-2/r}E\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(Y_{nk}-EY_{nk})\right|^{2}\right)\\&\le C\sum_{n=1}^{\infty}n^{-1-2/r}\sum_{k=1}^{n}E|Y_{nk}|^{2}\\&\le C\sum_{n=1}^{\infty}n^{-1-2/r}\sum_{k=1}^{n}a_{nk}^{2}E|X_k|^{2}I(a_{nk}|X_k|\le n^{1/r})\\&\quad+C\sum_{n=1}^{\infty}n^{-1}\sum_{k=1}^{n}P(a_{nk}|X_k|>n^{1/r})\\&\le C\sum_{n=1}^{\infty}n^{-1-2/r}\sum_{k=1}^{n}a_{nk}^{2}E|X|^{2}I(a_{nk}|X|\le n^{1/r})\\&\quad+C\sum_{n=1}^{\infty}n^{-1}\sum_{k=1}^{n}P(a_{nk}|X|>n^{1/r}).\end{aligned}$$

From $1/r = 1/\alpha + 1/\beta$ and $1 < r < 2$, we know that $\alpha \le 2$ and $\beta \le 2$ will not hold simultaneously. Hence, we can obtain

$$
\begin{aligned}
(4.6)\quad & \sum_{n=1}^{\infty} n^{-1-2/r} \sum_{k=1}^{n} a_{nk}^2 E|X|^2 I(a_{nk}|X| \le n^{1/r}) \\
&\le \begin{cases} C \sum_{n=1}^{\infty} n^{-1-2/r+2/\alpha} E|X|^2, & \alpha \le 2 < \beta, \\ C \sum_{n=1}^{\infty} n^{-1-\beta/r} \sum_{k=1}^{n} a_{nk}^{\beta} E|X|^{\beta} I(a_{nk}|X| \le n^{1/r}), & \beta \le 2 < \alpha, \\ C \sum_{n=1}^{\infty} n^{-2/r} E|X|^2, & \beta > 2,\ \alpha > 2, \end{cases} \\
&\le \begin{cases} C \sum_{n=1}^{\infty} n^{-1-2/\beta} E|X|^2 < \infty, & \alpha \le 2 < \beta, \\ C \sum_{n=1}^{\infty} n^{-\beta/r} E|X|^{\beta} < \infty, & \beta \le 2 < \alpha, \\ C \sum_{n=1}^{\infty} n^{-2/r} E|X|^2 < \infty, & \beta > 2,\ \alpha > 2. \end{cases}
\end{aligned}
$$

Therefore, we can conclude that (2.1) holds by (4.5) and (4.6). The prove is completed. □

Proof of Corollary 2.1. Taking $a_{nk} \equiv a_k$ for $1 \le k \le n$ in Theorem 2.1, we obtain (2.2) immediately. We will prove (2.3). For all $\varepsilon > 0$, we have

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P\left\{ \max_{1 \le m \le n} \left| \sum_{k=1}^{m} a_k X_k \right| > \varepsilon n^{1/r} \right\} \\
&\ge \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{-1} P\left\{ \max_{1 \le m \le n} \left| \sum_{k=1}^{m} a_k X_k \right| > \varepsilon n^{1/r} \right\} \\
&\ge \frac{1}{2} \sum_{j=0}^{\infty} P\left\{ \max_{1 \le m \le 2^j} \left| \sum_{k=1}^{m} a_k X_k \right| > \varepsilon 2^{(j+1)/r} \right\}, \qquad (4.7)
\end{aligned}
$$

which together with Borel-Cantelli lemma yields that,

$$
(4.8)\qquad \lim_{j\to\infty} \frac{1}{2^{(j+1)/r}} \max_{1 \le m \le 2^j} \left| \sum_{k=1}^{m} a_k X_k \right| \to 0 \text{ a.s.}, \quad \text{as} \quad n \to \infty.
$$

Noting that for any $2^{j-1} \le n \le 2^j$, we have

$$
\begin{aligned}
\frac{1}{n^{1/r}} \left| \sum_{k=1}^{n} a_k X_k \right| &\le \frac{1}{2^{(j-1)/r}} \max_{1 \le m \le 2^j} \left| \sum_{k=1}^{m} a_k X_k \right| \\
&= \frac{2^{(j+1)/r}}{2^{(j-1)/r}} \frac{1}{2^{(j+1)/r}} \max_{1 \le m \le 2^j} \left| \sum_{k=1}^{m} a_k X_k \right|
\end{aligned}
$$

$$= 4^{1/r} \frac{1}{2^{(j+1)/r}} \max_{1 \le m \le 2^j} \left| \sum_{k=1}^{m} a_k X_k \right| \to 0 \text{ a.s.}, \quad \text{as} \quad n \to \infty,$$

which together with (4.8) yields that

$$\frac{1}{n^{1/r}} \sum_{k=1}^{n} a_k X_k \to 0 \text{ a.s.}, \quad \text{as} \quad n \to \infty.$$

The proof is completed. □

Proof of Theorem 2.2. Without loss of generality, we assume that $a_{nn} \ge 0$, $\sup_{k \ge 1} a_{nk} \le n^{-\gamma}$ and $\sum_{n=1}^{\infty} a_{nk} \le n^{\alpha}$.

(a) If $1 + \alpha + \beta < 0$, then the result can be easily proved by

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\beta} P\left(\sup_{m \ge 1} \left| \sum_{k=1}^{m} a_{nk} X_{nk} \right| > \varepsilon \right) &\le C \sum_{n=1}^{\infty} n^{\beta} E\left(\sup_{m \ge 1} \left| \sum_{k=1}^{m} a_{nk} X_{nk} \right| \right) \\ &\le C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E\left| a_{nk} X_{nk} \right| \\ &\le C \sum_{n=1}^{\infty} n^{\alpha+\beta} E\left| X \right| < \infty. \end{aligned}$$

In the following, we will prove the result for $1 + \alpha + \beta \ge 0$. For fixed $k \ge 1$ and $n \ge 1$, we define

$$X_{nk}^{(1)} = a_{nk} X_{nk} I(|a_{nk} X_{nk}| \le 1) + I(a_{nk} X_{nk} > 1) - I(a_{nk} X_{nk} < -1).$$

It is easy to check that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\beta} P\left(\sup_{m \ge 1} \left| \sum_{k=1}^{m} a_{nk} X_{nk} \right| > \varepsilon \right) \le & \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} P\left(|a_{nk} X_{nk}| > 1 \right) \\ & + \sum_{n=1}^{\infty} n^{\beta} P\left(\sup_{m \ge 1} \left| \sum_{k=1}^{m} X_{nk}^{(1)} \right| > \varepsilon \right) \\ \doteq & \, J_1 + J_2. \end{aligned}$$

Hence, in order to prove (2.4), it suffices to prove that $J_1 < \infty$ and $J_2 < \infty$.

(b) If $1 + \alpha + \beta > 0$, then by Markov's inequality, stochastic domination, (1.3) and (1.4), we obtain

$$\begin{aligned} J_1 &\le \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} P(|a_{nk} X| > 1) \\ &\le C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk} X| I(|a_{nk} X| > 1) \\ &\le C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk} X| I(|X| > |a_{nk}|^{-1}) \end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{\alpha+\beta} E|X|I(|X|>n^{\gamma})\\
&\le C\sum_{n=1}^{\infty} n^{\alpha+\beta}\sum_{j=n}^{\infty} E|X|I(j^{\gamma}<|X|\le (k+1)^{\gamma})\\
&\le CE|X|^{1+(1+\alpha+\beta)/\gamma}<\infty. \qquad (4.9)
\end{aligned}
$$

To prove $J_2<\infty$, we first show that

$$
\sup_{m\ge 1}\left|\sum_{k=1}^{m} EX_{nk}^{(1)}\right| \to 0 \quad \text{as} \quad n\to\infty. \tag{4.10}
$$

Noting that $E|X_{nk}|^{a}I(|X_{nk}|>b)\le CE|X|^{a}I(|X|>b)$ for any $a>0$ and $b>0$, we have by (1.3), (1.4), $EX_{nk}=0$, $E|X|^{\nu}<\infty$ and $\beta>-1$, that

$$
\begin{aligned}
\sup_{m\ge 1}\left|\sum_{k=1}^{m} EX_{nk}^{(1)}\right| &\le \sum_{k=1}^{\infty} P(|a_{nk}X_{nk}|>1)+\sum_{k=1}^{\infty} E|a_{nk}X_{nk}|I(|a_{nk}X_{nk}|>1)\\
&\le 2\sum_{k=1}^{\infty} E|a_{nk}X_{nk}|^{\nu}I(|a_{nk}X_{nk}|>1)\\
&\le C\sup_{k\ge 1} a_{nk}^{\nu-1}\sum_{k=1}^{\infty} a_{nk}E|X|^{\nu}I(|X|>a_{nk}^{-1})\\
&\le Cn^{-(\nu-1)\gamma}n^{\alpha}E|X|^{\nu}I(|X|>n^{\gamma})\\
&= Cn^{-(1+\beta)}E|X|^{\nu}I(|X|>n^{\gamma})\to 0 \quad \text{as} \quad n\to\infty, \qquad (4.11)
\end{aligned}
$$

which implies (4.10). Hence, to prove $J_2<\infty$, we only need to show that for all $\varepsilon>0$,

$$
J_2^{*} \doteq \sum_{n=1}^{\infty} n^{\beta} P\left(\sup_{m\ge 1}\left|\sum_{k=1}^{m}(X_{nk}^{(1)}-EX_{nk}^{(1)})\right|>\varepsilon\right)<\infty.
$$

By Lemma 3.1, we know that $\{X_{nk}^{(1)}-EX_{nk}^{(1)}, k\ge 1,\ n\ge 1\}$ is still an array of rowwise AANA random variables. Thus, by Markov's inequality and Lemma 3.2, we have for $p\ge 2$ that

$$
\begin{aligned}
J_2^{*} &\le C\sum_{n=1}^{\infty} n^{\beta} E\left(\sup_{m\ge 1}\left|\sum_{k=1}^{m}(X_{nk}^{(1)}-EX_{nk}^{(1)})\right|\right)^{p}\\
&\le C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty} E\left|X_{nk}^{(1)}\right|^{p}+C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{k=1}^{\infty} E|X_{nk}^{(1)}|^{2}\right)^{p/2}.
\end{aligned}
$$

Noting that

$$
p>\max(2,\frac{2+2\beta}{\gamma-\alpha}),
$$

we have $\beta-(1+\beta)p/2<-1$ and $\beta-(\gamma-\alpha)p/2<-1$.

When $1<\nu<2$, by Lemma 3.3, $E|X|^\nu<\infty$, (1.3) and (1.4), we obtain

$$\begin{aligned}\sum_{k=1}^\infty E|X_{nk}^{(1)}|^2 &\le \sum_{k=1}^\infty P(|a_{nk}X_{nk}|>1)+\sum_{k=1}^\infty E|a_{nk}X_{nk}|^2 I(|a_{nk}X_{nk}|\le 1)\\ &\le \sum_{k=1}^\infty P(|a_{nk}X|>1)+\sum_{k=1}^\infty E|a_{nk}X|^2 I(|a_{nk}X|\le 1)\\ &\le \sup_{k\ge 1}|a_{nk}|^{\nu-1}\sum_{k=1}^\infty |a_{nk}|E|X|^\nu \le Cn^{-(1+\beta)},\end{aligned}$$

which yields that

$$\sum_{n=1}^\infty n^\beta\left(\sum_{k=1}^\infty E|X_{nk}^{(1)}|^2\right)^{p/2}\le C\sum_{n=1}^\infty n^{\beta-(1+\beta)p/2}<\infty. \tag{4.12}$$

When $\nu\ge 2$, by Lemma 3.3, $E|X|^\nu<\infty$, (1.3) and (1.4), we obtain

$$\begin{aligned}\sum_{k=1}^\infty E|X_{nk}^{(1)}|^2 &\le \sum_{k=1}^\infty P(|a_{nk}X_{nk}|>1)+\sum_{k=1}^\infty E|a_{nk}X_{nk}|^2 I(|a_{nk}X_{nk}|\le 1)\\ &\le \sum_{k=1}^\infty P(|a_{nk}X|>1)+\sum_{k=1}^\infty E|a_{nk}X|^2 I(|a_{nk}X|\le 1)\\ &\le \sup_{k\ge 1}|a_{nk}|\sum_{k=1}^\infty |a_{nk}|E|X|^2 \le Cn^{-\gamma+\alpha},\end{aligned}$$

which together with $\beta-(\gamma-\alpha)p/2<-1$ yields that

$$\sum_{n=1}^\infty n^\beta\left(\sum_{k=1}^\infty E|X_{nk}^{(1)}|^2\right)^{p/2}\le C\sum_{n=1}^\infty n^{\beta-(\gamma-\alpha)p/2}<\infty. \tag{4.13}$$

On the other hand, by (4.9) and Lemma 3.3, we get that

$$\begin{aligned}&\sum_{n=1}^\infty n^\beta\sum_{k=1}^\infty E\left|X_{nk}^{(1)}\right|^p\\ &\le C\sum_{n=1}^\infty n^\beta\sum_{k=1}^\infty P(|a_{nk}X|>1)+C\sum_{n=1}^\infty n^\beta\sum_{k=1}^\infty E|a_{nk}X|^p I(|a_{nk}X|\le 1)\\ &\le C+C\sum_{n=1}^\infty n^\beta\sum_{k=1}^\infty E|a_{nk}X|^p I(|a_{nk}X|\le 1).\end{aligned}$$

Take $I_{ni}=\{k:(ni)^\gamma\le 1/a_{nk}<(n(i+1))^\gamma\}$, $i\ge 1$, $n\ge 1$. Then $\bigcup_{j=1}^\infty I_{nj}=N$, for all $n\ge 1$ from (1.3), where N is the set of positive integers. Hence,

$$\sum_{n=1}^\infty n^\beta\sum_{k=1}^\infty E|a_{nk}X|^p I(|a_{nk}X|\le 1)$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} \sum_{k\in I_{nj}} E|a_{nk}X|^p I(|a_{nk}X| \le 1) \\
&\le \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} (\sharp I_{nj})(nj)^{-\gamma p} E|X|^p I(|X| \le n(j+1)^{\gamma}) \\
&\le \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} (\sharp I_{nj})(nj)^{-\gamma p} \sum_{i=0}^{n(j+1)} E|X|^p I(i^{\gamma} < |X| \le (i+1)^{\gamma}) \\
&\le \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} (\sharp I_{nj})(nj)^{-\gamma p} \sum_{i=0}^{2n} E|X|^p I(i^{\gamma} < |X| \le (i+1)^{\gamma}) \\
&\quad + \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} (\sharp I_{nj})(nj)^{-\gamma p} \sum_{i=2n+1}^{n(j+1)} E|X|^p I(i^{\gamma} < |X| \le (i+1)^{\gamma}) \\
&\doteq J_3 + J_4.
\end{aligned}
$$

Noting that for all $m \ge 1$, we have

$$
\begin{aligned}
n^{\alpha} &\ge \sum_{k=1}^{\infty} |a_{nk}| = \sum_{j=1}^{\infty} \sum_{k\in I_{nj}} |a_{nk}| \ge \sum_{j=1}^{\infty} \sharp I_{nj} (n(j+1))^{-\gamma} \\
&\ge \sum_{j=m}^{\infty} \sharp I_{nj} (n(j+1))^{-\gamma} \ge \sum_{j=m}^{\infty} \sharp I_{nj} (n(j+1))^{-\gamma p} (n(m+1))^{\gamma(p-1)},
\end{aligned}
$$

which yields that $\sum_{j=m}^{\infty} \sharp I_{nj}(nj)^{-\gamma p} \le C n^{\alpha-\gamma(p-1)} m^{-\gamma(p-1)}$ for each $m \ge 1$.

Noting that $p > v = 1 + \frac{\alpha+\beta+1}{\gamma}$, we have

$$
\begin{aligned}
J_3 &\le \sum_{n=1}^{\infty} n^{\beta} n^{\alpha-\gamma(p-1)} \sum_{i=0}^{2n} E|X|^p I(i^{\gamma} < |X| \le (i+1)^{\gamma}) \\
&\le \sum_{i=1}^{2} \sum_{n=\lfloor i/2 \rfloor}^{\infty} n^{\beta+\alpha-\gamma(p-1)} E|X|^p I(i^{\gamma} < |X| \le (i+1)^{\gamma}) \\
&\quad + \sum_{i=2}^{\infty} \sum_{n=\lfloor i/2 \rfloor}^{\infty} n^{\beta+\alpha-\gamma(p-1)} E|X|^p I(i^{\gamma} < |X| \le (i+1)^{\gamma}) \\
&\le C + CE|X|^{1+\frac{\alpha+\beta+1}{\gamma}} < \infty, \qquad (4.14)
\end{aligned}
$$

and

$$
\begin{aligned}
J_4 &\le \sum_{n=1}^{\infty} n^{\beta} \sum_{i=2n+1}^{\infty} \sum_{j \ge \frac{i}{n}-1} (\sharp I_{nj})(nj)^{-\gamma p} E|X|^p I(i^{\gamma} < |X| \le (i+1)^{\gamma}) \\
&\le C \sum_{n=1}^{\infty} n^{\beta} \sum_{i=2n+1}^{\infty} n^{\alpha-\gamma(p-1)} \left(\frac{i}{n}\right)^{-\gamma(p-1)} E|X|^p I(i^{\gamma} < |X| \le (i+1)^{\gamma})
\end{aligned}
$$

$$\leq C\sum_{i=2}^{\infty} i^{-\gamma(p-1)} E|X|^p I(i^\gamma < |X| \leq (i+1)^\gamma) \sum_{n=1}^{\lfloor i/2 \rfloor} n^{\beta+\alpha}$$
$$\leq C\sum_{i=2}^{\infty} i^{\beta+\alpha+1-\gamma(p-1)} E|X|^p I(i^\gamma < |X| \leq (i+1)^\gamma)$$
$$\leq CE|X|^{1+\frac{\beta+\alpha+1}{\gamma}} < \infty. \quad (4.15)$$

Hence, when $\alpha+\beta+1>0$, (2.4) follows from (4.9)-(4.15) immediately.

Finally, we will prove (2.4) under the case $1+\alpha+\beta=0$. At first, to prove $J_1<\infty$, we have

$$\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} P(|a_{nk}X_{nk}|>1)$$
$$\leq C\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} P(|a_{nk}X|>1) \leq CE|X|\log(1+|X|) < \infty. \quad (4.16)$$

Note that $1+\beta=-\alpha\geq 0$ if $1+\alpha+\beta=0$. Similar to the proof of (4.11), we have $\nu=1$ that

$$\sup_{m\geq 1}\left|\sum_{k=1}^{m} EX_{nk}^{(1)}\right| \leq Cn^{-(1+\beta)}E|X|I(|X|>n^\gamma) \to 0 \quad \text{as} \quad n\to\infty. \quad (4.17)$$

Next, according to Lemma 3.3, we have by (4.16) and (4.17) that

$$\sum_{n=1}^{\infty} n^\beta P\left(\sup_{m\geq 1}\left|\sum_{k=1}^{m}(X_{nk}^{(1)}-EX_{nk}^{(1)})\right|>\varepsilon\right)$$
$$\leq C\sum_{n=1}^{\infty} n^\beta E\left(\sup_{m\geq 1}\left|\sum_{k=1}^{m}(X_{nk}^{(1)}-EX_{nk}^{(1)})\right|\right)^2$$
$$\leq C\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} E|X_{nk}^{(1)}|^2$$
$$\leq C\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} P(|a_{nk}X|>1) + C\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} E|a_{nk}X|^2 I(|a_{nk}X|\leq 1)$$
$$\leq C + J_3^* + J_4^*,$$

where J_3^* and J_4^* are J_3 and J_4 when $p=2$, respectively. Similar to the estimations of J_3 and J_4, we get

$$J_3^* \leq \sum_{n=1}^{\infty} n^{-1-\gamma} \sum_{i=0}^{2n} E|X|^2 I(i^\gamma < |X| \leq (i+1)^\gamma)$$
$$\leq \sum_{i=1}^{2} \sum_{n=\lfloor i/2 \rfloor}^{\infty} n^{-1-\gamma} E|X|^2 I(i^\gamma < |X| \leq (i+1)^\gamma)$$

$$+\sum_{i=2}^{\infty}\sum_{n=\lfloor i/2\rfloor}^{\infty} n^{-1-\gamma}E|X|^2 I(i^\gamma<|X|\le (i+1)^\gamma)$$
$$\le C+CE|X|<\infty,$$

and

$$\begin{aligned}J_4^* &\le \sum_{n=1}^{\infty} n^\beta \sum_{i=2n+1}^{\infty}\sum_{j\ge \frac{i}{n}-1}(\sharp I_{nj})(nj)^{-\gamma 2}E|X|^2 I(i^\gamma<|X|\le (i+1)^\gamma)\\ &\le C\sum_{n=1}^{\infty} n^\beta \sum_{i=2n+1}^{\infty} n^{\alpha-\gamma}\left(\frac{i}{n}\right)^{-\gamma} E|X|^2 I(i^\gamma<|X|\le (i+1)^\gamma)\\ &\le C\sum_{i=2}^{\infty} i^{-\gamma}E|X|^2 I(i^\gamma<|X|\le (i+1)^\gamma)\sum_{n=1}^{\lfloor i/2\rfloor} n^{\beta+\alpha}\\ &\le C\sum_{i=2}^{\infty}(\log i) i^{-\gamma}E|X|^2 I(i^\gamma<|X|\le (i+1)^\gamma)\\ &\le CE|X|\log|X|<\infty.\end{aligned}$$

This completes the proof of the theorem. □

Proof of Corollary 2.2. (a) For the case $0<r<1$, by Definition 1.4, the result can be easily proved by

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(\sup_{m\ge 1}\left|\sum_{k=1}^{m} a_{nk}X_{nk}\right|>\varepsilon n^{1/r}\right) &\le C\sum_{n=1}^{\infty} n^{-1/r}E\left(\sup_{m\ge 1}\left|\sum_{k=1}^{m} a_{nk}X_{nk}\right|\right)\\ &\le C\sum_{n=1}^{\infty} n^{-1/r}\sum_{k=1}^{\infty} E\left|a_{nk}X_{nk}\right|\\ &\le C\sum_{n=1}^{\infty} n^{-1/r}E\left|X\right|<\infty.\end{aligned}$$

(b) For the case $r>1$, we let $b_{nk}=a_{nk}n^{-1/r}$. Observe that

$$\sup_{m\ge 1}|b_{nk}|=O(n^{-\delta}),\ \sum_{k=1}^{\infty}|b_{nk}|=O(n^{-1/r}). \tag{4.18}$$

Hence, to prove (2.6), it suffices to show that

$$\sum_{n=1}^{\infty} P\left(\sup_{m\ge 1}\left|\sum_{k=1}^{m} b_{nk}X_{nk}\right|>\varepsilon\right)<\infty.$$

We will apply Theorem 2.2 with $\alpha=-1/r$, $\beta=0$ and $\gamma=\delta$ such that $1+\alpha+\beta=1-1/r>0$. For fixed $k\ge 1$ and $n\ge 1$, we define

$$Z_{nk}^{(1)}=b_{nk}X_{nk}I(|b_{nk}X_{nk}|\le 1)+I(b_{nk}X_{nk}>1)-I(b_{nk}X_{nk}<-1).$$

According to the proof of Theorem 2.2, it suffices to show

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} P\left(|b_{nk}X_{nk}|>1\right)<\infty, \tag{4.19}$$

and

$$\sum_{n=1}^{\infty} P\left(\sup_{m\geq 1}\left|\sum_{k=1}^{m} Z_{nk}^{(1)}\right|>\varepsilon\right)<\infty. \tag{4.20}$$

Noting that $E|X|^{1+(1-1/r)/\delta}<\infty$, it follows by the definition of stochastic domination and (4.18) that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} P(|b_{nk}X_{nk}|>1) &\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} E|b_{nk}X|I(|b_{nk}X|>1)\\ &\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |b_{nk}|E|X|I(|X|>|b_{nk}|^{-1})\\ &\leq C\sum_{n=1}^{\infty} n^{-1/r}E|X|I(|X|>n^{\delta})\\ &\leq C\sum_{n=1}^{\infty} n^{-1/r}\sum_{k=n}^{\infty} E|X|I(k^{\delta}\leq |X|<(k+1)^{\delta})\\ &\leq CE|X|^{1+(1-1/r)/\delta}<\infty.\end{aligned}$$

Next, we will show that

$$\sup_{m\geq 1}\left|\sum_{k=1}^{m} EZ_{nk}^{(1)}\right|\to 0 \quad \text{as} \quad n\to\infty. \tag{4.21}$$

It follows by $EX_{nk}=0$, Lemma 3.3 and Markov's inequality that

$$\begin{aligned}\sup_{m\geq 1}\left|\sum_{k=1}^{m} EZ_{nk}^{(1)}\right| &\leq \sum_{k=1}^{\infty} E|b_{nk}X_{nk}|I(|b_{nk}X_{nk}|>1)+\sum_{k=1}^{\infty} P(|b_{nk}X_{nk}|>1)\\ &\leq C\sum_{k=1}^{\infty} E|b_{nk}X|I(|b_{nk}X|>1)+C\sum_{k=1}^{\infty} P(|b_{nk}X|>1)\\ &\leq C\sum_{k=1}^{\infty} |b_{nk}|E|X|\leq Cn^{-1/r}\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

which implies (4.21). Thus, we can see that (4.20) still holds.

(c) Let $b_{nk}=a_{nk}n^{-1}$. By the condition (2.5), we have $\sup_{k\geq 1}|b_{nk}|=O(n^{-\delta})$ for some $\delta>1$, and $\sum_{k=1}^{\infty}|b_{nk}|=O(n^{-1})$. Hence, the result (2.4) follows by Theorem 2.2 with $\alpha=-1$, $\beta=0$, $\gamma=\delta$ and a_{nk} replaced by b_{nk}. Therefore, the desired result (2.7) follows from Theorem 2.2(c) immediately. This completes the proof of the corollary. □

5. An application to a nonparametric regression model

In the section, we will apply the result of Corollary 2.2 to a nonparametric regression model and investigate the complete convergence for the nonparametric regression estimator based on AANA errors.

Consider the following nonparametric regression model:

$$Y_{nk} = g(x_{nk}) + \varepsilon_{nk},\ k = 1, 2, \ldots, n,\ n \geq 1, \tag{5.1}$$

where x_{nk} are known fixed design points from A, where $A \subset \mathbb{R}^m$ is a given compact set for some $m \geq 1$, $g(\cdot)$ is an unknown regression function on A and ε_{nk} are random errors. As an estimator of $g(\cdot)$, the following weighted regression estimator will be considered:

$$g_n(x) = \sum_{k=1}^{n} W_{nk}(x) Y_{nk}, \tag{5.2}$$

where $W_{nk}(x) = W_{nk}(x, x_{n1}, \ldots, x_{nn})$, $k = 1, 2, \ldots, n$ are the weight functions.

The above estimator was first proposed by Stone [18] and next adapted by Georgiev (2005) to the fixed design case. Since then, many interesting results for the weighted estimator were provided. See for example, when ε_{nk} are assumed to be independent, consistency and asymptotic normality have been investigated by Georgiev and Greblicki [8], Georgiev [7], Müller [12] among others. Results for the case when ε_{nk} are dependent have also been studied by many authors in recent years. One can refer to Fan [6], Roussas [14], Tran et al. [19], Shen et al. [17], Wang et al. [20, 23], Wu et al. [25] among others. The main purpose of this paper is to further investigate the complete consistency of the estimator $g_n(x)$ based on AANA errors by using the complete convergence that we have obtained in Section 2.

Unless otherwise specified, we assume throughout the paper that $g_n(x)$ is defined by (5.2). For any function $g(x)$, we use $c(g)$ to denote all continuity points of the function g on A. The norm $\|x\|$ is the Euclidean norm. For any fixed design point $x \in A$, the following assumptions on weight functions $W_{nk}(x) = W_{nk}(x, x_{n1}, \ldots, x_{nn})$ will be used:

(H_1) $\left|\sum_{k=1}^{n} W_{nk}(x) - 1\right| \to 0$ as $n \to \infty$;

(H_2) $\sum_{k=1}^{n} |W_{nk}(x)| \leq C$ for all $n \geq 1$;

(H_3) $\sum_{k=1}^{n} |W_{nk}(x)| \cdot |g(x_{nk}) - g(x)| I(||x_{nk} - x|| > a) \to \infty$ as $n \to \infty$, for all $a > 0$.

Based on the assumptions above, we can get the following results on complete consistency for the nonparametric regression estimator $g_n(x)$.

Theorem 5.1. *Let $r > 1$ and $1/r < \delta < 1$, $\{\varepsilon_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise zero mean AANA random variables with mixing coefficients $\{q(n),\ n \geq 1\}$ which is stochastically dominated by a random variable*

X with $E|X|^{1+(1-1/r)/\delta} < \infty$. Assume that conditions (H_1)–(H_3) hold and $\sum_{n=1}^{\infty} q^{q/p}(n) < \infty$ for some $p \in (3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}]$, where integer numbers $k \geq 1$ and $1/p + 1/q = 1$. If

$$\sup_{k\geq 1} |W_{nk}(x)| = O(n^{-\delta}), \tag{5.3}$$

then for any $x \in c(g)$,

$$g_n(x) \to g(x) \quad \textit{completely.} \tag{5.4}$$

Proof. For $x \in c(g)$, we have by (5.1) and (5.2) that

$$\begin{aligned} |Eg_n(x) - g(x)| \leq & \sum_{k=1}^{n} |W_{nk}(x)| \cdot |g(x_{nk}) - g(x)| \, I(\|x_{nk} - x\| \leq a) \\ & + \sum_{k=1}^{n} |W_{nk}(x)| \cdot |g(x_{nk}) - g(x)| \, I(\|x_{nk} - x\| > a) \\ & + |g(x)| \left| \sum_{k=1}^{n} W_{nk}(x) - 1 \right|. \end{aligned} \tag{5.5}$$

Since $x \in c(g)$, for any $\varepsilon > 0$, there exists a $\eta > 0$ such that $|g(x^{'}) - g(x)| < \varepsilon$ when $\|x^{'} - x\| < \eta$. Thus, by setting $0 < a < \eta$ in (5.5), we can get that

$$\begin{aligned} |Eg_n(x) - g(x)| \leq & \, \varepsilon \sum_{k=1}^{n} |W_{nk}(x)| + |g(x)| \left| \sum_{k=1}^{n} W_{nk}(x) - 1 \right| \\ & + \sum_{k=1}^{n} |W_{nk}(x)| \cdot |g(x_{nk}) - g(x)| \, I(\|x_{nk} - x\| > a). \end{aligned}$$

Therefore, we have by conditions (H_1)–(H_3) that

$$\lim_{n\to\infty} Eg_n(x) = g(x), \; x \in c(g). \tag{5.6}$$

For fixed design point $x \in c(g)$, without loss of generality, we assume that $W_{nk}(x) \geq 0$. By (5.6), we can see that in order to prove (5.4), we only need to show that

$$g_n(x) - Eg_n(x) = \sum_{k=1}^{n} W_{nk}(x)\varepsilon_{nk} \to 0, \text{ completely.} \tag{5.7}$$

That is to say, it suffices to show that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left| \sum_{k=1}^{n} W_{nk}(x)\varepsilon_{nk} \right| > \varepsilon \right) < \infty.$$

We will apply Corollary 2.2 with $a_{nk} = W_{nk} n^{1/r}$. By Corollary 2.2(b), we have

$$\sum_{n=1}^{\infty} P\left(\sup_{m\geq} \left| \sum_{k=1}^{m} W_{nk}(x)\varepsilon_{nk} \right| > \varepsilon \right) < \infty. \tag{5.8}$$

The proof is completed. □

As an application of Theorem 5.1, we give the complete consistency for the nearest neighbor estimator of $g(x)$. Without loss of generality, let $A = [0,1]$ and $x_{nk} = \frac{k}{n}$, $k = 1, 2, \ldots, n$. For any $x \in A$, we rewrite $|x_{n1} - x|$, $|x_{n2} - x|, \ldots, |x_{nn} - x|$ as follows:

$$\left|x_{T_1(x)}^{(x)} - x\right| \leq \left|x_{T_2(x)}^{(x)} - x\right| \leq \cdots \leq \left|x_{T_n(x)}^{(x)} - x\right|,$$

if $|x_{nk} - x| = |x_{nm} - x|$, then $|x_{nk} - x|$ is permuted before $|x_{nm} - x|$ when $x_{nk} < x_{nm}$.

Let $1 \leq k_n \leq n$, the nearest neighbor weight function estimator of $g(x)$ in model (5.1) is defined as follows:

$$\tilde{g}_n(x) = \sum_{k=1}^{n} \tilde{W}_{nk}(x) Y_{nk}, \tag{5.9}$$

where

$$\tilde{W}_{nk}(x) = \begin{cases} 1/k_n, & |x_{nk} - x| \leq |x_{T_{k_n}(x)}^{(n)} - x|, \\ 0, & \text{otherwise.} \end{cases} \tag{5.10}$$

Corollary 5.1. *Let $\{\varepsilon_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise zero mean AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$ such that $\sum_{n=1}^{\infty} q^{q/p}(n) < \infty$ for some $p \in (3\cdot 2^{k-1}, 4\cdot 2^{k-1}]$, where integer numbers $k \geq 1$ and $1/p + 1/q = 1$, which is stochastically dominated by a random variable X. Assume that conditions (H_1)–(H_3) hold and $g(x)$ is continuous on the compact set A. If there exist some $r > 1$ and $\delta \in (1/r, 1)$ such that $E|X|^{1+(1-1/r)/\delta} < \infty$ and $k_n = \lfloor n^\delta \rfloor$, then for any $x \in A$ and any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} P\left(|\tilde{g}_n(x) - g(x)| > \varepsilon\right) < \infty. \tag{5.11}$$

Proof. For any $x \in [0,1]$, if follows from the definitions of $T_k(x)$ and $\tilde{W}_{nk}(x)$ that

$$\sum_{k=1}^{n} \tilde{W}_{nk}(x) = \sum_{k=1}^{n} \tilde{W}_{nT_k(x)}(x) = \sum_{k=1}^{k_n} \frac{1}{k_n} = 1,$$
$$\sup_{k\geq 1} \tilde{W}_{nk}(x) = \frac{1}{k_n} \leq C n^{-\delta}, \ \tilde{W}_{nk}(x) \geq 0,$$

and

$$\sum_{k=1}^{n} \left|\tilde{W}_{nk}(x)\right| \cdot |g(x_{nk}) - g(x)| \, I(\|x_{nk} - x\| > a)$$
$$\leq C \sum_{k=1}^{n} \left|\tilde{W}_{nk}(x)\right| \cdot \frac{|x_{nk} - x|^2}{a^2}$$

$$\leq C\sum_{k=1}^{k_n}\frac{1}{k_n}\cdot\frac{|x^{(n)}_{T_k(x)}-x|^2}{a^2}$$
$$\leq C\left(\frac{k_n}{na}\right)^2\to 0 \quad\text{as}\quad n\to\infty.$$

Hence, conditions (H_1)–(H_3) and (5.3) are satisfied. The desired result (5.11) follows from Theorem 5.1 immediately. The proof is completed. □

6. Simulations

In this section, we will present some simulations to study the numerical performance of the consistency for the nearest neighbor weight function estimator $\tilde{g}_n(x)$ in nonparametric regression model. The data are generated from model (5.1).

For any fixed $n\geq 3$, let normal random vector $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\sim N_n(\mathbf{0},\mathbf{\Sigma})$, where $\mathbf{0}$ represents zero vector and

$$\mathbf{\Sigma}=\begin{bmatrix} 1+\theta^2 & -\theta & 0 & \cdots & 0 & 0 & 0\\ -\theta & 1+\theta^2 & -\theta & \cdots & 0 & 0 & 0\\ 0 & -\theta & 1+\theta^2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1+\theta^2 & -\theta & 0\\ 0 & 0 & 0 & \cdots & -\theta & 1+\theta^2 & -\theta\\ 0 & 0 & 0 & \cdots & 0 & -\theta & 1+\theta^2 \end{bmatrix}_{n\times n},$$

where $0<\theta<1$. From Joag-Dev and Proschan [11] one can see that $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)$ is a NA vector for each $n\geq 3$ with finite moment of any order, and thus an AANA vector with $q(n)=0$. Let $\theta=0.6$ and $k_n=\lfloor n^{2/3}\rfloor$. It is easy to show that the conditions of Corollary 5.1 are all satisfied. Take the points $x=0.3,0.6,0.9$ and the sample sizes n as $n=100,200,300,400$, respectively. We use R software to compute $\tilde{g}_n(x)-g(x)$ with $g(x)=x-\sin x$ and $g(x)=1-\cos x$, respectively, for 1000 times and obtain the boxplots of $\tilde{g}_n(x)-g(x)$ in Figures 1–6 and the Mean Square Error (MSE, in short) of $\tilde{g}_n(x)$ in Table 1.

Table 1. MSE of the estimator $\tilde{g}_n(x)$

$g(x)$	x	$n=100$	$n=200$	$n=300$	$n=400$
	0.3	0.01039417	0.005427781	0.004417772	0.003194583
$x-\sin x$	0.6	0.009586905	0.005739558	0.004255095	0.003317827
	0.9	0.01118775	0.005605034	0.004478383	0.003278699
	0.3	0.01022655	0.00571078	0.004629933	0.003430464
$1-\cos x$	0.6	0.0102188	0.006019389	0.004525906	0.003341986
	0.9	0.01043029	0.005811146	0.004347353	0.003464602

Figures 1–3 are the boxplots of $\tilde{g}_n(x)-g(x)$ for $g(x)=x-\sin x$ and Figures 4–6 are the boxplots of $\tilde{g}_n(x)-g(x)$ for $g(x)=1-\cos x$ with $x=0.3,0.6,0.9$,

respectively. We can see that no matter $g(x) = x - \sin x$ or $g(x) = 1 - \cos x$, with $x = 0.3, 0.6, 0.9$, the differences $\tilde{g}_n(x) - g(x)$ fluctuate to zero and the variation ranges decrease as the sample n increases. To be more specific, from Table 1 we can see that the MSE of $\tilde{g}_n(x)$ decrease as n increases.

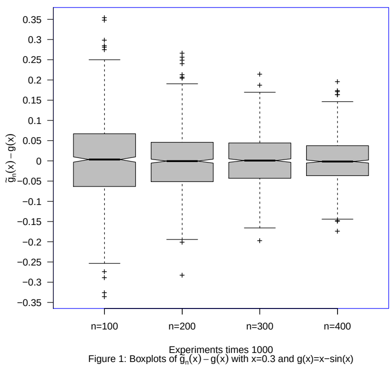

Figure 1: Boxplots of $\tilde{g}_n(x)-g(x)$ with x=0.3 and g(x)=x−sin(x)

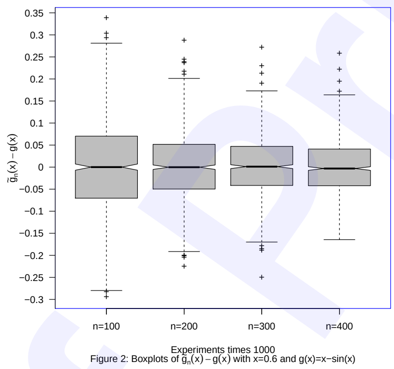

Figure 2: Boxplots of $\tilde{g}_n(x)-g(x)$ with x=0.6 and g(x)=x−sin(x)

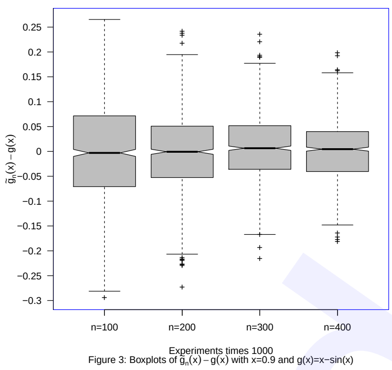

Figure 3: Boxplots of $\tilde{g}_n(x)-g(x)$ with x=0.9 and g(x)=x−sin(x)

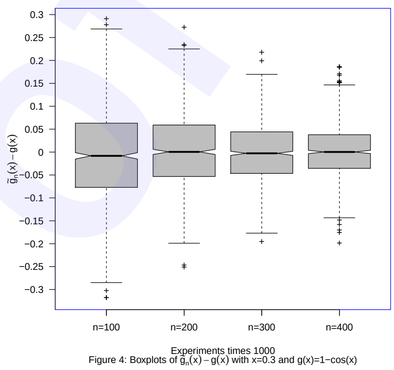

Figure 4: Boxplots of $\tilde{g}_n(x)-g(x)$ with x=0.3 and g(x)=1−cos(x)

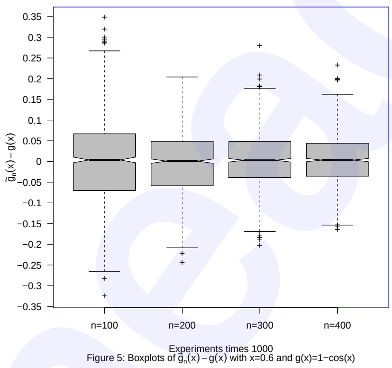

Figure 5: Boxplots of $\tilde{g}_n(x)-g(x)$ with x=0.6 and g(x)=1−cos(x)

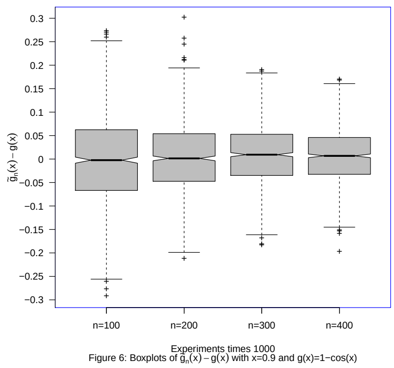

Figure 6: Boxplots of $\tilde{g}_n(x)-g(x)$ with x=0.9 and g(x)=1−cos(x)

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