

PARAMETER DEPENDENCE OF SMOOTH STABLE MANIFOLDS

LUIS BARREIRA AND CLAUDIA VALLS

ABSTRACT. We establish the existence of C^1 stable invariant manifolds for differential equations $u' = A(t)u + f(t, u, \lambda)$ obtained from sufficiently small C^1 perturbations of a *nonuniform* exponential dichotomy. Since any linear equation with nonzero Lyapunov exponents has a nonuniform exponential dichotomy, this is a very general assumption. We also establish the C^1 dependence of the stable manifolds on the parameter λ . We emphasize that our results are optimal, in the sense that the invariant manifolds are as regular as the vector field. We use the fiber contraction principle to establish the smoothness of the invariant manifolds. In addition, we can also consider linear perturbations, and thus our results can be readily applied to the robustness problem of nonuniform exponential dichotomies.

1. Introduction

The existence of an exponential dichotomy for a linear equation

$$(1) \quad u' = A(t)u$$

ensures the existence of stable and unstable invariant manifolds under sufficiently small perturbations. More generally, we can consider *nonuniform* exponential dichotomies. It turns out that the classical notion of (uniform) exponential dichotomy is very stringent for the dynamics and it is of interest to look for more general types of hyperbolic behavior. These generalizations can be much more typical. For example, almost all linear variational equations with nonzero Lyapunov exponents obtained from a measure-preserving flow have a nonuniform exponential dichotomy. We refer to [1] for a related discussion. Moreover, it is easy to show that if an autonomous linear equation has a nonuniform exponential dichotomy, then in fact the dichotomy must be uniform. This is why in the study of nonuniform exponential behavior we are only interested in perturbations of nonautonomous linear differential equations.

Received July 3, 2018; Accepted October 12, 2018.

2010 *Mathematics Subject Classification*. Primary 37D10, 37D25, 34D99.

Key words and phrases. growth rates, nonuniform exponential dichotomies, parameter dependence, stable manifolds.

Supported by FCT/Portugal through UID/MAT/04459/2013.

Our main objective is to show that the stable invariant manifolds for perturbations of equation (1) are of class C^1 provided that the vector field is of class C^1 , thus establishing their optimal regularity in the nonuniform setting. More precisely, we consider the perturbed equation

$$(2) \quad u' = A(t)u + f(t, u, \lambda),$$

where A and f are C^1 functions. We assume that $f(t, 0, \lambda) = 0$ for every $t \geq 0$, $u \in X = \mathbb{R}^p$ with $\|u\| \geq c$, and λ in an open ball $Y \subset \mathbb{R}^q$ for some constant $c > 0$.

The following is a consequence of our stable manifold theorem.

Theorem 1. *If equation (1) has a nonuniform exponential dichotomy, and*

$$\left\| \frac{\partial f}{\partial u}(t, u, \lambda) \right\| \leq \kappa e^{-t/\kappa} \quad \text{and} \quad \left\| \frac{\partial f}{\partial \lambda}(t, u, \lambda) \right\| \leq \kappa e^{-t/\kappa} \|u\|$$

for every $t \geq 0$, $u \in X$, and $\lambda \in Y$, for some sufficiently small $\kappa > 0$, then for each $\lambda \in Y$ the zero solution of equation (2) has a C^1 stable invariant manifold \mathcal{V}_λ . In addition, the map $\lambda \mapsto \mathcal{V}_\lambda$ is of class C^1 .

Reversing time we can also obtain invariant unstable manifolds with optimal regularity. The invariance of the stable manifolds means that for each $\lambda \in Y$ the set \mathcal{V}_λ is invariant under the flow defined by the autonomous equation

$$t' = 1, \quad u' = A(t)u + f(t, u, \lambda).$$

More generally, we also consider linear equations (1) that may exhibit stable and unstable behavior with respect to arbitrary growth rates $e^{c\rho(t)}$ determined by a function $\rho(t)$. We note that the usual exponential behavior with $\rho(t) = t$ is included as a very special case. These arbitrary growth rates include for example situations in which all Lyapunov exponents of equation (1) are infinite (either $+\infty$ or $-\infty$).

Our results are also a contribution to the theory of nonuniform hyperbolicity. We refer to [1] for a detailed exposition of the theory, which goes back to the landmark works of Oseledets [6] and particularly Pesin [7, 8]. Among the most important properties due to nonuniform hyperbolicity is precisely the existence of stable and unstable invariant manifolds, established by Pesin in [7]. In [10] Ruelle obtained a proof of the stable manifold theorem based on the study of perturbations of products of matrices in Oseledets' multiplicative ergodic theorem [6]. Another proof was given by Pugh and Shub in [9] using graph transform techniques. In [4] Fathi, Herman and Yoccoz provided a detailed exposition of the stable manifold theorem essentially following the approaches of Pesin and Ruelle. In [11] Ruelle established a version of the theorem in Hilbert spaces, following his approach in [10]. In [5] Mañé considered transformations in Banach spaces under certain compactness and invertibility assumptions. We note that in all these works the dynamics is assumed to be of class $C^{1+\varepsilon}$ for some $\varepsilon > 0$. Pugh and Shub first obtained in [9] an optimal regularity of the stable manifolds for diffeomorphisms on finite-dimensional manifolds. Namely,

they showed that the stable manifolds are of class $C^{1+\varepsilon}$ if the dynamics is of class $C^{1+\varepsilon}$. More recently, we showed in [2] that the stable manifolds are of class C^1 with Lipschitz derivative if the dynamics has this same regularity, which provides another optimal result.

In strong contrast, here we only assume that the dynamics is of class C^1 and we show that there exist stable manifolds with the optimal C^1 regularity. Moreover, we establish the C^1 dependence of the stable manifolds on a parameter assuming that the dynamics is C^1 on this parameter (see Theorem 1). The proof is based on earlier work of ours in [3] where we obtained stable manifolds with the optimal C^1 regularity, but without considering a dependence on a parameter. We emphasize that although the present approach follows analogous steps, the required changes to consider a dependence on a parameter are nontrivial in particular since it is impossible to know a priori whether there are appropriate estimates for some new associated operators. Incidentally, it should be noted that although all works [2, 3, 9] establish the optimal regularity of the stable manifolds under corresponding assumptions for the dynamics (assumed to be, respectively, C^1 with Lipschitz derivative, C^1 and $C^{1+\varepsilon}$), the three methods of proof are quite different and each of them seems to be of no use in the other two developments.

More precisely, the proof of the C^1 regularity of the stable manifolds uses what is usually called the fiber contraction principle (unlike, besides [3], all the works mentioned above). Given metric spaces $X = (X, d_X)$ and $Y = (Y, d_Y)$, we define a distance in $X \times Y$ by

$$d((x, y), (\bar{x}, \bar{y})) = d_X(x, \bar{x}) + d_Y(y, \bar{y}).$$

We consider transformations $S: X \times Y \rightarrow X \times Y$ of the form

$$S(x, y) = (T(x), A(x, y))$$

for some functions $T: X \rightarrow X$ and $A: X \times Y \rightarrow Y$. We say that S is a *fiber contraction* if there exists $\lambda \in (0, 1)$ such that

$$d_Y(A(x, y), A(x, \bar{y})) \leq \lambda d_Y(y, \bar{y})$$

for every $x \in X$ and $y, \bar{y} \in Y$. For each $x \in X$ we define a transformation $A_x: Y \rightarrow Y$ by $A_x(y) = A(x, y)$. We also say that a fixed point $x_0 \in X$ of T is *attracting* if $T^n(x) \rightarrow x_0$ when $n \rightarrow \infty$, for every $x \in X$.

Proposition 1 (Fiber contraction principle). *If S is a continuous fiber contraction, $x_0 \in X$ is an attracting fixed point of T , and $y_0 \in Y$ is a fixed point of A_{x_0} , then (x_0, y_0) is an attracting fixed point of S .*

A nontrivial consequence of Theorem 1 concerns the robustness problem of nonuniform exponential dichotomies. Namely, consider the linear equation

$$(3) \quad u' = [A(t) + B(t, \lambda)]u$$

in $X = \mathbb{R}^p$, where $t \mapsto A(t)$ and $(t, \lambda) \mapsto B(t, \lambda)$ are C^1 functions. The robustness problem asks under what assumptions the exponential behavior of

a nonuniform exponential dichotomy for equation (1) with $a < 0 < b$ in (5) persists under such a linear perturbation. The following is an immediate consequence of Theorem 1.

Theorem 2. *If equation (1) has a nonuniform exponential dichotomy with $a < 0 < b$ in (5), and*

$$\|B(t, \lambda)\| \leq \kappa e^{-t/\kappa} \quad \text{and} \quad \|(\partial B/\partial \lambda)(t, \lambda)\| \leq \kappa e^{-t/\kappa}$$

for every $t \geq 0$, $u \in X$, and $\lambda \in Y$, for some sufficiently small $\kappa > 0$, then equation (3) has stable and unstable invariant subspaces $E_\lambda^s(t)$ and $E_\lambda^u(t)$ for each $t \geq 0$ and $\lambda \in Y$. In addition, the functions $\lambda \mapsto E_\lambda^s(t)$ and $\lambda \mapsto E_\lambda^u(t)$ are of class C^1 for each $t \geq 0$.

For each $t, \tau \geq 0$, the subspaces $E_\lambda^s(t)$ and $E_\lambda^u(t)$ satisfy $X = E_\lambda^s(t) \oplus E_\lambda^u(t)$,

$$T_\lambda(t, \tau)E_\lambda^s(\tau) = E_\lambda^s(t) \quad \text{and} \quad T_\lambda(t, \tau)E_\lambda^u(\tau) = E_\lambda^u(t),$$

where $T_\lambda(t, \tau)$ is the linear evolution operator associated to equation (3). Moreover, for each $\lambda \in Y$ there exist constants $a < 0 < b$ and $\varepsilon, D > 0$ such that

$$\|T_\lambda(t, \tau)E_\lambda^s(\tau)\| \leq D e^{a(t-\tau)+\varepsilon\tau}, \quad \|T_\lambda(t, \tau)E_\lambda^u(\tau)\| \leq D e^{-b(t-\tau)+\varepsilon\tau}$$

for each $t \geq \tau \geq 0$.

2. Standing assumptions

The following are standing assumptions in the paper. Let X be a Banach space, and let $A: \mathbb{R}_0^+ \rightarrow B(X)$ be a C^1 function, where $B(X)$ is the set of bounded linear operators in X . We consider the initial value problem

$$(4) \quad u' = A(t)u, \quad u(s) = u_s$$

for each $s \geq 0$ and $u_s \in X$. Its unique solution is defined for every $t > 0$, and we write it in the form $u(t) = T(t, s)u(s)$, where $T(t, s)$ is the associated linear evolution operator. Given an increasing differentiable function $\rho: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ with $\rho(t) \rightarrow +\infty$ when $t \rightarrow +\infty$, we say that equation (4) admits a ρ -nonuniform exponential dichotomy if there exist constants

$$(5) \quad a < 0 \leq b, \quad \varepsilon, D > 0,$$

and a continuous function $P: \mathbb{R}_0^+ \rightarrow B(X)$ such that $P(t)$ is a projection for $t \geq 0$, and for each $t \geq s \geq 0$ we have

$$P(t)T(t, s) = T(t, s)P(s),$$

$$(6) \quad \|T(t, s)P(s)\| \leq D e^{a(\rho(t)-\rho(s))+\varepsilon\rho(s)},$$

and

$$(7) \quad \|T(t, s)^{-1}Q(t)\| \leq D e^{-b(\rho(t)-\rho(s))+\varepsilon\rho(t)},$$

where $Q(t) = \text{Id} - P(t)$ is the complementary projection of $P(t)$. We then define the stable and unstable subspaces at time s by

$$E(s) = P(s)X \quad \text{and} \quad F(s) = Q(s)X.$$

Now let Y be an open subset of a Banach space (the parameter space), and let $f: \mathbb{R}_0^+ \times X \times Y \rightarrow X$ be a C^1 function with $f(t, 0, \lambda) = 0$ for every $t \geq 0$ and $\lambda \in Y$. We assume that there is a constant $\delta > 0$ such that

$$(8) \quad \left\| \frac{\partial f}{\partial u}(t, u, \lambda) \right\| \leq \delta \min\{1, \rho'(t)\} e^{-3\varepsilon\rho(t)},$$

and

$$(9) \quad \left\| \frac{\partial f}{\partial \lambda}(t, u, \lambda) \right\| \leq \delta \min\{1, \rho'(t)\} e^{-3\varepsilon\rho(t)} \|u\|$$

for every $t \geq 0$, $u \in X$, and $\lambda \in Y$.

Given $s \geq 0$ and $u_s = (\xi, \eta) \in E(s) \times F(s)$, let $(x(t), y(t)) \in E(t) \times F(t)$ be the unique solution of the initial value problem

$$(10) \quad u' = A(t)u + f(t, u, \lambda), \quad u(s) = u_s,$$

or equivalently of the problem

$$\begin{cases} x(t) = T(t, s)\xi + \int_s^t P(t)T(t, s)f(\tau, x(\tau), y(\tau), \lambda) d\tau, \\ y(t) = T(t, s)\eta + \int_s^t Q(t)T(t, s)f(\tau, x(\tau), y(\tau), \lambda) d\tau. \end{cases}$$

By (8), each solution of this problem is defined for every $t > 0$. Indeed, it follows from (10) that

$$u(t) = u(s) + \int_s^t A(\tau)u(\tau) d\tau + \int_s^t f(\tau, u(\tau), \lambda) d\tau$$

and thus, by (8),

$$\|u(t)\| \leq \|u(s)\| + \int_s^t \|A(\tau)\| \cdot \|u(\tau)\| d\tau + \delta \int_s^t \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \|u(\tau)\| d\tau.$$

It follows from Gronwall's lemma that

$$\begin{aligned} \|u(t)\| &\leq \|u(s)\| \exp \int_s^t (\|A(\tau)\| + \delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)}) d\tau \\ &= \|u(s)\| \exp \left(\int_s^t \|A(\tau)\| d\tau - \frac{\delta}{3\varepsilon} e^{-3\varepsilon(\rho(t) - \rho(s))} \right), \end{aligned}$$

and each solution is global. For each $\tau \geq 0$ we also write

$$\Psi_\tau^\lambda(s, u_s) = (s + \tau, x(s + \tau), y(s + \tau)),$$

where $(x(t), y(t))$ is the solution of equation (10). We note that this is the semiflow defined by the autonomous equation

$$t' = 1, \quad u' = A(t)u + f(t, u, \lambda)$$

for each given $\lambda \in Y$.

3. Invariant stable manifolds

We establish in this section the existence of C^1 stable manifolds for equation (10) assuming that equation (4) has a nonuniform exponential dichotomy. We also show that the stable manifolds are C^1 in λ . We emphasize that since the vector field is of class C^1 both results are optimal.

Let \mathcal{X} be the space of families $\phi = (\phi_\lambda)_{\lambda \in Y}$ of continuous functions

$$\phi_\lambda: \{(s, \xi) \in \mathbb{R}_0^+ \times X : \xi \in E(s)\} \rightarrow X$$

such that for each $s \geq 0$, $\xi, \bar{\xi} \in E(s)$, and $\lambda, \mu \in Y$:

1. $\phi_\lambda(s, 0) = 0$ and $\phi_\lambda(s, E(s)) \subset F(s)$;
- 2.

$$(11) \quad \|\phi_\lambda(s, \xi) - \phi_\lambda(s, \bar{\xi})\| \leq \|\xi - \bar{\xi}\|,$$

and

$$(12) \quad \|\phi_\lambda(s, \xi) - \phi_\mu(s, \xi)\| \leq \|\lambda - \mu\| \cdot \|\xi\|.$$

Given $\phi \in \mathcal{X}$ and $\lambda \in Y$ we consider the graph

$$V_{\phi, \lambda} = \{(s, \xi, \phi_\lambda(s, \xi)) : (s, \xi) \in \mathbb{R}_0^+ \times E(s)\}$$

of ϕ_λ . The stable manifolds of equation (10) are obtained in this form.

The following is our stable manifold theorem.

Theorem 3. *If the equation $u' = A(t)u$ admits a ρ -nonuniform exponential dichotomy with*

$$(13) \quad a - b + \varepsilon < 0,$$

$f(t, 0, \lambda) = 0$ for every $t \geq 0$ and $\lambda \in Y$, and (8) and (9) hold with δ sufficiently small, then there is a unique function $\phi \in \mathcal{X}$ such that

$$(14) \quad \Psi_\tau^\lambda(V_{\phi, \lambda}) = V_{\phi, \lambda} \quad \text{for every } \tau \geq 0, \lambda \in Y.$$

Moreover,

1. *there exists $D' > 0$ such that for every $s \geq 0$, $\lambda, \mu \in Y$, $\xi, \bar{\xi} \in E(s)$, and $\tau \geq 0$ we have*

$$\begin{aligned} & \|\Psi_{t-s}^\lambda(s, \xi, \phi_\lambda(s, \xi)) - \Psi_{t-s}^\lambda(s, \bar{\xi}, \phi_\lambda(s, \bar{\xi}))\| \\ & \leq D' e^{(a+2\delta D)(\rho(t)-\rho(s))+\varepsilon\rho(s)} \|\xi - \bar{\xi}\|, \end{aligned}$$

and

$$\begin{aligned} & \|\Psi_{t-s}^\lambda(s, \xi, \phi_\lambda(s, \xi)) - \Psi_{t-s}^\mu(s, \xi, \phi_\mu(s, \xi))\| \\ & \leq D' e^{(a+2\delta D)(\rho(t)-\rho(s))+\varepsilon\rho(s)} \|\xi\| \cdot \|\lambda - \mu\|; \end{aligned}$$

2. *for $X = \mathbb{R}^p$ and $Y \subset \mathbb{R}^q$ an open ball, if $f(t, u, \lambda) = 0$ for every $t \geq 0$, $u \in X$ with $\|u\| \geq c$, and $\lambda \in Y$, for some constant $c > 0$, then:*

- (a) *the function $(\xi, \lambda) \mapsto \phi_\lambda(s, \xi)$ is of class C^1 for each $s \geq 0$;*

(b) if in addition $(\partial f / \partial u)(t, 0, \lambda) = 0$ for every $t \geq 0$ and $\lambda \in Y$, then

$$(\partial \phi_\lambda / \partial \xi)(s, 0) = 0 \quad \text{for every } (s, \lambda) \in \mathbb{R}_0^+ \times Y.$$

Proof. We separate the proof into several steps.

Step 1: Solution in the stable direction. Given $s \geq 0$ we consider the space $\mathcal{B} = \mathcal{B}_s$ of continuous functions

$$x: \{(t, \xi, \lambda) : t \geq s, \xi \in E(s), \text{ and } \lambda \in Y\} \rightarrow X$$

such that:

1. for every $t \geq s$, $\xi \in E(s)$, and $\lambda \in Y$ we have

$$x(s, \xi, \lambda) = \xi \quad \text{and} \quad x(t, \xi, \lambda) \in E(t);$$

- 2.

$$(15) \quad \alpha(x) := \sup \left\{ \frac{\|x(t, \xi, \lambda)\|}{\|\xi\| e^{a(\rho(t) - \rho(s)) + \varepsilon \rho(s)}} \right\} \leq 2D,$$

with the supremum taken over $t \geq s$, $\xi \in E(s) \setminus \{0\}$, and $\lambda \in Y$.

By (15) and the continuity of x we have

$$(16) \quad x(t, 0, \lambda) = 0 \quad \text{for every } t \geq s.$$

We can easily verify that \mathcal{B} is a complete metric space with the distance induced by the norm α in (15).

So that (14) holds we must have $y(t) = \phi(t, x(t))$ for every $t \geq s$, that is,

$$(17) \quad x(t) = T(t, s)\xi + \int_s^t P(t)T(t, s)f(\tau, x(\tau), \phi_\lambda(\tau, x(\tau)), \lambda) d\tau,$$

and

$$(18) \quad \phi_\lambda(t, x(t)) = T(t, s)\phi_\lambda(s, \xi) + \int_s^t Q(t)T(t, s)f(\tau, x(\tau), \phi_\lambda(\tau, x(\tau)), \lambda) d\tau.$$

Lemma 1. For every $\delta > 0$ sufficiently small, given $\phi \in \mathcal{X}$ and $s \geq 0$ there is a unique function $x = x_\phi \in \mathcal{B}$ satisfying (17) for every $t \geq s$, $\xi \in E(s)$, and $\lambda \in Y$. Furthermore,

$$(19) \quad \|x_\phi(t, \xi, \lambda) - x_\phi(t, \xi, \mu)\| \leq \frac{3D^2\delta}{\varepsilon} e^{(a+2\delta D)(\rho(t) - \rho(s))} \|\xi\| \cdot \|\lambda - \mu\|$$

for every $\phi \in \mathcal{X}$, $t \geq s$, $\xi \in E(s)$, and $\lambda, \mu \in Y$.

Proof. Given $\phi \in \mathcal{X}$, we define an operator J on \mathcal{B} by

$$(Jx)(t, \xi, \lambda) = T(t, s)\xi + \int_s^t P(t)T(t, \tau)f(\tau, x(\tau, \xi, \lambda), \phi_\lambda(\tau, x(\tau, \xi, \lambda)), \lambda) d\tau$$

for each $t \geq s$, $\xi \in E(s)$, and $\lambda \in Y$. Clearly, Jx is a continuous function, and $(Jx)(s, \xi, \lambda) = \xi$.

By (8), (11), and (15), for each $\tau \geq s$, we have

$$K(\tau) := \|f(\tau, x(\tau, \xi, \lambda), \phi_\lambda(\tau, x(\tau, \xi, \lambda)), \lambda)\|$$

$$\begin{aligned}
& - f(\tau, y(\tau, \xi, \lambda), \phi_\lambda(\tau, y(\tau, \xi, \lambda)), \lambda) \| \\
& \leq \delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \| (x(\tau, \xi, \lambda), \phi_\lambda(\tau, x(\tau, \xi, \lambda))) \\
& \quad - (y(\tau, \xi, \lambda), \phi_\lambda(\tau, y(\tau, \xi, \lambda))) \| \\
& \leq 2\delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \| x(\tau, \xi, \lambda) - y(\tau, \xi, \lambda) \| \\
(20) \quad & \leq 2\delta \rho'(\tau) e^{\alpha(\rho(\tau) - \rho(s)) + \varepsilon\rho(s)} e^{-3\varepsilon\rho(\tau)} \|\xi\| \alpha(x - y).
\end{aligned}$$

By (6) we obtain

$$\begin{aligned}
& \| (Jx)(t, \xi, \lambda) - (Jy)(t, \xi, \lambda) \| \\
& \leq \int_s^t \| P(t)T(t, \tau) \| K(\tau) d\tau \\
& \leq 2\delta D \|\xi\| \alpha(x - y) \int_s^t e^{\alpha(\rho(t) - \rho(\tau)) + \varepsilon\rho(\tau)} e^{\alpha(\rho(\tau) - \rho(s)) + \varepsilon\rho(s)} e^{-3\varepsilon\rho(\tau)} \rho'(\tau) d\tau \\
& \leq 2\delta D \|\xi\| \alpha(x - y) e^{\alpha(\rho(t) - \rho(s)) + \varepsilon\rho(s)} \int_s^\infty e^{-2\varepsilon\rho(\tau)} \rho'(\tau) d\tau \\
& \leq \frac{\delta D}{\varepsilon} \|\xi\| \alpha(x - y) e^{\alpha(\rho(t) - \rho(s)) + \varepsilon\rho(s)},
\end{aligned}$$

and hence,

$$\alpha(Jx - Jy) \leq \frac{\delta D}{\varepsilon} \alpha(x - y).$$

Taking δ sufficiently small so that $\delta D/\varepsilon < 1$ the operator J becomes a contraction. In addition, by (6) we have $\alpha(J0) \leq D$, and hence,

$$\begin{aligned}
\alpha(Jx) & \leq \alpha(J0) + \alpha(Jx - J0) \\
& \leq D + (\delta D/\varepsilon)\alpha(x) \\
& \leq D + D = 2D.
\end{aligned}$$

Therefore, $J(\mathcal{B}) \subset \mathcal{B}$, and there is a unique $x = x_\phi \in \mathcal{B}$ such that $Jx = x$.

Now we establish (19). Writing $y_\lambda = x_\phi(\cdot, \xi, \lambda)$, we have

$$\| (y_\lambda(\tau), \phi_\lambda(\tau, y_\lambda(\tau))) \| \leq 2\|y_\lambda(\tau)\|,$$

and

$$\begin{aligned}
& \| (y_\lambda(\tau), \phi_\lambda(\tau, y_\lambda(\tau))) - (y_\mu(\tau), \phi_\mu(\tau, y_\mu(\tau))) \| \\
& \leq \| y_\lambda(\tau) - y_\mu(\tau) \| + \| \phi_\lambda(\tau, y_\lambda(\tau)) - \phi_\lambda(\tau, y_\mu(\tau)) \| \\
& \quad + \| \phi_\lambda(\tau, y_\mu(\tau)) - \phi_\mu(\tau, y_\mu(\tau)) \| \\
(21) \quad & \leq 2\|y_\lambda(\tau) - y_\mu(\tau)\| + \|y_\mu(\tau)\| \cdot \|\lambda - \mu\|.
\end{aligned}$$

Hence, by (8), (9), (11)–(12), and (15) we obtain

$$\begin{aligned}
a(\tau) & := \| f(\tau, y_\lambda(\tau), \phi_\lambda(\tau, y_\lambda(\tau)), \lambda) - f(\tau, y_\mu(\tau), \phi_\mu(\tau, y_\mu(\tau)), \mu) \| \\
& \leq \| f(\tau, y_\lambda(\tau), \phi_\lambda(\tau, y_\lambda(\tau)), \lambda) - f(\tau, y_\lambda(\tau), \phi_\lambda(\tau, y_\lambda(\tau)), \mu) \| \\
& \quad + \| f(\tau, y_\lambda(\tau), \phi_\lambda(\tau, y_\lambda(\tau)), \mu) - f(\tau, y_\mu(\tau), \phi_\mu(\tau, y_\mu(\tau)), \mu) \|
\end{aligned}$$

$$\begin{aligned}
&\leq \delta e^{-3\varepsilon\rho(\tau)} \rho'(\tau) \|(y_\lambda(\tau), \phi_\lambda(\tau, y_\lambda(\tau)))\| \cdot \|\lambda - \mu\| \\
&\quad + \delta e^{-3\varepsilon\rho(\tau)} \|(y_\lambda(\tau), \phi_\lambda(\tau, y_\lambda(\tau))) - (y_\mu(\tau), \phi_\mu(\tau, y_\mu(\tau)))\| \\
&\leq 2\delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \|y_\lambda(\tau)\| \cdot \|\lambda - \mu\| \\
&\quad + 2\delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \|y_\lambda(\tau) - y_\mu(\tau)\| \\
&\quad + \delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \|y_\mu(\tau)\| \cdot \|\lambda - \mu\| \\
&\leq 6D\delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} e^{a(\rho(\tau)-\rho(s))+\varepsilon\rho(s)} \|\xi\| \cdot \|\lambda - \mu\| \\
(22) \quad &\quad + 2\delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \|y_\lambda(\tau) - y_\mu(\tau)\|.
\end{aligned}$$

Moreover, by (6) we have

$$\begin{aligned}
\|y_\lambda(t) - y_\mu(t)\| &\leq \int_s^t \|P(t)T(t, \tau)\| a(\tau) d\tau \\
&\leq 6D^2\delta \|\xi\| \cdot \|\lambda - \mu\| e^{a(\rho(t)-\rho(s))+\varepsilon\rho(s)} \int_s^t \rho'(\tau) e^{-2\varepsilon\rho(\tau)} d\tau \\
&\quad + 2\delta D \int_s^t \rho'(\tau) e^{a(\rho(t)-\rho(\tau))-2\varepsilon\rho(\tau)} \|y_\lambda(\tau) - y_\mu(\tau)\| d\tau \\
&\leq \frac{3D^2\delta}{\varepsilon} \|\xi\| \cdot \|\lambda - \mu\| e^{a(\rho(t)-\rho(s))} \\
&\quad + 2\delta D e^{a(\rho(t)-\rho(s))} \int_s^t \rho'(\tau) e^{-a(\rho(\tau)-\rho(s))} \|y_\lambda(\tau) - y_\mu(\tau)\| d\tau.
\end{aligned}$$

Setting $\Gamma(t) = e^{-a(\rho(t)-\rho(s))} \|y_\lambda(t) - y_\mu(t)\|$, we obtain

$$\Gamma(t) \leq \frac{3D^2\delta}{\varepsilon} \|\xi\| \cdot \|\lambda - \mu\| + 2\delta D \int_s^t \rho'(\tau) \Gamma(\tau) d\tau,$$

and it follows by Gronwall's lemma that

$$\begin{aligned}
\Gamma(t) &\leq \frac{3D^2\delta}{\varepsilon} \|\xi\| \cdot \|\lambda - \mu\| e^{2\delta D \int_s^t \rho'(\tau) d\tau} \\
&= \frac{3D^2\delta}{\varepsilon} \|\xi\| \cdot \|\lambda - \mu\| e^{2\delta D(\rho(t)-\rho(s))}.
\end{aligned}$$

This yields inequality (19). \square

Step 2: Auxiliary properties. Now we describe several additional properties of the function x_ϕ . We equip the space \mathcal{X} with the distance

$$d(\phi, \psi) = \sup \left\{ \frac{\|\phi_\lambda(t, x) - \psi_\lambda(t, x)\|}{\|x\|} : t \geq 0, x \in E(t) \setminus \{0\}, \text{ and } \lambda \in Y \right\}.$$

We can easily verify that \mathcal{X} is a complete metric space with this distance.

Lemma 2. *For every $\delta > 0$ sufficiently small, given $\phi, \psi \in \mathcal{X}$, $t \geq s$, $\xi, \bar{\xi} \in E(s)$, and $\lambda \in Y$ we have*

$$(23) \quad \|x_\phi(t, \xi, \lambda) - x_\phi(t, \bar{\xi}, \lambda)\| \leq D e^{(a+2\delta D)(\rho(t)-\rho(s))+\varepsilon\rho(s)} \|\xi - \bar{\xi}\|,$$

and

$$(24) \quad \|x_\phi(t, \xi, \lambda) - x_\psi(t, \xi, \lambda)\| \leq \frac{2\delta D^2}{\varepsilon} e^{(a+4\delta D)(\rho(t)-\rho(s))} \|\xi\| d(\phi, \psi).$$

Proof. Proceeding in a similar manner to that in (20), for each $\tau \geq s$ we have

$$\begin{aligned} & \|f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) - f(\tau, \bar{x}_\phi(\tau, \bar{\xi}, \lambda), \phi_\lambda(\tau, \bar{x}_\phi(\tau, \bar{\xi}, \lambda)), \lambda)\| \\ & \leq 2\delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \|x_\phi(\tau, \xi, \lambda) - \bar{x}_\phi(\tau, \bar{\xi}, \lambda)\|. \end{aligned}$$

Setting $\Gamma(t) = \|x_\phi(t, \xi, \lambda) - x_\phi(t, \bar{\xi}, \lambda)\|$, and using (6) we obtain

$$\begin{aligned} \Gamma(t) & \leq \|P(t)T(t, s)\| \cdot \|\xi - \bar{\xi}\| + \int_s^t \|P(t)T(t, \tau)\| 2\delta \rho'(\tau) e^{-2\varepsilon\rho(\tau)} \rho'(\tau) \Gamma(\tau) d\tau \\ & \leq D e^{a(\rho(t)-\rho(s))+\varepsilon\rho(s)} \|\xi - \bar{\xi}\| + 2\delta D \int_s^t e^{a(\rho(t)-\rho(\tau))-2\varepsilon\rho(\tau)} \rho'(\tau) \Gamma(\tau) d\tau \\ & \leq e^{a(\rho(t)-\rho(s))} \left(D e^{\varepsilon\rho(s)} \|\xi - \bar{\xi}\| + 2\delta D \int_s^t e^{-a(\rho(\tau)-\rho(s))} \rho'(\tau) \Gamma(\tau) d\tau \right). \end{aligned}$$

It follows from Gronwall's lemma applied to the function $e^{-a(\rho(t)-\rho(s))}\Gamma(t)$ that

$$\Gamma(t) \leq D e^{(a+2\delta D)(\rho(t)-\rho(s))+\varepsilon\rho(s)} \|\xi - \bar{\xi}\|.$$

This establishes inequality (23).

Similarly, we have

$$\begin{aligned} & \|f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) - f(\tau, x_\psi(\tau, \xi, \lambda), \psi_\lambda(\tau, x_\psi(\tau, \xi, \lambda)), \lambda)\| \\ & \leq 2\delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \\ & \quad \times \|(x_\phi(\tau, \xi, \lambda) - x_\psi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)) - \psi_\lambda(\tau, x_\psi(\tau, \xi, \lambda)))\|, \end{aligned}$$

and

$$\begin{aligned} & \|\phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)) - \psi_\lambda(\tau, x_\psi(\tau, \xi, \lambda))\| \\ & \leq \|\phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)) - \psi_\lambda(\tau, x_\phi(\tau, \xi, \lambda))\| \\ & \quad + \|\phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)) - \psi_\lambda(\tau, x_\psi(\tau, \xi, \lambda))\| \\ & \leq \|x_\phi(\tau, \xi, \lambda)\| d(\phi, \psi) + \|x_\phi(\tau, \xi, \lambda) - x_\psi(\tau, \xi, \lambda)\|. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) - f(\tau, x_\psi(\tau, \xi, \lambda), \psi_\lambda(\tau, x_\psi(\tau, \xi, \lambda)), \lambda)\| \\ & \leq 2\delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} (\|x_\phi(\tau, \xi, \lambda)\| d(\phi, \psi) + 2\|x_\phi(\tau, \xi, \lambda) - x_\psi(\tau, \xi, \lambda)\|). \end{aligned}$$

Setting $\Gamma(t) = \|x_\phi(t, \xi, \lambda) - x_\psi(t, \xi, \lambda)\|$, we obtain

$$\begin{aligned} \Gamma(t) & \leq 2\delta \int_s^t \|P(t)T(t, \tau)\| \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \|x_\phi(\tau, \xi, \lambda)\| d(\phi, \psi) d\tau \\ & \quad + 4\delta \int_s^t \|P(t)T(t, \tau)\| e^{-3\varepsilon\rho(\tau)} \rho'(\tau) \|x_\phi(\tau, \xi, \lambda) - x_\psi(\tau, \xi, \lambda)\| d\tau \end{aligned}$$

$$\begin{aligned} &\leq 4\delta D^2 \|\xi\| d(\phi, \psi) \int_s^t e^{a(\rho(t)-\rho(\tau))-2\varepsilon\rho(\tau)} e^{a(\rho(\tau)-\rho(s))+\varepsilon\rho(s)} \rho'(\tau) d\tau \\ &\quad + 4\delta D \int_s^t e^{a(\rho(t)-\rho(\tau))-2\varepsilon\rho(\tau)} \rho'(\tau) \Gamma(\tau) d\tau, \end{aligned}$$

and thus,

$$\begin{aligned} e^{-a(\rho(t)-\rho(s))} \Gamma(t) &\leq 4\delta D^2 \|\xi\| d(\phi, \psi) \int_s^\infty e^{-2\varepsilon(\rho(\tau)-\rho(s))} d\tau \\ &\quad + 4\delta D \int_s^t e^{-a(\rho(\tau)-\rho(s))} \Gamma(\tau) d\tau \\ &\leq \frac{2\delta D^2}{\varepsilon} \|\xi\| d(\phi, \psi) + 4\delta D \int_s^t e^{-a(\rho(\tau)-\rho(s))} \Gamma(\tau) d\tau. \end{aligned}$$

Inequality (24) follows now from Gronwall's lemma applied to $e^{-a(\rho(t)-\rho(s))} \Gamma(t)$. \square

Step 3: Equivalent problem. Now we describe an equivalent problem to (18).

Lemma 3. *For every $\delta > 0$ sufficiently small, given $\phi \in \mathcal{X}$ and $\lambda \in Y$ the following properties are equivalent:*

1. *for every $s \geq 0$, $\xi \in E(s)$ and $t \geq s$ we have*

$$(25) \quad \begin{aligned} \phi_\lambda(t, x_\phi(t, \xi, \lambda)) &= T(t, s) \phi_\lambda(s, \xi) \\ &\quad + \int_s^t Q(t) T(t, \tau) f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) d\tau; \end{aligned}$$

2. *for every $s \geq 0$ and $\xi \in E(s)$ we have*

$$(26) \quad \phi_\lambda(s, \xi) = - \int_s^\infty Q(s) T(\tau, s)^{-1} f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) d\tau.$$

Proof. For each $\tau \geq s$ we have

$$\begin{aligned} &\|f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda)\| \\ &\leq 2\delta \rho'(\tau) e^{-3\varepsilon\rho(\tau)} \|x_\phi(\tau, \xi, \lambda)\| \\ &\leq 4\delta D \rho'(\tau) e^{a(\rho(\tau)-\rho(s))+\varepsilon\rho(s)} e^{-3\varepsilon\rho(\tau)} \|\xi\|. \end{aligned}$$

It follows from (7) that

$$\begin{aligned} &\int_s^\infty \|Q(s) T(\tau, s)^{-1} f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda)\| d\tau \\ &\leq 4\delta D^2 \|\xi\| \int_s^\infty e^{(a-b-\varepsilon)(\rho(\tau)-\rho(s))} \rho'(\tau) d\tau < \infty, \end{aligned}$$

and thus the integral in (26) is well-defined. Now we assume that identity (25) holds, and we write it in the equivalent form

$$\phi_\lambda(s, \xi) = T(t, s)^{-1} \phi_\lambda(t, x_\phi(t, \xi, \lambda))$$

$$(27) \quad - \int_s^t Q(s)T(\tau, s)^{-1} f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) d\tau.$$

By (7), for every $t \geq s$ we have

$$\begin{aligned} \|T(t, s)^{-1} \phi_\lambda(t, x_\phi(t, \xi, \lambda))\| &\leq D e^{-b(\rho(t)-\rho(s))+\varepsilon\rho(t)} \|x_\phi(t, \xi, \lambda)\| \\ &\leq 2D^2 \|\xi\| e^{(a-b)(\rho(t)-\rho(s))+\varepsilon\rho(s)+\varepsilon\rho(t)} \\ &\leq 2D^2 \|\xi\| e^{2\varepsilon\rho(s)} e^{(a-b+\varepsilon)(\rho(t)-\rho(s))}. \end{aligned}$$

By (13), we have $a - b + \varepsilon < 0$, and letting $t \rightarrow +\infty$ in (27) yields (26).

Now we assume that identity (26) holds. We obtain

$$(28) \quad \begin{aligned} &T(t, s)\phi_\lambda(s, \xi) + \int_s^t Q(t)T(t, \tau)f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) d\tau \\ &= - \int_t^\infty Q(t)T(\tau, t)^{-1} f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) d\tau. \end{aligned}$$

It follows from (26) with (s, ξ) replaced by $(t, x_\phi(t, \xi, \lambda))$ that

$$\begin{aligned} &\phi_\lambda(t, x_\phi(t, \xi, \lambda)) \\ &= - \int_t^\infty Q(t)T(\tau, t)^{-1} f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) d\tau \end{aligned}$$

for every $t \geq s$. Together with (28) this yields identity (25). \square

Step 4: Construction of a Lipschitz manifold. Now we solve problem (26). In view of Lemma 3 this corresponds to the construction of a Lipschitz stable manifold.

Lemma 4. *For every $\delta > 0$ sufficiently small, there exists a unique $\phi \in \mathcal{X}$ satisfying (26) for every $s \geq 0$, $\xi \in E(s)$, and $\lambda \in Y$.*

Proof. We consider the operator T in \mathcal{X} defined for each $\phi \in \mathcal{X}$ by

$$(29) \quad (T\phi)_\lambda(s, \xi) = - \int_s^\infty Q(s)T(\tau, s)^{-1} f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) d\tau$$

for $(s, \xi, \lambda) \in \mathbb{R}_0^+ \times E(s) \times Y$. One can verify that the function $(T\phi)_\lambda$ is continuous for each $\phi \in \mathcal{X}$ and $\lambda \in Y$. Since $x_\phi(t, 0, \lambda) = 0$ for every $t \geq s$ and $\lambda \in Y$ (see (16)), it follows from (29) that $(T\phi)_\lambda(s, 0) = 0$ for every $s \geq 0$ and $\lambda \in Y$. By Lemma 2 we have

$$\begin{aligned} &\|f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda))) - f(\tau, x_\phi(\tau, \bar{\xi}, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \bar{\xi}, \lambda)), \lambda)\| \\ &\leq 2\delta D \rho'(\tau) e^{-3\varepsilon\rho(\tau)} e^{(a+2\delta D)(\rho(\tau)-\rho(s))+\varepsilon\rho(s)} \|\xi - \bar{\xi}\|, \end{aligned}$$

and using (7) we obtain

$$\begin{aligned} &\|(T\phi)_\lambda(s, \xi) - (T\phi)_\lambda(s, \bar{\xi})\| \\ &\leq 2\delta D^2 \|\xi - \bar{\xi}\| \int_s^\infty \rho'(\tau) e^{(a-b+2\delta D)(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)+\varepsilon\rho(s)-3\varepsilon\rho(\tau)} d\tau \end{aligned}$$

$$\begin{aligned}
&= 2\delta D^2 \|\xi - \bar{\xi}\| \int_s^\infty \rho'(\tau) e^{(a-b+2\delta D-\varepsilon)(\rho(\tau)-\rho(s))} d\tau \\
&= \frac{2\delta D^2}{|a-b+2\delta D-\varepsilon|} \|\xi - \bar{\xi}\|,
\end{aligned}$$

where we have chosen δ sufficiently small so that

$$(30) \quad a - b + 2\delta D - \varepsilon < 0 \quad \text{and} \quad 2\delta D^2 / |a - b + 2\delta D - \varepsilon| < 1.$$

In particular,

$$\|(T\phi)_\lambda(s, \xi) - (T\phi)_\lambda(s, \bar{\xi})\| \leq \|\xi - \bar{\xi}\|$$

for every $s \geq 0$ and $\xi, \bar{\xi} \in \mathcal{X}$. Moreover, by (19) and (22) we have

$$\begin{aligned}
a(\tau) &\leq 6D\delta\rho'(\tau)e^{-3\varepsilon\rho(\tau)}e^{a(\rho(\tau)-\rho(s))+\varepsilon\rho(s)}\|\xi\| \cdot \|\lambda - \mu\| \\
&\quad + 2\delta\rho'(\tau)e^{-3\varepsilon\rho(\tau)}\|y_\lambda(\tau) - y_\mu(\tau)\| \\
&\leq K'\delta\rho'(\tau)e^{-3\varepsilon\rho(\tau)}e^{(a+2\delta D)(\rho(\tau)-\rho(s))+\varepsilon\rho(s)}\|\xi\| \cdot \|\lambda - \mu\|,
\end{aligned}$$

and hence,

$$\begin{aligned}
&\|(T\phi)_\lambda(s, \xi) - (T\phi)_\mu(s, \xi)\| \\
&\leq \int_s^\infty \|Q(s)T(\tau, s)^{-1}\| a(\tau) d\tau \\
&\leq K'\delta D \|\xi\| \cdot \|\lambda - \mu\| \\
&\quad \times \int_s^\infty \rho'(\tau) e^{(a+2\delta D)(\rho(\tau)-\rho(s))+\varepsilon\rho(s)-3\varepsilon\rho(\tau)} e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} d\tau \\
&\leq K'\delta D \|\xi\| \cdot \|\lambda - \mu\| \int_s^\infty \rho'(\tau) e^{(a+2\delta D-b-\varepsilon)(\rho(\tau)-\rho(s))} d\tau.
\end{aligned}$$

By (30) this yields

$$\|(T\phi)_\lambda(s, \xi) - (T\phi)_\mu(s, \xi)\| \leq \|\lambda - \mu\| \cdot \|\xi\|$$

provided that δ is sufficiently small. Therefore, $T(\mathcal{X}) \subset \mathcal{X}$.

Now we show that T is a contraction. By (11) and Lemma 2 we have

$$\begin{aligned}
L(\tau) &:= \|f(\tau, x_\phi(\tau, \xi, \lambda), \phi_\lambda(\tau, x_\phi(\tau, \xi, \lambda)), \lambda) \\
&\quad - f(\tau, x_\psi(\tau, \xi, \lambda), \psi_\lambda(\tau, x_\psi(\tau, \xi, \lambda)), \lambda)\| \\
&\leq \delta\rho'(\tau)e^{-3\varepsilon\rho(\tau)}\|x_\psi(\tau, \xi, \lambda)\|d(\phi, \psi) \\
&\quad + \frac{8\delta D^2}{\varepsilon}\rho'(\tau)e^{-3\varepsilon\rho(\tau)}e^{(a+4\delta D)(\rho(\tau)-\rho(s))}\|\xi\|d(\phi, \psi) \\
&\leq 2\delta\rho'(\tau)De^{-3\varepsilon\rho(\tau)}e^{a(\rho(\tau)-\rho(s))+\varepsilon\rho(s)}\|\xi\|d(\phi, \psi) \\
&\quad + \frac{8\delta D^2}{\varepsilon}\rho'(\tau)e^{-3\varepsilon\rho(\tau)}e^{(a+4\delta D)(\rho(\tau)-\rho(s))}\|\xi\|d(\phi, \psi) \\
&\leq L'\delta\rho'(\tau)e^{-3\varepsilon\rho(\tau)}e^{(a+4\delta D)(\rho(\tau)-\rho(s))+\varepsilon\rho(s)}\|\xi\|d(\phi, \psi)
\end{aligned}$$

for some constant $L' > 0$. Hence,

$$\begin{aligned}
& \|(T\phi)_\lambda(s, \xi) - (T\psi)_\lambda(s, \xi)\| \\
& \leq \int_s^\infty \|Q(s)T(\tau, s)^{-1}\|L(\tau) d\tau \\
& \leq DL'\delta\|\xi\|d(\phi, \psi) \int_s^\infty \rho'(\tau)e^{(a-b+4\delta D)(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)+\varepsilon\rho(s)-3\varepsilon\rho(\tau)} d\tau \\
& \leq DL'\delta\|\xi\|d(\phi, \psi) \int_s^\infty \rho'(\tau)e^{(a-b+4\delta D-\varepsilon)(\rho(\tau)-\rho(s))} d\tau \\
& = \frac{DL'\delta}{|a-b+4\delta D-\varepsilon|}\|\xi\|d(\phi, \psi),
\end{aligned}$$

taking δ sufficiently small so that

$$a - b + 4\delta D - \varepsilon < 0 \quad \text{and} \quad DL'\delta/|a - b + 4\delta D - \varepsilon| < 1.$$

Then T is a contraction, and there is a unique $\phi \in \mathcal{X}$ satisfying $T\phi = \phi$. \square

Step 5: Additional properties. We obtain the remaining properties in the theorem. By Lemma 2 we have

$$\begin{aligned}
& \|\Psi_{t-s}^\lambda(s, \xi, \phi_\lambda(s, \xi)) - \Psi_{t-s}^\lambda(s, \bar{\xi}, \phi_\lambda(s, \bar{\xi}))\| \\
& = \|(t, x(t, \xi, \lambda), \phi_\lambda(t, x(t, \xi, \lambda))) - (t, x(t, \bar{\xi}, \lambda), \phi_\lambda(t, x(t, \bar{\xi}, \lambda)))\| \\
& \leq 2\|x(t, \xi, \lambda) - x(t, \bar{\xi}, \lambda)\| \\
& \leq 2De^{(a+2\delta D)(\rho(t)-\rho(s))+\varepsilon\rho(s)}\|\xi - \bar{\xi}\|,
\end{aligned}$$

and by (21) and Lemma 1,

$$\begin{aligned}
& \|\Psi_{t-s}^\lambda(s, \xi, \phi_\lambda(s, \xi)) - \Psi_{t-s}^\mu(s, \xi, \phi_\mu(s, \xi))\| \\
& = \|(t, x(t, \xi, \lambda), \phi_\lambda(t, x(t, \xi, \lambda))) - (t, x(t, \xi, \mu), \phi_\mu(t, x(t, \xi, \mu)))\| \\
& \leq 2\|x(t, \xi, \lambda) - x(t, \xi, \mu)\| + \|x_\phi(t, \xi, \mu)\| \cdot \|\lambda - \mu\| \\
& \leq \left(\frac{12D^2\delta}{\varepsilon} + 2D\right)e^{(a+2\delta D)(\rho(t)-\rho(s))+\varepsilon\rho(s)}\|\xi\| \cdot \|\lambda - \mu\|.
\end{aligned}$$

This establishes property 1 in the theorem.

Step 6: Preliminaries for the regularity. Now we establish the C^1 regularity of the stable manifolds. We consider the space \mathcal{F} of continuous functions

$$\Phi: \{(s, \xi, \lambda) \in \mathbb{R}_0^+ \times X \times Y : \xi \in E(s)\} \rightarrow \prod_{s \geq 0} L(s),$$

where $L(s)$ is the family of linear transformations from $E(s)$ to $F(s)$, such that $\Phi(s, \xi, \lambda) \in L(s)$ for every $s \geq 0$, $\xi \in E(s)$, and $\lambda \in Y$, with

$$(31) \quad \|\Phi\| := \sup \left\{ \|\Phi(s, \xi, \lambda)\| : (s, \xi, \lambda) \in \mathbb{R}_0^+ \times E(s) \times Y \right\} \leq 1.$$

We can easily verify that \mathcal{F} is a complete metric space with the distance induced by this norm. We also consider the space \mathcal{G} of continuous functions

$$U: \{(s, \xi, \lambda) \in \mathbb{R}_0^+ \times X \times Y : \xi \in E(s)\} \rightarrow \prod_{s \geq 0} L(s)$$

such that $U(s, \xi, \lambda) \in L(s)$ for every $s \geq 0$, $\xi \in E(s)$, and $\lambda \in Y$, with

$$(32) \quad \|U\| := \sup \left\{ \frac{\|U(s, \xi, \lambda)\|}{\|\xi\|} : (s, \xi, \lambda) \in \mathbb{R}_0^+ \times (E(s) \setminus \{0\}) \times Y \right\} \leq 1.$$

Again, we can easily verify that \mathcal{G} is a complete metric space with the distance induced by this norm.

We observe that the function $x = x_\phi$ given by Lemma 1 is the solution of the differential equation

$$(33) \quad x' = P(t)A(t)x + P(t)f(t, x, \phi_\lambda(t, x), \lambda)$$

with $x(s) = \xi$, for each $\lambda \in Y$. By the continuous dependence of the solutions of a differential equation on the initial conditions and on parameters, and Lemma 2, the function $(t, \phi, s, \xi, \lambda) \mapsto x_\phi(t, \xi, \lambda)$ is continuous.

Step 7: Auxiliary operators. Now we define a linear operator $A(\phi, \Phi)$ for each $(\phi, \Phi) \in \mathcal{X} \times \mathcal{F}$ by

$$\begin{aligned} & A(\phi, \Phi)(s, \xi, \lambda) \\ &= - \int_s^\infty Q(s)T(\tau, s)^{-1} \left(\frac{\partial f}{\partial x}(y_\phi(\tau))W(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Phi(z_\phi(\tau))W(\tau) \right) d\tau, \end{aligned}$$

with the notations

$$y_\phi(t) = (t, x_\phi(t, \xi, \lambda), \phi_\lambda(t, x_\phi(t, \xi, \lambda)), \lambda) \quad \text{and} \quad z_\phi(t) = (t, x_\phi(t, \xi, \lambda), \lambda),$$

where $(x, y) \in E(s) \times F(s)$, and where $W = W_{\phi, \Phi, \xi, \lambda}$ satisfies

$$(34) \quad \begin{aligned} & W(t) = P(t)T(t, s) \\ & + \int_s^t P(t)T(t, \tau) \left(\frac{\partial f}{\partial x}(y_\phi(\tau))W(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Phi(z_\phi(\tau))W(\tau) \right) d\tau \end{aligned}$$

for every $\tau \geq s$. We note that $W(t)$ is a linear operator from $E(s)$ to $E(t)$, with $W(s) = \text{Id}_{E(s)}$.

We also define a linear operator $B(\phi, \Phi, U)$ for each $(\phi, \Phi, U) \in \mathcal{X} \times \mathcal{F} \times \mathcal{G}$ by

$$\begin{aligned} & B(\phi, \Phi, U)(s, \xi, \lambda) \\ &= - \int_s^\infty Q(s)T(\tau, s)^{-1} \left(\frac{\partial f}{\partial x}(y_\phi(\tau))Z(\tau) \right. \\ & \quad \left. + \frac{\partial f}{\partial y}(y_\phi(\tau)) \left(\Phi(z_\phi(\tau))Z(\tau) + U(z_\phi(\tau)) \right) + \frac{\partial f}{\partial \lambda}(y_\phi(\tau)) \right) d\tau, \end{aligned}$$

where $Z = Z_{\phi, \Phi, U, \xi, \lambda}$ satisfies

$$(35) \quad \begin{aligned} Z(t) &= \int_s^t P(t)T(t, \tau) \left(\frac{\partial f}{\partial x}(y_\phi(\tau))Z(\tau) \right. \\ &\quad \left. + \frac{\partial f}{\partial y}(y_\phi(\tau)) \left(\Phi(z_\phi(\tau))Z(\tau) + U(z_\phi(\tau)) \right) + \frac{\partial f}{\partial \lambda}(y_\phi(\tau)) \right) d\tau \end{aligned}$$

for every $t \geq s$. By the continuity of the solutions of a differential equation with respect to parameters, and the continuity of $(t, \phi, s, \xi, \lambda) \mapsto x_\phi(t, \xi, \lambda)$, ϕ , Φ and U , the functions

$$(36) \quad (t, \phi, s, \xi, \lambda) \mapsto W_{\phi, \Phi, \xi, \lambda}(t) \quad \text{and} \quad (t, \phi, s, \xi, \lambda) \mapsto Z_{\phi, \Phi, U, \xi, \lambda}(t)$$

are also continuous.

Lemma 5. *The operator A is well-defined, and $A(\mathcal{X} \times \mathcal{F}) \subset \mathcal{F}$.*

Proof. Set

$$C = \int_s^\infty \left\| Q(s)T(\tau, s)^{-1} \left(\frac{\partial f}{\partial x}(y_\phi(\tau))W(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Phi(z_\phi(\tau))W(\tau) \right) \right\| d\tau.$$

It follows from (7) and (8) that

$$(37) \quad \begin{aligned} C &\leq 2\delta D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)-3\varepsilon\rho(\tau)} \rho'(\tau) \|W(\tau)\| d\tau \\ &= 2\delta D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \rho'(\tau) \|W(\tau)\| d\tau. \end{aligned}$$

On the other hand, by (35) we have

$$(38) \quad \begin{aligned} \|W(t)\| &\leq D e^{a(\rho(t)-\rho(s))+\varepsilon\rho(s)} \\ &\quad + 2\delta D \int_s^t e^{a(\rho(t)-\rho(\tau))+\varepsilon\tau-3\varepsilon\rho(\tau)} \rho'(\tau) \|W(\tau)\| d\tau. \end{aligned}$$

Setting $\Gamma(t) = e^{-a(\rho(t)-\rho(s))} \|W(t)\|$ we obtain

$$\begin{aligned} \Gamma(t) &\leq D e^{\varepsilon\rho(s)} + 2\delta D \int_s^t e^{-2\varepsilon\rho(\tau)} \rho'(\tau) \Gamma(\tau) d\tau \\ &\leq D e^{\varepsilon\rho(s)} + 2\delta D \int_s^t \rho'(\tau) \Gamma(\tau) d\tau, \end{aligned}$$

and it follows from Gronwall's lemma that

$$(39) \quad \|W(t)\| \leq D e^{\varepsilon\rho(s)} e^{(a+2\delta D)(\rho(t)-\rho(s))}.$$

By (30) and (37) we have

$$\begin{aligned} C &\leq 2\delta D^2 \int_s^\infty \rho'(\tau) e^{(-b+a-\varepsilon+2\delta D)(\rho(\tau)-\rho(s))} d\tau \\ &= \frac{2\delta D^2}{|-b+a-\varepsilon+2\delta D|} < 1. \end{aligned}$$

Therefore, $A(\phi, \Phi)$ is well-defined, and since

$$\|A(\phi, \Phi)(s, \xi, \lambda)\| \leq C < 1$$

for every $s \geq 0$, $\xi \in E(s)$, and $\lambda \in Y$, we obtain $\|A(\phi, \Phi)\| \leq 1$. This shows that $A(\mathcal{X} \times \mathcal{F}) \subset \mathcal{F}$. \square

Lemma 6. *The operator B is well-defined, and $B(\mathcal{X} \times \mathcal{F} \times \mathcal{G}) \subset \mathcal{G}$.*

Proof. Set

$$\begin{aligned} C = & \int_s^\infty \left\| Q(s)T(\tau, s)^{-1} \left(\frac{\partial f}{\partial x}(y_\phi(\tau))W(\tau) \right. \right. \\ & \left. \left. + \frac{\partial f}{\partial y}(y_\phi(\tau)) \left(\Phi(z_\phi(\tau))W(\tau) + U(z_\phi(\tau)) \right) + \frac{\partial f}{\partial \lambda}(y_\phi(\tau)) \right) \right\| d\tau. \end{aligned}$$

It follows from (7), (8), (9), (31), and (32) that

$$\begin{aligned} (40) \quad C \leq & 2\delta D \int_s^\infty \rho'(\tau) e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)-3\varepsilon\rho(\tau)} \|Z(\tau)\| d\tau \\ & + 4\delta D^2 \|\xi\| \int_s^\infty \rho'(\tau) e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)-3\varepsilon\rho(\tau)+a(\rho(\tau)-\rho(s))+\varepsilon\rho(s)} d\tau \\ = & 2\delta D \int_s^\infty \rho'(\tau) e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|Z(\tau)\| d\tau \\ & + 4\delta D^2 \|\xi\| \int_s^\infty \rho'(\tau) e^{(a-b-\varepsilon)(\rho(\tau)-\rho(s))} d\tau. \end{aligned}$$

On the other hand, by (31), (32), and (35) we have

$$\begin{aligned} \|Z(t)\| \leq & 2\delta D \int_s^t \rho'(\tau) e^{a(\rho(t)-\rho(\tau))+\varepsilon\rho(\tau)-3\varepsilon\rho(\tau)} \|Z(\tau)\| d\tau \\ & + 4\delta D^2 \|\xi\| \int_s^t \rho'(\tau) e^{a(\rho(t)-\rho(\tau))-2\varepsilon\rho(\tau)+a(\rho(\tau)-\rho(s))+\varepsilon\rho(s)} d\tau. \end{aligned}$$

Setting $\Gamma(t) = e^{-a(\rho(t)-\rho(s))} \|Z(t)\|$ we obtain

$$\begin{aligned} \Gamma(t) \leq & \frac{4\delta D^2}{\varepsilon} \|\xi\| + 2\delta D \int_s^t e^{-2\varepsilon\rho(\tau)} \rho'(\tau) \Gamma(\tau) d\tau \\ \leq & \frac{4\delta D^2}{\varepsilon} \|\xi\| + 2\delta D \int_s^t \rho'(\tau) \Gamma(\tau) d\tau, \end{aligned}$$

and it follows from Gronwall's lemma that

$$(41) \quad \|Z(t)\| \leq \frac{4\delta D^2}{\varepsilon} e^{(a+2\delta D)(\rho(t)-\rho(s))} \|\xi\|.$$

By (30) and (40) we obtain

$$C \leq \frac{8\delta^2 D^3}{\varepsilon} \|\xi\| \int_s^\infty \rho'(\tau) e^{(a-b+2\delta D)(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} d\tau$$

$$\begin{aligned}
& + 4\delta D^2 \|\xi\| \int_s^\infty \rho'(\tau) e^{(a-b-\varepsilon)(\rho(\tau)-\rho(s))} d\tau \\
& = \frac{8\delta^2 D^3 \|\xi\|}{|a-b+2\delta D|} + \frac{4\delta D^2 \|\xi\|}{|a-b-\varepsilon|} \leq \|\xi\|,
\end{aligned}$$

provided that δ is sufficiently small. This shows that $B(\phi, \Phi, U)$ is well-defined, and that $\|B(\phi, \Phi, U)\| \leq 1$. Therefore, $B(\mathcal{X} \times \mathcal{F} \times \mathcal{G}) \subset \mathcal{G}$. \square

Step 8: Construction of a fiber contraction. We consider the transformation $S: \mathcal{X} \times \mathcal{F} \times \mathcal{G} \rightarrow \mathcal{X} \times \mathcal{F} \times \mathcal{G}$ defined by

$$S(\phi, \Phi, U) = (T\phi, A(\phi, \Phi), B(\phi, \Phi, U)),$$

where we have set $(T\phi)(s, \xi, \lambda) = (T\phi)_\lambda(s, \xi)$ with T as in (29).

Lemma 7. *For every $\delta > 0$ sufficiently small, the operator S is a fiber contraction.*

Proof. Given $\phi \in \mathcal{X}$, $\Phi, \Psi \in \mathcal{F}$, $\xi \in E(s)$, and $\lambda \in Y$, set

$$W_\Phi = W_{\phi, \Phi, \xi, \lambda} \quad \text{and} \quad W_\Psi = W_{\phi, \Psi, \xi, \lambda}.$$

We have

$$\begin{aligned}
& \|A(\phi, \Phi)(s, \xi, \lambda) - A(\phi, \Psi)(s, \xi, \lambda)\| \\
& \leq D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \\
& \quad \times \left\| \frac{\partial f}{\partial x}(y_\phi(\tau)) W_\Phi(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau)) \Phi(z_\phi(\tau)) W_\Phi(\tau) \right. \\
& \quad \left. - \frac{\partial f}{\partial x}(y_\phi(\tau)) W_\Psi(\tau) - \frac{\partial f}{\partial y}(y_\phi(\tau)) \Psi(z_\phi(\tau)) W_\Psi(\tau) \right\| d\tau \\
& \leq \delta D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \rho'(\tau) \\
& \quad \times (\|W_\Phi(\tau) - W_\Psi(\tau)\| + \|\Phi(z_\phi(\tau)) W_\Phi(\tau) - \Psi(z_\phi(\tau)) W_\Psi(\tau)\|) d\tau \\
& \leq \delta D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \rho'(\tau) (\|W_\Phi(\tau) - W_\Psi(\tau)\| \\
& \quad + \|\Phi(z_\phi(\tau))\| \cdot \|W_\Phi(\tau) - W_\Psi(\tau)\| + \|\Phi - \Psi\| \cdot \|W_\Psi(\tau)\|) d\tau \\
& \leq \delta D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \rho'(\tau) \\
(42) \quad & \times (2\|W_\Phi(\tau) - W_\Psi(\tau)\| + \|\Phi - \Psi\| \cdot \|W_\Psi(\tau)\|) d\tau.
\end{aligned}$$

In a similar manner to that in (38) and using (39) we obtain

$$\begin{aligned}
& \|W_\Phi(t) - W_\Psi(t)\| \\
& \leq 2\delta D \int_s^t e^{a(\rho(t)-\rho(\tau))+\varepsilon\rho(\tau)-3\varepsilon\rho(\tau)} \rho'(\tau) \|W_\Phi(\tau) - W_\Psi(\tau)\| d\tau
\end{aligned}$$

$$\begin{aligned}
& + \delta D \|\Phi - \Psi\| \int_s^t e^{a(\rho(t)-\rho(\tau))+\varepsilon\rho(\tau)-3\varepsilon\rho(\tau)} \rho'(\tau) \|W_\Psi(\tau)\| d\tau \\
\leq & 2\delta D e^{a(\rho(t)-\rho(s))} \int_s^t e^{-a(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \rho'(\tau) \|W_\Phi(\tau) - W_\Psi(\tau)\| d\tau \\
& + \delta D^2 e^{a(\rho(t)-\rho(s))} \|\Phi - \Psi\| \\
& \times \int_s^t e^{-(a+\varepsilon)(\rho(\tau)-\rho(s))} e^{(a+2\delta D)(\rho(\tau)-\rho(s))} e^{-\varepsilon\rho(\tau)} \rho'(\tau) d\tau \\
= & 2\delta D e^{a(\rho(t)-\rho(s))} \int_s^t e^{-a(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \rho'(\tau) \|W_\Phi(\tau) - W_\Psi(\tau)\| d\tau \\
& + \delta D^2 e^{a(\rho(t)-\rho(s))} \|\Phi - \Psi\| \int_s^t e^{-(\varepsilon-2\delta D)(\rho(\tau)-\rho(s))} \rho'(\tau) d\tau.
\end{aligned}$$

Setting $\Gamma(t) = e^{-a(\rho(t)-\rho(s))} \|W_\Phi(t) - W_\Psi(t)\|$, we obtain

$$\Gamma(t) \leq \frac{\delta D^2}{|\varepsilon - 2\delta D|} \|\Phi - \Psi\| + 2\delta D \int_s^t \rho'(\tau) \Gamma(\tau) d\tau,$$

provided that δ is sufficiently small, and it follows from Gronwall's lemma that

$$(43) \quad \|W_\Phi(t) - W_\Psi(t)\| \leq \frac{\delta D^2}{|\varepsilon - 2\delta D|} \|\Phi - \Psi\| e^{(a+2\delta D)(\rho(t)-\rho(s))}.$$

By (39) and (43), and in view of (30), it follows from (42) that

$$\begin{aligned}
& \|A(\phi, \Phi)(s, \xi, \lambda) - A(\phi, \Psi)(s, \xi, \lambda)\| \\
\leq & C_1 \delta \|\Phi - \Psi\| \int_s^\infty e^{(a-b-\varepsilon+2\delta D)(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \rho'(\tau) d\tau \\
& + \delta D^2 \|\Phi - \Psi\| \int_s^\infty e^{(a-b-\varepsilon+2\delta D)(\rho(\tau)-\rho(s))} \rho'(\tau) d\tau \\
\leq & K_1 \delta \|\Phi - \Psi\| \int_s^\infty e^{(a-b-\varepsilon+2\delta D)(\rho(\tau)-\rho(s))} \rho'(\tau) d\tau \\
(44) \quad & \leq \frac{K_1 \delta}{|a-b-\varepsilon+2\delta D|} \|\Phi - \Psi\|
\end{aligned}$$

for some constants $C_1, K_1 > 0$.

Now we consider the operator B . Given $\phi \in \mathcal{X}$, $\Phi, \Psi \in \mathcal{F}$, $U, V \in \mathcal{G}$, $\xi \in E(s)$, and $\lambda \in Y$, set

$$Z_{\Phi, U} = Z_{\phi, \Phi, U, \xi, \lambda} \quad \text{and} \quad Z_{\Psi, V} = W_{\phi, \Psi, V, \xi, \lambda}.$$

We have

$$\begin{aligned}
& \|B(\phi, \Phi, U)(s, \xi, \lambda) - B(\phi, \Psi, V)(s, \xi, \lambda)\| \\
\leq & D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)}
\end{aligned}$$

$$\begin{aligned}
& \times \left\| \frac{\partial f}{\partial x}(y_\phi(\tau))Z_{\Phi,U}(\tau) - \frac{\partial f}{\partial x}(y_\phi(\tau))Z_{\Psi,V}(\tau) \right. \\
& \quad + \frac{\partial f}{\partial y}(y_\phi(\tau))\left(\Phi(z_\phi(\tau))Z_{\Phi,U}(\tau) + U(z_\phi(\tau))\right) \\
& \quad \left. - \frac{\partial f}{\partial y}(y_\phi(\tau))\left(\Psi(z_\phi(\tau))Z_{\Psi,V}(\tau) + V(z_\phi(\tau))\right) \right\| d\tau \\
& \leq \delta D \int_s^\infty \rho'(\tau)e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)}(2\|Z_{\Phi,U}(\tau) - Z_{\Psi,V}(\tau)\| \\
(45) \quad & + \|\Phi - \Psi\| \cdot \|Z_{\Phi,U}(\tau)\| + \|z_\phi(\tau)\| \cdot \|U - V\|) d\tau.
\end{aligned}$$

In a similar manner to that in (45) and using (41) we obtain

$$\begin{aligned}
& \|Z_{\Phi,U}(t) - Z_{\Psi,V}(t)\| \\
& \leq \delta D \int_s^\infty \rho'(\tau)e^{a(\rho(t)-\rho(\tau))-2\varepsilon\rho(\tau)}(2\|Z_{\Phi,U}(\tau) - Z_{\Psi,V}(\tau)\| \\
& \quad + \|\Phi - \Psi\| \cdot \|Z_{\Phi,U}(\tau)\| + \|z_\phi(\tau)\| \cdot \|U - V\|) d\tau \\
& \leq 2\delta D e^{a(\rho(t)-\rho(s))} \int_s^t \rho'(\tau)e^{-a(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)}\|Z_{\Phi,U}(\tau) - Z_{\Psi,V}(\tau)\| d\tau \\
& \quad + \frac{4\delta^2 D^3}{\varepsilon} \|\xi\| \cdot \|\Phi - \Psi\| e^{a(\rho(t)-\rho(s))} \int_s^t \rho'(\tau)e^{2\delta D(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} d\tau \\
& \quad + 2\delta D^2 \|\xi\| \cdot \|U - V\| e^{a(\rho(t)-\rho(s))} \int_s^t \rho'(\tau)e^{-\varepsilon(\rho(\tau)-\rho(s))} d\tau \\
& \leq 2\delta D e^{a(\rho(t)-\rho(s))} \int_s^t \rho'(\tau)e^{-a(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)}\|Z_{\Phi,U}(\tau) - Z_{\Psi,V}(\tau)\| d\tau \\
& \quad + \frac{2\delta D^2}{\varepsilon} e^{(a+2\delta D)(\rho(t)-\rho(s))} \|\xi\| \cdot \|\Phi - \Psi\| + \frac{2\delta D^2}{\varepsilon} e^{a(\rho(t)-\rho(s))} \|\xi\| \cdot \|U - V\|.
\end{aligned}$$

Setting

$$\Gamma(t) = e^{-a(\rho(t)-\rho(s))}\|Z_{\Phi,U}(t) - Z_{\Psi,V}(t)\|,$$

yields

$$\begin{aligned}
(46) \quad \Gamma(t) & \leq \frac{2\delta D^2}{\varepsilon} e^{2\delta D(\rho(t)-\rho(s))} \|\xi\| \cdot \|\Phi - \Psi\| + \frac{2\delta D^2}{\varepsilon} \|\xi\| \cdot \|U - V\| \\
& \quad + 2\delta D \int_s^t \rho'(\tau)\Gamma(\tau) d\tau.
\end{aligned}$$

We have

$$\begin{aligned}
& \frac{d}{dt} \left(e^{-2\delta D(\rho(t)-\rho(s))} \int_s^t \rho'(\tau)\Gamma(\tau) d\tau \right) \\
& = e^{-2\delta D(\rho(t)-\rho(s))} \rho'(t) \left(\Gamma(t) - 2\delta D \int_s^t \rho'(\tau)\Gamma(\tau) d\tau \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2\delta D^2}{\varepsilon} \rho'(t) \|\xi\| (\|U - V\| e^{-2\delta D(\rho(t) - \rho(s))} + \|\Phi - \Psi\|) \\
&\leq \frac{2\delta D^2}{\varepsilon} \rho'(t) \|\xi\| (\|U - V\| + \|\Phi - \Psi\|),
\end{aligned}$$

and integrating,

$$\int_s^t \rho'(\tau) \Gamma(\tau) d\tau \leq \frac{2\delta D^2}{\varepsilon} \|\xi\| (\|U - V\| + \|\Phi - \Psi\|) e^{2\delta D(\rho(t) - \rho(s))} (\rho(t) - \rho(s)).$$

By (46) this yields

$$\begin{aligned}
\Gamma(t) &\leq \frac{2\delta D^2}{\varepsilon} \|\xi\| (\|U - V\| + \|\Phi - \Psi\|) e^{2\delta D(\rho(t) - \rho(s))} [1 + 2\delta D(\rho(t) - \rho(s))] \\
&\leq \frac{2\delta D^2}{\varepsilon} \|\xi\| (\|U - V\| + \|\Phi - \Psi\|) e^{4\delta D(\rho(t) - \rho(s))},
\end{aligned}$$

and hence,

$$(47) \quad \|Z_{\Phi, U}(t) - Z_{\Psi, V}(t)\| \leq \frac{2\delta D^2}{\varepsilon} \|\xi\| (\|\Phi - \Psi\| + \|U - V\|) e^{(a+4\delta D)(\rho(t) - \rho(s))}.$$

By (47) and (30), it follows from (45) that

$$\begin{aligned}
&\|B(\phi, \Phi, U)(s, \xi, \lambda) - B(\phi, \Psi, V)(s, \xi, \lambda)\| \\
&\leq \frac{4\delta^2 D^3}{\varepsilon} \|\xi\| \cdot (\|\Phi - \Psi\| + \|U - V\|) \int_s^\infty \rho'(\tau) e^{(a-b+4\delta D)(\rho(\tau) - \rho(s))} d\tau \\
&\quad + \frac{4\delta^2 D^3}{\varepsilon} \|\xi\| \|\Phi - \Psi\| \int_s^\infty \rho'(\tau) e^{(a-b+2\delta D)(\rho(\tau) - \rho(s))} d\tau \\
&\quad + 2\delta D^2 \|\xi\| \cdot \|U - V\| \int_s^\infty \rho'(\tau) e^{(a-b-\varepsilon)(\rho(\tau) - \rho(s))} d\tau \\
&= \frac{4\delta^2 D^3}{\varepsilon |a-b+4\delta D|} \|\xi\| (\|\Phi - \Psi\| + \|U - V\|) \\
&\quad + \frac{4\delta^2 D^3}{\varepsilon |a-b+2\delta D|} \|\xi\| \cdot \|\Phi - \Psi\| + \frac{2\delta D^2}{|a-b-\varepsilon|} \|\xi\| \cdot \|U - V\| \\
(48) \quad &\leq \frac{2\delta D^2}{|a-b+4\delta D|} \left(4\delta D + \frac{1}{\varepsilon}\right) \|\xi\| (\|\Phi - \Psi\| + \|U - V\|).
\end{aligned}$$

It follows from (44) and (48) that for δ sufficiently small the operator S is a fiber contraction. \square

Step 9: Continuity of the fiber contraction.

Lemma 8. *For every $\delta > 0$ sufficiently small, the operator S is continuous.*

Proof. Setting

$$W_\phi = W_{\phi, \Phi, \xi} \quad \text{and} \quad W_\psi = W_{\psi, \Phi, \xi},$$

we obtain

$$\|A(\phi, \Phi)(s, \xi, \lambda) - A(\psi, \Phi)(s, \xi, \lambda)\|$$

$$\begin{aligned} &\leq D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \\ &\quad \times \left\| \frac{\partial f}{\partial x}(y_\phi(\tau))W_\phi(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Phi(z_\phi(\tau))W_\phi(\tau) \right. \\ &\quad \left. - \frac{\partial f}{\partial x}(y_\psi(\tau))W_\psi(\tau) - \frac{\partial f}{\partial y}(y_\psi(\tau))\Phi(z_\psi(\tau))W_\psi(\tau) \right\| d\tau. \end{aligned}$$

It follows from (8) and (39) that

$$\begin{aligned} &\|A(\phi, \Phi)(s, \xi, \lambda) - A(\psi, \Phi)(s, \xi, \lambda)\| \\ &\leq D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial x}(y_\phi(\tau)) - \frac{\partial f}{\partial x}(y_\psi(\tau)) \right\| \cdot \|W_\phi(\tau)\| d\tau \\ &\quad + D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial x}(y_\psi(\tau)) \right\| \cdot \|W_\phi(\tau) - W_\psi(\tau)\| d\tau \\ &\quad + D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial y}(y_\phi(\tau)) - \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| \cdot \|\Phi(z_\phi(\tau))W_\phi(\tau)\| d\tau \\ &\quad + D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| \cdot \|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\| \cdot \|W_\phi(\tau)\| d\tau \\ &\quad + D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| \cdot \|\Phi(z_\psi(\tau))\| \cdot \|W_\phi(\tau) - W_\psi(\tau)\| d\tau, \end{aligned}$$

and setting $\chi(\tau) = \min \{1, \rho'(\tau)\}$, this yields

$$\begin{aligned} &\|A(\phi, \Phi)(s, \xi, \lambda) - A(\psi, \Phi)(s, \xi, \lambda)\| \\ &\leq D^2 e^{2\varepsilon\rho(s)} \int_s^\infty e^{(a+2\delta D+\varepsilon-b)(\rho(\tau)-\rho(s))} \left\| \frac{\partial f}{\partial x}(y_\phi(\tau)) - \frac{\partial f}{\partial x}(y_\psi(\tau)) \right\| d\tau \\ &\quad + \delta D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \chi(\tau) \|W_\phi(\tau) - W_\psi(\tau)\| d\tau \\ &\quad + D^2 e^{2\varepsilon\rho(s)} \int_s^\infty e^{(a+2\delta D+\varepsilon-b)(\rho(\tau)-\rho(s))} \left\| \frac{\partial f}{\partial y}(y_\phi(\tau)) - \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| d\tau \\ &\quad + \delta D^2 \int_s^\infty e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))} e^{-\varepsilon\rho(\tau)} \|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\| \chi(\tau) d\tau \\ &\quad + \delta D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \chi(\tau) \|W_\phi(\tau) - W_\psi(\tau)\| d\tau \\ &\leq 2D^2 e^{2\varepsilon\rho(s)} \int_s^\infty e^{(a+2\delta D+\varepsilon-b)(\rho(\tau)-\rho(s))} \left\| \frac{\partial f}{\partial u}(y_\phi(\tau)) - \frac{\partial f}{\partial u}(y_\psi(\tau)) \right\| d\tau \\ &\quad + 2\delta D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|W_\phi(\tau) - W_\psi(\tau)\| \chi(\tau) d\tau \\ (49) \quad &\quad + \delta D^2 \int_s^\infty e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))-\varepsilon\rho(\tau)} \|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\| \chi(\tau) d\tau. \end{aligned}$$

Again by (8) and (39), and in view of (30), given $\gamma > 0$ there exists $\sigma > 0$ (independent of s and ξ) such that, setting $\eta = \rho^{-1}(\rho(s) + \rho(\sigma))$,

$$\begin{aligned}
& 2D^2 e^{2\varepsilon\rho(s)} \int_{\eta}^{\infty} e^{(a+2\delta D+\varepsilon-b)(\rho(\tau)-\rho(s))} \left\| \frac{\partial f}{\partial u}(y_{\phi}(\tau)) - \frac{\partial f}{\partial u}(y_{\psi}(\tau)) \right\| d\tau \\
& \leq 4\delta D^2 \int_{\eta}^{\infty} e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))} \rho'(\tau) d\tau \\
(50) \quad & = \frac{4\delta D^2 e^{(a+2\delta D-\varepsilon-b)\rho(\sigma)}}{|a+2\delta D-\varepsilon-b|} < \gamma,
\end{aligned}$$

$$\begin{aligned}
& 2\delta D \int_{\eta}^{\infty} e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|W_{\phi}(\tau) - W_{\psi}(\tau)\| \chi(\tau) d\tau \\
& \leq 2\delta D \int_{\eta}^{\infty} e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|W_{\phi}(\tau) - W_{\psi}(\tau)\| \rho'(\tau) d\tau \\
(51) \quad & \leq 4\delta D^2 \int_{\eta}^{\infty} e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))} \rho'(\tau) d\tau < \gamma,
\end{aligned}$$

and

$$\begin{aligned}
& \delta D^2 \int_{\eta}^{\infty} e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))-\varepsilon\rho(\tau)} \|\Phi(z_{\phi}(\tau)) - \Phi(z_{\psi}(\tau))\| \chi(\tau) d\tau \\
& \leq \delta D^2 \int_{\eta}^{\infty} e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))} \|\Phi(z_{\phi}(\tau)) - \Phi(z_{\psi}(\tau))\| \rho'(\tau) d\tau \\
(52) \quad & \leq 2\delta D^2 \int_{\eta}^{\infty} e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))} \rho'(\tau) d\tau < \gamma.
\end{aligned}$$

Now we consider the integrals from s to η . We show that given $\gamma > 0$ there exists $\eta > 0$ (independent of s and ξ) such that each integral from s to η is bounded by γ whenever $d(\phi, \psi) < \gamma$. Setting $p = \rho(\tau) - \rho(s)$, we consider the functions

$$\begin{aligned}
B(p, \phi)(s, \xi, \lambda) &= \frac{2D^2 e^{2\varepsilon\rho(s)}}{\rho'(\rho^{-1}(p + \rho(s)))} e^{(a+2\delta D+\varepsilon-b)p} \frac{\partial f}{\partial u}(y_{\phi}(\rho^{-1}(p + \rho(s)))), \\
C(p, \phi)(s, \xi, \lambda) &= 2\delta D e^{-bp-2\varepsilon(p+\rho(s))} W_{\phi}(\rho^{-1}(p + \rho(s))), \\
D(p, \phi)(s, \xi, \lambda) &= \delta D^2 e^{(a+2\delta D-\varepsilon-b)p} e^{-\varepsilon(p+\rho(s))} \Phi(z_{\phi}(\rho^{-1}(p + \rho(s))),
\end{aligned}$$

for each $p \in [0, \rho(\sigma)]$ and $\phi \in \mathcal{X}$. We note that

$$\begin{aligned}
& 2D^2 e^{\varepsilon\rho(s)} \int_s^{\rho^{-1}(\rho(s)+\rho(\sigma))} e^{(a+2\delta D+\varepsilon-b)(\rho(\tau)-\rho(s))} \frac{\partial f}{\partial u}(y_{\phi}(\tau)) d\tau \\
& + 2\delta D \int_s^{\rho^{-1}(\rho(s)+\rho(\sigma))} e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} W_{\phi}(\tau) \rho'(\tau) d\tau \\
& + \delta D^2 \int_s^{\rho^{-1}(\rho(s)+\rho(\sigma))} e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))} e^{-\varepsilon\rho(\tau)} \Phi(z_{\phi}(\tau)) \rho'(\tau) d\tau
\end{aligned}$$

$$= \int_0^{\rho(\sigma)} [B(p, \phi) + C(p, \phi) + D(p, \phi)](s, \xi, \lambda) dp.$$

Therefore, by (49), it is sufficient to show that

$$(53) \quad \phi \mapsto \int_0^{\rho(\sigma)} [B(p, \phi) + C(p, \phi) + D(p, \phi)] dp$$

is continuous. Since the functions Φ ,

$$(t, \phi, s, \xi, \lambda) \mapsto x_\phi(t, \xi) \quad \text{and} \quad (t, \phi, s, \xi, \lambda) \mapsto W_{\phi, \Phi, \xi}(t)$$

are continuous, the functions

$$(54) \quad (p, \phi, s, \xi, \lambda) \mapsto B(p, \phi)(s, \xi, \lambda), \quad C(p, \phi)(s, \xi, \lambda), \quad D(p, \phi)(s, \xi, \lambda)$$

are also continuous. Furthermore, by (8), (30), and (39), given $p \in [0, \rho(\sigma)]$ and $\phi \in \mathcal{X}$ we have

$$\begin{aligned} \|B(p, \phi)\| &\leq 2\delta D^2 e^{(a+2\delta D-\varepsilon-b)p} e^{-\varepsilon(p+\rho(s))} \leq 2\delta D^2 e^{-\varepsilon\rho(s)}, \\ \|C(p, \phi)\| &\leq 2\delta D^2 e^{(a+2\delta D-\varepsilon-b)p} e^{-\varepsilon(p+\rho(s))} \leq 2\delta D^2 e^{-\varepsilon\rho(s)}, \\ \|D(p, \phi)\| &\leq \delta D^2 e^{(a+2\delta D-\varepsilon-b)p} e^{-\varepsilon(p+\rho(s))} \leq \delta D^2 e^{-\varepsilon\rho(s)}. \end{aligned}$$

Here we are using the norm $\|\cdot\|$ in (31). In particular, $B(p, \phi)$, $C(p, \phi)$, and $D(p, \phi)$ are in \mathcal{F} provided that δ is sufficiently small. We proceed with the proof of the continuity of the integral in (53). We first note that there exists $R > 0$ such that

$$\begin{aligned} \|B(p, \phi)(s, \xi, \lambda) - B(q, \psi)(s, \xi, \lambda)\| &\leq 4\delta D^2 e^{-\varepsilon\rho(s)} < \gamma, \\ \|C(p, \phi)(s, \xi, \lambda) - C(q, \psi)(s, \xi, \lambda)\| &\leq 4\delta D^2 e^{-\varepsilon\rho(s)} < \gamma, \\ \|D(p, \phi)(s, \xi, \lambda) - D(q, \psi)(s, \xi, \lambda)\| &\leq 4\delta D^2 e^{-\varepsilon\rho(s)} < \gamma \end{aligned}$$

for every $s > R$, $p \in [0, \rho(\sigma)]$, $\xi \in E(s)$, and $\lambda \in Y$. It remains to consider the case when $s \leq R$. Given $s \in \mathbb{R}_0^+$ and $(\phi, \xi, \lambda) \in \mathcal{X} \times E(s) \times Y$, we observe that due to the continuity in (54) there exists $\delta > 0$ such that

$$\|B(p, \phi)(s, \xi, \lambda) - B(q, \psi)(\bar{s}, \bar{\xi}, \bar{\lambda})\| < \gamma$$

whenever $d(\phi, \psi) < \delta$ and $\|(p, s, \xi, \lambda) - (q, \bar{s}, \bar{\xi}, \bar{\lambda})\| < \delta$. Since $u \mapsto f(t, u, \lambda)$ vanishes for $\|u\| \geq c$, given s it is sufficient to establish the desired continuity for ξ inside a certain ball in $E(s)$, possibly depending (continuously) on p and s . This shows that it is sufficient to consider (ξ, λ) in some compact set K . We can cover $[0, \rho(\sigma)] \times [0, R] \times K$ with a finite number of balls B_i , $i = 1, \dots, r$ centered at points in this set, such that

$$\|B(p, \phi)(s, \xi, \lambda) - B(\bar{p}, \bar{\psi})(\bar{s}, \bar{\xi}, \bar{\lambda})\| < \gamma$$

whenever $d(\phi, \psi) < \delta_i$ and $(p, s, \xi, \lambda), (\bar{p}, \bar{s}, \bar{\xi}, \bar{\lambda}) \in B_i$, for $i = 1, \dots, r$ and some numbers $\delta_i > 0$. Therefore,

$$\|B(p, \phi)(s, \xi, \lambda) - B(p, \psi)(s, \xi, \lambda)\| < \gamma$$

whenever $d(\phi, \psi) < \delta = \min\{\delta_1, \dots, \delta_r\}$, for every $p \in [0, \rho(\sigma)]$, $s \leq R$, and $(\xi, \lambda) \in K$. This shows that

$$\sup_{s \leq R} \sup_{(\xi, \lambda) \in K} \|B(p, \phi)(s, \xi, \lambda) - B(p, \psi)(s, \xi, \lambda)\| \leq \gamma$$

whenever $d(\phi, \psi) < \delta$. The argument is identical for the operators $C(p, \phi)$ and $D(p, \phi)$. It follows from these inequalities that the map in (53) is continuous. Together with (50), (51), and (52) this implies that $\phi \mapsto A(\phi, \Phi)$ is continuous.

Now we consider the operator B . Setting

$$Z_\phi = Z_{\phi, \Phi, U, \xi, \lambda} \quad \text{and} \quad Z_\psi = Z_{\psi, \Phi, U, \xi, \lambda},$$

we obtain

$$\begin{aligned} & \|B(\phi, \Phi, U)(s, \xi, \lambda) - B(\psi, \Phi, U)(s, \xi, \lambda)\| \\ & \leq D \int_s^\infty e^{-b(\rho(\tau) - \rho(s)) + \varepsilon \rho(\tau)} \\ & \quad \times \left\| \frac{\partial f}{\partial x}(y_\phi(\tau)) Z_\phi(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau)) \Phi(z_\phi(\tau)) Z_\phi(\tau) \right. \\ & \quad + \frac{\partial f}{\partial y}(y_\phi(\tau)) U(z_\phi(\tau)) + \frac{\partial f}{\partial \lambda}(y_\phi(\tau)) - \frac{\partial f}{\partial x}(y_\psi(\tau)) Z_\psi(\tau) \\ & \quad \left. - \frac{\partial f}{\partial y}(y_\psi(\tau)) \Phi(z_\psi(\tau)) Z_\psi(\tau) - \frac{\partial f}{\partial y}(y_\psi(\tau)) U(z_\psi(\tau)) - \frac{\partial f}{\partial \lambda}(y_\psi(\tau)) \right\| d\tau \\ & \leq D \int_s^\infty e^{-b(\rho(\tau) - \rho(s)) + \varepsilon \rho(\tau)} \left\| \frac{\partial f}{\partial x}(y_\phi(\tau)) - \frac{\partial f}{\partial x}(y_\psi(\tau)) \right\| \cdot \|Z_\phi(\tau)\| d\tau \\ & \quad + D \int_s^\infty e^{-b(\rho(\tau) - \rho(s)) + \varepsilon \rho(\tau)} \left\| \frac{\partial f}{\partial x}(y_\psi(\tau)) \right\| \cdot \|Z_\phi(\tau) - Z_\psi(\tau)\| d\tau \\ & \quad + D \int_s^\infty e^{-b(\rho(\tau) - \rho(s)) + \varepsilon \rho(\tau)} \left\| \frac{\partial f}{\partial y}(y_\phi(\tau)) - \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| \|\Phi(z_\phi(\tau))\| \cdot \|Z_\phi(\tau)\| d\tau \\ & \quad + D \int_s^\infty e^{-b(\rho(\tau) - \rho(s)) + \varepsilon \rho(\tau)} \left\| \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| \|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\| \cdot \|Z_\phi(\tau)\| d\tau \\ & \quad + D \int_s^\infty e^{-b(\rho(\tau) - \rho(s)) + \varepsilon \rho(\tau)} \left\| \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| \|\Phi(z_\psi(\tau))\| \cdot \|Z_\phi(\tau) - Z_\psi(\tau)\| d\tau \\ & \quad + D \int_s^\infty e^{-b(\rho(\tau) - \rho(s)) + \varepsilon \rho(\tau)} \left\| \frac{\partial f}{\partial y}(y_\phi(\tau)) - \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| \cdot \|U(z_\phi(\tau))\| d\tau \\ & \quad + D \int_s^\infty e^{-b(\rho(\tau) - \rho(s)) + \varepsilon \rho(\tau)} \left\| \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| \cdot \|U(z_\phi(\tau)) - U(z_\psi(\tau))\| d\tau \\ & \quad + D \int_s^\infty e^{-b(\rho(\tau) - \rho(s)) + \varepsilon \rho(\tau)} \left\| \frac{\partial f}{\partial \lambda}(y_\phi(\tau)) - \frac{\partial f}{\partial \lambda}(y_\psi(\tau)) \right\| d\tau. \end{aligned}$$

Setting again $\chi(\tau) = \min\{1, \rho'(\tau)\}$, it follows from (8) and (41) that

$$\begin{aligned}
& \|B(\phi, \Phi, U)(s, \xi, \lambda) - B(\psi, \Phi, U)(s, \xi, \lambda)\| \\
& \leq \frac{4\delta D^3}{\varepsilon} \|\xi\| \int_s^\infty e^{(a-b+2\delta D)(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial x}(y_\phi(\tau)) - \frac{\partial f}{\partial x}(y_\psi(\tau)) \right\| d\tau \\
& \quad + \delta D \int_s^\infty \chi(\tau) e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|Z_\phi(\tau) - Z_\psi(\tau)\| d\tau \\
& \quad + \frac{4\delta D^3}{\varepsilon} \|\xi\| \int_s^\infty e^{(a-b+2\delta D)(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial y}(y_\phi(\tau)) - \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| d\tau \\
& \quad + \frac{4\delta^2 D^3}{\varepsilon} \|\xi\| \int_s^\infty \chi(\tau) e^{(a-b+2\delta D)(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\| d\tau \\
& \quad + \delta D \int_s^\infty \chi(\tau) e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|Z_\phi(\tau) - Z_\psi(\tau)\| d\tau \\
& \quad + 2D^2 \|\xi\| e^{\varepsilon\rho(s)} \int_s^\infty e^{(a-b)(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial y}(y_\phi(\tau)) - \frac{\partial f}{\partial y}(y_\psi(\tau)) \right\| d\tau \\
& \quad + D\delta \int_s^\infty \chi(\tau) e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|U(z_\phi(\tau)) - U(z_\psi(\tau))\| d\tau \\
& \quad + D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial \lambda}(y_\phi(\tau)) - \frac{\partial f}{\partial \lambda}(y_\psi(\tau)) \right\| d\tau \\
& \leq L \|\xi\| e^{\varepsilon\rho(s)} \int_s^\infty e^{(a-b+2\delta D)(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial u}(y_\phi(\tau)) - \frac{\partial f}{\partial u}(y_\psi(\tau)) \right\| d\tau \\
& \quad + 2\delta D \int_s^\infty \chi(\tau) e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|Z_\phi(\tau) - Z_\psi(\tau)\| d\tau \\
& \quad + \frac{4\delta^2 D^3}{\varepsilon} \|\xi\| \int_s^\infty \chi(\tau) e^{(a-b+2\delta D)(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\| d\tau \\
& \quad + D\delta \int_s^\infty \chi(\tau) e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|U(z_\phi(\tau)) - U(z_\psi(\tau))\| d\tau \\
& \quad + D \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial \lambda}(y_\phi(\tau)) - \frac{\partial f}{\partial \lambda}(y_\psi(\tau)) \right\| d\tau
\end{aligned}$$

for some constant $L > 0$. Again by (8) and (41), and in view of (30), given $\gamma > 0$ there exists $\sigma > 0$ (independent of s and ξ) such that

$$\begin{aligned}
& L \|\xi\| e^{2\varepsilon\rho(s)} \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^\infty e^{(a+2\delta D+\varepsilon-b)(\rho(\tau)-\rho(s))} \left\| \frac{\partial f}{\partial u}(y_\phi(\tau)) - \frac{\partial f}{\partial u}(y_\psi(\tau)) \right\| d\tau \\
& \leq 2\delta L \|\xi\| \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^\infty \rho'(\tau) e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))} d\tau \\
(55) \quad & = \frac{2\delta L e^{(a+2\delta D-\varepsilon-b)\rho(\sigma)}}{|a+2\delta D-\varepsilon-b|} \|\xi\| \leq \gamma \|\xi\|, \\
& \quad 2\delta D \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^\infty \chi(\tau) e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|Z_\phi(\tau) - Z_\psi(\tau)\| d\tau
\end{aligned}$$

$$\begin{aligned}
& \leq 2\delta D \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^{\infty} \rho'(\tau) e^{-b(\rho(\tau)-\rho(s))-\varepsilon\rho(\tau)} \|Z_\phi(\tau) - Z_\psi(\tau)\| d\tau \\
(56) \quad & \leq \frac{16\delta^2 D^3}{\varepsilon} \|\xi\| \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^{\infty} \rho'(\tau) e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))} d\tau \leq \gamma \|\xi\|,
\end{aligned}$$

$$\begin{aligned}
& \frac{4\delta^2 D^3}{\varepsilon} \|\xi\| \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^{\infty} \chi(\tau) e^{(a-b+2\delta D)(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \\
& \quad \times \|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\| d\tau \\
& \leq \frac{4\delta^2 D^3}{\varepsilon} \|\xi\| \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^{\infty} \rho'(\tau) e^{(a-b+2\delta D)(\rho(\tau)-\rho(s))-\varepsilon\rho(\tau)} \\
& \quad \times \|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\| d\tau \\
(57) \quad & \leq \frac{8\delta^2 D^3}{\varepsilon} \|\xi\| \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^{\infty} \rho'(\tau) e^{(a+2\delta D-\varepsilon-b)(\rho(\tau)-\rho(s))} d\tau \leq \gamma \|\xi\|,
\end{aligned}$$

$$\begin{aligned}
& D\delta \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^{\infty} \chi(\tau) e^{-b(\rho(\tau)-\rho(s))-2\varepsilon\rho(\tau)} \|U(z_\phi(\tau)) - U(z_\psi(\tau))\| d\tau \\
& \leq D\delta \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^{\infty} \rho'(\tau) e^{-b(\rho(\tau)-\rho(s))-\varepsilon\rho(\tau)} \|U(z_\phi(\tau)) - U(z_\psi(\tau))\| d\tau \\
(58) \quad & \leq 4\delta D^2 \|\xi\| \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^{\infty} \rho'(\tau) e^{(a-b-\varepsilon)(\rho(\tau)-\rho(s))} d\tau \leq \gamma \|\xi\|,
\end{aligned}$$

and

$$\begin{aligned}
& D \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^{\infty} e^{-b(\rho(\tau)-\rho(s))+\varepsilon\rho(\tau)} \left\| \frac{\partial f}{\partial \lambda}(y_\phi(\tau)) - \frac{\partial f}{\partial \lambda}(y_\psi(\tau)) \right\| d\tau \\
(59) \quad & \leq 4\delta D^2 \|\xi\| \int_{\rho^{-1}(\rho(s)+\rho(\sigma))}^{\infty} \rho'(\tau) e^{(a-b-\varepsilon)(\rho(\tau)-\rho(s))} d\tau \leq \gamma \|\xi\|.
\end{aligned}$$

Now we consider the functions

$$\begin{aligned}
C_1(p, \phi)(s, \xi, \lambda) &= \frac{L\|\xi\| e^{2\varepsilon\rho(s)} e^{(a+2\delta D+\varepsilon-b)p}}{\rho'(\rho^{-1}(p+\rho(s)))} \cdot \frac{\partial f}{\partial u}(y_\phi(\rho^{-1}(p+\rho(s)))), \\
C_2(p, \phi)(s, \xi, \lambda) &= 2\delta D e^{-bp-2\varepsilon(p+\rho(s))} Z_\phi(\rho^{-1}(p+\rho(s))), \\
C_3(p, \phi)(s, \xi, \lambda) &= \frac{4\delta^2 D^3}{\varepsilon} \|\xi\| e^{(a+2\delta D-b)p-2\varepsilon(p+\rho(s))} \Phi(\rho^{-1}(p+\rho(s))), \\
C_4(p, \phi)(s, \xi, \lambda) &= D\delta e^{-bp-2\varepsilon(p+\rho(s))} U(z_\phi(\rho^{-1}(p+\rho(s))), \\
C_5(p, \phi)(s, \xi, \lambda) &= \frac{D e^{-bp-\varepsilon(p+\rho(s))}}{\rho'(\rho^{-1}(p+\rho(s)))} \cdot \frac{\partial f}{\partial \lambda}(y_\phi(\rho^{-1}(p+\rho(s))))
\end{aligned}$$

for each $p \in [0, \rho(\sigma)]$ and $\phi \in \mathcal{X}$. By (49) and (55), (56), (57), (58), and (59), it remains to show that the integral

$$\int_0^{\rho(\sigma)} \sum_{i=1}^5 C_i(p, \phi) dp$$

is continuous in ϕ . Since the function $(t, \phi, s, \xi, \lambda) \mapsto x_\phi(t, \xi, \lambda)$, those in (36), together with Φ and U are continuous, the functions

$$(p, \phi, s, \xi, \lambda) \mapsto C_i(p, \phi)(s, \xi, \lambda)$$

are also continuous. Furthermore, by (8), (41), and (30), for each $p \in [0, \rho(\sigma)]$ and $\phi \in \mathcal{X}$ we have

$$\begin{aligned} \|C_1(p, \phi)\| &\leq L\delta e^{(a+2\delta D-\varepsilon-b)p} e^{-\varepsilon(p+\rho(s))} \leq L\delta e^{-\varepsilon\rho(s)}, \\ \|C_2(p, \phi)\| &\leq \frac{8\delta^2 D^3}{\varepsilon} e^{(a+2\delta D-\varepsilon-b)p} e^{-\varepsilon(p+\rho(s))} \leq \frac{8\delta^2 D^3}{\varepsilon} e^{-\varepsilon\rho(s)}, \\ \|C_3(p, \phi)\| &\leq \frac{4\delta^2 D^3}{\varepsilon} e^{(a+2\delta D-\varepsilon-b)p} e^{-\varepsilon(p+\rho(s))} \leq \frac{4\delta^2 D^3}{\varepsilon} e^{-\varepsilon\rho(s)}, \\ \|C_4(p, \phi)\| &\leq \frac{6D^3\delta^2}{\varepsilon} e^{(a+2\delta D-b-\varepsilon)p} e^{-\varepsilon(p+\rho(s))} \leq \frac{6D^3\delta^2}{\varepsilon} e^{-\varepsilon\rho(s)}, \\ \|C_5(p, \phi)\| &\leq \frac{12D^3\delta^2}{\varepsilon} e^{(a+2\delta D-b-\varepsilon)p} e^{-\varepsilon\rho(s)} \leq \frac{12D^3\delta^2}{\varepsilon} e^{-\varepsilon\rho(s)}, \end{aligned}$$

with the norm $\|\cdot\|$ in (32). We can now show in a similar manner to that for A that the operator $\phi \mapsto B(\phi, \Phi)$ is continuous, and we conclude that S is also continuous (the operator T in (29) is a contraction, and thus it is continuous). \square

Step 10: C^1 regularity of the stable manifold. Now we establish the C^1 regularity of the function $\phi = (\phi_\lambda)_{\lambda \in Y}$ in Theorem 3, or more precisely of the function $\bar{\phi}$ defined by $\bar{\phi}(s, \xi, \lambda) = \phi_\lambda(s, \xi)$. We start with an auxiliary statement.

Lemma 9. *Given $\phi \in \mathcal{X}$, if $\bar{\phi}$ is of class C^1 in ξ and λ , then $\overline{T\bar{\phi}}$ is also of class C^1 in ξ and λ , and*

$$(60) \quad \partial(\overline{T\bar{\phi}})/\partial\xi = A(\phi, \partial\bar{\phi}/\partial\xi) \quad \text{and} \quad \partial(\overline{T\bar{\phi}})/\partial\lambda = B(\phi, \partial\bar{\phi}/\partial\xi, \partial\bar{\phi}/\partial\lambda).$$

Proof. Since $\bar{\phi}$ is of class C^1 in ξ and λ , the function y defined by $y(t, \xi, \lambda) = x_\phi(t, \xi, \lambda)$ is also of class C^1 (when $\bar{\phi}$ is of class C^1 the right-hand side of (33) is also of class C^1 , and thus the solutions are C^1 in the initial conditions and on the parameters). Furthermore, for $\Phi = \partial\bar{\phi}/\partial\xi$ and $U = \partial\bar{\phi}/\partial\lambda$ the solutions of equations (34) and (35) are given respectively by $W(t) = \partial y/\partial\xi$ and $Z(t) = \partial y/\partial\lambda$. Therefore, repeating arguments in the proofs of Lemmas 5 and 6 we can apply Leibnitz's rule to obtain

$$\begin{aligned} A(\phi, \partial\bar{\phi}/\partial\xi) &= - \int_s^\infty \frac{\partial}{\partial\xi} \left[Q(s)T(\tau, s)^{-1} f(\tau, x(\tau), \phi_\lambda(\tau, x(\tau)), \lambda) \right] d\tau \\ &= \partial(\overline{T\bar{\phi}})/\partial\xi, \end{aligned}$$

and

$$\begin{aligned} B(\phi, \partial\bar{\phi}/\partial\xi, \partial\bar{\phi}/\partial\lambda) &= - \int_s^\infty \frac{\partial}{\partial\lambda} \left[Q(s)T(\tau, s)^{-1} f(\tau, x(\tau), \phi_\lambda(\tau, x(\tau)), \lambda) \right] d\tau \\ &= \partial(\overline{T\bar{\phi}})/\partial\lambda, \end{aligned}$$

where we have written for simplicity $x_\phi(\tau, \xi, \lambda) = x(\tau)$. \square

Finally, we consider the triple $(\phi_1, \Phi_1, U_1) = (0, 0, 0) \in \mathcal{X} \times \mathcal{F} \times \mathcal{G}$. Clearly, $\Phi_1 = \partial\phi_1/\partial\xi$ and $U_1 = \partial\phi_1/\partial\lambda$. We define recursively a sequence $(\phi_n, \Phi_n, U_n) \in \mathcal{X} \times \mathcal{F} \times \mathcal{G}$ by

$$(61) \quad \begin{aligned} (\phi_{n+1}, \Phi_{n+1}, U_{n+1}) &= S(\phi_n, \Phi_n, U_n) \\ &= (T\phi_n, A(\phi_n, \Phi_n), B(\phi_n, \Phi_n, U_n)). \end{aligned}$$

Assuming that ϕ_n is of class C^1 in ξ and λ , with $\Phi_n = \partial\phi_n/\partial\xi$ and $U_n = \partial\phi_n/\partial\lambda$, it follows from Lemma 9 that $\bar{\phi}_{n+1} = \overline{T\phi_n}$ is of class C^1 in ξ and λ , and by (60) we have

$$(62) \quad \partial\bar{\phi}_{n+1}/\partial\xi = \partial(\overline{T\phi_n})/\partial\xi = A(\phi_n, \Phi_n) = \Phi_{n+1},$$

and

$$(63) \quad \partial\bar{\phi}_{n+1}/\partial\lambda = \partial(\overline{T\phi_n})/\partial\lambda = B(\phi_n, \Phi_n, U_n) = U_{n+1}.$$

Now let ϕ_0 be the unique fixed point of T (the unique function ϕ in Theorem 3), and let (Φ_0, U_0) be the unique fixed point of

$$(\Phi, U) \mapsto (A(\phi_0, \Phi), B(\phi_0, \Phi, U)).$$

By Proposition 1 the sequences ϕ_n , Φ_n , and U_n converge uniformly respectively to ϕ_0 , Φ_0 , and U_0 on bounded subsets. For example, although the norm in \mathcal{X} is not the supremum norm, for each $c > 0$ we have

$$\|\phi(t, x) - \psi(t, x)\| \leq \|x\|d(\phi, \psi) \leq cd(\phi, \psi)$$

whenever $t \geq 0$ and $x \in E(t)$ with norm $\|x\| \leq c$. This yields the desired uniform convergence on bounded subsets. It follows from (62) and (63) that ϕ_0 is of class C^1 in ξ and λ , and that

$$(64) \quad (\partial\phi_0/\partial\xi, \partial\phi_0/\partial\lambda) = (\Phi_0, U_0)$$

(we recall that if a sequence f_n of C^1 functions converges uniformly, and the sequence f'_n of derivatives also converges uniformly, then the limit of f_n is of class C^1 , and its derivative is the limit of f'_n).

Now we assume that $(\partial f/\partial u)(t, 0, \lambda) = 0$ for every $t \geq 0$ and $\lambda \in Y$. We consider the subset $\mathcal{F}_0 \subset \mathcal{F}$ composed of the functions $\Phi \in \mathcal{F}$ such that $\Phi(s, 0, \lambda) = 0$ for every $s \geq 0$ and $\lambda \in Y$. We can easily verify that \mathcal{F}_0 is a complete metric space with the distance induced by the norm of \mathcal{F} . Since the triple $(\phi_1, \Phi_1, U_1) = (0, 0, 0)$ is in $\mathcal{X} \times \mathcal{F}_0 \times \mathcal{G}$, and $S(\mathcal{X} \times \mathcal{F}_0 \times \mathcal{G}) \subset \mathcal{X} \times \mathcal{F}_0 \times \mathcal{G}$, the sequence (ϕ_n, Φ_n, U_n) defined in (61) is also in $\mathcal{X} \times \mathcal{F}_0 \times \mathcal{G}$. Therefore, $\Phi_0(s, 0, \lambda) = 0$ for every $s \geq 0$ and $\lambda \in Y$, and it follows from (64) that in this case $(\partial\phi_0/\partial\xi)(s, 0, \lambda) = 0$ for every $s \geq 0$ and $\lambda \in Y$. \square

References

- [1] L. Barreira and Y. Pesin, *Nonuniform Hyperbolicity*, Encyclopedia of Mathematics and its Applications, **115**, Cambridge University Press, Cambridge, 2007.
- [2] L. Barreira and C. Valls, *Characterization of stable manifolds for nonuniform exponential dichotomies*, Discrete Contin. Dyn. Syst. **21** (2008), no. 4, 1025–1046.
- [3] ———, *Optimal regularity of stable manifolds of nonuniformly hyperbolic dynamics*, Topol. Methods Nonlinear Anal. **38** (2011), no. 2, 333–362.
- [4] A. Fathi, M.-R. Herman, and J.-C. Yoccoz, *A proof of Pesin's stable manifold theorem*, in Geometric dynamics (Rio de Janeiro, 1981), 177–215, Lecture Notes in Math., **1007**, Springer, Berlin, 1983.
- [5] R. Mañé, *Lyapounov exponents and stable manifolds for compact transformations*, in Geometric dynamics (Rio de Janeiro, 1981), 522–577, Lecture Notes in Math., **1007**, Springer, Berlin, 1983.
- [6] V. I. Oseledec, *A multiplicative ergodic theorem. Characteristic Ljapunov exponents of dynamical systems*, Trudy Moskov. Mat. Obšč. **19** (1968), 179–210.
- [7] Ya. Pesin, *Families of invariant manifolds corresponding to nonzero characteristic exponents*, Math. USSR-Izv. **40** (1976), 1261–1305.
- [8] ———, *Characteristic Ljapunov exponents, and smooth ergodic theory*, Uspehi Mat. Nauk **32** (1977), no. 4 (196), 55–112, 287.
- [9] C. Pugh and M. Shub, *Ergodic attractors*, Trans. Amer. Math. Soc. **312** (1989), no. 1, 1–54.
- [10] D. Ruelle, *Ergodic theory of differentiable dynamical systems*, Inst. Hautes Études Sci. Publ. Math. No. 50 (1979), 27–58.
- [11] ———, *Characteristic exponents and invariant manifolds in Hilbert space*, Ann. of Math. (2) **115** (1982), no. 2, 243–290.

LUIS BARREIRA
CENTRO DE ANÁLISE MATEMÁTICA, GEOMETRIA E SISTEMAS DINÂMICOS
DEPARTAMENTO DE MATEMÁTICA
INSTITUTO SUPERIOR TÉCNICO
UNIVERSIDADE DE LISBOA
1049-001 LISBOA, PORTUGAL
Email address: barreira@math.tecnico.ulisboa.pt

CLAUDIA VALLS
CENTRO DE ANÁLISE MATEMÁTICA, GEOMETRIA E SISTEMAS DINÂMICOS
DEPARTAMENTO DE MATEMÁTICA
INSTITUTO SUPERIOR TÉCNICO
UNIVERSIDADE DE LISBOA
1049-001 LISBOA, PORTUGAL
Email address: cvals@math.tecnico.ulisboa.pt