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ON THE BETTI NUMBERS OF THREE FAT POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$

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Abstract. In these notes we introduce a numerical function which allows us to describe explicitly (and nonrecursively) the Betti numbers, and hence, the Hilbert function of a set Z of three fat points whose support is an almost complete intersection (ACI) in $\mathbb{P}^1 \times \mathbb{P}^1$. A nonrecursively formula for the Betti numbers and the Hilbert function of these configurations is hard to give even for the corresponding set of five points on a special support in $\mathbb{P}^2$ and we did not find any kind of this result in the literature. Moreover, we also give a criterion that allows us to characterize the Hilbert functions of these special set of fat points.

1. Introduction

The computation of the homological invariants of a scheme is a challenging problem which involves a strong interaction between Algebraic Geometry and Commutative Algebra.

Let $R := k[\mathbb{P}^n] = k[x_0, \ldots, x_n]$ be the standard polynomial ring over an infinite field. Given $P_1, \ldots, P_s$ distinct points of $\mathbb{P}^n$ and $m_1, \ldots, m_s$ non negative integers we call *set of fat points* $Z = m_1P_1 + \cdots + m_sP_s$ and *support* of Z is $\operatorname{Supp}(Z) = P_1 + \cdots + P_s$.

Given a set of fat points Z, the homogeneous ideal in R defining Z is generated by the homogeneous forms vanishing at each point P_i of Z with multiplicity at least m_i,

$$I_Z := \bigcap_{i=1}^{s} I_{P_i}^{m_i} \subseteq R,$$

where we denoted by I_{P_i} the homogeneous ideal defining P_i. The Hilbert function of Z computes the dimension of the homogeneous components of degree t of R/I_Z for all $t \in \mathbb{N}$, i.e.,

$$H_Z(t) := \dim_k(R/I_Z)_t \quad \text{for all } t > 0,$$

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i.e., of R/I_Z. There are a lot of papers studying the homological invariant of fat points schemes, mostly in $\mathbb{P}^2$, see for example the surveys [9, 16].

Changing the ambient space from a single projective space $\mathbb{P}^n$ to a multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ the description of the Hilbert Function and the multigraded Betti numbers of (fat) points scheme became more tricky. An obstacle is that the coordinate ring of a set of (fat) points in a multiprojective space is not necessarily Cohen-Macaulay. Several characterizations describe the Hilbert Function and the bigraded Betti numbers of ACM set of (fat) points in $\mathbb{P}^1 \times \mathbb{P}^1$, see for instance [10, 14]. But, without the hypothesis of Cohen-Macaulayness, we are still far away from a complete understanding of these homological invariants even for distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$. So, any result in this direction could be interesting. Recently, the first author (see [4]) gave a characterization of the Hilbert functions of bigraded algebras in $k[\mathbb{P}^1 \times \mathbb{P}^1]$. In [1], the authors compute the bigraded Hilbert function of general triple points in $\mathbb{P}^1 \times \mathbb{P}^1$ and in [6, 7] the authors focus on the aritmetically Cohen-Macaulay property for set of points in multiprojective spaces.

In this paper, we focus on a set Z of fat points whose support $Supp(X)$ is an *almost complete intersection* (ACI for short), i.e., the number of minimal generators of $I_{Supp(X)}$ is one more the codimension. Thus, in $\mathbb{P}^1 \times \mathbb{P}^1$ an ideal of an ACI set of points has exactly three minimal generators.

The motivation of the study of fat points whose support is an ACI in $\mathbb{P}^1 \times \mathbb{P}^1$ is also related to the study of the comparison of symbolic and regular powers of ideals of codimension 2 in $\mathbb{P}^n$, that was started in [3]. This can be useful when studying asymptotic properties of fat point ideals (such as the resurgence, the symbolic defect and the Waldschmidt constant). Combining together Theorem 1.1 in [13] with Corollary 4.4 in [3], we have that:

Theorem 1.1. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be an ACM set of points. Then the following conditions are equivalent*:

- $I_X^m = I_X^{(m)}$ *for all* $m \geq 1$;
- $I_X^3 = I_X^{(3)}$;
- X *is either a complete intersection or an almost complete intersection.*

As a consequence of the above result we have that the description of an ACI set of points should be of interest. In [5] the authors give a recursive description of a minimal free resolution of I_Z and show that $I_Z^{(m)} = I_Z^m$, for a fat points scheme $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ whose support is an ACI.

In this paper, we compute explicitly the Betti numbers and the Hilbert function of an ideal of a set of three fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support is an ACI by introducing a specific numerical function (see Definition 3.1). We point out that this numerical function also allows us to explicitly describe the Betti numbers and the Hilbert function of an ideal of a set of five fat points in $\mathbb{P}^2$ supported on some special configuration (see [11]). In fact, some homological invariants of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ can be computed looking at a suitable set of points in $\mathbb{P}^2$ (see, for example, Section 2.1, Remark 2.3.1 in [12]). In particular, the Hilbert function in degree (a, b) of a set Z of three fat points whose support is an ACI in $\mathbb{P}^1 \times \mathbb{P}^1$ corresponds to the Hilbert function of a set of five fat points in $\mathbb{P}^2$ as pictured in Figure 1 and Figure 2.

$Z =$

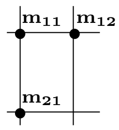

FIGURE 1. The set of 3 fat points in $\mathbb{P}^1 \times \mathbb{P}^1$

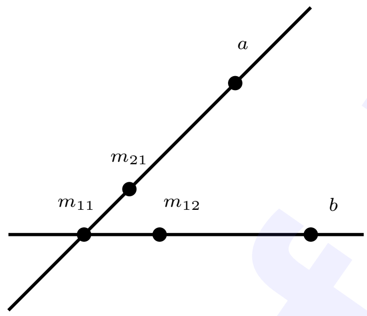

FIGURE 2. The set of 5 fat points in $\mathbb{P}^2$

The configuration in Figure 2 was studied (in a more general setting) in [11]. Precisely, it corresponds to configurations of Type 6, Type 8, Type 9 and Type 10 in Fig. 1 in [11] where one of the points has multiplicity 0. We use these previous known results in the present paper (see Example 5.6). Five points in $\mathbb{P}^2$ always lie on a conic, so such configurations of points have been studied in detail. For instance, the case of fat point subschemes supported at five general points was studied by Catalisano in [2] and the case of fat points (or infinitely near points) supported on a reducible conic was handled by Harbourne in [15]. Additional special configurations, with points of multiplicity 2, are investigated in [8].

An explicit description of the Betti numbers and the Hilbert function of these configurations is hard to give even in $\mathbb{P}^2$, and we did not find this result in the literature. All known results are algorithmically computed.

The paper is structured as follow. In Section 2 we describe the connection between points in $\mathbb{P}^1 \times \mathbb{P}^1$ and points in $\mathbb{P}^2$. We also recall some results and notation from [5]. In Section 3 we introduce and study a numerical function which will be related with the homological invariant of such sets of points. In Section 4, first, we give a formula to compute the graded Betti numbers and the Hilbert Function of a set of three fat points on an ACI support.

We point out that the only characterizations of Hilbert functions of set of points, both reduced or fat, are known only when the set of points is supported on an aritmetically Cohen-Macaulay set of points. Here we give a criterion (Theorem 5.5) that allows us to characterize the Hilbert function for a set of 3 fat points supported on an ACI in $\mathbb{P}^1 \times \mathbb{P}^1$.

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2. Notation and preliminary results

Throughout this paper $R := k[x_0, x_1, x_2, x_3]$ is the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ over an infinite field of characteristic 0, with the bidegree induced by $\deg x_0 = \deg x_1 = (1, 0)$ and $\deg x_2 = \deg x_3 = (0, 1)$. We will denote by H_i the *horizontal lines*, i.e., the lines defined by a form of degree $(1, 0)$ and by V_j the *vertical lines*, that are defined by a form of degree $(0, 1)$. We denote the point in $\mathbb{P}^1 \times \mathbb{P}^1$ intersection of H_i and V_j by $P_{ij} = H_i \times V_j$. With an abuse of notation we use H_i and V_j also to denote the correspondent linear forms in R. Then $I_{P_{ij}} = (H_i, V_j)$ will be the ideal defining the point P_{ij}.

Remark 2.1. It can be useful to reinterpret problems involving points of $\mathbb{P}^1 \times \mathbb{P}^1$ as problems involving points of $\mathbb{P}^2$ (see, for example, Section 2.1, Remark 2.3.1 in [12]).

As for 3 points in $\mathbb{P}^1 \times \mathbb{P}^1$ corresponding to 5 points in $\mathbb{P}^2$, this depends on thinking of the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 1 point as being the blow up of $\mathbb{P}^2$ at 2 points.

Let E_{ij} be the blow up of P_{ij} on $\mathbb{P}^1 \times \mathbb{P}^1$. So we have $rH + sV - (m_{11}E_{11} + m_{21}E_{21} + m_{12}E_{12})$. This becomes $r(L - E_2) + s(L - E_1) - m_{11}(L - E_1 - E_2) - m_{21}E_3 - m_{21}E_4$, where $V = L - E_1$, $H = L - E_2$, $E_{11} = L - E_1 - E_2$, $E_1 = H - E_{11}$ is the blow up of a point P_1 on $\mathbb{P}^2$, $E_2 = V - E_{11}$ is the blow up of a point P_2 on $\mathbb{P}^2$, E_3 is the blow up of a point P_3 on E_1 (so $N_1 = E_1 - E_3$ is effective and irreducible), and E_4 is the blow up of a point P_4 on E_2 (so $N_2 = E_2 - E_4$ is effective and irreducible).

Alternatively, if we do not to use P_{11} as the point of $\mathbb{P}^1 \times \mathbb{P}^1$ you blow up to get a 2 point blow up of $\mathbb{P}^2$, then let P be a general point of $\mathbb{P}^1 \times \mathbb{P}^1$. Blow up P to get E and blow down $V - E$ to a point P_1 of $\mathbb{P}^2$ and $H - E$ to a point P_2 of $\mathbb{P}^2$. Let E_i be the blow up of P_i, so $E_1 = V - E$ and $E_2 = H - E$. Let E_{ij} be the blow up of P_{ij}. As before $V = L - E_1$, $H = L - E_2$, and $E = L - E_1 - E_2$. Then the points P_{11}, P_{21} and P_{12} become points on $\mathbb{P}^2$, where P_{21} is a point on the line through P_1 and P_{11}, and P_{12} is a point on the line through P_2 and P_{11}. And $rH + sV - (m_{11}E_{11} + m_{21}E_{21} + m_{12}E_{12}) = r(L - E_2) + s(L - E_1) - (m_{11}E_{11} + m_{21}E_{21} + m_{12}E_{12})$. Thus here you get 5 points of $\mathbb{P}^2$ on two lines, where P_{11} is where the lines cross, and P_{21}, P_1 and P_{11} are on one line and P_{12}, P_2 and P_{11} are on the other. The line through P_1 and P_2 is the blow up of the point P on $\mathbb{P}^1 \times \mathbb{P}^1$ (see Fig. 1 and Fig. 2). In particular, it is

$$\begin{aligned} &H^0(X, rH + sV - (m_{11}E_{11} + m_{21}E_{21} + m_{12}E_{12})) \\ &= H^0(X, (r+s)L - (m_{11}E_{11} + m_{21}E_{21} + m_{12}E_{12}) - rP_1 - sP_2). \end{aligned} \tag{2.1}$$

Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a set of distinct points and positive integers m_{ij}, we call $Z = \sum_{P_{ij} \in X} m_{ij} P_{ij}$ a set of fat points supported at X. The associated ideal to Z is $I_Z := \bigcap_{P_{ij} \in X} I_{P_{ij}}^{m_{ij}}$.

We also recall that if X is a set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$, then two points of X are collinear if there is a ruling, either vertical V_j or horizontal H_i, that contains both of the points. For instance, $X = \{P_{11}, P_{12}\}$ is a set of two collinear points since $P_{11} = H_1 \times V_1$ and $P_{12} = H_1 \times V_2$, i.e., the points lie in the same ruling H_1 or, according to our setting, they lie in the same *horizontal line* H_1. Analougously, $X = \{P_{11}, P_{22}\}$ is a set of two non collinear points since $P_{11} = H_1 \times V_1$ and $P_{22} = H_2 \times V_2$, i.e., there does not exist a ruling, either vertical V_j or horizontal H_i, that contains both of the points.

In these note we use the following notation as in [5].

Notation 2.2. We set $Z := m_{11}P_{11} + m_{12}P_{12} + m_{21}P_{21}$, where $m_{ij} \geq 0$ and, without loss of generality, $m_{12} \geq m_{21}$. We denote by $Z_1 = (m_{11} - 1)_+ P_{11} + m_{12}P_{12} + (m_{21} - 1)_+ P_{21}$, where $(n)_+ := \max\{n, 0\}$.

The following results were proven in [5] in a more general setting. For the convenience of reader, we recall them in a version which is useful to our focus.

Lemma 2.3 (Lemma 2.2, [5])**.** *Let $Z = m_{11}P_{11} + m_{12}P_{12} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of two collinear fat points. Set $M := \max\{m_{11}, m_{12}\}$, then a minimal free resolution of I_Z is*

$$\begin{aligned} 0 \to &\bigoplus_{t=1}^{M} R(-t, -(m_{11} - t + 1)_+ - (m_{12} - t + 1)_+) \\ \to &\bigoplus_{t=0}^{M} R(t, -(m_{11} - t)_+ - (m_{12} - t)_+) \to I_Z \to 0. \end{aligned}$$

Lemma 2.4 (Lemma 3.4, [5])**.** *Let $Z := m_{12}P_{12} + m_{21}P_{21}$ be a set of two non collinear fat points. Then a minimal free resolution of I_Z is*

$$\begin{aligned} 0 \to \bigoplus_{(a,b,c,d) \in \mathcal{D}_2} R(-a-b, -c-d) \to \bigoplus_{(a,b,c,d) \in \mathcal{D}_1} R(-a-b, -c-d) \to \\ \to \bigoplus_{(a,b,c,d) \in \mathcal{D}_0} R(-a-b, -c-d) \to I_Z \to 0 \end{aligned}$$

where

$\mathcal{D}_0 := \{(a, b, c, d) \,|\, 0 \leq a, d \leq m_{12},\ 0 \leq b, c \leq m_{21},\ a + d = m_{12},\ b + c = m_{21}\}$,
$\mathcal{D}_1 := \{(a, b, c, d) \,|\, 0 \leq a, d \leq m_{12},\ 0 \leq b, c \leq m_{21},$

$$(a+d=m_{12}+1,\ b+c=m_{21}) \vee (a+d=m_{12},\ b+c=m_{21}+1)\},$$
$$\mathcal{D}_2 := \{(a,b,c,d) \mid 0 \le a,d \le m_{12},\ 0 \le b,c \le m_{21},\ a+d=m_{12}+1,$$
$$b+c=m_{21}+1\}.$$

These two lemmas describe the resolution of Z in the degenerate case when one of the multiplicities is 0. The next result allows us to recursively compute the resolution of Z in the remaining cases.

Lemma 2.5 (Remark 2.10, Theorem 2.12, [5])**.** *Let Z be as in Notation 2.2, let $0 \to L_2 \to L_1 \to L_0$ be a minimal free resolution of I_{Z_1}. Then a minimal free resolution for I_Z is*

$$0 \to \bigoplus_{(a,b)\in A_2(Z)} R(-a,-b) \oplus L_2(0,-1) \tag{2.2}$$
$$\to \bigoplus_{(a,b)\in A_1(Z)} R(-a,-b)^2 \oplus R(-m_{11}-m_{21}, -(m_{12}-m_{11})_+ - 1) \oplus L_1(0,-1)$$
$$\to \bigoplus_{(a,b)\in A_0(Z)} R(-a,-b) \oplus L_0(0,-1) \to I_Z \to 0$$

where

$$A_0(Z) := \{(a,b) \mid a+b = m_{11}+m_{21}+(m_{12}-m_{11})_+ \text{ and } 0 \le b \le (m_{12}-m_{11})_+\},$$
$$A_1(Z) := \{(a,b) \mid a+b = 1+m_{11}+m_{21}+(m_{12}-m_{11})_+ \text{ and } 1 \le b \le (m_{12}-m_{11})_+\},$$
$$A_2(Z) := \{(a,b) \mid a+b = 2+m_{11}+m_{21}+(m_{12}-m_{11})_+ \text{ and } 2 \le b \le (m_{12}-m_{11})_+ + 1\}.$$

3. Numerical facts

We introduce a numerical function depending on a parameter $t \in \mathbb{Z}$ to determine explicitly the graded Betti numbers of Z.

Definition 3.1. We define inductively the function $\varphi_t : \mathbb{Z} \to \mathbb{Z}$ as follows:

$$\varphi_1(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and, for $t > 1$

$$\varphi_t(n) = \begin{cases} \varphi_{t-1}(n) & \text{if } 0 \le n < t-1, \\ \varphi_{t-1}(n)+1 & \text{if } t-1 \le n < 2t-1, \\ 0 & \text{otherwise.} \end{cases}$$

We will use the convention $\varphi_t(n) = 0$ if $t \le 0$.

Remark 3.2. One can inductively check that $\varphi_t(n) = \left(\left\lfloor \frac{\min\{n,\ 2t-2-n\}}{2} \right\rfloor + 1\right)_+$.

Definition 3.3. Let $t, d \in \mathbb{Z}$ be two integers such that $t \geq d$, we define the following function $\varphi_{t,d}(n) : \mathbb{Z} \to \mathbb{Z}$:

$$\varphi_{t,d}(n) = \varphi_t(n+d) - \varphi_d(n+d).$$

We give an example in order to clarify the notation.

Example 3.4. To shorten the notation, we represent the functions as tuples, where the first entry is their value in 0, the second entry is their value in 1 and so on.

$$\begin{aligned}
\varphi_1 &= (1, 0, 0, \ldots) \\
\varphi_2 &= (1, 1, 1, 0, 0, \ldots) \\
\varphi_3 &= (1, 1, 2, 1, 1, 0, \ldots) \\
\varphi_4 &= (1, 1, 2, 2, 2, 1, 1, 0, \ldots) \\
&\ldots \\
\varphi_7 &= (1, 1, 2, 2, 3, 3, 4, 3, 3, 2, 2, 1, 1, 0, \ldots) \\
\varphi_7 - \varphi_4 &= (0, 0, 0, 0, 1, 2, 3, 3, 3, 2, 2, 1, 1, 0, \ldots) \\
\varphi_{7,4} &= (1, 2, 3, 3, 3, 2, 2, 1, 1, 0, \ldots) \\
\varphi_{4,-3} &= (0, 0, 0, 1, 1, 2, 2, 2, 1, 1, 0, \ldots)
\end{aligned}$$

We have the following property.

Proposition 3.5. *Let $t \geq d$ be two integers. Then*

$$\varphi_{t,d+1}(n-1) = \begin{cases} \varphi_{t,d}(n) - 1 & \textit{if } 0 \leq n \leq d, \\ \varphi_{t,d}(n) & \textit{otherwise.} \end{cases}$$

Proof. By definition, we have $\varphi_{t,d+1}(n-1) = \varphi_t(n+d) - \varphi_{d+1}(n+d)$, hence

$$\begin{aligned}
\varphi_{t,d+1}(n-1) &= \begin{cases} \varphi_t(n+d) - \varphi_d(n+d) & \text{if } 0 \leq n+d < d, \\ \varphi_t(n+d) - \varphi_d(n+d) - 1 & \text{if } d \leq n+d < 2d+1, \\ \varphi_t(n+d) - 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \varphi_t(n+d) - \varphi_d(n+d) - 1 & \text{if } 0 \leq n \leq d, \\ \varphi_t(n+d) - \varphi_d(n+d) & \text{otherwise.} \end{cases}
\end{aligned}$$

□

4. The graded Betti numbers of I_Z

Let X be a set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ and let $I_X \subseteq R := k[\mathbb{P}^1 \times \mathbb{P}^1]$ be the bihomogeneous ideal associated to X. Then we can associate to I_X a minimal bigraded free resolution of the form

$$\begin{aligned}
0 \to \bigoplus R(-i,-j)^{\beta_{2,(i,j)}(X)} &\to \bigoplus R(-i,-j)^{\beta_{1,(i,j)}(X)} \\
\to \bigoplus R(-i,-j)^{\beta_{0,(i,j)}(X)} &\to R \to R/I_X \to 0,
\end{aligned}$$

where $R(-i,-j)$ is the free R-module obtained by shifting the degrees of R by (i,j). The graded Betti number $\beta_{u,(i,j)}(X)$ of R/I_X counts the number of a minimal set of generators of degree (i,j) in the u-th syzygy module of R/I_X.

Using the same strategy as in [5] we split the description in two cases.

4.1. First case $m_{11} \le m_{21}$

Theorem 4.1. *With the Notation 2.2, if $m_{11} \le m_{21}$, then the bigraded Betti numbers of I_Z are:*

$$\beta_{0,(a,b)}(Z) = \begin{cases} (\min\{a, b-m_{11}, m_{21}-m_{11}\}+1)_+ + \varphi_{m_{12},m_{12}-m_{11}}(b) & \textit{if } a+b = m_{21}+m_{12}, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\beta_{1,(a,b)}(Z) = \left(\beta_{0,(a,b-1)}(Z) + \beta_{0,(a-1,b)}(Z) - 1\right)_+,$$

$$\beta_{2,(a,b)}(Z) = \left(\beta_{0,(a-1,b-1)}(Z) - 1\right)_+.$$

Proof. We proceed by induction on m_{11}. Since $\varphi_{m_{12},m_{12}}(b) = 0$, if $m_{11} = 0$ the statement is true by Lemma 2.4. Assume $m_{11} > 0$, by Lemma 2.5 we get

$$\beta_{0,(a,b)}(Z) = \begin{cases} \beta_{0,(a,b-1)}(Z_1)+1 & \text{if } a+b = m_{12}+m_{21} \text{ and } b \le m_{12}-m_{11}, \\ \beta_{0,(a,b-1)}(Z_1) & \text{if } a+b = m_{12}+m_{21} \text{ and } b > m_{12}-m_{11}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, set $S := m_{12}+m_{21}$ and $B := m_{12}-m_{11}$, we have

$$\beta_{0,(a,b)}(Z) = \begin{cases} (\min\{a, b-m_{11}, m_{21}-m_{11}\}+1)_+ + \varphi_{m_{12},B+1}(b-1)+1 \\ \text{if } a+b = S \text{ and } b \le B, \\ (\min\{a, b-m_{11}, m_{21}-m_{11}\}+1)_+ + \varphi_{m_{12},B+1}(b-1) \\ \text{if } a+b = S \text{ and } b > B, \\ \ 0 \quad \text{otherwise} \end{cases}$$

and by using Proposition 3.5 it is

$$\beta_{0,(a,b)}(Z) = \begin{cases} (\min\{a, b-m_{11}, m_{21}-m_{11}\}+1)_+ + \varphi_{m_{12},B}(b) \\ \text{if } a+b = S \text{ and } b \le B, \\ (\min\{a, b-m_{11}, m_{21}-m_{11}\}+1)_+ + \varphi_{m_{12},B}(b) \\ \text{if } a+b = S \text{ and } b > B, \\ \ 0 \quad \text{otherwise.} \end{cases}$$

The computation of $\beta_{1,(a,b)}(Z)$ also follows by induction and Lemma 2.5.

$$\beta_{1,(a,b)}(Z)$$
$$= \begin{cases} \beta_{1,(a,b-1)}(Z_1)+2 & \text{if } a+b-1 = m_{21}+m_{12} \text{ and } 1 \le b \le m_{12}-m_{11}, \\ \beta_{1,(a,b)}(Z)+1 & \text{if } (a,b) = (m_{11}+m_{21}, m_{12}-m_{11}+1), \\ 0 & \text{otherwise} \end{cases}$$

$$
= \begin{cases} \beta_{0,(a,b-2)}(Z_1) + \beta_{0,(a-1,b-1)}(Z_1) + 1 \\ \quad \text{if } a+b-1 = m_{21}+m_{12} \text{ and } 1 \le b \le m_{12}-m_{11}, \\ \beta_{0,(a,b-2)}(Z_1) + \beta_{0,(a-1,b-1)}(Z_1) \\ \quad \text{if } (a,b) = (m_{11}-m_{21}, m_{12}-m_{11}+1), \\ 0 \quad \text{otherwise} \end{cases}
$$

$$
= \begin{cases} \beta_{0,(a,b-1)}(Z) + \beta_{0,(a-1,b)}(Z) - 1 \\ \quad \text{if } a+b-1 = m_{21}+m_{12} \text{ and } 1 \le b \le m_{12}-m_{11}, \\ \beta_{0,(a,b-1)}(Z) + \beta_{0,(a-1,b)}(Z) - 1 \\ \quad \text{if } (a,b) = (m_{11}+m_{21}, m_{12}-m_{11}+1), \\ 0 \quad \text{otherwise} \end{cases}
$$

The computation of $\beta_{2,(a,b)}(Z)$ requires the same procedure as above by using the inductive hypotheses and Lemma 2.5. □

We show in the following example how to compute the graded Betti numbers of a set of three fat points using Theorem 4.1.

Example 4.2. Consider $Z = 2P_{11} + 5P_{12} + 4P_{21}$, to compute $\beta_{0,(a,b)}(Z)$ we first need to compute the bigraded Betti numbers of $Z' := (2-2)P_{11} + 5P_{12} + (4-2)P_{21} = 5P_{12} + 2P_{21}$. By Lemma 2.4, the non zero bigraded Betti numbers of $R/I_{Z'}$ are:

$$
\begin{array}{lll}
\beta_{0,(7,0)}(I_{Z'}) = 1 & \beta_{1,(7,1)}(I_{Z'}) = 2 & \beta_{2,(7,2)}(I_{Z'}) = 1 \\
\beta_{0,(6,1)}(I_{Z'}) = 2 & \beta_{1,(6,2)}(I_{Z'}) = 4 & \beta_{2,(6,3)}(I_{Z'}) = 2 \\
\beta_{0,(5,2)}(I_{Z'}) = 3 & \beta_{1,(5,3)}(I_{Z'}) = 5 & \beta_{2,(5,4)}(I_{Z'}) = 2 \\
\beta_{0,(4,3)}(I_{Z'}) = 3 & \beta_{1,(4,4)}(I_{Z'}) = 5 & \beta_{2,(4,5)}(I_{Z'}) = 2 \\
\beta_{0,(3,4)}(I_{Z'}) = 3 & \beta_{1,(3,5)}(I_{Z'}) = 5 & \beta_{2,(3,6)}(I_{Z'}) = 2 \\
\beta_{0,(2,5)}(I_{Z'}) = 3 & \beta_{1,(2,6)}(I_{Z'}) = 4 & \beta_{2,(2,7)}(I_{Z'}) = 1 \\
\beta_{0,(1,6)}(I_{Z'}) = 2 & \beta_{1,(1,7)}(I_{Z'}) = 2 & \\
\beta_{0,(0,7)}(I_{Z'}) = 1. & &
\end{array}
$$

Moreover we have $\varphi_{5,3} = (1,2,2,2,1,1,0\ldots)$. Hence, if $a+b=9$ we have

$$
\beta_{0,(a,b)}(Z) = \beta_{0,(a,b-2)}(Z') + \varphi_{5,3}(b).
$$

Then the non zero bigraded Betti numbers of R/I_Z are:

$$
\begin{array}{lll}
\beta_{0,(9,0)}(Z) = 1 & \beta_{1,(9,1)}(Z) = 2 & \beta_{2,(9,2)}(Z) = 1 \\
\beta_{0,(8,1)}(Z) = 2 & \beta_{1,(8,2)}(Z) = 4 & \beta_{2,(8,3)}(Z) = 2 \\
\beta_{0,(7,2)}(Z) = 3 & \beta_{1,(7,3)}(Z) = 6 & \beta_{2,(7,4)}(Z) = 3 \\
\beta_{0,(6,3)}(Z) = 4 & \beta_{1,(6,4)}(Z) = 7 & \beta_{2,(6,5)}(Z) = 3 \\
\beta_{0,(5,4)}(Z) = 4 & \beta_{1,(5,5)}(Z) = 7 & \beta_{2,(5,6)}(Z) = 3 \\
\beta_{0,(4,5)}(Z) = 4 & \beta_{1,(4,6)}(Z) = 6 & \beta_{2,(4,7)}(Z) = 2
\end{array}
$$

$$\begin{array}{lll}
\beta_{0,(3,6)}(Z)=3 & \beta_{1,(3,7)}(Z)=5 & \beta_{2,(3,8)}(Z)=2\\
\beta_{0,(2,7)}(Z)=3 & \beta_{1,(2,8)}(Z)=4 & \beta_{2,(2,9)}(Z)=1\\
\beta_{0,(1,8)}(Z)=2 & \beta_{1,(1,9)}(Z)=2 & \\
\beta_{0,(0,9)}(Z)=1. & &
\end{array}$$

4.2. Second case $m_{11} > m_{21}$

To conclude the description of the graded Betti numbers of R/I_Z we need some preliminaries.

Definition 4.3. Let $Z = m_{11}P_{11} + m_{12}P_{12} + m_{21}P_{21}$ be a set of three fat points in $\mathbb{P}^1 \times \mathbb{P}^1$, set $B_Z := m_{12} - m_{11}$, we define the following sets of integers associated to Z:

$$D_1(Z) = \{(a, b) \in \mathbb{N}^2 \mid 0 \le b < (-B_Z)_+ - (-B_Z - m_{21})_+ \text{ and } a + 2b = m_{11} + m_{21}\},$$

$$D_2(Z) = \{(a, b) \in \mathbb{N}^2 \mid (-B_Z)_+ - (-B_Z - m_{21})_+ \le b < m_{21} \text{ and } a + b = \max\{m_{11}, m_{12} + m_{21}\}\},$$

$$D_3(Z) = \{(a, b) \in \mathbb{N}^2 \mid m_{21} \le b \le m_{21} + |B_Z + m_{21}| \text{ and } a + b = \max\{m_{11}, m_{12} + m_{21}\}\},$$

$$D_4(Z) = \{(a, b) \in \mathbb{N}^2 \mid b > m_{21} + |B_Z + m_{21}| \text{ and } 2a + b = m_{11} + m_{12}\}.$$

As immediate consequences we have:

Lemma 4.4.

i) $B_{Z_1} = B_Z + 1$;
ii) $D_1(Z) = \emptyset$ *if and only if* $B_Z \ge 0$;
iii) $D_2(Z) = \emptyset$ *if and only if* $B + m_{21} \le 1$ (*in this case* Z *is ACM by Theorem 6.21* [14]);
iv) *If* $B_Z < 0$ *and* $B_Z + m_{21} = 1$, *then* $(a, -B_Z) \notin D_3(Z)$ *for any* a;
v) *If* $(a, b-1) \in D_i(Z_1)$, *then* $(a, b) \in D_i(Z)$ *for* $i = 1, 2, 3, 4$;
vi) *If* $(a, b) \in D_i(Z)$ *for some* i, *then* $(a, \bar{b}), (\bar{a}, b) \notin D_j$ *for any* $\bar{a} \ne a$, $\bar{b} \ne b$ *and* $j = 1, 2, 3, 4$.
vii) *If* $(a-1, b) \in D_4(Z)$, *then* $(a, b-1) \notin D_3(Z)$.

Now we can complete the second case.

Theorem 4.5. *If* $m_{11} > m_{21}$, *then the bigraded Betti numbers of* R/I_Z *are*:

$$\beta_{0,(a,b)}(Z) = \begin{cases} 1 & \text{if } (a,b) \in D_1(Z),\\ \varphi_{m_{21}+B,B}(b) & \text{if } (a,b) \in D_2(Z),\\ 1 + \varphi_{m_{21}+B,B}(b) & \text{if } (a,b) \in D_3(Z),\\ 1 & \text{if } (a,b) \in D_4(Z),\\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_{1,(a,b)}(Z) = \begin{cases} 1 & \textit{if } (a, b-1) \in D_1(Z), \\ \beta_{0,(a,b-1)}(Z) + \beta_{0,(a-1,b)}(Z) - 1 & \textit{if } (a, b-1) \in D_2(Z) \cup D_3(Z), \\ 1 & \textit{if } (a-1, b) \in D_4(Z), \\ 0 & \textit{otherwise}, \end{cases}$$

$$\beta_{2,(a,b)}(Z) = \begin{cases} \beta_{0,(a-1,b-1)}(Z) - 1 & \textit{if } (a-1, b-1) \in D_2(Z) \cup D_3(Z), \\ 0 & \textit{otherwise}, \end{cases}$$

where the $D_i(Z)$ are defined in Definition 4.3.

Proof. We proceed by induction on m_{21}. If $m_{21} = 0$, then Z is a set of 2 collinear (fat) points, $D_1 = D_2 = \emptyset$ and $\varphi_{B,B}(b) = 0$. Therefore the statement follows by Lemma 2.3. Assume now $m_{21} > 0$, by Lemma 2.5 we have

$$\beta_{0,(a,b)}(Z) = \begin{cases} \beta_{0,(a,b-1)}(Z_1) + 1 & \text{if } a + b = m_{11} + m_{21} + (m_{12} - m_{11})_+ \\ & \text{and } b \le (m_{12} - m_{11})_+, \\ \beta_{0,(a,b-1)}(Z_1) & \text{otherwise.} \end{cases}$$

If $m_{12} \le m_{11}$, i.e., $B_Z < 0$ we get

$$\beta_{0,(a,b)}(Z) = \begin{cases} 1 & \text{if } (a, b) = (m_{11} + m_{21}, 0), \\ \beta_{0,(a,b-1)}(Z_1) & \text{otherwise.} \end{cases}$$

So, by the inductive hypothesis and using Remark 4.4, we have

$$\beta_{0,(a,b)}(Z) = \begin{cases} 1 & \text{if } (a, b) = (m_{11} + m_{21}, 0), \\ 1 & \text{if } (a, b-1) \in D_1(Z_1), \\ \varphi_{m_{21}+B_Z, B_Z+1}(b-1) & \text{if } (a, b-1) \in D_2(Z_1), \\ 1 + \varphi_{m_{21}+B_Z, B_Z+1}(b-1) & \text{if } (a, b-1) \in D_3(Z_1), \\ 1 & \text{if } (a-1, b) \in D_4(Z_1), \\ 0 & \text{otherwise.} \end{cases}$$

By using Proposition 2.5 we are done. Consider now $m_{12} > m_{11}$, i.e., $B_Z > 0$. We get

$$\beta_{0,(a,b)}(Z) = \begin{cases} \beta_{0,(a,b-1)}(Z_1) + 1 & \text{if } a + b = m_{21} + m_{12} \text{ and } b \le B, \\ \beta_{0,(a,b-1)}(Z_1) & \text{otherwise,} \end{cases}$$

where, by inductive hypothesis, the bigraded Betti numbers of degree zero of R/I_{Z_1} are

$$\beta_{0,(a,b-1)}(Z_1) = \begin{cases} \varphi_{m_{21}+B_Z, B_Z+1}(b-1) & \text{if } (a, b-1) \in D_2(Z_1), \\ 1 + \varphi_{m_{21}+B_Z, B_Z+1}(b-1) & \text{if } (a, b-1) \in D_3(Z_1), \\ 1 & \text{if } (a-1, b) \in D_4(Z_1), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by using Lemma 2.5 and Remark 4.4, we are done. Finally the computation of $\beta_{1,(a,b)}(Z)$ and $\beta_{2,(a,b)}(Z)$ requires the same procedure as above by using the inductive hypothesis and Lemma 2.5. □

Corollary 4.6. *In $\mathbb{P}^1 \times \mathbb{P}^1$, let $Z = mP_{11} + mP_{12} + mP_{21}$ be a set of fat points where all the points have the same multiplicity. Then the bigraded Betti numbers of I_Z are:*

$$\beta_{0,(a,b)}(Z) = \begin{cases} \varphi_{m+1}(b) & \text{if } a+b=2m, \\ 0 & \text{otherwise}, \end{cases}$$
$$\beta_{1,(a,b)}(Z) = \left(\beta_{0,(a,b-1)}(Z) + \beta_{0,(a-1,b)}(Z) - 1\right)_+,$$
$$\beta_{2,(a,b)}(Z) = \left(\beta_{0,(a-1,b-1)}(Z) - 1\right)_+.$$

Proof. The proof is an immediate consequence of Theorem 4.1 and Proposition 3.5 since we have

$$\beta_{0,(a,b)}(Z) = \begin{cases} 1+\varphi_m(b) & \text{if } b \geq m \text{ and } a+b=2m, \\ \varphi_m(b) & \text{if } b < m \text{ and } a+b=2m, \\ 0 & \text{otherwise.} \end{cases}$$

□

5. The Hilbert function of Z

In this section we explicitly compute the Hilbert function of set of fat points Z supported on an ACI. Recall that the Hilbert function of Z is a numeric function $H_Z := H_{R/I_Z} : \mathbb{N}^2 \to \mathbb{N}$, defined by

$$H_Z(a,b) = \dim_K(R/I_Z)_{(a,b)} = \dim_k R_{(a,b)} - \dim_k I_Z(a,b).$$

The first difference of the Hilbert function is defined as

$$\Delta H_Z(a,b) = H_Z(a,b) + H_Z(a-1,b-1) - H_Z(a-1,b) - H_Z(a,b-1).$$

From now on we will write $(i,j) \leq (a,b)$ if and only if both $i \leq a$ and $j \leq b$.

The following results are the multigraded version of well known results for standard graded algebras. The next lemma shows how we can compute the Hilbert function of Z from a minimal free resolution of R/I_Z.

Lemma 5.1. *Let Z be a set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then*

$$H_Z(a,b) = (a+1)(b+1) - \sum_{(i,j)\leq(a,b)} (a-i+1)(b-j+1)(\beta_{0,(i,j)}(Z) - \beta_{1,(i,j)}(Z) + \beta_{2,(i,j)}(Z)).$$

Proof. Since a minimal free resolution of R/I_Z

$$0 \to \bigoplus R(-i,-j)^{\beta_{2,(i,j)}(Z)} \to \bigoplus R(-i,-j)^{\beta_{1,(i,j)}(Z)} \to \bigoplus R(-i,-j)^{\beta_{0,(i,j)}(Z)} \to R \to R/I_Z \to 0$$

has bigraded morphisms, we get the following exact sequence of vector spaces

$$0 \to \bigoplus \left(R(-i,-j)^{\beta_{2,(i,j)}(Z)}\right)_{(a,b)} \to \bigoplus \left(R(-i,-j)^{\beta_{1,(i,j)}(Z)}\right)_{(a,b)}$$
$$\to \bigoplus \left(R(-i,-j)^{\beta_{0,(i,j)}(Z)}\right)_{(a,b)} \to (R)_{(a,b)} \to (R/I)_{(a,b)} \to 0.$$

Moreover

$$\begin{aligned} &\dim_k \left(\bigoplus R(-i,-j)^{\beta_{u,(i,j)}(Z)}\right)_{(a,b)} \\ =& \sum_{(i,j)\leq(a,b)} \dim_k \left(R(-i,-j)^{\beta_{u,(i,j)}(Z)}\right)_{(a,b)} \\ =& \sum_{(i,j)\leq(a,b)} \beta_{u,(i,j)}(Z)(a-i+1)(b-j+1). \end{aligned}$$

□

Corollary 5.2. *Let* $B_{u,(a,b)} := \sum_{(i,j)\leq(a,b)} \beta_{u,(a,b)}(Z)$. *Then*

$$\Delta H_Z(a,b) = 1 - B_{0,(a,b)} + B_{1,(a,b)} - B_{2,(a,b)}$$

Proof. This follows from Lemma 5.1. □

In the following two propositions we explicitly compute the first difference of the Hilbert Function of Z using Corollary 5.2.

Proposition 5.3. *Let* $Z = m_{11}P_{11} + m_{12}P_{12} + m_{21}P_{21}$. *If* $m_{11} \leq m_{21}$, *then*

$$\Delta H_Z(a,b) = \begin{cases} 1 & \text{if } a+b < m_{12}+m_{21}, \\ 1-\beta_{0,(a,b)}(Z) & \text{if } a+b = m_{12}+m_{21}, \\ 0 & \text{if } a+b > m_{12}+m_{21}. \end{cases}$$

Proof. By Theorem 4.1, we have $\beta_{1,(i,j)}(Z) = \beta_{0,(i-1,j)}(Z)+\beta_{0,(i,j-1)}(Z)-1$ for $i+j = m_{12}+m_{21}+1$ and zero elsewhere, and $\beta_{2,(i,j)}(Z) = \beta_{0,(i-1,j-1)}(Z)-1$ if and only if $i+j = m_{12}+m_{21}+2$ (otherwise zero). So, by applying Corollary 5.2 and a machinery computation we are done. □

Proposition 5.4. *Let* $Z = m_{11}P_{11} + m_{12}P_{12} + m_{21}P_{21}$. *If* $m_{11} > m_{21}$, *then*

$$\Delta H_Z(a,b) \begin{cases} 1 & \text{if } (a,b) < (i,j) \text{ for some } (i,j) \in \cup D_i(Z), \\ 1-\beta_{0,(a,b)}(Z) & \text{if } (a,b) \in \cup D_i(Z), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof uses the same argument as in Proposition 5.3. □

In the last part of these notes we give a criterion to verify if an admissible function, $H : \mathbb{N}^2 \to N$, as introduced in [10] Definition 2.2, is the Hilbert function of a set of at most three (fat) points on an ACI support.

Theorem 5.5. *Let $H : \mathbb{N}^2 \to N$ be an admissible function, and let $H(i,j) = \gamma$ for $(i,j) \gg (0,0)$. We denote by $h_{ij} := \Delta H(i,j)$, moreover we set*

$$A_i^{(d)} := \sum_{j=0}^{d-1} h_{ij}, \quad B_j^{(d)} := \sum_{i=0}^{d-1} h_{ij}$$

and $\alpha := \max\{i \mid A_i^{(d)} \neq 0\} + 1$, and $\beta := \max\{j \mid B_j^{(d)} \neq 0\} + 1$. Then we have the following cases:
Case (1) There exist $d, d_1, d_2 \in \mathbb{N}$ such that if $i < d_1$ and $j < d_2$, then $h_{ij} = 1$ iff $i + j < d$. If H is the Hilbert function of a set of points $Z := m_{11}P_{11} + m_{12}P_{12} + m_{21}P_{21}$, then (m_{11}, m_{12}, m_{21}) is the solution of one of the following systems:

$$\begin{cases} x + y = \alpha \\ x + z = \beta \\ y + z = d \end{cases} \qquad \begin{cases} x + y = \alpha \\ y = \beta \\ y + z = d \end{cases}$$

$$\begin{cases} z = \alpha \\ x + z = \beta \\ y + z = d \end{cases} \qquad \begin{cases} z = \alpha \\ y = \beta \\ \binom{x+1}{2} + \binom{y+1}{2} + \binom{z+1}{2} = \gamma. \end{cases}$$

Case (2) Assume the first case does not occur. Then H is not the Hilbert function of any set of points $m_{11}P_{11} + m_{12}P_{12} + m_{21}P_{21}$.

Proof. Case (1). The condition $h_{ij} = 1$ if and only if $i+j < d$ ($i < d_1$, $j < d_2$) is always verified for sets of three points on an ACI support (see Theorem 4.1 and Theorem 4.5). In both cases we have $m_{12} + m_{21} = d$. Moreover, from Theorem 2.12 in [10], α and β respectively count the maximum number of point on a line of type $(0,1)$ and $(1,0)$ that are respectively $\max\{m_{11} + m_{21}, m_{12}\}$ and $\max\{m_{11} + m_{12}, m_{21}\}$. These conditions give arise to four linear systems:

$$\begin{cases} x + y = \alpha \\ x + z = \beta \\ y + z = d \end{cases} \quad \begin{cases} x + y = \alpha \\ y = \beta \\ y + z = d \end{cases} \quad \begin{cases} z = \alpha \\ x + z = \beta \\ y + z = d \end{cases} \quad \begin{cases} z = \alpha \\ y = \beta \\ y + z = d. \end{cases}$$

But the last system, since in that case $\alpha = m_{12}$ and $\beta = m_{12}$ is not determined. So we need to replace one equation with $\binom{x+1}{2} + \binom{y+1}{2} + \binom{z+1}{2} = \gamma$, that is the degree of a set of three fat points.

Moreover, from Proposition 5.3 and Proposition 5.4 we can see that Case (2) does not lead to any set of at most three fat points on an ACI support. □

Given an admissible numerical function H, if Case (1) of Theorem 5.5 occurs, we are able to construct a set Z of three fat points, that could have Hilbert function equal to H, by solving the systems. Then, by using Theorem 4.1 and Proposition 5.3 (or Theorem 4.5 and Proposition 5.4), we check if $H_Z = H$. The next example shows this procedure.

Example 5.6. Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function such that

$\Delta H =$

	0	1	2	3	4	5	6	7	$\cdots$
0	1	1	1	1	1	1	1	0	$\cdots$
1	1	1	1	1	1	0	0	0	$\cdots$
2	1	1	1	1	0	0	0	0	$\cdots$
3	1	1	1	−1	0	0	0	0	$\cdots$
4	1	1	−2	0	0	0	0	0	$\cdots$
5	1	−1	0	0	0	0	0	0	$\cdots$
6	0	0	0	0	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Note that we are in Case (1) of Theorem 5.5. In particular, we have $d = 6$. Thus $\gamma = \sum \Delta H(i,j) = 18$, $\alpha = 4$ and $\beta = 7$. So we need to investigate on the solutions of the following systems:

$$\text{(i)} \quad \begin{cases} x + y = 4 \\ x + z = 7 \\ y + z = 6 \end{cases} \qquad \text{(ii)} \quad \begin{cases} x + y = 4 \\ y = 7 \\ y + z = 6 \end{cases}$$

$$\text{(iii)} \quad \begin{cases} z = 4 \\ x + z = 7 \\ y + z = 6 \end{cases} \qquad \text{(iv)} \quad \begin{cases} z = 4 \\ y = 7 \\ x^2 + x + 18 = 0. \end{cases}$$

Note that (i), (ii), (iv) have not solution in $\mathbb{N}^3$, i.e., H is the Hilbert function of a set of fat points $Z = m_{11}P_{11} + m_{12}P_{12} + m_{21}P_{21}$ if and only if $(m_{11}, m_{12}, m_{21}) = (3, 2, 4)$, that is the solution of (iii). But from Proposition 5.4 and Theorem 4.5 we have that $Z = 3P_{11} + 2P_{12} + 4P_{21}$ has the first difference of the Hilbert function equal to

$\Delta H_Z =$

	0	1	2	3	4	5	6	7	$\cdots$
0	1	1	1	1	1	1	1	0	$\cdots$
1	1	1	1	1	1	0	0	0	$\cdots$
2	1	1	1	1	0	0	0	0	$\cdots$
3	1	1	1	−1	0	0	0	0	$\cdots$
4	1	1	−1	0	0	0	0	0	$\cdots$
5	1	−1	0	0	0	0	0	0	$\cdots$
6	0	0	0	0	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

so $H \neq H_Z$ and hence H is not the Hilbert function of any set of at most three fat points on an ACI support.

We can also use (2.1) and run the script from [11] available at

http://www.math.unl.edu/∼bharbour/6ptres/6reswebsite.html

to see that in $\mathbb{P}^2$ the set of five fat points $Z = 3P_{11} + 2P_{12} + 4P_{21} + 4P_1 + 2P_2$ has Hilbert function equal $\dim(I_Z)_6 = 2 \neq 15 - H(4,2) = 3$.

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