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LIOUVILLE THEOREMS FOR GENERALIZED SYMPHONIC MAPS

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Abstract. In this paper, we introduce the notion of the generalized symphonic map with respect to the functional Φ_ε. Then we use the stress-energy tensor to obtain some monotonicity formulas and some Liouville results for these maps. We also obtain some Liouville type results by assuming some conditions on the asymptotic behavior of the maps at infinity.

1. Introduction

Liouville type theorems for harmonic maps, p-harmonic maps, F-harmonic maps, and F-stationary maps were investigated by several authors ([1, 6, 10, 16, 18, 28, 29, 31, 32] and the references therein). It is well known that the stress-energy tensor is a useful tool to investigate the energy behavior and some vanishing results of related energy functional. Most Liouville results have established by assuming either the finiteness of the energy of the map or the smallness of whole image of the domain manifold under the map. In [18], Jin proved several interesting Liouville theorems for harmonic maps from complete manifolds, whose assumptions concern the asymptotic behavior of the maps at infinity. One special case of his results is that if $u : (\mathbb{R}^m, g_0) \to (N^n, h)$ is a harmonic map, and $u(x) \to p_0 \in N^n$ as $|x| \to \infty$, then u is a constant map. In [9], Dong, Lin and Yang, generalized Jin's method to F-harmonic maps and obtained some Liouville theorems and their applications.

Let $\Omega \subset R^2$ be a smooth bounded simply connected domain. Consider the following functional defined for maps $u \in H^1(\Omega, \mathbf{C})$:

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dv + \frac{1}{4\varepsilon^2}\int_\Omega (1 - |u|^2)^2 dv.$$

Ginzburg-Landau introduced this functional in study of phase transition problems and it plays an important role ever since, especially in superconductivity, superfluidity and XY-magnetism (see details for [21, 25, 27]). A lot of paper

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devote to the asymptotic behavior of minimizers u_ε of $E_\varepsilon(u,\Omega)$ in $H^1(\Omega,C)$ as $\varepsilon\to 0$. It was shown in those cases that u_ε converges strongly to a harmonic map u_0 on any compact subset away from the zeros. Readers can refer to [3–5, 30] for the progress in this field. For general case, the p-Ginzburg-Landau functional has been introduced. Hong in [17] and Lei in [22] investigated the convergence of a p-Ginzburg-Landau type functional when parameter goes to zero. In [7], Chong, Cheng, Dong and Zhang investigated the critical points of the p-Ginzburg-Landau type functional and obtained some Liouville theorems for these maps.

Let (M^m,g) and (N^n,h) be two Riemannian manifolds, and let u be a smooth map from M to N. In [19], S. Kawai, and N. Nakauchi introduced a functional

$$\Phi(u)=\int_M \frac{||u^*h||^2}{4}dv_g,$$

where dv_g is the volume form on (M,g), u^*h is the symmetric 2-tensor (pullback metric) defined by $u^*h(X,Y)=h(du(X),du(Y))$ for any vector fields X,Y on M and $||u^*h||$, its norm as $||u^*h||^2=\sum_{i,j=1}^m\left[h(du(e_i),du(e_j))\right]^2$, with respect to $\{e_i\}$ which is a local orthonormal frame on (M,g). The map u is a symphonic (or stationary) map if it is a critical point of $\Phi(u)$ with respect to any compact supported variation of u and u is symphonic (or stationary) stable if the second variation for the functional $\Phi(u)$ is nonnegative. When M and N are compact without boundary, the same authors showed the non-existence of non-constant stable symphonic map for Φ, if M (respectively N) is a standard sphere S^m (respectively S^n). Readers can refer to [1, 13, 14, 20, 23, 24] for the progress in this field.

In this paper, we can consider a smooth map $u:(M^m,g)\to(\mathbb{R}^n,h)$ from a Riemannian manifold to the standard Euclidean space and the following functional

$$\Phi_\varepsilon(u)=\int_M\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]dv_g, \tag{1}$$

where ε is any small positive number. We call u a generalized symphonic map for the functional $\Phi_\varepsilon(u)$, if

$$\frac{d}{dt}\Phi_\varepsilon(u_t)|_{t=0}=0$$

for any compactly supported variation $u_t:(M,g)\to(\mathbb{R}^n,h)$ with $u_0=u$. To generalized the Liouville type results for harmonic maps to the generalized symphonic maps, we first introduce the stress-energy tensor S_{Φ_ε} associated with the functional $\Phi_\varepsilon(u)$. We prove that the generalized symphonic map satisfies the conservation law, that is, $divS_{\Phi_\varepsilon}=0$. By using the stress-energy tensor, we obtain some monotonicity formulas for these maps, then we can prove some Liouville type results from these monotonicity formulas under suitable growth conditions on the functional $\Phi_\varepsilon(u)$.

Next we generalize Jin's method and results to the generalized symphonic maps. The procedure consists of two steps. The first step is to use the stress-energy tensor to establish the monotonicity formula which gives a lower bound for the growth rates of the functional $\Phi_\varepsilon(u)$. The second step is to use the asymptotic assumption of the maps at infinity to obtain the upper functional growth rates of the generalized symphonic maps. Under suitable conditions on u and the Hessian of the distance functions of the domain manifolds, one may show that these two growth rates are contradictory unless the generalized symphonic map is constant. In this way, we establish some Liouville theorems for the generalized maps with asymptotic property at infinity from some complete manifolds.

2. The first variation formula

Let ∇ and ${}^{\mathbb{R}^n}\nabla$ be always denote the Levi-Civita connections of (M^m, g) and $(\mathbb{R}^n, h)$ respectively. Let $\widetilde{\nabla}$ be the induced connection on $u^{-1}T\mathbb{R}^n$ defined by $\widetilde{\nabla}_X W = {}^{\mathbb{R}^n}\nabla_{du(X)}W$, where X is a tangent vector of M^m and W is a section of $u^{-1}T\mathbb{R}^n$. We can choose a local orthonormal frame field $\{e_i\}_{i=1}^m$ on M^m. We define the Φ_ε-tension field $\tau_{\Phi_\varepsilon}(u)$ by

$$\tau_{\Phi_\varepsilon}(u) = \mathrm{div}_g(\sigma_u) + \frac{1}{\varepsilon^n}(1-|u|^2)u, \tag{2}$$

where $\sigma_u(X) = \sum_{i=1}^n h(du(e_i), du(X))du(e_i)$, for any smooth vector field X on M. Let $u_t : (M^m, g \to (\mathbb{R}^n, h))$, $|t| < k$ with $u_0 = u$ and $v = \frac{\partial u}{\partial t}|_{t=0}$ be a one parameter compactly supported variation, we have the following lemma.

Lemma 2.1 (The first variation formula)**.** *Let $u : (M^m, g) \to (\mathbb{R}^n, h)$ be a C^2 map. Then we have*

$$\frac{d}{dt}|_{t=0}\Phi_\varepsilon(u_t) = -\int_M h(\tau_{\Phi_\varepsilon}(u), v)dv_g. \tag{3}$$

Proof. Let $\{e_i\}$ be a local orthonormal frame of TM. Since the target manifold is the Standard Euclidean space $\mathbb{R}^n$, we can perform the following calculations,

$$\begin{aligned}
\frac{d}{dt}|_{t=0}\Phi_\varepsilon(u_t) &= \int_M \left[\frac{\partial}{\partial t}|_{t=0}\left(\frac{||u_t^*h||^2}{4}\right) + \frac{\partial}{\partial t}|_{t=0}\left[\frac{1}{4\varepsilon^n}(1-|u_t|^2)^2\right]\right]dv_g \\
&= \int_M \sum_{i,j=1}^m h(\widetilde{\nabla}_{\frac{\partial}{\partial t}} du_t(e_i), du_t(e_j))h(du_t(e_i), du_t(e_j))|_{t=0}dv_g \\
&\quad - \int_M \frac{1}{\varepsilon^n}(1-|u|^2)h(v,u)dv_g \\
&= \int_M \sum_{i=1}^m h(\widetilde{\nabla}_{e_i} du_t(\frac{\partial}{\partial t}), \sigma_{u_t}(e_i))|_{t=0}dv_g \\
&\quad - \int_M \frac{1}{\varepsilon^n}(1-|u|^2)h(v,u)dv_g
\end{aligned}$$

$$= \int_M \sum_{i=1}^m h(\widetilde{\nabla}_{e_i} v, \sigma_u(e_i)) dv_g - \int_M \frac{1}{\varepsilon^n}(1-|u|^2) h(v,u) dv_g$$
$$= -\int_M h(\mathrm{div}_g(\sigma_u) + \frac{1}{\varepsilon^n}(1-|u|^2)u, v) dv_g$$
$$= -\int_M h(\tau_{\Phi_\varepsilon}(u), v) dv_g.$$

Here we have used the Green's theorem in the fifth equality. □

The first variation formula allows us to define the notion of generalized symphonic map for the functional $\Phi_\varepsilon(u)$.

Definition 2.2. A C^2 map u is called generalized symphonic map for the functional $\Phi_\varepsilon(u)$ if it is a solution of the Euler-Lagrange equation,

$$\tau_{\Phi_\varepsilon}(u) = \mathrm{div}_g(\sigma_u) + \frac{1}{\varepsilon^n}(1-|u|^2)u = 0. \tag{4}$$

3. Stress-energy tensor

Following Baird [2], for a smooth map $u : (M^m, g) \to (\mathbb{R}^n, h)$, we associate a symmetric 2-tensor S_{Φ_ε} to the functional Φ_ε called the stress-energy tensor

$$S_{\Phi_\varepsilon}(X,Y) = \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] g(X,Y) - h(du(X), \sigma_u(Y)), \tag{5}$$

where X, Y are smooth vector field on M.

Proposition 3.1. *Let $u : (M^m, g) \to (\mathbb{R}^n, h)$ be a smooth map and S_{Φ_ε} be the associated stress-energy tensor, then for each vector field X on M, we have*

$$(divS_{\Phi_\varepsilon})(X) = -h(div_g\sigma_u + \frac{1}{\varepsilon^n}(1-|u|^2)u, du(X)).$$

Proof. Let ∇ and ${}^{\mathbb{R}^n}\nabla$ always denote the Levi-Civita connections of M and $\mathbb{R}^n$ respectively. Let $\widetilde{\nabla}$ be the induced connection on $u^{-1}T\mathbb{R}^n$. We choose a local orthonormal frame field $\{e_i\}$ around a point P on M with $\nabla_{e_i} e_j|_P = 0$.

Let X be a vector field on M. At P, we compute

$$(divS_{\Phi_\varepsilon})(X)$$
$$= \sum_{i=1}^m (\nabla_{e_i} S_{\Phi_\varepsilon})(e_i, X)$$
$$= \sum_{i=1}^m [e_i S_{\Phi_\varepsilon}(e_i, X) - S_{\Phi_\varepsilon}(e_i, \nabla_{e_i} X)]$$
$$= \sum_{i=1}^m \left\{ e_i \left(\left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] g(e_i, X) \right) - e_i h(\sigma_u(e_i), du(X)) \right.$$
$$\left. - \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] g(e_i, \nabla_{e_i} X) + h(\sigma_u(e_i), du(\nabla_{e_i} X)) \right\}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}[e_i\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]g(e_i,X)-h(\widetilde{\nabla}_{e_i}\sigma_u(e_i),du(X))\\
&\quad -h(\sigma_u(e_i),\widetilde{\nabla}_{e_i}du(X))+h(\sigma_u(e_i),du(\nabla_{e_i}X))]\\
&= \sum_{i=1}^{m}\left[X\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]-h(\sigma_u(e_i),(\nabla_{e_i}du)(X))\right]\\
&\quad -h(div(\sigma_u),du(X))\\
&= \sum_{i=1}^{m}\left[-\frac{1}{\varepsilon^n}(1-|u|^2)h(u,du(X))\right.\\
&\quad +\sum_{j=1}^{m}h((\nabla_X du)(e_i),du(e_j))h(du(e_i),du(e_j))\\
&\quad -\ h(\sigma_u(e_i),(\nabla_{e_i}du)(X)]-h(div(\sigma_u),du(X))\\
&= \sum_{i=1}^{m}[h((\nabla_X du)(e_i),\sigma_u(e_i))-h(\sigma_u(e_i),(\nabla_{e_i}du)(X)]\\
&\quad -h(div(\sigma_u)+\frac{1}{\varepsilon^n}(1-|u|^2)u,du(X)).
\end{aligned}
$$

Since $(\nabla_X du)(e_i)=(\nabla_{e_i}du)(X)$, we have

$$(divS_{\Phi_\varepsilon})(X)=-h\left(div\sigma_u+\frac{1}{\varepsilon^n}(1-|u|^2)u,du(X)\right).$$

This completes the proof. □

Definition 3.2. We say that u satisfies the conversation law if $divS_{\Phi_\varepsilon}=0$.

By the above proposition, we can obtain the following result.

Corollary 3.3. *If $u:(M,g)\to(\mathbb{R}^n,h)$ is a generalized symphonic map, then u satisfies the conservation law, i.e., $divS_{\Phi_\varepsilon}=0$.*

Recall that for two 2-tensors T_1, $T_2\in\Gamma(T^*M\otimes T^*M)$, their inner product is defined as follows;

$$\langle T_1,T_2\rangle=\sum_{i,j=1}^{m}T(e_i,e_j)T_2(e_i,e_j), \tag{6}$$

where $\{e_i\}$ is an orthonormal basis with respect to g. For a vector field $X\in\Gamma(TM)$, we denote by θ_X its dual one form, i.e., $\theta_X(Y)=g(X,Y)$, where $Y\in\Gamma(TM)$. The covariant derivative of θ_X gives a 2-tensor field $\nabla\theta_X$:

$$(\nabla\theta_X)(Y,Z)=(\nabla_Y\theta_X)(Z)=g(\nabla_Y X,Z). \tag{7}$$

If $X=\nabla\varphi$ is the gradient field of some C^2 function φ on M, then $\theta_X=d\varphi$ and $\nabla\theta_X=Hess\varphi$.

Lemma 3.4 (cf. [2,10])**.** *Let T be a symmetric $(0,2)$-type tensor field and let X be a vector field, then*

$$div(i_X T) = (divT)(X) + \langle T, \nabla\theta_X\rangle = (divT)(X) + \frac{1}{2}\langle T, L_X g\rangle, \tag{8}$$

where L_X is the Lie derivative of the metric g in the direction of X. Indeed, let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame field on M. Then

$$\begin{aligned}\frac{1}{2}\langle T, L_X g\rangle &= \sum_{i,j=1}^{m} \frac{1}{2}\langle T(e_i, e_j), L_X g(e_i, e_j)\rangle \\ &= \sum_{i,j=1}^{m} T(e_i, e_j) g(\nabla_{e_i} X, e_j) = \langle T, \nabla\theta_X\rangle.\end{aligned}$$

Let D be any bounded domain of M with C^1 boundary. By using the stokes' theorem, we have the following integral formula:

$$\int_{\partial D} T(X,\nu) ds_g = \int_D [\langle T, \frac{1}{2} L_X g\rangle + (divT)(X)] dv_g, \tag{9}$$

where ν is the unit outward normal vector field along ∂D. By the definition of generalized symphonic map and (9), we have

$$\int_{\partial D} S_{\Phi_\varepsilon}(X,\nu) ds_g = \int_D \langle S_{\Phi_\varepsilon}, \frac{1}{2} L_X g\rangle dv_g. \tag{10}$$

4. Monotonicity formulas

Let (M^m, g) be a complete Riemannian manifold with a pole x_0. A pole $x_0 \in M$ is a point such that the exponential map from the tangent space to M at x_0 is a diffeomorphism. Denote by $r(x)$ the g-distance function relative to the pole x_0, that is, $r(x) = dist_g(x, x_0)$. Set $B(r) = \{x \in M^m : r(x) \le r\}$. It is known that $\frac{\partial}{\partial r}$ is always an eigenvector of $Hess_g(r^2)$ associated to eigenvalue 2. Denote by $\lambda_{\max}$ (resp. $\lambda_{\min}$) the maximum (resp. minimal) eigenvalues of $Hess_g(r^2) - 2dr \otimes dr$ at each point of $M\backslash\{x_0\}$.

Theorem 4.1. *Let $u : (M^m, g) \to (\mathbb{R}^n, h)$ be a generalized symphonic map from a complete Riemannian manifold (M^m, g) to $(\mathbb{R}^n, h)$. If there exists a constant $\sigma > 0$ such that*

$$1 + \frac{m-1}{2}\lambda_{\min} - 2\max\{2, \lambda_{\max}\} \ge \sigma, \tag{11}$$

then

$$\frac{\int_{B(\rho_1)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g}{\rho_1^\sigma} \le \frac{\int_{B(\rho_2)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g}{\rho_2^\sigma}$$

for any $0 < \rho_1 \le \rho_2$. In particular, if $\int_{B(R)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g = o(R^\sigma)$, then u is constant and $u \in S^{n-1}$.

Proof. We take $D = B(R)$ and $X = r\frac{\partial}{\partial r} = \frac{1}{2}\nabla r^2$ in (10), where ∇ denotes the covariant derivative determined by g, we have

$$\int_{\partial B(R)} S_{\Phi_\varepsilon}(r\frac{\partial}{\partial r}, \nu)ds_g = \int_{B(R)} \langle S_{\Phi_\varepsilon}, \frac{1}{2}L_{r\frac{\partial}{\partial r}}g\rangle dv_g. \tag{12}$$

Let $\{e_i\}_{i=1}^m$ ba an orthonormal basis with respect to g and $e_m = \nu = \frac{\partial}{\partial r}$. We may assume that $Hess_g(r^2)$ becomes a diagonal matrix with respect to $\{e_i\}_{i=1}^m$. Now we compute,

$$\begin{aligned}
&\langle S_{\Phi_\varepsilon}, L_{r\frac{\partial}{\partial r}}g\rangle = \sum_{i,j=1}^{m} S_{\Phi_\varepsilon}(e_i, e_j)(L_{r\frac{\partial}{\partial r}}g)(e_i, e_j)\\
&= \sum_{i,j=1}^{m}\Bigg[\left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] g(e_i, e_j)(L_{r\frac{\partial}{\partial r}}g)(e_i, e_j)\\
&\quad - h(du(e_i), \sigma_u(e_j))(L_{r\frac{\partial}{\partial r}}g)(e_i, e_j)\Bigg]\\
&= \sum_{i=1}\left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] Hess_g(r^2)(e_i, e_i)\\
&\quad - \sum_{i,j=1}^{m} h(du(e_i), \sigma_u(e_j))Hess_g(r^2)(e_i, e_j)\\
&\geq \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right][2 + (m-1)\lambda_{\min}]\\
&\quad - \max\{2, \lambda_{\max}\}\sum_{i=1}^{m} h(du(e_i), \sigma_u(e_i))\\
&= \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right][2 + (m-1)\lambda_{\min}]\\
&\quad - 4\max\{2, \lambda_{\max}\}\frac{||u^*h||^2}{4}\\
&\geq \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right][2 + (m-1)\lambda_{\min} - 4\max\{2, \lambda_{\max}\}]. \qquad (13)
\end{aligned}$$

From (11) and (13), we have

$$\langle S_{\Phi_\varepsilon}, \frac{1}{2}L_{r\frac{\partial}{\partial r}}g\rangle \geq \sigma\left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]. \tag{14}$$

On the other hand, by the coarea formula, we have

$$\begin{aligned}
&\int_{\partial B(r)} S_{\Phi_\varepsilon}(r\frac{\partial}{\partial r}, \frac{\partial}{\partial r})ds_g\\
&= r\int_{\partial B(r)}\left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] ds_g
\end{aligned}$$

$$
\begin{aligned}
&-r\int_{\partial B(r)} h(du(\frac{\partial}{\partial r}),\sigma_u(\frac{\partial}{\partial r}))\\
&= r\int_{\partial B(r)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]ds_g\\
&\quad -r\int_{\partial B(r)}[\sum_{j=1}^m h^2(du(\frac{\partial}{\partial r}),du(e_j))]\\
&\le r\int_{\partial B(r)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]ds_g\\
&= r\frac{d}{dr}\int_{B(r)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]dv_g. \qquad (15)
\end{aligned}
$$

From (12), (14) and (15), we have

$$
\begin{aligned}
&\sigma\int_{B(r)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]dv_g\\
&\le r\frac{d}{dr}\int_{B(r)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]dv_g,
\end{aligned}
$$

so we have

$$
\frac{\int_{B(\rho_1)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]dv_g}{\rho_1^\sigma}\le\frac{\int_{B(\rho_2)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]dv_g}{\rho_2^\sigma}
$$

for any $0<\rho_1\le\rho_2$. □

Lemma 4.2 ([8, 10–13, 15])**.** *Let (M^m,g) be a complete Riemannian manifold with a pole x_0. Denote by K_r the radial curvature of M^m.*

(1) *If $-\alpha^2\le K_r\le-\beta^2$ with $\alpha\ge\beta\ge0$ and $(m-1)\beta-4\alpha>0$, then*

$$
1+\frac{m-1}{2}\lambda_{\min}-2\max\{2,\lambda_{\max}\}\ge m-\frac{4\alpha}{\beta}.
$$

(2) *If $-\frac{A}{(1+r^2)^{1+\epsilon}}\le K_r\le\frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon>0$, $A\ge0$ and $0\le B\le2\epsilon$, then*

$$
1+\frac{m-1}{2}\lambda_{\min}-2\max\{2,\lambda_{\max}\}\ge1+(m-1)(1-\frac{B}{2\epsilon})-4e^{\frac{A}{2\epsilon}}.
$$

(3) *If $-\frac{a^2}{c^2+r^2}\le K_r\le\frac{b^2}{c^2+r^2}$ with $a\ge0$, $b^2\in[0,\frac14]$ and $c^2\ge0$, then*

$$
\begin{aligned}
&1+\frac{m-1}{2}\lambda_{\min}-2\max\{2,\lambda_{\max}\}\\
&\ge1+(m-1)\frac{1+\sqrt{1-4b^2}}{2}-4\frac{1+\sqrt{1-4a^2}}{2}.
\end{aligned}
$$

From Theorem 4.1 and Lemma 4.2, we have the following result.

Corollary 4.3. *Let (M^m, g) be an m-dimensional complete manifold with a pole x_0. Assume that the radial curvature K_r of M satisfies one of the following three conditions*:

(1) *If* $-\alpha^2 \leq K_r \leq -\beta^2$ *with* $\alpha \geq \beta \geq 0$ *and* $(m-1)\beta - 4\alpha > 0$.

(2) *If* $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ *with* $\epsilon > 0$, $A \geq 0$, $0 \leq B \leq 2\epsilon$ *and* $1 + (m-1)(1 - \frac{B}{2\epsilon}) - 4e^{\frac{A}{2\epsilon}} > 0$.

(3) *If* $-\frac{a^2}{c^2+r^2} \leq K_r \leq \frac{b^2}{c^2+r^2}$ *with* $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ *and* $1+(m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\frac{1+\sqrt{1-4a^2}}{2} > 0$. *If* $u : (M^m, g) \to (\mathbb{R}^n, h)$ *is a generalized symphonic map, then*

$$\frac{\int_{B(\rho_1)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g}{\rho_1^\Lambda} \leq \frac{\int_{B(\rho_2)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g}{\rho_2^\Lambda}$$

for $0 < \rho_1 \leq \rho_2$, *where* Λ *is given as follows,*

$$\Lambda = \begin{cases} m - \frac{4\alpha}{\beta} & \text{if} \quad K_r \quad \text{satisfies } (1), \\ 1 + (m-1)(1 - \frac{B}{2\epsilon}) - 4e^{\frac{A}{2\epsilon}} & \text{if} \quad K_r \quad \text{satisfies } (2), \\ 1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\frac{1+\sqrt{1-4a^2}}{2} & \text{if} \quad K_r \quad \text{satisfies } (3). \end{cases}$$

In particular, if $\int_{B(R)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g = o(R^\Lambda)$, *then* u *is constant and* $u \in S^{n-1}$.

5. Liouville theorems

We first give the lower Φ_ε functional growth rates for generalized symphonic map.

Proposition 5.1. *Let* $u : (M^m, g) \to (\mathbb{R}^n, h)$ *be a generalized symphonic map from a Riemannian manifold with a pole* x_0 *to a standard Euclidean space. It* $u(M)$ *is not contained in* S^{n-1} *and* $r(x)$ *satisfies the condition* (11), *then*

$$\int_{B(R)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g \geq C(u)R^\sigma, \quad \text{as} \quad R \to \infty,$$

where $C(u)$ *is a positive constant only depending on* u.

Proof. Since u satisfies the condition in Theorem 4.1, we have

$$\frac{\int_{B(\rho)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g}{\rho^\sigma} \leq \frac{\int_{B(R)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g}{R^\sigma}$$

for any $0 < \rho < R$. Note that $u(M)$ is not contained in S^{n-1}, there exist some $\rho > 0$ such that

$$\int_{B(\rho)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g > 0.$$

Set $C(u) = \frac{\int_{B(\rho)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]dv_g}{\rho^\sigma}$, then

$$\int_{B(R)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]dv_g \geq C(u)R^\sigma.$$

This completes the proof of the proposition. □

Corollary 5.2. *Let (M^m, g) be an m-dimensional complete manifold with a pole x_0. Assume that the radial curvature K_r satisfies the condition in Corollary 4.3. If $u : (M, g) \to (\mathbb{R}^n, h)$ be a generalized symphonic map and $u(M)$ is not contained in S^{n-1}, then we have*

$$\int_{B(R)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]dv_g \geq C(u)R^\Lambda, \quad as \quad R \to \infty,$$

where $C(u)$ is a positive constant only depending on u and Λ is given in Corollary 4.3.

Next we will use the assumption for the map at infinity to derivative an upper bound for the growth rate. The condition that we will assume for u is as follow:
(P_1) There exists a positive constant $\widetilde{\sigma}$ strictly less than σ in (11) such that

$$\left[\max_{r(x)=r} h^2(u(x), P_0)\right]^{\frac{2}{3}} \leq r^{\frac{\widetilde{\sigma}}{3}} \int_r^\infty \frac{ds}{[vol(\partial B(s))]^{\frac{1}{3}}} \quad \text{for} \quad r(x) \gg 1,$$

where P_0 is a fixed point in S^{p-1}.

Theorem 5.3. *Let $u : (M^m, g) \to (\mathbb{R}^n, h)$ be a generalized symphonic map. Suppose that $r(x)$ satisfies the condition (11). If $u(x) \to P_0 \in S^{n-1}$ as $r(x) \to \infty$ and u satisfies the condition (P_1), then u must be a constant map.*

Proof. Suppose u is not constant, then by Proposition 5.1, the Φ_ε functional of u must be infinite, that is, $\Phi_\varepsilon^R(u) = \int_{B(R)}\left[\frac{||u^*h||^2}{4}+\frac{1}{4\varepsilon^n}(1-|u|^2)^2\right]dv_g \to \infty$, as $R \to \infty$.

Since $P_0 = (c_1, \ldots, c_n) \in S^{n-1}$, then $\sum_{\alpha=1}^n c_\alpha^2 = 1$. It is clear that we can choose an orthogonal matrix A such that $AP_0 = \widetilde{P_0} = (\widetilde{c}_1, \ldots, \widetilde{c}_n)$, $\widetilde{c_\alpha} \neq 0$, for $\alpha = 1, \ldots, n$. Clearly if u is a generalized map, then Au is also the generalized map. Hence without loss of generality, we may assume that $u(x) \to P_0$ as $r(x) \to \infty$, where $P_0 = (c_1, \ldots, c_n)$, where $c_\alpha \neq 0$, for $\alpha = 1, \ldots, n$.

Now the assumption that $u(x) \to P_0$ as $r(x) \to \infty$ implies that there exists a $R_1 > 0$ and a neighborhood U of P_0 such that for $r(x) > R_1$, $u(x) \in U$ and $u_\alpha \neq 0$ for $\alpha = 1, \ldots, n$.

For $\omega \in C_0^2(M\backslash B(R_1), U)$, we consider the variation $u + t\omega : M \to \mathbb{R}^n$ defined as follows:

$$(u+t\omega)(q) = \begin{cases} u(q) & q \in B(R_1), \\ (u+t\omega)(q) & q \in M\backslash B(R_1) \end{cases}$$

for sufficiently small t. Since u is a generalized symphonic map, we have

$$\frac{d}{dt}|_{t=0}\Phi_\varepsilon(u+t\omega)=0,$$

that is,

$$\int_{M\setminus B(R_1)}\left[\sum_{i,j,k,l=1}^{m}\sum_{\alpha,\beta=1}^{n}g^{ik}g^{jl}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial \omega_\alpha}{\partial x_j}\frac{\partial u_\beta}{\partial x_k}\frac{\partial u_\beta}{\partial x_l}\right.$$
$$\left.-\frac{1}{\varepsilon^n}(1-\sum_{\alpha=1}^{n}u_\alpha^2)\sum_{\alpha=1}^{n}u_\alpha\omega_\alpha\right]dv_g=0.$$

Choosing $\omega=\phi(r(x))\widetilde{u}$ in the above equation for $\phi(t)\in C_0^\infty(R_1,\infty)$, $\widetilde{u}_\alpha=\frac{u_\alpha^2-c_\alpha^2}{u_\alpha}$, we obtain

$$\int_{M\setminus B(R_1)}\sum_{i,j,k,l=1}^{m}\sum_{\alpha,\beta=1}^{n}g^{ik}g^{jl}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial \widetilde{u}_\alpha}{\partial x_j}\frac{\partial u_\beta}{\partial x_k}\frac{\partial u_\beta}{\partial x_l}\phi(r(x))dv_g$$
$$-\int_{M\setminus B(R_1)}\frac{1}{\varepsilon^n}\left(1-\sum_{\alpha=1}^{n}u_\alpha^2\right)\phi(r(x))\sum_{\alpha=1}^{n}u_\alpha\widetilde{u}_\alpha dv_g$$
$$=-\int_{M\setminus B(R_1)}\sum_{i,j,k,l=1}^{m}\sum_{\alpha,\beta=1}^{n}g^{ik}g^{jl}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial \phi(r(x))}{\partial x_j}\widetilde{u}_\alpha\frac{\partial u_\beta}{\partial x_k}\frac{\partial u_\beta}{\partial x_l}dv_g. \tag{16}$$

By a standard approximation argument, (16) holds for any Lipschitz function ϕ with compact support.

For $0\le\epsilon\le1$, define

$$\varphi_\epsilon(t)=\begin{cases}1 & t\le 1,\\ 1+\frac{1-t}{\epsilon} & 1<t<1+\epsilon,\\ 0 & t\ge 1+\epsilon\end{cases}$$

and choose the Lipschitz function $\phi(r(x))$ to be

$$\phi(r(x))=\varphi_\epsilon\left(\frac{r(x)}{R}\right)\left(1-\varphi_1\left(\frac{r(x)}{R_1}\right)\right),\quad R>2R_1.$$

Then the first term on the left hand side of (16) becomes

$$\int_{M\setminus B(R_1)}\sum_{i,j,k,l=1}^{m}\sum_{\alpha,\beta=1}^{n}g^{ik}g^{jl}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial \widetilde{u}_\alpha}{\partial x_j}\frac{\partial u_\beta}{\partial x_k}\frac{\partial u_\beta}{\partial x_l}\phi(r(x))dv_g$$
$$=\int_{B(R_2)\setminus B(R_1)}\sum_{\alpha,\beta=1}^{n}g^{ik}g^{jl}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial \widetilde{u}_\alpha}{\partial x_j}\frac{\partial u_\beta}{\partial x_k}\frac{\partial u_\beta}{\partial x_l}\left(1-\varphi_1\left(\frac{r(x)}{R_1}\right)\right)dv_g$$
$$+\int_{B(R)\setminus B(R_2)}\sum_{\alpha,\beta=1}^{n}g^{ik}g^{jl}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial \widetilde{u}_\alpha}{\partial x_j}\frac{\partial u_\beta}{\partial x_k}\frac{\partial u_\beta}{\partial x_l}dv_g$$

$$+\int_{B((1+\epsilon)R)\setminus B(R)} \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \widetilde{u}_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \varphi_\epsilon \left(\frac{r(x)}{R} \right) dv_g, \tag{17}$$

where $R_2 = 2R_1$. The second term on the left hand side of (16) becomes

$$\begin{aligned}
&-\int_{M^m\setminus B(R_1)} \frac{1}{\varepsilon^n} \left(1 - \sum_{\alpha=1}^n u_\alpha^2\right) \phi(r(x)) \sum_{\alpha=1}^n u_\alpha \widetilde{u}_\alpha dv_g \\
=& -\int_{B(R_2)\setminus B(R_1)} \frac{1}{\varepsilon^n} \left(1 - \sum_{\alpha=1}^n u_\alpha^2\right) \left(1 - \varphi_1\left(\frac{r(x)}{R_1}\right)\right) \sum_{\alpha=1}^n u_\alpha \widetilde{u}_\alpha dv_g \\
&-\int_{B(R)\setminus B(R_2)} \frac{1}{\varepsilon^n} \left(1 - \sum_{\alpha=1}^n u_\alpha^2\right) \sum_{\alpha=1}^n u_\alpha \widetilde{u}_\alpha dv_g \\
&-\int_{B((1+\epsilon)R)\setminus B(R)} \frac{1}{\varepsilon^n} \left(1 - \sum_{\alpha=1}^n u_\alpha^2\right) \varphi_\epsilon\left(\frac{r(x)}{R}\right) \sum_{\alpha=1}^n u_\alpha \widetilde{u}_\alpha dv_g.
\end{aligned} \tag{18}$$

The term on the right hand of (16) becomes,

$$\begin{aligned}
&-\int_{M^m\setminus B(R_1)} \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \phi(r(x))}{\partial x_j} \widetilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g \\
=& \int_{B(R_2)\setminus B(R_1)} \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \varphi_1\left(\frac{r(x)}{R_1}\right)}{\partial x_j} \widetilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g \\
&-\int_{B((1+\epsilon)R)\setminus B(R)} \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \varphi_\epsilon\left(\frac{r(x)}{R}\right)}{\partial x_j} \widetilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g \\
=& \int_{B(R_2)\setminus B(R_1)} \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \varphi_1\left(\frac{r(x)}{R_1}\right)}{\partial x_j} \widetilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g \\
&+\frac{1}{R\epsilon} \int_{B((1+\epsilon)R)\setminus B(R)} \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \widetilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g.
\end{aligned} \tag{19}$$

From (16)-(19) and let $\epsilon \to 0$, we have

$$\begin{aligned}
&\int_{B(R)\setminus B(R_2)} \left[\sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \widetilde{u}_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \right. \\
&\qquad \left. -\frac{1}{\varepsilon^n} \left(1 - \sum_{\alpha=1}^n u_\alpha^2\right) \sum_{\alpha=1}^n u_\alpha \widetilde{u}_\alpha \right] dv_g + D(R_1) \\
=& \int_{\partial B(R)} \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \widetilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g,
\end{aligned} \tag{20}$$

where $D(R_1)$ is defined as follows:

$$\begin{aligned}
D(R_1) = & \int_{B(R_2)\setminus B(R_1)} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \widetilde{u}_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \left(1 - \varphi_1 \left(\frac{r(x)}{R_1}\right)\right) dv_g \\
& - \int_{B(R)\setminus B(R_2)} \frac{1}{\varepsilon^n} \left(1 - \sum_{\alpha=1}^{n} u_\alpha^2\right) \sum_{\alpha=1}^{n} u_\alpha \widetilde{u}_\alpha dv_g \\
& - \int_{B(R_2)\setminus B(R_1)} \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \varphi_1 \left(\frac{r(x)}{R_1}\right)}{\partial x_j} \widetilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g.
\end{aligned}$$

Note that $\widetilde{u}_\alpha = \frac{u_\alpha^2 - c_\alpha^2}{u_\alpha}$. Thus we have

$$\frac{\partial \widetilde{u}_\alpha}{\partial x_j} = \left(1 + \frac{c_\alpha^2}{u_\alpha^2}\right) \frac{\partial u_\alpha}{\partial x_j} \tag{21}$$

and

$$\sum_{\alpha=1}^{n} u_\alpha \widetilde{u}_\alpha = \sum_{\alpha=1}^{n} [u_\alpha^2 - c_\alpha^2] = -\left(1 - \sum_{\alpha=1}^{n} u_\alpha^2\right). \tag{22}$$

From (20), (21) and (22), we have

$$\begin{aligned}
& \int_{B(R)\setminus B(R_2)} \left[\sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial u_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} (1 + \frac{c_\alpha^2}{u_\alpha^2}) \right. \\
& \qquad \left. + \frac{1}{\varepsilon^n} \left(1 - \sum_{\alpha=1}^{n} u_\alpha^2\right)^2 \right] dv_g + D(R_1) \\
= & \int_{\partial B(R)} \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \widetilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g.
\end{aligned} \tag{23}$$

Now we estimate the term on the right hand of (23). Take any point $p \in \partial B(R)$. Since the term $\sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \widetilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l}$ does not depend on the coordinate system on M. At the point p, we can take the adapt coordinate system, such that $g_{ij}(p) = \delta_{ij}$ and $g^{ij}(p) = \delta^{ij}$. We compute at p.

$$\begin{aligned}
& \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \widetilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \\
= & \sum_{i,j=1}^{m} \left[\sum_{\alpha}^{n} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \widetilde{u}_\alpha\right] \left[\sum_{\beta=1}^{n} \frac{\partial u_\beta}{\partial x_i} \frac{\partial u_\beta}{\partial x_j}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[\sum_{i,j=1}^{m}\left(\sum_{\alpha}^{n}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j}\widetilde{u}_\alpha\right)^2\right]^{\frac{1}{2}}\left[\sum_{i,j=1}^{m}\left(\sum_{\beta=1}^{n}\frac{\partial u_\beta}{\partial x_i}\frac{\partial u_\beta}{\partial x_j}\right)^2\right]^{\frac{1}{2}}\\
&= \left[\sum_{i=1}^{m}\left(\sum_{\alpha}^{n}\frac{\partial u_\alpha}{\partial x_i}\widetilde{u}_\alpha\right)^2\right]^{\frac{1}{2}}\left[\sum_{i,j=1}^{m}\left(\sum_{\beta=1}^{n}\frac{\partial u_\beta}{\partial x_i}\frac{\partial u_\beta}{\partial x_j}\right)^2\right]^{\frac{1}{2}}\\
&= \left[\sum_{i=1}^{m}\left(\sum_{\alpha}^{n}\frac{\partial u_\alpha}{\partial x_i}\widetilde{u}_\alpha\right)^2\right]^{\frac{1}{2}}\left[\sum_{i,j=1}^{m}\left(h(du(\frac{\partial}{\partial x_i}),du(\frac{\partial}{\partial x_j}))\right)^2\right]^{\frac{1}{2}}\\
&= \left[\sum_{i=1}^{m}\left(\sum_{\alpha}^{n}\frac{\partial u_\alpha}{\partial x_i}\widetilde{u}_\alpha\right)^2\right]^{\frac{1}{2}}||u^*h||\\
&\leq \left[\sum_{i=1}^{m}\left(\sum_{\alpha}^{n}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\alpha}{\partial x_i}\right)\left(\sum_{\alpha=1}^{m}\widetilde{u}_\alpha^2\right)\right]^{\frac{1}{2}}||u^*h||\\
&\leq \sqrt[4]{m}\left[\left[\sum_{i=1}^{m}\left(h(du(\frac{\partial}{\partial x_i}),du(\frac{\partial}{\partial x_i}))\right)^2\right]^{\frac{1}{2}}\left(\sum_{\alpha=1}^{m}\widetilde{u}_\alpha^2\right)\right]^{\frac{1}{2}}||u^*h||\\
&\leq \sqrt[4]{m}\left[\left[\sum_{i,j=1}^{m}\left(h(du(\frac{\partial}{\partial x_i}),du(\frac{\partial}{\partial x_j}))\right)^2\right]^{\frac{1}{2}}\left(\sum_{\alpha=1}^{m}\widetilde{u}_\alpha^2\right)\right]^{\frac{1}{2}}||u^*h||\\
&= \sqrt[4]{m}||u^*h||^{\frac{3}{2}}\left(\sum_{\alpha=1}^{m}\widetilde{u}_\alpha^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Here we have used $|\nabla r|^2 = 1$ in the second equality and

$$
||u^*h||^2 = \sum_{i,j=1}^{m}\left(h(du(\frac{\partial}{\partial x_i}),du(\frac{\partial}{\partial x_j}))\right)^2
$$

at p. So we have

$$
\sum_{i,j,k,l=1}^{m}\sum_{\alpha,\beta=1}^{n} g^{ik}g^{jl}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j}\widetilde{u}_\alpha\frac{\partial u_\beta}{\partial x_k}\frac{\partial u_\beta}{\partial x_l} \leq \sqrt[4]{m}||u^*h||^{\frac{3}{2}}\left(\sum_{\alpha=1}^{m}\widetilde{u}_\alpha^2\right)^{\frac{1}{2}}
$$

and

$$
\int_{\partial B(R)}\sum_{i,j,k,l=1}^{m}\sum_{\alpha,\beta=1}^{n} g^{ik}g^{jl}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j}\widetilde{u}_\alpha\frac{\partial u_\beta}{\partial x_k}\frac{\partial u_\beta}{\partial x_l}dv_g
$$

$$\leq \int_{\partial B(R)} \sqrt[4]{m}||u^*h||^{\frac{3}{2}} \left(\sum_{\alpha=1}^{m} \widetilde{u}_\alpha^2\right)^{\frac{1}{2}} dv_g$$

(24)

$$\leq \sqrt[4]{m}\left[\int_{\partial B(R)} ||u^*h||^2 dv_g\right]^{\frac{3}{4}} \left[\int_{\partial B(R)} \left(\sum_{\alpha=1}^{m} \widetilde{u}_\alpha^2\right)^2 dv_g\right]^{\frac{1}{4}}$$

$$\leq \sqrt[4]{m}\left[\int_{\partial B(R)} \left[||u^*h||^2 + \frac{1}{\varepsilon^n}(1-|u|^2)^2\right] dv_g\right]^{\frac{3}{4}} \left[\int_{\partial B(R)} \left(\sum_{\alpha=1}^{m} \widetilde{u}_\alpha^2\right)^2 dv_g\right]^{\frac{1}{4}}.$$

Set

$$\begin{aligned} Z(R) = &\int_{B(R)\setminus B(R_2)} \left[\sum_{\alpha,\beta=1}^{n} g^{ik}g^{jl}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\alpha}{\partial x_j}\frac{\partial u_\beta}{\partial x_k}\frac{\partial u_\beta}{\partial x_l}\right. \\ &\left. +\frac{1}{\varepsilon^n}\left(1-\sum_{\alpha=1}^{n} u_\alpha^2\right)^2\right] dv_g + D(R_1) \\ = &\int_{B(R)\setminus B(R_2)} \left[||u^*h||^2 + \frac{1}{\varepsilon^n}\left(1-|u|^2\right)^2\right] dv_g + D(R_1). \end{aligned}$$

Then

$$\frac{\partial}{\partial R}Z(R) = \int_{\partial B(R)} \left[||u^*h||^2 + \frac{1}{\varepsilon^n}\left(1-|u|^2\right)^2\right] ds_g. \tag{25}$$

From (23), (24) and the fact that $1+\frac{c_\alpha^2}{u_\alpha^2} \geq 1$ for any $\alpha = 1, \ldots, n$, we have

$$Z(R) \leq \sqrt[4]{m}\,[Z'(R)]^{\frac{3}{4}} \left[\int_{\partial B(R)} \left(\sum_{\alpha=1}^{m} \widetilde{u}_\alpha^2\right)^2 dv_g\right]^{\frac{1}{4}},$$

that is

$$Z(R)^{\frac{4}{3}} \leq m^{\frac{1}{3}} Z'(R) \left[\int_{\partial B(R)} \left(\sum_{\alpha=1}^{m} \widetilde{u}_\alpha^2\right)^2 dv_g\right]^{\frac{1}{3}}. \tag{26}$$

On the other hand, we have

$$Z(R) - D(R_1) = \int_{B(R)\setminus B(R_2)} \left[||u^*h||^2 + \frac{1}{\varepsilon^n}\left(1-|u|^2\right)^2\right] dv_g. \tag{27}$$

Since $\Phi_\varepsilon(u)$ is infinity, there is an $R_3 \geq R_2$, such that $Z(R) > 0$ for any $R > R_3$. Denote

$$M(R) = \left[\int_{\partial B(R)} \left(\sum_{\alpha=1}^m \widetilde{u}_\alpha^2\right)^2 dv_g\right]^{\frac{1}{3}}.$$

Then we have

$$Z(R)^{\frac{4}{3}} \leq m^{\frac{1}{3}} Z'(R) M(R).$$

For any $R_4 > R > R_3$, we have

$$\int_R^{R_4} \frac{Z'(s)}{Z^{\frac{4}{3}}(s)} ds \geq \frac{1}{\sqrt[3]{m}} \int_R^{R_4} \frac{1}{M(s)} ds.$$

Let $R_4 \to \infty$ and notice that $Z(R) > 0$, we have

$$\frac{1}{Z^{\frac{1}{3}}(R)} \geq \frac{1}{3\sqrt[3]{m}} \int_R^\infty \frac{ds}{M(s)}$$

which implies that

$$Z(R) \leq \left[3\sqrt[3]{m}\right]^3 \left(\frac{1}{\int_R^\infty \frac{ds}{M(s)}}\right)^3 \quad \text{for} \quad R > R_3. \tag{28}$$

Using the condition (P_1) and $u(x) \to P_0$ as $r(x) \to \infty$, we obtain

$$\begin{aligned} M(r) &= \left[\int_{\partial B(r)} \left(\sum_{\alpha=1}^m \widetilde{u}_\alpha^2\right)^2 dv_g\right]^{\frac{1}{3}} \\ &\leq \left[\int_{\partial B(r)} \eta(r) dv_g\right]^{\frac{1}{3}} \\ &= \eta^{\frac{1}{3}}(r) \left[vol(\partial B(r))\right]^{\frac{1}{3}}, \end{aligned}$$

where $\eta(r)$ is chosen in such a way that

(i) $\eta(r)$ is nonincreasing on (R_3, ∞) and $\eta(r) \to 0$ as $r \to \infty$,

(ii) $\eta(r) \geq \max_{r(x)=r} \left(\sum_{\alpha=1}^m \widetilde{u}_\alpha^2\right)^2$,

(iii) $\eta^{\frac{1}{3}}(r) \leq C r^{\frac{\widetilde{\sigma}}{3}} \int_r^\infty \frac{ds}{[vol(\partial B(s))]^{\frac{1}{3}}}$,

where C is a constant only depending on P_0. Then we have

$$\int_R^\infty \frac{dr}{M(r)} \geq \frac{1}{\eta^{\frac{1}{3}}(R)} \int_R^\infty \frac{dr}{(vol(\partial B(r)))^{\frac{1}{3}}} \geq \frac{1}{C R^{\frac{\widetilde{\sigma}}{3}}}. \tag{29}$$

Hence we have $Z(R) \leq C_1 R^{\widetilde{\sigma}}$, for any $R > R_3$. Therefore, by the definition of $Z(R)$ and (28), we obtain

$$\Phi_\varepsilon^R(u) = \int_{B(R)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1 - |u|^2)^2\right] dv_g$$

$$\leq (3\sqrt[3]{m}C)^3 R^{\widetilde{\sigma}} - \frac{D(R_1)}{4} + \int_{B(R_2)} \left[\frac{||u^*h||^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right] dv_g$$

$$\leq (CR^{\widetilde{\sigma}-\sigma} + \frac{C(u)}{R^\sigma})R^\sigma, \tag{30}$$

where C is positive constant only depending on m, P_0, and $C(u)$ is a positive constant only depending on u. Since $\widetilde{\sigma} < \sigma$, it contradicts with Proposition 5.1. □

Corollary 5.4. *Let $u : (M^m, g) \to (\mathbb{R}^n, h)$, $(m > 4)$ be a generalized symphonic map. Assume that the radial curvature K_r of M satisfies the following condition*

$$-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$$

with $\epsilon > 0$, $A \geq 0$, $0 \leq B \leq 2\epsilon$ and $1 + (m-1)(1-\frac{B}{2\epsilon}) - 4e^{\frac{A}{2\epsilon}} > 0$. If $u(x) \to P_0 \in S^{n-1}$ as $r(x) \to \infty$, and

$$\left[\max_{r(x)=r} h^2(u(x), P_0)\right]^2 \leq \left[\frac{3}{m-4}\right]^3 \frac{r^{\widetilde{\sigma}-(m-4)}}{\omega_m e^{\frac{(m-1)A}{2\epsilon}}},$$

then u must be a constant map. Here ω_m is the $(m-1)$-volume of the unit sphere in $\mathbb{R}^m$ and $\widetilde{\sigma}$ is a positive constant such that $\widetilde{\sigma} < 1+(m-1)(1-\frac{B}{2\epsilon})-4e^{\frac{A}{2\epsilon}}$.

Proof. By the assumption, we have

$$Ric^M(x) \geq -\frac{(m-1)A}{(1+r^2(x))^{1+\epsilon}}, \quad \forall x \in M^m.$$

Since

$$\int_0^\infty \frac{Ar}{(1+r^2)^{1+\epsilon}} dr = \frac{A}{2\epsilon},$$

Then the volume comparison theorem (cf.[26]) implies that

$$vol(\partial B(R)) \leq \omega_m e^{\frac{(m-1)A}{2\epsilon}} R^{m-1},$$

where ω_m is the $(m-1)$-volume of the unit sphere in $\mathbb{R}^m$, and thus

$$\left[\int_R^\infty \frac{dr}{(vol(\partial B(r)))^{\frac{1}{3}}}\right]^{-3} \leq \left[\frac{m-4}{3}\right]^3 \omega_m e^{\frac{(m-1)A}{2\epsilon}} R^{m-4}$$

for $R \gg 1$. By using Corollary 5.2 and Theorem 5.3, we can obtain the result. □

Corollary 5.5. *Let $u : (M^m, g) \to (\mathbb{R}^n, h)$, $(m > 4)$ be a generalized symphonic map. Assume that the radial curvature K_r of M satisfies the following condition*

$$-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$$

with $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ *and* $1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\frac{1+\sqrt{1-4a^2}}{2} > 0$. *If* $u(x) \to P_0 \in S^{n-1}$ *as* $r(x) \to \infty$, *and*

$$\left[\max_{r(x)=r} h^2(u(x), P_0)\right]^2 \leq C^{-1}\left[\frac{3}{(m-1)\frac{1+\sqrt{1+4a^2}}{2}}\right]^3 R^{\widetilde{\sigma}-(m-1)\frac{1+\sqrt{1+4a^2}}{2}+3},$$

then u *must be a constant map. Here* $\widetilde{\sigma}$ *is a positive constant such that* $\widetilde{\sigma} < 1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\frac{1+\sqrt{1-4a^2}}{2}$ *and* C *is a positive constant only depending on* m.

Proof. By the assumption, we have

$$Ric^M(x) \geq -\frac{(m-1)a^2}{1+r^2(x)}, \quad \forall x \in M^m.$$

Then the volume comparison theorem (cf.[26]) implies that

$$vol(\partial B(R)) \leq CR^{(m-1)\frac{1+\sqrt{1+4a^2}}{2}},$$

where C is a suitable constant. Thus we have

$$\left[\int_R^\infty \frac{dr}{(vol(\partial B(r)))^{\frac{1}{3}}}\right]^{-3} \leq C\left[\frac{(m-1)\frac{1+\sqrt{1+4a^2}}{2}}{3}\right]^3 R^{(m-1)\frac{1+\sqrt{1+4a^2}}{2}-3}$$

for $R \gg 1$. By using Corollary 5.2 and Theorem 5.3, we can obtain the result. □

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