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AREA DISTORTION UNDER MEROMORPHIC MAPPINGS WITH NONZERO POLE HAVING QUASICONFORMAL EXTENSION

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Abstract. Let $\Sigma_k(p)$ be the class of univalent meromorphic functions defined on the unit disc $\mathbb{D}$ with k-quasiconformal extension to the extended complex plane $\widehat{\mathbb{C}}$, where $0 \leq k < 1$. Let $\Sigma_k^0(p)$ be the class of functions $f \in \Sigma_k(p)$ having expansion of the form $f(z) = 1/(z-p) + \sum_{n=1}^{\infty} b_n z^n$ on $\mathbb{D}$. In this article, we obtain sharp area distortion and weighted area distortion inequalities for functions in $\Sigma_k^0(p)$. As a consequence of the obtained results, we present a sharp upper bound for the Hilbert transform of characteristic function of a Lebesgue measurable subset of $\mathbb{D}$.

1. Introduction

Let $\mathbb{C}$ denote the complex plane and $\widehat{\mathbb{C}}$ be the extended complex plane $\mathbb{C} \cup \{\infty\}$. Throughout the discussion in this article, we shall use the following notations: $\mathbb{D} = \{z : |z| < 1\}$, $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$, $\mathbb{D}^* = \{z : |z| > 1\}$, $\overline{\mathbb{D}^*} = \{z : |z| \geq 1\}$. Let Σ be the class of univalent meromorphic functions defined on $\mathbb{D}$ having simple pole at the origin with residue 1 and therefore each $f \in \Sigma$ has the following expansion

$$f(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

The class Σ and its various subclasses have been studied by a number of function theorists till date. Let $\Sigma(p)$ be the class of functions that are univalent, meromorphic on $\mathbb{D}$ having a simple pole at $z = p \in [0,1)$ with residue 1 with the following expansion

$$f(z) = (z-p)^{-1} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D}. \tag{1.2}$$

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Let $\Sigma^0(p) := \{f \in \Sigma(p) : b_0 = 0\}$. We emphasise here that merely considering the pole of a meromorphic function at a nonzero point not only changes the normalization but provides us with the Taylor expansion of the same function inside the disc $\{z : |z| < p\}$ along with its other Laurent expansions. Now since for $f \in \Sigma^0(p)$, we have $f(0) = -1/p$ and $f'(0) = b_1 - 1/p^2$, the function g defined as

$$g(z) = \frac{f(z) + 1/p}{b_1 - 1/p^2},\ (p \neq 0),$$

belongs to the class $S(p)$, where $S(p)$ is the class of meromorphic, univalent functions defined on $\mathbb{D}$, having a simple pole at $z = p$, with the normalization $f(0) = 0 = f'(0) - 1$. Thus there is a one-to-one correspondence between the classes $\Sigma^0(p)$ and $S(p)$. It is clear that if $f \in S(p)$, then it will have a Taylor series expansion as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ |z| < p,$$

about the origin. The class $S(p)$ and its various subclasses have also been the object of study for many mathematicians over the years (see [4,5,12,17,18] and the references therein). It is well-known that the univalent functions defined in $\mathbb{D}$ that admit a quasiconformal extension to the sphere $\widehat{\mathbb{C}}$ play an important role in Teichmüller space theory. It is therefore of interest to study such class of functions.

Let Σ_k be the class of functions in Σ that have k-quasiconformal extension $(0 \leq k < 1)$ to $\widehat{\mathbb{C}}$. Here, a mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called k-quasiconformal if f is a homeomorphism and has locally L^2-derivatives on $\mathbb{C} \setminus \{f^{-1}(\infty)\}$ (in the sense of distribution) satisfying $|\bar{\partial} f| \leq k|\partial f|$ a.e., where $\partial f = \partial f/\partial z$ and $\bar{\partial} f = \partial f/\partial \bar{z}$. Note that such an f is also be called K-quasiconformal, where $K = (1+k)/(1-k) \geq 1$, in the literature. The quantity $\mu = \bar{\partial} f/\partial f$ is called the complex dilatation of f. The functions in the class Σ_k has primarily been studied by O. Lehto, (compare [15]) and later R. Kühnau and W. Niske ([14]), and S. Krushkal ([13]) continued the research in this direction. More precisely, they obtained distortion theorems, coefficient estimates, area theorem for functions in this class. This motivates us to study the class of functions belonging to $\Sigma^0(p)$ that have quasiconformal extension to $\widehat{\mathbb{C}}$, namely the class $\Sigma_k^0(p)$. This function class $\Sigma_k^0(p)$, defined above, has been introduced recently in [7]. The area theorem, coefficient estimates and distortion inequalities for this class have also been studied recently (compare [6,7]).

In 1955, Bojarski considered the area distortion problem for quasiconformal mappings (see [8]). Thereafter further improvements on this problem were made by Gehring and Reich (compare [11, Theorem 1]) in a more precise form and they conjectured that:

Theorem A. *If $f : \mathbb{D} \to \mathbb{D}$ be a k-quasiconformal mapping with $f(0) = 0$, then*

$$|f(E)| \leq M(K)|E|^{1/K}$$

for all Lebesgue measurable sets $E \subset \mathbb{D}$*, where* $|\cdot|$ *stands for the area,* $K = (1+k)/(1-k) \geq 1$*, and the constant* $M(K) = 1 + O(K-1)$ *as* $K \to 1$*.*

This conjecture was proved by K. Astala ([1, Theorem 1.1]) in 1994 using thermodynamic formalism and holomorphic motion theory. Later, Eremenko and Hamilton in [10, Theorem 1] gave a direct and much simpler proof of the above result. They assumed f to be a k-quasiconformal mapping of the plane which is conformal on $\mathbb{C} \setminus \Delta$, where Δ is a compact set of transfinite diameter 1 and f has the normalization $f(z) = z + o(1)$ near ∞. Therefore, we consider the class Σ_k^0 that consists of functions defined on $\mathbb{D}^*$, having k-quasiconformal extension in $\mathbb{D}$ such that they have pole at the point $z = \infty$ and have the following form

$$f(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in \mathbb{D}^*. \tag{1.3}$$

In [10, Theorem 1] , if we assume $\Delta = \overline{\mathbb{D}}$, then $f \in \Sigma_k^0$. We state this result below:

Theorem B. *Let* $f \in \Sigma_k^0$ *having the expansion of the form* (1.3)*, so that* $f(z) - z \to 0$ *as* $z \to \infty$ *and let* E *be a Lebesgue measurable subset of* $\mathbb{D}$*.*

(i) *If* f *is conformal on* E*, i.e.,* $\bar{\partial} f = 0$ *a.e. on* E*, then*

$$|f(E)| \leq \pi^{1-1/K} |E|^{1/K}.$$

(ii) *If* f *is conformal on* $\mathbb{C} \setminus E$*, then*

$$|f(E)| \leq K|E|.$$

(iii) *Hence, for an arbitrary Lebesgue measurable subset* E *of* $\mathbb{D}$*,*

$$|f(E)| \leq K\pi^{1-1/K} |E|^{1/K}.$$

All the constants in the above inequalities are best possible.

In particular, equality holds in Theorem B(i) (see [3, p. 344]) for the function

$$f_r(z) = \begin{cases} r^{1/K-1} z, & |z| < r, \\ z|z|^{1/K-1}, & r \leq |z| \leq 1, \\ z, & |z| > 1, \end{cases} \tag{1.4}$$

where $0 < r < 1$ and $f_r \in \Sigma_k^0$ is conformal on $E = \{z : |z| < r\}$. Next, the inequality in Theorem B(ii) is sharp for the function $f_r^{-1} \in \Sigma_k^0$ and $E = \{z : r^{1/K} \leq |z| \leq 1\}$ (compare [2, p. 324]). Also the inequality in Theorem B(iii) is sharp as the inequalities in Theorem B(i) and Theorem B(ii) are also so. Further, Astala and Nesi proved the weighted area distortion inequality ([3, Theorem 1.6]), where they considered a non negative weight function w defined on a Lebesgue measurable set $E \subset \mathbb{D}$. We state the result below:

Theorem C. *Suppose $f \in \Sigma_k^0$ having expansion of the form* (1.3) *and let E be a Lebesgue measurable subset of $\mathbb{D}$ such that f is conformal on E, i.e., $\bar{\partial} f = 0$ a.e. on E. Let $w(z) \geq 0$ be a (measurable) weight function defined on E, then*

$$\pi^{1-K}\left(\int_E w(z)^{1/K}\,dm\right)^K \leq \int_E w(z) J_f(z)\,dm \leq \pi^{1-1/K}\left(\int_E w(z)^K\,dm\right)^{1/K}.$$

The inequalities are sharp. Here, $dm = dxdy$ denotes the two dimensional Lebesgue measure on the plane with $z = x + iy$.

We note here that, when $w(z) = 1$ for all $z \in E$, second inequality of the above theorem yields Theorem B(i). Area distortion results for quasiconformal mappings have several consequences. Firstly, they give the precise degree of integrability of the partial derivatives of a K-quasiconformal mapping. The precise regularity of quasiconformal mappings also controls the distortion of Hausdorff dimension of a set under a K-quasiregular mapping. Area distortion inequality also provides sharp bounds for the Hilbert transformation of characteristic function of a set lying in the domain of a quasiconformal mapping. See [2, Chap. 13, 14] for details.

In this article, we prove an area distortion inequality for functions in the class $\Sigma_k^0(p)$. This is discussed in Theorem 1 in the next section. Further, we obtain weighted area distortion inequality for theses functions. This is the content of Theorem 2 in the next section. We point out here that Theorem 1 and Theorem 2 coincide with Theorem B and Theorem C respectively, for $p = 0$, i.e., when $f \in \Sigma_k^0$. Finally as an application of Theorem 1, we present a sharp estimate for the Hilbert transform of the characteristic function χ_E, where E is a Lebesgue measurable subset of $\mathbb{D}$.

2. Main results

We start the Section with area distortion inequality for functions in the class $\Sigma_k^0(p)$.

Theorem 1. *Let $k \in (0,1)$ and each $f \in \Sigma_k^0(p)$ has the expansion of the form* (1.2) *and let E be a Lebesgue measurable subset of $\mathbb{D}^*$.*

(i) *If f is conformal on E, i.e., $\bar{\partial} f = 0$ a.e. on E, then*

$$|f(E)| \leq \left[\pi(1-p^2)^{-2}\right]^{1-1/K} |f_0(E)|^{1/K}. \tag{2.1}$$

(ii) *If f is conformal on $\mathbb{C} \setminus E$, then*

$$|f(E)| \leq K|f_0(E)|. \tag{2.2}$$

(iii) *Hence, for an arbitrary Lebesgue measurable subset E of $\mathbb{D}^*$,*

$$|f(E)| \leq K\left[\pi(1-p^2)^{-2}\right]^{1-1/K} |f_0(E)|^{1/K}.$$

Here $K = (1+k)/(1-k)$ and $f_0(z) = 1/(z-p)$, $z \in \mathbb{C}$. The constants appearing in the theorem are the best possible.

Proof. To prove the first part of the theorem, we assume that the set E to be open and for the second part of the theorem, we assume the set $\mathbb{C} \setminus E$ to be open. Now, for any Lebesgue measurable set E, the proof of the theorem can be completed by the standard approximation argument given in [3, p. 343].

(i) We see that if $z \in \mathbb{D}^*$, then each function $f \in \Sigma_k^0(p)$ has the expansion of the following form

$$f(1/z) = z(1 - pz)^{-1} + \sum_{n=1}^{\infty} b_n z^{-n}. \tag{2.3}$$

Let us define $g(z) := f(1/z)$, so that $g \in \Sigma_k^0(p)$ with the expansion of the form (2.3) in $\mathbb{D}^*$. As g is obtained by composing a Möbius transformation with a k-quasiconformal map f in $\widehat{\mathbb{C}}$, it is also k-quasiconformal in $\widehat{\mathbb{C}}$. Here, since f is conformal in $\mathbb{D}$, then g is also conformal in $\mathbb{D}^*$ and hence the dilatation of g has support in $\overline{\mathbb{D}}$ and it has the same modulus as that of f. Since f is conformal on $E \subset \mathbb{D}^*$, so g is again conformal on $\tilde{g}(E) = E' \subset \mathbb{D}$, where $\tilde{g}(z) = 1/z$. As a result, the dilatation μ of g satisfies $|\mu(z)| \le k$ for all $z \in \overline{\mathbb{D}} \setminus E'$ and vanishes on E'. Now we consider the dilatation

$$\mu_\lambda(z) = \frac{\lambda \mu(z)}{k}, \ \lambda \in \mathbb{D}. \tag{2.4}$$

Therefore by the Measurable Riemann Mapping Theorem (see [2, p. 168]), there exists a unique quasiconformal mapping $g(z, \lambda) = g_\lambda(z)$ (for each λ), whose dilatation is $\mu_\lambda(z)$. Now $g_\lambda \in \Sigma_{|\lambda|}^0(p)$ as $g \in \Sigma_k^0(p)$ and also g_λ satisfies the normalization, $g_\lambda(z) = z/(1 - pz) + o(1)$ as $z \to \infty$. Hence $g_\lambda|_{\mathbb{D}^*} \in \Sigma^0(p)$, so by Chichra's area theorem (see [9]), we have

$$|g_\lambda(\mathbb{D})| = \pi(1 - p^2)^{-2} - \pi \sum_{n=1}^{\infty} n|b_n|^2 \le \pi(1 - p^2)^{-2}.$$

Thus

$$\int_{\mathbb{D}} J_\lambda(z)\, dm \le \pi(1 - p^2)^{-2}, \quad (z = x + iy),$$

where J_λ denotes the Jacobian of the map g_λ. As $E' \subset \mathbb{D}$, it follows that

$$\int_{E'} (1 - p^2)^2 \pi^{-1} J_\lambda(z)\, dm \le 1. \tag{2.5}$$

Now by holomorphic dependence of the solution to the Beltrami equation, on parameter (see [16, II, Theorem 3.1]), the function $\lambda \to g(z, \lambda)$ is holomorphic in the variable $\lambda \in \mathbb{D}$, for each fixed $z \in \mathbb{D}$. This dependency also happens for the function $\partial g(z, \lambda)$ where $g(z, \lambda)$ is analytic in z. As $g(z)$ is conformal in E', so is $g(z, \lambda)$, hence we can say that the function $\lambda \to \partial g(z, \lambda)$ is holomorphic in $\lambda \in \mathbb{D}$, for each fixed $z \in E'$. Since $g(z, \lambda)$ is $|\lambda|$-quasiconformal with dilatation $\mu_\lambda(z)$ in the variable $z \in \mathbb{D}$, for each fixed λ, we can write

$$J_\lambda(z) = |\partial g(z, \lambda)|^2 - |\bar{\partial} g(z, \lambda)|^2 = |\partial g(z, \lambda)|^2 (1 - |\mu_\lambda(z)|^2).$$

Thus for $z \in E'$ we have $J_\lambda(z) = |\partial g(z,\lambda)|^2$. As $g(z,\lambda)$ is quasiconformal in $\mathbb{D}$, the Jacobian $J_\lambda(z)$ never vanishes in $\mathbb{D}$ and in particular in E'. Hence, the function $\partial g(z,\lambda)$ is a non vanishing analytic function on $E' \times \mathbb{D}$ and so is the function $(1-p^2)^2\pi^{-1}\partial g(z,\lambda)^2$. Now if we define

$$a(z,\lambda) = (1-p^2)^2\pi^{-1}|\partial g(z,\lambda)|^2,$$

then $\log a(z,\lambda)$ is harmonic in $\lambda \in \mathbb{D}$, for $z \in E'$. Thus from (2.5) we see that the function $a(z,\lambda)$ satisfies the conditions of the continuous version of Lemma 1 in [10], consequently we have

$$\begin{aligned}(1-p^2)^2\pi^{-1}\int_{E'}|\partial g(z,\lambda)|^2\,dm &\le \left[(1-p^2)^2\pi^{-1}\int_{E'}|\partial g(z,0)|^2\,dm\right]^{\frac{1-|\lambda|}{1+|\lambda|}}\\ &= \left[(1-p^2)^2\pi^{-1}\int_{E'}J_0(z)\,dm\right]^{\frac{1-|\lambda|}{1+|\lambda|}}\\ &= \left[(1-p^2)^2\pi^{-1}|g_0(E')|\right]^{\frac{1-|\lambda|}{1+|\lambda|}}.\end{aligned}$$

Using the fact that for $z \in E'$, $J_\lambda(z) = |\partial g(z,\lambda)|^2$, we get from the above inequality

$$(1-p^2)^2\pi^{-1}|g_\lambda(E')| \le \left[(1-p^2)^2\pi^{-1}|g_0(E')|\right]^{\frac{1-|\lambda|}{1+|\lambda|}}.$$

Now for $\lambda = k$, we have $g_\lambda = g$, which yields after simplification

$$|g(E')| \le \left[\pi(1-p^2)^{-2}\right]^{1-1/K}|g_0(E')|^{1/K}. \tag{2.6}$$

Now since $f(z) = g(1/z)$, we get inequality (2.1), where $E \subset \mathbb{D}^*$ and g_0 is replaced by f_0. We now find explicitly the function $g(z,0) = g_0(z)$. For $\lambda = 0$, the function g_0 is conformal on the whole sphere $\widehat{\mathbb{C}}$ onto itself as well as it satisfies the normalization of the class $\Sigma^0(p)$ on $\mathbb{D}^*$, viz.

(i) $g_0(z) - z/(1-pz) \to 0$ as $z \to \infty$,
(ii) $g_0(1/p) = \infty$,
(iii) $(1-pz)^2 g_0'(z)\big|_{z=1/p} = 1$.

It is now easy to see that $g_0(z) = z/(1-pz)$ for all $z \in \mathbb{C}$, is the only choice and hence $f_0(z) = g_0(1/z) = 1/(z-p)$ for all $z \in \mathbb{C}$, which proves the theorem.

Now we consider the equality case. We observe that equality holds in (2.1) if it does hold in (2.6) and to establish this, we consider the following function:

$$g(z) = \begin{cases} \frac{r^{1/K-1}}{1-p^2}\left(\frac{z-p}{1-pz}\right) + \frac{p}{1-p^2}, & z \in B(r),\\ \frac{1}{1-p^2}\left(\frac{z-p}{1-pz}\right)\left|\frac{z-p}{1-pz}\right|^{1/K-1} + \frac{p}{1-p^2}, & z \in \overline{\mathbb{D}}\setminus B(r),\\ \frac{z}{1-pz}, & z \in \mathbb{D}^*,\end{cases} \tag{2.7}$$

where $0 < r < 1$ and $B(r)$ ($\subsetneq \mathbb{D}$) is the disc given by

$$B(r) = \left\{z : \left|z - \frac{p(1-r^2)}{1-p^2r^2}\right| < \frac{r(1-p^2)}{1-p^2r^2}\right\}.$$

It is easy to verify that g is a member of $\Sigma_k^0(p)$ and that g is conformal on the set $E' = B(r) \subset \mathbb{D}$. To establish the equality case, we again observe that the Möbius transformations $(z-p)/(1-pz)$ and $g_0(z) = z/(1-pz)$ maps the above disc $B(r)$ onto the discs $\{w : |w| < r\}$ and $\{w : |w - p(1-p^2)^{-1}| < r(1-p^2)^{-1}\}$ respectively. Hence the right hand side of (2.6) becomes $\pi r^{2/K}(1-p^2)^{-2}$. Again g in (2.7) maps the disc $B(r)$ onto the disc $\{w : |w - p(1-p^2)^{-1}| < r^{1/K}(1-p^2)^{-1}\}$, which yields $|g(B(r))| = \pi r^{2/K}(1-p^2)^{-2}$. Hence equality holds in (2.6) for the above g and $E' = B(r)$. Now as $f(z) = g(1/z)$, we obtain the following extremal function for the inequality (2.1):

$$f(z) = \begin{cases} \frac{r^{1/K-1}}{1-p^2}\left(\frac{1-pz}{z-p}\right) + \frac{p}{1-p^2}, & z \in \tilde{B}(r), \\ \frac{1}{1-p^2}\left(\frac{1-pz}{z-p}\right)\left|\frac{1-pz}{z-p}\right|^{1/K-1} + \frac{p}{1-p^2}, & z \in \overline{\mathbb{D}^*} \setminus \tilde{B}(r), \\ \frac{1}{z-p}, & z \in \mathbb{D}, \end{cases}$$

where we assume $0 \le p < r < 1$. Here $\tilde{B}(r)(\subsetneq \mathbb{D}^*)$ is the image of the disc $B(r)$ under the map $\tilde{g}(z) = 1/z$, given by

$$\tilde{B}(r) = \left\{ z \in \mathbb{C} : \left| z + \frac{p(1-r^2)}{r^2 - p^2} \right| > \frac{r(1-p^2)}{r^2-p^2} \right\}.$$

Hence equality holds in (2.1) for the above f and $E = \tilde{B}(r)$.

(ii) As before we start the proof of this part with the transformation $g(z) = f(1/z)$. Since $g \in \Sigma_k^0(p)$ of the form (2.3) in $\mathbb{D}^*$, the dilatation μ of g vanishes outside the compact set $\overline{\mathbb{D}}$ and hence by equation (1.7) of [6, p. 3], we get

$$g(z) = z/(1-pz) + T[\overline{\partial} g](z), \quad z \in \mathbb{C}.$$

Taking partial derivative of both sides w.r.t. z and using $\partial T[\omega] = H[\omega]$, we have

$$\partial g(z) = 1/(1-pz)^2 + H[\overline{\partial} g](z), \tag{2.8}$$

where T and H denote two dimensional Cauchy and Hilbert transform respectively (see [16, I §4.3]). Since $\overline{\partial} g = \mu \partial g$, the above equation takes the following form

$$\overline{\partial} g(z) = \mu/(1-pz)^2 + \mu H[\overline{\partial} g](z). \tag{2.9}$$

It is also known that

$$\begin{aligned} w &= \overline{\partial} g \\ &= \mu(1-pz)^{-2} + \mu H\left[\mu(1-pz)^{-2}\right] + \mu H\left[\mu H\left[\mu(1-pz)^{-2}\right]\right] + \cdots \end{aligned} \tag{2.10}$$

satisfies equation (2.9) (see [6, p. 5]). By our assumption, $w = \bar{\partial} g$ vanishes outside E'. Hence using (2.8) and the fact that the Hilbert transform is a linear isometry on $L^2(\mathbb{C})$, we get

$$|g(E')| = \int_{E'} J_g(z)\, dm$$

$$
\begin{aligned}
&= \int_{E'} (|\partial g|^2 - |\overline{\partial} g|^2)\, dm \\
&= \int_{E'} \left(\left| (1-pz)^{-2} + H[w] \right|^2 - |w|^2 \right) dm \\
&= \int_{E'} \left(|1-pz|^{-4} + 2\mathrm{Re}\left((1-p\overline{z})^{-2} H[w] \right) \right) dm \\
&\quad + \int_{E'} \left(|H[w]|^2 - |w|^2 \right) dm \qquad (2.11) \\
&\leq \int_{E'} |1-pz|^{-4}\, dm + 2 \int_{E'} \left| (1-pz)^{-2} H[w] \right| dm \\
&\quad + \int_{\mathbb{C}} \left(|H[w]|^2 - |w|^2 \right) dm \\
&= |g_0(E')| + 2 \int_{E'} \left| (1-pz)^{-2} H[w] \right| \, dm, \qquad (2.12)
\end{aligned}
$$

where $g_0(z) = z/(1-pz)$, as mentioned earlier. Now using the fact that the Hilbert transformation is linear, we get from the identity (2.10) that

$$
\begin{aligned}
(1-pz)^{-2} H[w] = {} & (1-pz)^{-2} H\left[\mu (1-pz)^{-2} \right] \\
& + (1-pz)^{-2} H\left[\mu H\left[\mu (1-pz)^{-2} \right] \right] + \cdots .
\end{aligned}
$$

This gives

$$
\begin{aligned}
\int_{E'} \left| (1-pz)^{-2} H[w] \right| \, dm \leq {} & \int_{E'} |1-pz|^{-2} \left| H\left[\mu (1-pz)^{-2} \right] \right| \, dm \\
& + \int_{E'} |1-pz|^{-2} \left| H\left[\mu H\left[\mu (1-pz)^{-2} \right] \right] \right| \, dm + \cdots . \qquad (2.13)
\end{aligned}
$$

We now apply the Cauchy-Schwarz inequality and the isometry property of the Hilbert transformation to the n-th term of the right hand side of (2.13) to get an upper bound for this term. We show below the computational details:

$$
\begin{aligned}
& \int_{E'} |1-pz|^{-2} \Big| \underbrace{H\left[\mu H \cdots \mu H \right.}_{n \text{ terms}} \left[\mu (1-pz)^{-2} \right] \left. \right] \Big| \, dm \\
\leq & \left(\int_{E'} |1-pz|^{-4}\, dm \right)^{1/2} \left(\int_{E'} \Big| \underbrace{H\left[\mu H \cdots \mu H \right.}_{n \text{ terms}} \left[\mu (1-pz)^{-2} \right] \left. \right] \Big|^2 dm \right)^{1/2} \\
\leq & \, |g_0(E')|^{1/2} \left(\int_{\mathbb{C}} \Big| \underbrace{H\left[\mu H \cdots \mu H \right.}_{n \text{ terms}} \left[\mu (1-pz)^{-2} \right] \left. \right] \Big|^2 dm \right)^{1/2} \\
= & \, |g_0(E')|^{1/2} \left(\int_{\mathbb{C}} \Big| \underbrace{\mu H\left[\mu H \cdots \mu H \right.}_{(n-1) \text{ terms}} \left[\mu (1-pz)^{-2} \right] \left. \right] \Big|^2 dm \right)^{1/2}
\end{aligned}
$$

$$\leq \|\mu\|_\infty |g_0(E')|^{1/2} \left(\int_{E'} \Big| \underbrace{H\left[\mu H \cdots \mu H\right.}_{(n-1)\text{ terms}} \left.\left[\mu(1-pz)^{-2}\right]\right] \Big|^2 dm \right)^{1/2}$$

$$\vdots$$

$$\leq \|\mu\|_\infty^n |g_0(E'|)^{1/2} \left(\int_{E'} |1-pz|^{-4}\, dm \right)^{1/2}$$

$$= k^n |g_0(E')|,$$

where $\|\mu\|_\infty = k < 1$. Using this estimate, we get from (2.13) that

$$\int_{E'} \left|(1-pz)^{-2} H[w]\right| dm \leq \sum_{n=1}^{\infty} |g_0(E')| k^n = k(1-k)^{-1} |g_0(E')|.$$

Plugging the above estimate in (2.12), we finally obtain

$$|g(E')| \leq \left(\frac{1+k}{1-k}\right) |g_0(E')| = K|g_0(E')|. \tag{2.14}$$

Now applying $f(z) = g(1/z)$, we get inequality (2.2), where $E \subset \mathbb{D}^*$ and $f_0(z) = 1/(z-p),\ z \in \mathbb{C}$. Next we show that the constant K in Theorem 1(ii) is the best possible. This can be verified if we can show that the constant K in (2.14) is the best possible. We consider the following example:

$$h(z) = \begin{cases} \frac{r^{1-1/K}}{1-p^2}\left(\frac{z-p}{1-pz}\right) + \frac{p}{1-p^2}, & z \in B_0(r), \\ \frac{1}{1-p^2}\left(\frac{z-p}{1-pz}\right)\left|\frac{z-p}{1-pz}\right|^{K-1} + \frac{p}{1-p^2}, & z \in \overline{\mathbb{D}} \setminus B_0(r), \\ \frac{z}{1-pz}, & z \in \mathbb{D}^*, \end{cases} \tag{2.15}$$

where $B_0(r)$ ($\subsetneq \mathbb{D}$) is the disc given by

$$B_0(r) = \left\{ z : \left| z - \frac{p(1-r^{2/K})}{1-p^2 r^{2/K}} \right| < \frac{r^{1/K}(1-p^2)}{1-p^2 r^{2/K}} \right\} \quad \text{for } 0 < r < 1.$$

Similarly to example (2.7), the functions $z/(1-pz)(= g_0(z))$ and $(z-p)/(1-pz)$ maps the disc $B_0(r)$ onto the discs $\{w : |w-p(1-p^2)^{-1}| < r^{1/K}(1-p^2)^{-1}\}$ and $\{w : |w| < r^{1/K}\}$ respectively. This in turn implies $|g_0(B_0(r))| = \pi r^{2/K}(1-p^2)^{-2}$ and that the function h in (2.15) itself maps the disc $B_0(r)$ onto the disc $\{w : |w-p(1-p^2)^{-1}| < r(1-p^2)^{-1}\}$. To verify the assertion we set E' in this case, as $E' = \overline{\mathbb{D}} \setminus B_0(r)$. Then h is conformal on outside of the compact set E' and

$$|g_0(E')| = |g_0(\overline{\mathbb{D}})| - |g_0(B_0(r))| = \pi(1-p^2)^{-2}(1-r^{2/K}).$$

On the other hand,

$$|h(E')| = |h(\overline{\mathbb{D}})| - |h(B_0(r))| = \pi(1-p^2)^{-2}(1-r^2)$$

$$
\begin{aligned}
&= \pi(1-p^2)^{-2} - \pi(1-p^2)^{-2}\left[1-(1-r^{2/K})\right]^K \\
&= \pi(1-p^2)^{-2}\left[K(1-r^{2/K}) - (K/2)(K-1)(1-r^{2/K})^2 + \cdots\right] \\
&= K|g_0(E')| + O\left(|g_0(E')|^2\right) \quad \text{as } |g_0(E')| \to 0.
\end{aligned}
$$

Hence the constant K can not be improved as equality holds in (2.14) for $|g_0(E')|$ small enough. Composing h with the inverse mapping $\tilde{g}(z) = 1/z$ and taking inversion of the disc $B_0(r)$ (for $p < r$), extremality of (2.2) follows easily, as similar to Theorem 1(i).

(iii) To prove the last part of the theorem, we consider the following change of variable $g(z) = f(1/z)$. Hence $g \in \Sigma_k^0(p)$ such that it is conformal on $\mathbb{D}^*$ and k-quasiconformal on $\overline{\mathbb{D}}$. We write $g = g_1 \circ g_2$, where g_2 is conformal on $E \subset \mathbb{D}$, k-quasiconformal on $\overline{\mathbb{D}} \backslash E$ and $g_2 \in \Sigma_k^0(p)$. We assume that the function g_1 is k-quasiconformal on $g_2(E)$ and hence on $g_2(\overline{E})$ (as a set of area zero is removable for quasiconformality), so that g_1 is conformal outside the compact set $g_2(\overline{E})$ and satisfies the conditions of Theorem B(ii). Applying Theorem 1(i) to g_2 and Theorem B(ii) to g_1, we get

$$|g(E)| = |g_1(g_2(E))| \le K|g_2(E)| \le K\left[\pi(1-p^2)^{-2}\right]^{1-1/K}|g_0(E)|^{1/K}.$$

Putting $f(z) = g(1/z)$ we obtain the theorem in terms of f and g_0 is replaced by $f_0(z) = 1/(z-p)$. As the constants in corresponding theorems for g_1 and g_2 are best possible, hence for Theorem 1(iii) also. □

Remark. For the case $p = 0$, i.e., whenever $f \in \Sigma_k^0$, the inequality (2.6) reduces to that of Theorem B(i), and the extremal function g defined in (2.7) becomes f_r, as defined in (1.4). This coincidence also occurs for Theorem 1(ii), when $p = 0$, as can be seen from the inequality (2.14) and the extremal function h defined in (2.15). In this case h reduces to f_r^{-1} for $p = 0$, which is the extremal case for Theorem B(ii). Although, in our case h is not the inverse mapping of g.

Next we consider the weighted area distortion problem for a function in the class $\Sigma_k^0(p)$, where we consider a nonnegative weight function w defined on a Lebesgue measurable subset E of $\mathbb{D}^*$.

Theorem 2. *Suppose $f \in \Sigma_k^0(p)$ with the expansion of the form* (1.2) *and E be a Lebesgue measurable subset of $\mathbb{D}^*$ such that f is conformal on E, i.e., $\bar{\partial} f = 0$ a.e. on E. Let $w(z) \ge 0$ be a (measurable) weight function defined on E, then*

$$
\begin{aligned}
&\left[\frac{\pi}{(1-p^2)^2}\right]^{1-K}\left(\int_E w(z)^{1/K} J_0(z)\, dm\right)^K \\
&\le \int_E w(z) J_f(z)\, dm \\
&\le \left[\frac{\pi}{(1-p^2)^2}\right]^{1-1/K}\left(\int_E w(z)^K J_0(z)\, dm\right)^{1/K},
\end{aligned}
\tag{2.16}
$$

where J_f *and* J_0 *denotes Jacobian of the function* f *and* $f_0(z) = 1/(z-p)$, $z \in \mathbb{C}$ *respectively. The inequalities are sharp.*

Proof. We first prove the theorem for an open set E and for any measurable set E, we use the approximation argument given in [3, p. 343].

Here initially we consider the weight function $w \geq 0$ that are bounded away from 0 and ∞ on the set E. The case for general w follows from a limiting argument. Now, to establish the theorem we follow the lines of the proof of [3, Theorem 1.6]. For the sake of completeness, we provide here computation details. Let $g(z) = f(1/z)$ having expansion of the form (2.3) in $\mathbb{D}^*$. Next we consider the weight function $w_0(z) = w(1/z)$ defined on $\tilde{g}(E) = E' \subset \mathbb{D}$, where $\tilde{g}(z) = 1/z$. Therefore g is conformal on E' and k-quasiconformal on $\overline{\mathbb{D}} \setminus E'$. Similarly to (2.4), we consider the function $g_\lambda(z)$ with the dilatation $\lambda k^{-1}\mu(z)$ for $\lambda \in \mathbb{D}$. Again $g_\lambda(z)$ is conformal on E' (since g is so) and

$$g'_\lambda(z) \neq 0 \quad \text{for all } z \in E' \quad \text{and } \lambda \in \mathbb{D}. \tag{2.17}$$

Using the concavity of logarithm and Jensen's Inequality, we get for any function $a(z) > 0$ defined in E', that

$$\log\left(\int_{E'} a(z)\, dm\right) = \sup_{q(z)}\left[\int_{E'} q(z) \log\left(\frac{a(z)}{q(z)}\right) dm\right], \tag{2.18}$$

where the supremum is taken over all functions $q(z)$ defined on E', such that (i) $0 < q(z) < 1$, a.e. $z \in E'$ and (ii) $\int_{E'} q(z)\, dm = 1$. In our case, we take

$$a(z) = (1-p^2)^2\pi^{-1} w_0(z) J_\lambda(z) = (1-p^2)^2\pi^{-1} w_0(z)|g'_\lambda(z)|^2, \ z \in E',$$

since for $z \in E'$, $J_\lambda(z) = |\partial g_\lambda(z)|^2 = |g'_\lambda(z)|^2$. Hence using (2.18), we get

$$\begin{aligned} &\log\left(\int_{E'} (1-p^2)^2\pi^{-1} w_0(z)|g'_\lambda(z)|^2\, dm\right) \\ &= \sup_{q(z)}\left[\int_{E'} q(z) \log\left(\frac{(1-p^2)^2\pi^{-1} w_0(z)|g'_\lambda(z)|^2}{q(z)}\right) dm\right] \\ &= \sup_{q(z)}\left[\int_{E'} q(z) \log(w_0(z))\, dm + h_p(\lambda)\right], \end{aligned} \tag{2.19}$$

where

$$h_p(\lambda) = \int_{E'} q(z) \log\left(\frac{(1-p^2)^2\pi^{-1}|g'_\lambda(z)|^2}{q(z)}\right) dm$$

is harmonic in $\lambda \in \mathbb{D}$, by (2.17), for each $z \in E'$. Using (2.18) and (2.5) successively, we get

$$h_p(\lambda) \leq \log\left(\int_{E'} (1-p^2)^2\pi^{-1}|g'_\lambda(z)|^2\, dm\right) \leq 0.$$

So for each $z \in E'$, $h_p(\lambda)$ is harmonic and nonpositive in $\mathbb{D}$. Hence by using Harnack's Inequality and the fact that $g_0(z) = z/(1-pz)$ (as claimed in the proof of Theorem 1(i)), we have

$$\begin{aligned} h_p(\lambda) &\le (1-|\lambda|)(1+|\lambda|)^{-1} h_p(0) \\ &= (1-|\lambda|)(1+|\lambda|)^{-1} \int_{E'} q(z) \log\left(\frac{(1-p^2)^2\pi^{-1}|g_0'(z)|^2}{q(z)}\right) dm. \end{aligned}$$

For $\lambda = k$, we have $g_\lambda = g$ and $(1+k)/(1-k) = K$. Thus using above inequality (for $\lambda = k$) in (2.19), and also using (2.18) once more, we get

$$\begin{aligned} &\log\left(\int_{E'} (1-p^2)^2\pi^{-1} w_0(z) J_g(z)\, dm\right) \\ \le\ & \sup_{q(z)} \left[\int_{E'} q(z)\log w_0(z)\, dm + \frac{1}{K}\int_{E'} q(z)\log\left(\frac{(1-p^2)^2\pi^{-1}J_{g_0}(z)}{q(z)}\right) dm\right] \\ =\ & \frac{1}{K}\sup_{q(z)} \left[\int_{E'} q(z)\log\left(\frac{(1-p^2)^2\pi^{-1}w_0(z)^K J_{g_0}(z)}{q(z)}\right) dm\right] \\ =\ & \log\left(\int_{E'} (1-p^2)^2\pi^{-1} w_0(z)^K J_{g_0}(z)\, dm\right)^{1/K}. \end{aligned}$$

Taking exponentiation and doing a rearrangement, we obtain

$$\int_{E'} w_0(z) J_g(z)\, dm \le \left[\frac{\pi}{(1-p^2)^2}\right]^{1-1/K} \left(\int_{E'} w_0(z)^K J_{g_0}(z)\, dm\right)^{1/K}. \tag{2.20}$$

Now putting $w(z) = w_0(1/z), f(z) = g(1/z)$ and observing that $J_g(z) = J_f(1/z)|z|^{-4}$, $J_{g_0}(z) = J_{f_0}(1/z)|z|^{-4}$, second inequality of (2.16) follows from above. Here E' and J_{g_0} is replaced by E and $J_{f_0} = J_0$ respectively, where $f_0(z) = 1/(z-p)$. To obtain the first inequality we use the other part of the Harnack's Inequality in (2.19) viz.

$$h_p(\lambda) \ge (1+|\lambda|)(1-|\lambda|)^{-1} h_p(0)$$

and proceed in a similar fashion. Next we show that the second inequality of Theorem 2 is sharp. To verify this, it is sufficient to show that the inequality (2.20) is sharp. We follow the arguments given in [3, Example 2.1]. First we choose the numbers w_j, p_j, r_j, ρ_j for $j = 1, \ldots, n$, suitably as $1 \le w_1 < w_2 < \cdots < w_n$ and $0 < p_j, r_j < 1$ such that

$$w_j = \left(\prod_{l=1}^{j} r_l\right)^{-2/K} \quad \text{and} \quad \sum_{j=1}^{n} p_j w_j^K = 1. \tag{2.21}$$

We now consider the function

$$g = f_{r_1}^{\rho_1} \circ \cdots \circ f_{r_n}^{\rho_n}, \text{ where } f_{r_j}^{\rho_j}(z) = \rho_j f_{r_j}(z/\rho_j),\ j = 1, \ldots, n, \tag{2.22}$$

where $0 < \rho_j < 1$ for $j = 2, \ldots, n$ with $\rho_1 = 1$ and f_r defined in (1.4). Next we consider the weight function $w_0(z) = \sum\limits_{j=1}^{n} w_j \chi_{E_j}(z)$, where

$$E_j = \{z : \rho_{j+1} < |z| < \rho_j r_j\},\ 1 \le j \le n-1; \quad E_n = \{z : |z| < \rho_n r_n\}.$$

The composition in (2.22) is well defined as we have

$$r_j^2\rho_j^2 - \rho_{j+1}^2 = p_j,\ 1 \le j \le n-1;\ r_n^2\rho_n^2 = p_n.$$

In our case, we define

$$G(z) = (1-p^2)^{-1} g\left(\frac{z-p}{1-pz}\right) + p/(1-p^2),\ z \in \mathbb{C}, \tag{2.23}$$

and the weight function as

$$W_0(z) = \sum_{j=1}^{n} w_j \chi_{\tilde{E}_j}(z),\ \tilde{E}_j = \tilde{f}^{-1}(E_j),\ \text{where } \tilde{f}(z) = (z-p)/(1-pz).$$

Now the function G defined in (2.23) belongs to the class $\Sigma_k^0(p)$, as the function g defined in (2.22) belongs to the class Σ_k^0. If we now take $\tilde{E} = \cup_{j=1}^n \tilde{E}_j$, then G is conformal on $\tilde{E}$. Hence using first relation of (2.21), it is easy to see that

$$J_{G|_{\tilde{E}}} = W_0(z)^{K-1}|1-pz|^{-4} = W_0(z)^{K-1}|g_0(\tilde{E})|,\ z \in \tilde{E}.$$

Again, using second relation of (2.21), we get

$$\begin{aligned}
\int_{\tilde{E}} W_0(z) J_G(z)\, dm &= \sum_{j=1}^{n} \left(w_j^K \int_{\tilde{E}_j} |1-pz|^{-4}\, dm \right) \\
&= \sum_{j=1}^{n} \left(w_j^K |g_0(\tilde{E}_j)| \right) \\
&= \pi(1-p^2)^{-2} \left[\sum_{j=1}^{n-1} w_j^K (r_j^2\rho_j^2 - \rho_{j+1}^2) + w_n^K r_n^2 \rho_n^2 \right] \\
&= \pi(1-p^2)^{-2} \sum_{j=1}^{n} p_j w_j^K \\
&= \pi(1-p^2)^{-2} \\
&= \left[\pi(1-p^2)^{-2}\right]^{1-1/K} \left(\int_{\tilde{E}} W_0(z)^K J_{g_0}(z)\, dm \right)^{1/K}.
\end{aligned}$$

As equality holds in (2.20), hence it also holds for the second inequality in (2.16). Optimality of the other inequality in (2.16) can be established by similar construction. □

Remark. (i) If $w(z) = 1$ for all $z \in E$, then the second inequality of Theorem 2 implies Theorem 1(i).

(ii) In Theorem C, we assumed $f \in \Sigma_k^0$ of the form (1.3) in $\mathbb{D}^*$, as taken in [3]. But if we take $f \in \Sigma_k^0$ of the form (1.1) (with $b_0 = 0$) in $\mathbb{D}$ and f is conformal on a Lebesgue measurable subset E of $\mathbb{D}^*$, then Theorem C can be restated as

$$\pi^{1-K}\left(\int_E w(z)^{1/K}|z|^{-4}\,dm\right)^K \le \int_E w(z)J_f(z)\,dm \le \pi^{1-1/K}\left(\int_E w(z)^K|z|^{-4}\,dm\right)^{1/K}.$$

This result coincides with Theorem 2 for $p=0$.

As an application of Theorem 1, we prove the next result. It deals with the bounds of the Hilbert transform of the characteristic function of a Lebesgue measurable set $E\subset\mathbb{D}$.

Theorem 3. *Let E be a Lebesgue measurable subset of $\mathbb{D}$, then*

$$\int_{\mathbb{D}\setminus E}\frac{1}{|1-pz|^2}\left|H\left[\frac{\chi_E}{(1-p\overline{z})^2}\right]\right|\,dm \le |g_0(E)|\log\left(\frac{\pi(1-p^2)^{-2}}{|g_0(E)|}\right), \tag{2.24}$$

where $g_0(z)=z/(1-pz), z\in\mathbb{C}$. The inequality is sharp.

Proof. For any function μ with $|\mu|=1$, supported in $\mathbb{D}\setminus E$, we define $\mu_\lambda(z)=\lambda\mu(z)$ for $\lambda\in\mathbb{D}$ and consider the corresponding family of quasiconformal mappings g_λ in $\widehat{\mathbb{C}}$, with dilatation μ_λ. We also assume that the functions g_λ are normalized such that they belong to the class $\Sigma^0(p)$, when restricted on $\mathbb{D}^*$, therefore each function g_λ belongs to the class $\Sigma^0_{|\lambda|}(p)$, for each $\lambda\in\mathbb{D}$. Now by the assumption each g_λ is conformal on E, which gives from (2.11) that

$$\begin{aligned}|g_\lambda(E)| &= \int_E |\partial g_\lambda(z)|^2\,dm\\ &= |g_0(E)| + 2\mathrm{Re}\int_E (1-p\overline{z})^{-2}H[\bar\partial g_\lambda]\,dm + \int_E |H[\bar\partial g_\lambda]|^2\,dm.\end{aligned} \tag{2.25}$$

Now from (2.10), $w=\bar\partial g_\lambda$ can be written as

$$\bar\partial g_\lambda = \lambda\mu(1-pz)^{-2}+h_\lambda(z), \tag{2.26}$$

where $\|h_\lambda\|_2\le C|\lambda|^2$, C is a constant. Using above identity it is easy to see that

$$\int_E |H[\bar\partial g_\lambda]|^2\,dm = O(|\lambda|^2)\quad \text{as}\ \lambda\to 0.$$

Again from (2.26) we get

$$\begin{aligned}&\mathrm{Re}\int_E (1-p\overline{z})^{-2}H[\bar\partial g_\lambda(z)]\,dm\\ &= \mathrm{Re}\int_E \lambda(1-p\overline{z})^{-2}H[\mu(1-pz)^{-2}]\,dm + O(|\lambda|^2),\quad \lambda\to 0.\end{aligned}$$

Now upon using the last two estimates obtained above, we get from (2.25) that

$$|g_\lambda(E)| = |g_0(E)| + 2\mathrm{Re}\int_E \lambda(1-p\overline{z})^{-2}H[\mu(1-pz)^{-2}]\,dm + O(|\lambda|^2). \tag{2.27}$$

Now as $g_\lambda \in \Sigma^0_{|\lambda|}(p)$, by the area distortion inequality (Theorem 1(i)), we get

$$|g_\lambda(E)| \le \left[\pi(1-p^2)^{-2}\right]^{1-1/K} |g_0(E)|^{1/K},$$

where $K = (1+|\lambda|)(1-|\lambda|)^{-1}$. Since $1-K^{-1} = 2|\lambda| + O(|\lambda|^2)$, therefore the above inequality can be written as

$$|g_\lambda(E)| \le |g_0(E)| + 2|\lambda||g_0(E)| \log\left(\pi(1-p^2)^{-2}|g_0(E)|^{-1}\right) + O(|\lambda|^2).$$

Comparing the coefficients of the terms which are linear in $|\lambda|$ of the above inequality and that of with (2.27), we get

$$\operatorname{Re}\left[\lambda \int_E (1-p\overline{z})^{-2} H[\mu(1-pz)^{-2}]\, dm\right] \le |\lambda||g_0(E)| \log\left(\pi(1-p^2)^{-2}|g_0(E)|^{-1}\right).$$

Now for a particular choice of λ, we have

$$\operatorname{Re}\left[\lambda \int_E (1-p\overline{z})^{-2} H[\mu(1-pz)^{-2}]\, dm\right] = |\lambda| \left|\int_E (1-p\overline{z})^{-2} H[\mu(1-pz)^{-2}]\, dm\right|.$$

From above two relations, we get

$$\left|\int_E (1-p\overline{z})^{-2} H[\mu(1-pz)^{-2}]\, dm\right| \le |g_0(E)| \log\left(\pi(1-p^2)^{-2}|g_0(E)|^{-1}\right). \tag{2.28}$$

Next, by using the symmetric property of H (see [2, p. 95]), we obtain

$$\begin{aligned}\int_E (1-p\overline{z})^{-2} H[\mu(1-pz)^{-2}]\, dm &= \int_{\mathbb{C}} \chi_E (1-p\overline{z})^{-2} H[\mu(1-pz)^{-2}]\, dm \\ &= \int_{\mathbb{C}} \mu(1-pz)^{-2} H[\chi_E(1-p\overline{z})^{-2}]\, dm \\ &= \int_{\mathbb{D}\setminus E} \mu(1-pz)^{-2} H[\chi_E(1-p\overline{z})^{-2}]\, dm,\end{aligned}$$

since μ has support in $\mathbb{D}\setminus E$. Using the inequality (2.28), we get

$$\left|\int_{\mathbb{D}\setminus E} \mu(1-pz)^{-2} H[\chi_E(1-p\overline{z})^{-2}]\, dm\right| \le |g_0(E)| \log\left(\pi(1-p^2)^{-2}|g_0(E)|^{-1}\right).$$

For a suitable choice of μ, we can take modulus inside the integral of the left hand side of the above inequality, which proves the theorem. Finally it remains to prove the sharpness of the inequality (2.24). To show this we consider

$$E = \left\{ z : \left| z - \frac{p(1-r^2)}{1-p^2r^2} \right| < \frac{r(1-p^2)}{1-p^2r^2} \right\}, \quad 0 < r < 1.$$

Clearly $E \subset \mathbb{D}$. Hence $|g_0(E)| = \pi r^2(1-p^2)^{-2}$, so that right hand side of (2.24) reduces to $2\pi(1-p^2)^{-2}r^2\log(r^{-1})$. Next in order to find the Hilbert transform of the function $\chi_E(1-p\overline{z})^{-2}$, we define

$$f(z) = \begin{cases} \frac{1}{1-p^2}\left(\frac{\overline{z}-p}{1-p\overline{z}}\right), & z \in E, \\ \frac{r^2}{1-p^2}\left(\frac{1-pz}{z-p}\right), & z \in \mathbb{C}\setminus E. \end{cases}$$

Here f is continuous on $\mathbb{C}$ and a little calculation reveals that $\bar{\partial} f = \chi_E(1-p\overline{z})^{-2}$ and $\partial f = -r^2(z-p)^{-2}\chi_{\mathbb{C}\setminus E}$. Using the relation $H[\bar{\partial} f] = \partial f$, we have

$$H[\chi_E(1-p\overline{z})^{-2}] = -r^2(z-p)^{-2}\chi_{\mathbb{C}\setminus E}.$$

Let $w = \tilde{f}(z) = (z-p)/(1-pz) = u+iv$. Therefore, $\tilde{f}(\mathbb{D}\setminus E) = \{w : r \leq |w| < 1\}$ and $J_{\tilde{f}}(z) = (1-p^2)^2|1-pz|^{-4}$. Hence we have,

$$\begin{aligned}
&\int_{\mathbb{D}\setminus E} |1-pz|^{-2} \left|H[\chi_E(1-p\overline{z})^{-2}]\right| \, dm \\
&= r^2 \int_{\mathbb{D}\setminus E} (|1-pz||z-p|)^{-2} \, dm \\
&= r^2(1-p^2)^{-2} \int_{\mathbb{D}\setminus E} \left|\frac{1-pz}{z-p}\right|^2 \frac{(1-p^2)^2}{|1-pz|^4} \, dm \\
&= r^2(1-p^2)^{-2} \int_{\tilde{f}(\mathbb{D}\setminus E)} |w|^{-2} \, dudv \\
&= 2\pi(1-p^2)^{-2} r^2 \log(r^{-1}).
\end{aligned}$$

Thus the inequality (2.24) is sharp and this completes proof of the theorem. □

Remark. For $p = 0$, the functions g_λ defined in the proof of Theorem 3 belong to the class $\Sigma^0_{|\lambda|}$ and the function g_0 is the identity function. Hence the inequality (2.24) reads as (compare Theorem 14.6.1 of [2, p. 385])

$$\int_{\mathbb{D}\setminus E} |H[\chi_E]| \, dm \leq |E| \log\left(\pi/|E|\right).$$

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