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COMBINATORIAL AUSLANDER-REITEN QUIVERS AND REDUCED EXPRESSIONS

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Abstract. In this paper, we introduce the notion of combinatorial Auslander-Reiten (AR) quivers for commutation classes $[\widetilde{w}]$ of w in a finite Weyl group. This combinatorial object is the Hasse diagram of the convex partial order $\prec_{[\widetilde{w}]}$ on the subset $\Phi(w)$ of positive roots. By analyzing properties of the combinatorial AR-quivers with labelings and reflection functors, we can apply their properties to the representation theory of KLR algebras and dual PBW-basis associated to any commutation class $[\widetilde{w}_0]$ of the longest element w_0 of any finite type.

Introduction

For a Dynkin quiver Q of finite type ADE, the Auslander-Reiten quiver Γ_Q encodes the representation theory of the path algebra $\mathbb{C}Q$ in the following sense: (i) the set of vertices corresponds to the set $\operatorname{Ind} Q$ of isomorphism classes of indecomposable $\mathbb{C}Q$-modules, (ii) the set of arrows corresponds to the set of irreducible morphisms between objects in $\operatorname{Ind} Q$. On the other hand, by reading the residues of vertices of Γ_Q in a *compatible way* ([2]), one can obtain reduced expressions $\widetilde{w}_0$ of the longest element w_0 in the Weyl group W. Such reduced expressions can be grouped into one class $[Q]$ via commutation equivalence $\sim$: $\widetilde{w}_0 \sim \widetilde{w}'_0$ if and only if $\widetilde{w}'_0$ can be obtained by applying the commutation relations $s_is_j = s_js_i$.

A reduced expression in $[Q]$ is called *adapted to* Q.

Another important role of Γ_Q in Lie theory is a realization of the convex partial order $\prec_Q$ on Φ^+, which has been used in representation theory intensively (see, for example, [7, 11, 13]). Here, the order $\prec_Q$ is defined as follows: For a reduced expression $\widetilde{w}_0 = s_{i_1}s_{i_2}\cdots s_{i_N} \in [Q]$, we denote a positive root $s_{i_1}s_{i_2}\cdots s_{i_{k-1}}\alpha_k \in \Phi^+$ by $\beta_k^{\widetilde{w}_0}$ and assign the *residue* i_k to $\beta_k^{\widetilde{w}_0}$. Then each reduced expression $\widetilde{w}_0 \in [Q]$ induces the total order $<_{\widetilde{w}_0}$ on Φ^+ such that

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$\beta_k^{\widetilde{w}_0} <_{\widetilde{w}_0} \beta_l^{\widetilde{w}_0} \iff k < l$. Using the total orders $<_{\widetilde{w}'_0}$ for $\widetilde{w}'_0 \in [Q]$, we obtain the convex partial order $\prec_Q$ on Φ^+:

$$\alpha \prec_Q \beta \text{ if and only if } \alpha <_{\widetilde{w}'_0} \beta \text{ for all } \widetilde{w}'_0 \in [Q]$$

such that $\alpha \prec_Q \beta$ and $\gamma = \alpha + \beta \in \Phi^+$ imply $\alpha \prec_Q \gamma \prec_Q \beta$ (the convexity).

As the definition itself, $\prec_Q$ is quite complicated since there are lots of reduced expressions in each $[Q]$. However, interestingly, Γ_Q realizes $\prec_Q$ in the sense that

$$\alpha \prec_Q \beta \text{ if and only if there exists a path from } \beta \text{ to } \alpha \text{ in } \Gamma_Q$$

and there exists a way of finding root labels[1] of vertices in Γ_Q only with its shape. Hence, Γ_Q is one of the most efficient tools for analyzing $\prec_Q$.

For the longest element w_0 in W of any finite type, it is proved in [20,27] that any convex total order $<$ on Φ^+ is $<_{\widetilde{w}_0}$ for some $\widetilde{w}_0$. Here, $\widetilde{w}_0$ is not necessarily adapted. Moreover, any order $<_{\widetilde{w}_0}$ is a convex order and each convex order $<_{\widetilde{w}_0}$ does a crucial role in the representation theory (see [4, 14] and Theorem 5.7). However, to the best of the authors' knowledge, properties of general $<_{\widetilde{w}_0}$ and $\prec_{[\widetilde{w}_0]}$ are not studied well, as much as $\prec_Q$ of type ADE. Inspired from the facts, in this article, we mainly deal with convex orders $<_{\widetilde{w}_0}$ and $\prec_{[\widetilde{w}_0]}$, for general $\widetilde{w}_0$ of any finite types.

To see orders $\prec_{[\widetilde{w}_0]}$ efficiently, we introduce the new quiver $\Upsilon_{[\widetilde{w}]}$ called the *combinatorial AR-quiver* for a reduced expression $\widetilde{w}$ of $w \in \mathsf{W}$, which realizes the convex partial order $\prec_{[\widetilde{w}]}$ on $\Phi(w)$; that is,

$$\alpha \prec_{[\widetilde{w}]} \beta \text{ if and only if there exists a path from } \beta \text{ to } \alpha \text{ in } \Upsilon_{[\widetilde{w}]}.$$

More precisely, we suggest a purely combinatorial algorithm for constructing the quiver $\Upsilon_{[\widetilde{w}]}$ associated with $\widetilde{w} = s_{i_1} \cdots s_{i_\ell}$ (Algorithm 2.1) and show, indeed, it is the Hasse diagram of $\prec_{[\widetilde{w}]}$. Thus $\Gamma_Q \simeq \Upsilon_{[Q]}$ and $\Upsilon_{[\widetilde{w}]}$ are distinct in the sense that $\Upsilon_{[\widetilde{w}]} \simeq \Upsilon_{[\widetilde{w}']}$ if and only if $[\widetilde{w}'] = [\widetilde{w}]$ (Theorem 2.21 and Theorem 2.22). In Section 3, we explain an efficient way to compute root labels, which are most useful in our applications. Since, via Algorithm 2.1, it requires a lot of computations to obtain labels, to avoid it, we show every vertex in a sectional path shares a *component* (Definition 3.5). As a consequence, the property allows us to find the labels with a little of computations.

Due to the results in Section 2 and Section 3, we can understand $\prec_{[\widetilde{w}_0]}$ completely using the quiver $\Upsilon_{[\widetilde{w}_0]}$. However, since there are too many classes $[\widetilde{w}_0]$ of reduced expressions to investigate $\prec_{[\widetilde{w}_0]}$ one by one, we aim to classify the classes. To this end, in Section 4, we consider another equivalence relation called a *reflection equivalence relation* on the set of commutation equivalence classes. An equivalence class induced from reflection equivalences is called an r-cluster point $[\![\widetilde{w}_0]\!]$. As one may expect, there are similarities between representation theories related to $[Q]$ and $[Q']$ (for example, [7, 11, 17–19], see also Corollary 5.15) and $\{[Q]\}$ forms an r-cluster point $[\![\Delta]\!]$, called the adapted

[1] elements in Φ^+ corresponding to vertices in Γ_Q

cluster point. In addition, we introduce the notion of Coxeter composition (Definition 4.10) with respect to a Dynkin diagram automorphism σ.

In Section 5, we apply our results in previous sections to the representation theory of KLR-algebras ([10,23]) and PBW-bases of quantum groups ([12,25]). It is well known that *proper standard modules* $\{\overrightarrow{S}_{\widetilde{w}_0}(\underline{m})\}$ of a KLR-algebra associated to $\widetilde{w}_0$ categorify the corresponding dual PBW-basis $\{P_{\widetilde{w}_0}(\underline{m})\}$ ([4, 7–9, 14]). Moreover, for finite type cases, $\{\overrightarrow{S}_{\widetilde{w}_0}(\underline{m})\}$ depends only on the commutation class $[\widetilde{w}_0]$, up to $q^{\mathbb{Z}}$, and so does $\{P_{\widetilde{w}_0}(\underline{m})\}$ (see [4, 14]). Note that this property is originated from the commutation relation between operators T_i and T_j in [12,25]. In Theorem 5.8, we give an alternative proof of the property using our observation on $\prec_{[\widetilde{w}_0]}$ and $\Upsilon_{[\widetilde{w}_0]}$.

If the Lie algebra $\mathfrak{g}$ is of finite simply laced type, the set of all simple modules of the KLR-algebra categorifies the dual canonical basis ([24, 26]). In [14], a transition map between a dual PBW-basis and the dual canonical basis was introduced (see (5.6)) and we consider a more refined transition map using $\prec_{[\widetilde{w}_0]}$ (see (5.7)). By the refined transition map, in Proposition 5.12, we prove that the root modules $S_{[\widetilde{w}_0]}(\beta)$ ($\beta \in \Phi^+$) for β's lying on the same sectional path *q-commute* to each other and hence so do the dual PBW-generators $P_{[\widetilde{w}_0]}(\beta)$'s. In addition, reflection functors on $[\![\widetilde{w}_0]\!]$ allow us to show similarities between $\{S_{[\widetilde{w}_0]}(\alpha)\}$ and $\{S_{[\widetilde{w}'_0]}(\alpha')\}$ for $[\widetilde{w}_0], [\widetilde{w}'_0] \in [\![\widetilde{w}_0]\!]$ (Corollary 5.15).

In Appendix, we give a table of r-cluster points of A_4 (Appendix A) and observations on the relations between $\Upsilon_{[\widetilde{w}']}$ and $\Upsilon_{[\widetilde{w}]}$ when $\widetilde{w}'$ is obtained from $\widetilde{w}$ by a braid relation (Appendix B).

1. Auslander-Reiten quivers

In this section, we recall properties of Auslander-Reiten quivers. We refer to [1, 6, 11, 15] for the basic theories on quiver representations and Auslander-Reiten quivers. For the combinatorial properties, we refer to [2, 18].

1.1. Auslander-Reiten quivers and related notions

Let $\mathsf{A} = (a_{ij})_{i,j \in I}$ for $I = \{1, \dots, n\}$ be a Cartan matrix of a finite-dimensional simple Lie algebra $\mathfrak{g}$. Let Δ be the Dynkin diagram associated to A. For vertices $i, j \in I$ in Δ, the minimal length of a path from i to j is called the *distance* between i and j and is denoted by $d_\Delta(i,j)$.

We denote by $\Pi = \{\alpha_i \mid i \in I\}$ the set of simple roots, Φ the set of roots, Φ^+ (resp. Φ^-) the set of positive roots (resp. negative roots). Let $\{\epsilon_i \mid 1 \le i \le m\}$ be the set of orthonormal basis of $\mathbb{C}^m$. The free abelian group $\mathsf{Q} := \oplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice*. Set $\mathsf{Q}^+ = \sum_{i \in I} \mathbb{Z}_{\ge 0}\alpha_i \subset \mathsf{Q}$ and $\mathsf{Q}^- = \sum_{i \in I} \mathbb{Z}_{\le 0}\alpha_i \subset \mathsf{Q}$. For $\beta = \sum_{i \in I} m_i\alpha_i \in \mathsf{Q}^+$, we set $\mathrm{ht}(\beta) = \sum_{i \in I} m_i$. Let $(\cdot,\cdot)$ be the the symmetric bilinear form on $\mathsf{Q} \times \mathsf{Q}$ (we refer [3, Plate I∼IX]).

A Dynkin quiver Q is obtained by adding an orientation to each edge in the Dynkin diagram Δ of a finite simply laced type. In other words, $Q = (Q^0, Q^1)$ where Q^0 is the set of vertices indexed by I and Q^1 is the set of oriented edges with the underlying graph Δ. We say that the vertex $i \in \Delta$ is a sink (resp. source) if every edge between i and j is oriented as follows: $j \to i$ (resp. $i \to j$).

1.1.1. *Auslander-Reiten quivers.* Let $\mathrm{Mod}(\mathbb{C}Q)$ be the category of finite dimensional modules over the path algebra $\mathbb{C}Q$. An object $M \in \mathrm{Mod}\,\mathbb{C}Q$ consists of the following data:

(1) a finite dimensional module M_i for each $i \in Q^0$,
(2) a linear map $\psi_{i\to j} : M_i \to M_j$ for each oriented edge $i \to j$.

The *dimension vector* of the module M is $\underline{\dim}\, M = \sum_{i\in I}(\dim\, M_i)\alpha_i$ and a simple object in $\mathrm{Mod}\,\mathbb{C}Q$ is $S(i)$ for some $i \in I$ where $\underline{\dim} S(i) = \alpha_i$. In $\mathrm{Mod}\,\mathbb{C}Q$, the set of isomorphism classes $[M]$ of indecomposable modules is denoted by $\mathrm{Ind}\,Q$.

Theorem 1.1 (Gabriel's theorem)**.** *Let Q and Φ^+ be a Dynkin quiver and the set of positive roots of finite type A_n, D_n or E_n. Then there is a bijection between* $\mathrm{Ind}\,Q$ *and* Φ^+ :

$$[M] \mapsto \underline{\dim} M.$$

Now we recall the Auslander-Reiten (AR) quiver Γ_Q associated to a Dynkin quiver Q of finite type A_n, D_n, or E_n. Let us denote by $\mathrm{Ind}\,Q$ the set of isomorphism classes $[M]$ of indecomposable modules in $\mathrm{Mod}\,\mathbb{C}Q$, where $\mathrm{Mod}\,\mathbb{C}Q$ is the category of finite dimensional modules over the path algebra $\mathbb{C}Q$.

Definition 1.2. The quiver $\Gamma_Q = (\Gamma_Q^0, \Gamma_Q^1)$ is called the Auslander-Reiten quiver (AR quiver) if

(i) each vertex V_M in Γ_Q^0 corresponds to an isomorphism class $[M]$ in $\mathrm{Ind}\,Q$,
(ii) an arrow $V_M \to V_{M'}$ in Γ_Q^1 corresponds to an *irreducible* morphism $M \to M'$.

Gabriel's theorem (Theorem 1.1) tells that there is a natural one-to-one correspondence between the set Γ_Q^0 of vertices in Γ_Q and the set Φ^+ of positive roots. Hence we use Φ^+ as the index set of Γ_Q^0.

1.1.2. *Adapted reduced expressions.* The Weyl group W of a finite type with rank n is generated by simple reflections $s_i \in \mathrm{Aut}(\mathsf{Q})$, $i \in I$, defined by $s_i(\alpha) := \alpha - \frac{(\alpha,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$. Note that $\big(w(\alpha), w(\beta)\big) = (\alpha, \beta)$ for any $w \in W$ and $\alpha, \beta \in \mathsf{Q}$. For $w \in \mathsf{W}$, the length of w is

$$\ell(w) = \min\{l \in \mathbb{Z}_{\geq 0} \,|\, s_{i_1}\cdots s_{i_l} = w,\ s_{i_k} \text{ are simple reflections}\}.$$

If $w = s_{i_1}s_{i_2}\cdots s_{i_{\ell(w)}}$, then the sequence of simple reflections $\widetilde{w} = (s_{i_1}, \ldots, s_{i_{\ell(w)}})$ is called a *reduced expression* associated to w. We denote by w_0 the longest element in W and by * the involution on I induced by w_0; i.e.,

$$w_0(\alpha_i) := -\alpha_{i^*} \quad \text{for all } i \in I. \tag{1.1}$$

For $w \in \mathsf{W}$ with a reduced expression $(s_{i_1}, \dots, s_{i_l})$, consider the subset ([3])

$$
\begin{aligned}
\Phi(w) &= \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\} \\
&= \{s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) \mid k = 1, \dots, \ell(w)\} \text{ such that } |\Phi(w)| = \ell(w).
\end{aligned} \tag{1.2}
$$

In particular, $\Phi(w_0) = \Phi^+$. Note that the definition of (1.2) does not depends on the choice of a reduced expression.

The action of a simple reflection s_i, $i \in I$, on the set of Dynkin quivers is defined by $s_i(Q) = Q'$, where $s_i(Q)$ is a quiver obtained by Q by reversing all the arrows incident with i.

Definition 1.3. A reduced expression $\widetilde{w} = (s_{i_1}, \dots, s_{i_l})$ of w is said to be *adapted* to a Dynkin quiver Q if

$$i_k \text{ is a sink of } Q_{k-1} = s_{i_{k-1}} \cdots s_{i_1}(Q).$$

Here, $Q_0 := Q$.

Remark 1.4. The followings are well known facts:

(1) A reduced expression $\widetilde{w}_0$ of w_0 is adapted to at most one Dynkin quiver Q.
(2) For each Dynkin quiver Q, there is a reduced expression $\widetilde{w}_0$ of w_0 adapted to Q.

Note that two different reduced expressions of w_0 can be adapted to the same Dynkin quiver Q. Actually, we can assign a *class* of reduced expressions of w_0 to each Dynkin quiver Q. (See Definition 1.5 and Proposition 1.6.)

Definition 1.5 ([2, 11])**.** Let $\widetilde{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_k})$ and $\widetilde{w}' = (s_{i'_1}, s_{i'_2}, \dots, s_{i'_k})$ be reduced expressions of $w \in W$. If $\widetilde{w}'$ can be obtained from $\widetilde{w}$ by a sequence of commutation relations, $s_i s_j = s_j s_i$ for $d_\Delta(i,j) > 1$, then we say $\widetilde{w}$ and $\widetilde{w}'$ are *commutation equivalent* and write $\widetilde{w} \sim \widetilde{w}'$. The *equivalence class* of $\widetilde{w}$ is denoted by $[\widetilde{w}]$.

Proposition 1.6 ([2, 11])**.** *Reduced expressions $\widetilde{w}_0 = (s_{i_1}, s_{i_2}, \dots, s_{i_l})$ and $\widetilde{w}'_0 = (s_{i'_1}, s_{i'_2}, \dots, s_{i'_l})$ of w_0 are adapted to the same quiver Q if and only if $\widetilde{w}_0 \sim \widetilde{w}'_0$ and $\widetilde{w}_0$ is adapted to Q.*

Thus we can denote by $[Q]$ the equivalence class of w_0 consisting of all reduced expressions adapted to the Dynkin quiver Q.

1.1.3. *Coxeter elements.* An element $\phi = s_{i_1} s_{i_2} \cdots s_{i_n} \in \mathsf{W}$ where $\{i_1, i_2, \dots, i_n\} = I$ is called a *Coxeter element.* There is the one-to-one correspondence between the set of Dynkin quivers and the set of Coxeter elements

$$Q \longleftrightarrow \phi_Q,$$

where ϕ_Q is the Coxeter element all of whose reduced expressions are adapted to Q.

1.1.4. *Partial orders on* $\Phi(w)$. Let w be an element in W of finite type. An order $\preceq$ on the set $\Phi(w)$ is said to be *convex* if

$$\alpha, \beta, \alpha+\beta \in \Phi(w) \text{ and } \alpha \preceq \beta \text{ implies } \alpha \preceq \alpha+\beta \preceq \beta.$$

Definition 1.7. The total order $<_{\widetilde{w}}$ on $\Phi(w)$ associated to $\widetilde{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_l})$ is defined by

$$\beta^{\widetilde{w}}_j <_{\widetilde{w}} \beta^{\widetilde{w}}_k \quad \text{if and only if} \quad j<k \quad \text{where } \beta^{\widetilde{w}}_j := s_{i_1}s_{i_2}\cdots s_{i_{j-1}}(\alpha_{i_j}).$$

Definition 1.8. Let $\alpha, \beta \in \Phi(w) \subset \Phi^+$. We define an order $\prec_{[\widetilde{w}]}$ on $\Phi(w)$ as follows:

$$\alpha \prec_{[\widetilde{w}]} \beta \quad \text{if and only if} \quad \alpha <_{\widetilde{w}'} \beta \quad \text{for any} \quad \widetilde{w}' \in [\widetilde{w}].$$

Proposition 1.9 ([20]). *The total order* $<_{\widetilde{w}}$ *and the partial order* $\prec_{[\widetilde{w}]}$ *are convex orders on* $\Phi(w)$.

Remark 1.10. Consider the adapted class $[Q]$ associated to the Dynkin quiver Q of type ADE. The convex partial order $\prec_{[Q]}$ is often denoted by $\prec_Q$ for the simplicity of notation.

1.2. Construction of AR-quivers

Consider the *height function* $\xi : I \to \mathbb{Z}$ associated to the Dynkin quiver Q, that is ξ satisfies

$$\text{if there exists an arrow } i \to j \text{ in } Q, \text{ then } \xi(j) = \xi(i) - 1 \in \mathbb{Z}.$$

Note that a height function exists and is unique (up to constant) since the Dynkin diagram do not have a cycle and connected.

The *repetition quiver* $\mathbb{Z}Q$ of Q associated to the height function ξ consists of the set of vertices

$$(\mathbb{Z}Q)^0 = \{(i,p) \in I \times \mathbb{Z} \mid p - \xi(i) \in 2\mathbb{Z}\}$$

and the set of arrows

$$(\mathbb{Z}Q)^1 = \{(j,p+1) \to (i,p),\ (i,p) \to (j,p-1) \,|\, i,j \in I \text{ such that } d_\Delta(i,j) = 1\}.$$

For $i \in I$, we define positive roots γ_i and θ_i in the following way:

$$\gamma_i = \alpha_i + \sum_{j \in \overleftarrow{i}} \alpha_j \quad \text{and} \quad \theta_i = \alpha_i + \sum_{j \in \overrightarrow{i}} \alpha_j, \tag{1.3}$$

where

- $\overleftarrow{i}$ is the set of vertices j in Q^0 such that there exists a path from i to j,
- $\overrightarrow{i}$ is the set of vertices j in Q^0 such that there exists a path from j to i.

Note that $\{\gamma_i \,|\, i \in I\} = \Phi(\phi_Q)$ and $\{\theta_i \,|\, i \in I\} = \Phi(\phi_Q^{-1})$. Consider the map $\pi_Q : \Phi^+ \to (\mathbb{Z}Q)^0$ such that

$$\gamma_i \mapsto (i, \xi(i)),\ \phi_Q(\alpha) \mapsto (i, p-2) \ \text{ if } \pi_Q(\alpha) = (i,p) \text{ and } \phi_Q(\alpha),\ \alpha \in \Phi^+. \tag{1.4}$$

Proposition 1.11 ([7]). *The subquiver of $\mathbb{Z}Q$ consisting of $\pi_Q(\Phi^+)$ is the same as the quiver Γ_Q by identifying their vertices as Φ^+.*

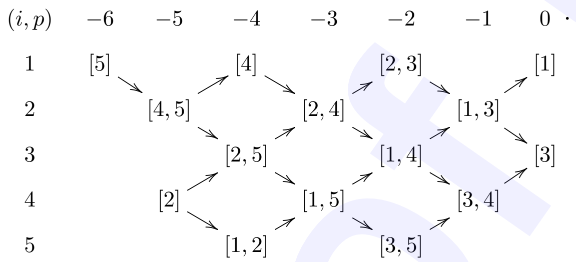
For a given Dynkin quiver Q and a root $\alpha \in \Phi^+$, (i,p) is the *coordinate* of α in Γ_Q and i is the *residue* of α in Γ_Q, when $\pi_Q(\alpha) = (i,p)$.

Proposition 1.12 ([2,21]). *Let $\widetilde{w}_0 = (s_{i_1}, s_{i_2}, \dots, s_{i_l}) \in [Q]$. The correspondence between coordinates of Γ_Q and roots in Φ^+ is given as follows:*

$$(i, \xi(i) + 2m) \leftrightarrow \beta = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_i) \in \Phi^+ \tag{1.5}$$

for $m = \#\{\, t \,|\, i_t = i,\ 1 \le t < k \}$ and $i = i_k$.

Example 1.13. Let $\widetilde{w}_0 = (s_1, s_3, s_2, s_4, s_1, s_3, s_5, s_2, s_4, s_1, s_3, s_5, s_2, s_4, s_1)$ of A_5, which is adapted to the Dynkin quiver $Q = \underset{1}{\circ} \leftarrow \underset{2}{\circ} \rightarrow \underset{3}{\circ} \leftarrow \underset{4}{\circ} \leftarrow \underset{5}{\circ}$. The AR quiver Γ_Q associated to Q is



Here $[a,b] := \sum_{i=a}^{b} \alpha_i \in \Phi^+$.

Definition 1.14. A path $\beta_0 \to \beta_1 \to \cdots \to \beta_s$ in Γ_Q is called a *sectional* path if, for each $0 \le k < l \le s$, $d_\Delta(i_k, i_l) = k - l$. Here i_t $(0 \le t \le s)$ denotes the residue of β_t in Γ_Q. Combinatorially, a path is sectional if the path is *upwards* or *downwards* in Γ_Q

1.3. Properties of AR-quivers

The AR quiver Γ_Q is the Hasse diagram of the convex partial order $\prec_Q$ when Q is a Dynkin quiver Q of type ADE in the following sense:

Theorem 1.15 ([22]). *For a Dynkin quiver Q and $\alpha, \beta \in \Phi^+$, we have $\alpha \prec_Q \beta$ if and only if there is a path from β to α in Γ_Q. Furthermore, there exists an arrow from β to α in Γ_Q if and only if β is a cover of α with respect to $\prec_Q$.*

Also, adapted reduced expressions to Q can be obtained from the AR-quiver Γ_Q by *compatible readings.* Here, a *compatible reading* of the AR quiver Γ_Q is the sequence $s_{i_1}, \dots, s_{i_N}$ (resp. $i_1, \dots, i_N$) of simple reflections (resp. indices) such that whenever there is an arrow from (i_q, n_q) to (i_r, n_r) in Γ_Q, read s_{i_r} before s_{i_q}.

Moreover, we have the following theorem.

Theorem 1.16 ([2]). *Let Q be a Dynkin quiver of finite type A_n, D_n, E_n. Then any reduced expression of $w_0 \in \mathsf{W}$ adapted to the quiver Q can be obtained by a compatible reading of the AR quiver Γ_Q.*

Note that, by Proposition 1.15, a compatible reading of Γ_Q gives a compatible reading of positive roots, in the sense that α is read before β if $\alpha \prec_Q \beta$ for $\alpha, \beta \in \Phi^+$.

2. Combinatorial AR-quivers and convex partial orders

In this section, we shall introduce combinatorial object $\Upsilon_{[\widetilde{w}]}$ which can be understood as the Hasse diagram of $\prec_{[\widetilde{w}]}$ on $\Phi(w)$ for a reduced expression $\widetilde{w}$ of *any* element w in *any* finite Weyl group W. First we suggest an algorithm for the object and then prove that the combinatorial object is distinct in the sense that $\Upsilon_{[\widetilde{w}]} = \Upsilon_{[\widetilde{w}']}$ if and only if $[\widetilde{w}] = [\widetilde{w}']$.

2.1. Combinatorial AR-quivers

Algorithm 2.1. *The quiver $\Upsilon_{\widetilde{w}} = (\Upsilon^0_{\widetilde{w}}, \Upsilon^1_{\widetilde{w}})$ associated to $\widetilde{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_{\ell(w)}})$ is constructed in the following algorithm*:

(Q1) *$\Upsilon^0_{\widetilde{w}}$ consists of $\ell(w)$ vertices labeled by $\beta^{\widetilde{w}}_1, \ldots, \beta^{\widetilde{w}}_{\ell(w)}$.*

(Q2) *There is an arrow from $\beta^{\widetilde{w}}_k$ to $\beta^{\widetilde{w}}_j$ if*

$$\text{(i) } k > j, \quad \text{(ii) } d_\Delta(i_k, i_j) = 1 \quad \textit{and} \quad \text{(iii) } \{\, t \mid j < t < k,\ i_t = i_j \text{ or } i_k \} = \emptyset.$$

(Q3) *Assign the color $m_{jk} = -(\alpha_{i_j}, \alpha_{i_k})$ to each arrow $\beta^{\widetilde{w}}_k \to \beta^{\widetilde{w}}_j$ in* (Q2)*; that is, $\beta^{\widetilde{w}}_k \xrightarrow{m_{jk}} \beta^{\widetilde{w}}_j$. Replace $\xrightarrow{1}$ by $\to$, $\xrightarrow{2}$ by $\Rightarrow$ and $\xrightarrow{3}$ by $\Rrightarrow$.*

We call the quiver $\Upsilon_{\widetilde{w}}$ the *combinatorial AR-quiver associated to $\widetilde{w}$*. Now we can define the notion of sectional paths in $\Upsilon_{\widetilde{w}}$ as in Definition 1.14. In $\Upsilon_{[\widetilde{w}]}$, the *residue* of the vertex labeled by $\beta^{\widetilde{w}}_k$ is i_k.

Remark 2.2.

1. To compute $\beta^{\widetilde{w}}_k$ from the reduced expression $\widetilde{w}$, we need lots of computations in general. So, we significantly deal with this problem separately, in Section 3.
2. The shape of $\Upsilon_{[\widetilde{w}]}$ can be obtained directly, without any computation, from Algorithm 2.1 (see (2.1) in Example 2.4).

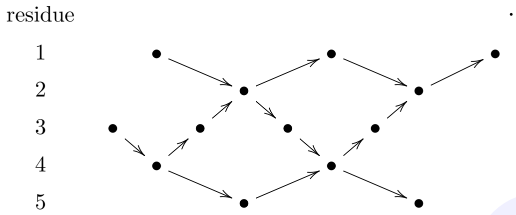
The following proposition follows from the construction of the quiver $\Upsilon_{\widetilde{w}}$:

Proposition 2.3. *If two reduced expressions $\widetilde{w}$ and $\widetilde{w}'$ are commutation equivalent, then $\Upsilon_{\widetilde{w}} = \Upsilon_{\widetilde{w}'}$. Hence we can define the combinatorial AR-quiver on $[\widetilde{w}]$:*

$$\Upsilon_{[\widetilde{w}]} := \Upsilon_{\widetilde{w}'} \quad \textit{for any } \widetilde{w}' \in [\widetilde{w}].$$

Example 2.4. Let $\widetilde{w} = (s_1, s_2, s_3, s_5, s_4, s_3, s_1, s_2, s_3, s_5, s_4, s_3, s_1)$ of A_5. Then one can easily check that $\widetilde{w}$ is *not* adapted to *any* Dynkin quiver Q of type A_5. According to Algorithm 2.1, the shape of $\Upsilon_{[\widetilde{w}]}$ is

(2.1)

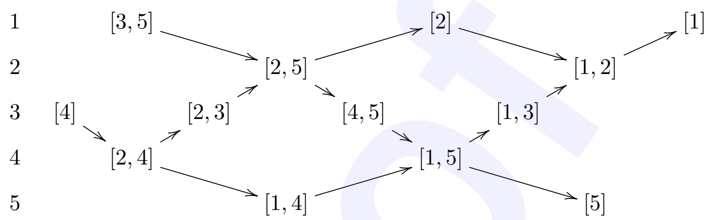


Labels of vertices of the combinatorial AR quiver $\Upsilon_{[\widetilde{w}]}$ are

$$\begin{aligned}&(\beta_k^{\widetilde{w}} \mid 1 \le k \le \ell(w) = 13)\\ &= ([1], [1,2], [1,3], [5], [1,5], [4,5], [2], [2,5], [2,3], [1,4], [2,4], [4], [3,5]).\end{aligned}$$

Hence $\Upsilon_{[\widetilde{w}]}$ is drawn as follows:

(2.2)

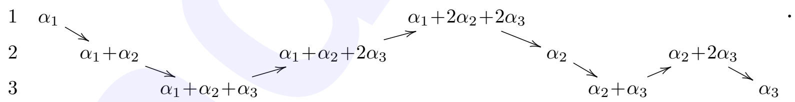


Here $[2,4]$ and $[2]$ are positive roots whose residues are 4 and 1, and lie in the sectional path:

$$[2,4] \to [2,4] \to [2,5] \to [2]$$

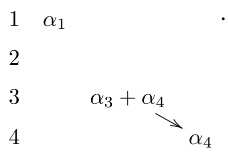
Example 2.5. Let $\widetilde{w}_0 = (s_3, s_2, s_3, s_2, s_1, s_2, s_3, s_2, s_1)$ of B_3. The combinatorial AR quiver of $[\widetilde{w}_0]$ is

$\Upsilon_{[\widetilde{w}_0]} =$

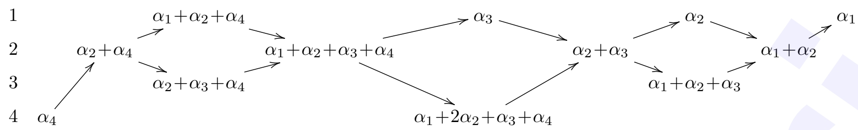


Example 2.6. A combinatorial AR quiver is not necessarily connected. For example, let $\widetilde{w} = (s_4, s_3, s_1)$ of A_4. Then

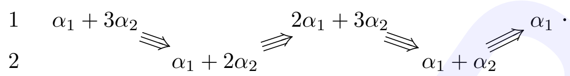
$\Upsilon_{[\widetilde{w}]} =$



Example 2.7. Let $\widetilde{w}_0 = (s_1, s_2, s_3, s_1, s_2, s_4, s_1, s_2, s_3, s_1, s_2, s_4)$ of D_4. Note that $\widetilde{w}_0$ is *not* adapted to any Dynkin quiver of type D_4. We can draw the combinatorial AR quiver $\Upsilon_{[\widetilde{w}_0]}$ as follows:



Example 2.8. Let $\widetilde{w} = (s_1, s_2, s_1, s_2, s_1)$ of G_2. Then

$\Upsilon_{[\widetilde{w}]} =$ 

Remark 2.9. A combinatorial AR quiver is not necessarily connected (see Example 2.6). However, when $\widetilde{w}$ is a reduced expression consisting of simple reflections $\{s_{i_1}, \ldots, s_{i_k}\}$, the quiver $\Upsilon_{[\widetilde{w}]}$ is connected if and only if the full subdiagram of Δ consisting of the set of indices $\{i_1, \ldots, i_k\}$ is connected.

2.2. Combinatorial AR-quivers and convex partial orders

In this subsection, we shall show each combinatorial AR-quiver gives rise to a distinct convex partial order $\prec_{[\widetilde{w}]}$ on $\Phi(w)$. To do this, we aim to show the converse (see Theorem 2.21):

$$\Upsilon_{[\widetilde{w}]} = \Upsilon_{[\widetilde{w}']} \text{ then } [\widetilde{w}] = [\widetilde{w}'] \tag{2.3}$$

of Proposition 2.3, by using the *level functions* (Definition 2.10, 2.12) of $\widetilde{w}$ and of $\Upsilon_{[\widetilde{w}]}$.

Definition 2.10 ([2])**.** Let $\widetilde{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_l})$ be a reduced expression of w. Given $\alpha \in \Phi(w)$, let

$$\beta_1, \beta_2, \ldots, \beta_k = \alpha \tag{2.4}$$

be a sequence of distinct elements of $\Phi(w)$ ending with α such that

$$\beta_{i-1} <_{\widetilde{w}} \beta_i \quad \text{and} \quad (\beta_i, \beta_{i-1}) \neq 0. \tag{2.5}$$

The function $\lambda_{\widetilde{w}} : \Phi(w) \to \mathbb{N}$ associated to the reduced expression $\widetilde{w}$ is defined as follows:

$$\lambda_{\widetilde{w}}(\alpha) = \max\{k \geq 1 \mid \beta_1, \beta_2, \ldots, \beta_k = \alpha \text{ is the sequence in (2.4)}\}. \tag{2.6}$$

We call it *the level function* associated to $\widetilde{w}$.

Proposition 2.11 ([2])**.** *Two reduced expressions $\widetilde{w}$ and $\widetilde{w}'$ of w are in the same commutation class if and only if $\lambda_{\widetilde{w}} = \lambda_{\widetilde{w}'}$.*

Definition 2.12. The level function $\lambda_{\Upsilon_{[\widetilde{w}]}} : \Phi^+(w) \to \mathbb{N}$ of $\Upsilon_{[\widetilde{w}]}$ is defined by

$$\lambda_{\Upsilon_{[\widetilde{w}]}}(\beta) = \text{the length of the longest path in } \Upsilon_{[\widetilde{w}]} \text{ from } \beta.$$

Remark 2.13. By Proposition 2.11 and (2.3), the converse (Theorem 2.21) of Proposition 2.3 can be re-written as

$$\Upsilon_{[\widetilde{w}]} = \Upsilon_{[\widetilde{w}']} \quad \text{then} \quad \lambda_{\widetilde{w}} = \lambda_{\widetilde{w}'}. \tag{2.7}$$

We shall prove (2.7) by showing $\lambda_{\Upsilon_{[\widetilde{w}]}} = \lambda_{\widetilde{w}}$ (Proposition 2.20).

The following lemmas (Lemma 2.14 and Lemma 2.19) will be used in Proposition 2.20. They explain the sequence $\beta_1, \beta_2, \ldots, \beta_k$ for the level function $\lambda_{\widetilde{w}}$ in (2.6), in terms of $\Upsilon_{[\widetilde{w}]}$.

Lemma 2.14. *Let α and β have residues i and j in the combinatorial Auslander-Reiten quiver $\Upsilon_{[\widetilde{w}]}$. If α and β are connected by one arrow, then we have $(\alpha, \beta) = -(\alpha_i, \alpha_j) > 0$.*

Proof. Take a reduced expression $\widetilde{w} = (s_{i_1}, \ldots, s_{i_{\ell(w)}}) \in [\widetilde{w}]$ and denote $\alpha = \beta_k^{\widetilde{w}}$ and $\beta = \beta_l^{\widetilde{w}}$ for $1 \le k < l \le \ell(w)$. Then the arrow is directed from β to α. If $l = k+1$, then our assertion follows from the formula below:

$$(\alpha, \beta) = (s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), s_{i_1} \cdots s_{i_k}(\alpha_{i_l})) = (-\alpha_{i_k}, \alpha_{i_l}).$$

Assume that $l > k+1$ and set $\widetilde{w}_{k \le \cdot \le l} := (s_{i_k}, \ldots, s_{i_l})$. It is enough to show that there exists a reduced expression $\widetilde{w}' \in [\widetilde{w}]$ such that $\beta_{k'}^{\widetilde{w}'} = \alpha$ and $\beta_{k'+1}^{\widetilde{w}'} = \beta$ for some $k' \in \{1, \ldots, \ell(w)-1\}$.

Observe that the following property is followed by the algorithm of combinatorial AR quivers

(i) $\{\, i_t \,|\, k < t < l,\ i_t = i\} = \{\, i_t \,|\, k < t < l,\ i_t = j\} = \emptyset$,
(ii) if $i' \neq i, j$, then $s_{i'} s_i = s_i s_{i'}$ or $s_{i'} s_j = s_j, s_{i'}$.

Hence we can find a reduced expression $\widetilde{w}' = (s_{i'_1}, \ldots, s_{i'_{\ell(w)}}) \in [\widetilde{w}]$ such that $\alpha = \beta_{k'}^{\widetilde{w}'}$ and $\beta = \beta_{k'+1}^{\widetilde{w}'}$ for some $1 \le k' < \ell(w)$. □

Proposition 2.15. *Let α and β have residues $i = i_0$ and $j = i_k$ in $\Upsilon_{[\widetilde{w}]}$. Suppose there is a sectional path in $\Upsilon_{[\widetilde{w}]}$*

$$\beta = \gamma_k \xrightarrow{m_{i_{k-1}, i_k}} \gamma_{k-1} \xrightarrow{m_{i_{k-2}, i_{k-1}}} \cdots \xrightarrow{m_{i_1, i_2}} \gamma_1 \xrightarrow{m_{i_0, i_1}} \gamma_0 = \alpha.$$

Then we have

$$(\alpha, \beta) = \begin{cases} \prod_{t=1}^{k-1} 2^{\delta_{3, i_t}} \prod_{t=0}^{k-1} m_{i_t, i_{t+1}} & \text{for Type } F_4, \\ \prod_{t=0}^{k-1} m_{i_t, i_{t+1}} & \text{otherwise,} \end{cases} \tag{2.8}$$

where i_t is the residue of γ_t and $m_{a,b} := -(\alpha_a, \alpha_b)$ for $a, b \in I$ (Algorithm 2.1). Hence

$$(\alpha, \beta) > 0.$$

Proof. Note that, by induction on k, we can see that

$$s_{i_0} s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) = \alpha_{i_k} + \sum_{p=1}^{k} (-2)^p \frac{\prod_{t=0}^{p-1} (\alpha_{i_{k-t-1}}, \alpha_{i_{k-t}})}{\prod_{t=0}^{p-1} (\alpha_{i_{k-t-1}}, \alpha_{i_{k-t-1}})} \alpha_{i_{k-p}}.$$

There exists $w \in W$ such that $\alpha = w(\alpha_i)$ and $\beta = ws_is_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_j)$. Hence we have

$$\begin{aligned}
&(w(\alpha_i), ws_is_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_j)) \\
&= \left(\alpha_{i_0}, (-2)^{k-1}\frac{\prod_{t=1}^{k-1}(\alpha_{i_t},\alpha_{i_{t+1}})}{\prod_{t=1}^{k-1}(\alpha_{i_t},\alpha_{i_t})}\,\alpha_{i_1} + (-2)^k\frac{\prod_{t=0}^{k-1}(\alpha_{i_t},\alpha_{i_{t+1}})}{\prod_{t=0}^{k-1}(\alpha_{i_t},\alpha_{i_t})}\,\alpha_{i_0}\right) \\
&= -(-2)^{k-1}\frac{\prod_{t=1}^{k-1}(\alpha_{i_t},\alpha_{i_{t+1}})}{\prod_{t=1}^{k-1}(\alpha_{i_t},\alpha_{i_t})}\,(\alpha_{i_0},\alpha_{i_1}) \\
&= \prod_{t=1}^{k-1}\frac{2}{(\alpha_{i_t},\alpha_{i_t})}\prod_{t=0}^{k-1}-(\alpha_{i_t},\alpha_{i_{t+1}})
\end{aligned} \tag{2.9}$$

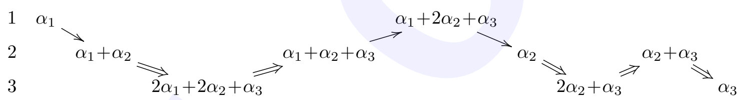
since $(\alpha_{i_0},\alpha_{i_a}) = 0$ for $a \neq 0, 1$. Here we note that only i_0 and i_k can be 1 or n. According to [3], except F_4 case, we can check that $(\alpha_{i_t},\alpha_{i_t}) = 2$ for all $t = 1, 2, \ldots, k-1$. In the case of type F_4, we have $(\alpha_2,\alpha_2) = 2$ and $(\alpha_3,\alpha_3) = 1$. Hence we get the formula (2.8). □

Remark 2.16. For any finite type other than F_4, we have

$$(\alpha,\beta) = \textstyle\prod_{t=0}^{k-1}(\gamma_t,\gamma_{t+1}) = \prod_{t=0}^{k-1} -(\alpha_{i_t},\alpha_{i_{t+1}}) = \prod_{t=0}^{k-1} m_{i_t,i_{t+1}} > 0.$$

Here we use notations in Proposition 2.15.

Example 2.17. Let us consider $\widetilde{w}_0 = (s_3, s_2, s_3, s_2, s_1, s_2, s_3, s_2, s_1)$ of type C_3. Then

$\Upsilon_{[\widetilde{w}_0]} =$ 

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One can check that Proposition 2.15 holds in the above quiver. For instance,

$$\begin{aligned}
2 &= (\alpha_1+2\alpha_2+\alpha_3, 2\alpha_1+2\alpha_2+\alpha_3) \\
&= (\alpha_1+2\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3, 2\alpha_1+2\alpha_2+\alpha_3) \\
&= (\alpha_1,\alpha_2)(\alpha_2,\alpha_3).
\end{aligned}$$

Lemma 2.18. *Let $\alpha, \beta \in \Phi(w)$ and $\widetilde{w}$ be a reduced expression of $w \in \mathsf{W}$. If there is no path between α and β in $\Upsilon_{[\widetilde{w}]}$, then there are two distinct reduced expressions $\widetilde{w}'$ and $\widetilde{w}''$ in $[\widetilde{w}]$ and two integers $k, l \in \mathbb{N}$ such that $\beta_k^{\widetilde{w}'} = \alpha$, $\beta_{k+1}^{\widetilde{w}'} = \beta$ and $\beta_{l+1}^{\widetilde{w}''} = \alpha$, $\beta_l^{\widetilde{w}''} = \beta$.*

Proof. Let $\alpha = \beta_s^{\widetilde{w}}$ and $\beta = \beta_t^{\widetilde{w}}$ have residues i and j, respectively, for $1 \le s < t \le \ell(w)$. Since there is no path from β to α in $\Upsilon_{[\widetilde{w}]}$, if there is a root $\gamma = \beta_{t'}^{\widetilde{w}}$ for $s < t' < t$ with residue i', then $s_{i'}s_i = s_is_{i'}$ or $s_{i'}s_j = s_js_{i'}$. Hence there is a reduced expression $\widetilde{w}' \in [\widetilde{w}]$ such that $\alpha = \beta_k^{\widetilde{w}'}$ and $\beta = \beta_{k+1}^{\widetilde{w}'}$. Also, since we know $s_is_j = s_js_i$, we have $\widetilde{w}'' \in [\widetilde{w}]$ such that $\alpha = \beta_{k+1}^{\widetilde{w}'}$ and $\beta = \beta_k^{\widetilde{w}'}$. □

Lemma 2.19. *Let $\alpha, \beta \in \Phi(w)$ and $\widetilde{w}$ be a reduced expression of $w \in \mathsf{W}$. Suppose there is no path between α and β in $\Upsilon_{[\widetilde{w}]}$. Then we have $(\alpha, \beta) = 0$.*

Proof. Since $<_{\widetilde{w}}$ is a total order, we can assume that $\beta^{\widetilde{w}}_k = \alpha$ and $\beta^{\widetilde{w}}_l = \beta$ for $k < l$ without loss of generality. If $l - k = 1$, then

$$\begin{aligned}(\alpha, \beta) &= (s_{i_1} \dots, s_{i_{k-1}}(\alpha_{i_k}), s_{i_1} \dots, s_{i_{k-1}} s_{i_k}(\alpha_{i_l})) \\ &= (\alpha_{i_k}, s_{i_k}(\alpha_{i_l})) = (\alpha_{i_k}, \alpha_{i_l}) = 0.\end{aligned}$$

Now our assertion follows from Lemma 2.18. □

Proposition 2.20. *Consider a reduced expression $\widetilde{w}$ of $w \in W$ of any finite type. We have*

$$\lambda_{\Upsilon_{[\widetilde{w}]}} = \lambda_{[\widetilde{w}]}.$$

Proof. Suppose $\lambda_{\Upsilon_{[\widetilde{w}]}}(\alpha) = k$ and it is obtained by a path $\alpha = \beta_k \to \beta_{k-1} \to \cdots \to \beta_2 \to \beta_1$ in$\Upsilon_{[\widetilde{w}]}$. Then $\beta_{i-1} \prec_{[\widetilde{w}]} \beta_i$ for $i = 2, \dots, k$ so that $\beta_{i-1} <_{\widetilde{w}} \beta_i$. Also, $(\beta_i, \beta_{i-1}) \neq 0$ by Lemma 2.14. Hence $\lambda_{\widetilde{w}}(\alpha) \geq \lambda_{\Upsilon_{[\widetilde{w}]}}(\alpha) = k$.

On the other hand, suppose $\lambda_{\widetilde{w}}(\alpha) = k$ is obtained by the sequence $\beta_1 <_{\widetilde{w}} \beta_2 <_{\widetilde{w}} \cdots <_{\widetilde{w}} \beta_{k-1} <_{\widetilde{w}} \beta_k = \alpha$ such that $(\beta_{i-1}, \beta_i) \neq 0$ for $i = 2, \dots, k$. Then $\beta_{i-1} \prec_{[\widetilde{w}]} \beta_i$ since otherwise $(\beta_{i-1}, \beta_i) = 0$ by Lemma 2.19. Hence there is a path $\alpha = \beta_k \to \beta_{k-1} \to \cdots \to \beta_2 \to \beta_1$ in$\Upsilon_{[\widetilde{w}]}$ which implies $k = \lambda_{\widetilde{w}}(\alpha) \leq \lambda_{\Upsilon_{[\widetilde{w}]}}(\alpha)$. As a consequence, we have $\lambda_{\Upsilon_{[\widetilde{w}]}} = \lambda_{[\widetilde{w}]}$. □

Theorem 2.21. *Two reduced expressions $\widetilde{w}$ and $\widetilde{w}'$ are in the same commutation class if and only if $\Upsilon_{[\widetilde{w}]} = \Upsilon_{[\widetilde{w}']}$.*

Proof. It is enough to show that if $\Upsilon_{[\widetilde{w}]} = \Upsilon_{[\widetilde{w}']}$, then $[\widetilde{w}] = [\widetilde{w}']$. However, since we know that $\lambda_{[\widetilde{w}]} = \lambda_{\Upsilon_{[\widetilde{w}]}} = \lambda_{\Upsilon_{[\widetilde{w}']}} = \lambda_{[\widetilde{w}']}$ and $\lambda_{[\widetilde{w}]} = \lambda_{[\widetilde{w}']}$ implies $[\widetilde{w}] = [\widetilde{w}']$ by Proposition 2.20, our assertion follows. □

The following theorem shows $\Upsilon_{[\widetilde{w}]}$ can be understood as a generalization of Γ_Q.

Theorem 2.22.

(1) *Every reduced expression of $w \in [\widetilde{w}]$ can be obtained by a compatible reading of $\Upsilon_{[\widetilde{w}]}$.*
(2) *The combinatorial AR quiver $\Upsilon_{[\widetilde{w}]}$ is the Hasse diagram of convex partial order $\preceq_{[\widetilde{w}]}$. That is $\alpha \preceq_{[\widetilde{w}]} \beta$ if and only if there is a path from β to α in $\Upsilon_{[\widetilde{w}]}$.*
(3) *If $\widetilde{w}_0 \in [Q]$, we have $\Upsilon_{[\widetilde{w}_0]} \simeq \Gamma_Q$.*

Proof. (1) In Algorithm 2.1, since the existence of arrow $\beta^{\widetilde{w}}_k \to \beta^{\widetilde{w}}_j$ in $\Upsilon_{[\widetilde{w}]}$ implies $k > j$, any reduced expression $\widetilde{w} \in [\widetilde{w}]$ can be obtained by a compatible reading of $\Upsilon_{[\widetilde{w}]}$.

(2) If there is a path from α to β in $\Upsilon_{[\widetilde{w}]}$, then any compatible reading of $\Upsilon_{[\widetilde{w}]}$ reads β before α. On the other hand, if there is no path from α to β or from β to α, then there are two compatible readings of $\Upsilon_{[\widetilde{w}]}$ such that one

is obtained by reading α before β and the other one is obtained by reading β before α (see Lemma 2.18). Hence $\Upsilon_{[\widetilde{w}]}$ is the Hasse diagram of $\prec_{[\widetilde{w}]}$.

(3) Since Γ_Q is the Hasse diagram of $\prec_Q$ and $\Upsilon_{[\widetilde{w}_0]}$ is the Hasse diagram of $\prec_{[\widetilde{w}_0]}$, if $[Q] = [\widetilde{w}_0]$, then $\Gamma_Q \simeq \Upsilon_{[\widetilde{w}_0]}$. □

Example 2.23. In Example 2.4, we can obtain the following reduced expression in $[\widetilde{w}_0]$ by compatible reading:

$$(s_1, s_2, s_5, s_3, s_4, s_3, s_1, s_2, s_5, s_1, s_3, s_4, s_3).$$

Theorem 2.22(3) shows a combinatorial AR-quiver is a generalization of an AR-quiver. As AR-quivers are used to investigate convex orders associated to adapted reduced expressions, combinatorial AR-quivers can be used to see convex orders associated to non-adapted reduced expressions.

3. Labeling of combinatorial AR quivers

In this section, we discuss finding labels of combinatorial AR quivers. For classical finite types, there is a more efficiency way to find the label of each vertex $\alpha \in \Phi^+$ in Γ_Q than direct computations. Similarly, for the labeling of $\Upsilon_{[\widetilde{w}]}$, there exists analogous way to avoid large amount of computations (see Remark 2.2(1)). We mainly focus on combinatorial AR quivers of type A_n and generalize the argument to other classical finite types.

3.1. Labeling of AR-quivers of type A

Let Γ_Q be an AR quiver of finite type A_n. Recall that we denote by $\pi_Q(\alpha)$ for $\alpha \in \Phi^+$ the coordinate of the vertex in Γ_Q labeled by α.

Lemma 3.1 ([2,8])**.** *We call the vertex k in the Dynkin quiver Q a left intermediate if Q has the subquiver* $\underset{k-1}{\circ} \longrightarrow \underset{k}{\circ} \longrightarrow \underset{k+1}{\circ}$ *and call the vertex k in the Dynkin quiver Q a right intermediate if Q has the subquiver* $\underset{k-1}{\circ} \longleftarrow \underset{k}{\circ} \longleftarrow \underset{k+1}{\circ}$. *Then we have the following properties.*

(1) *For a simple root α_k, we have*

$$\pi_Q(\alpha_k) = \begin{cases} (k, \xi_k), & \text{if } k \text{ is a sink in } Q, \\ (n+1-k, \xi_k - n + 1), & \text{if } k \text{ is a source in } Q, \\ (1, \xi_k - k + 1), & \text{if } k \text{ is a right intermediate}, \\ (n, \xi_k - n + k), & \text{if } k \text{ is a left intermediate}. \end{cases} \tag{3.1}$$

(2) *If $\beta \to \alpha$ is an arrow in Γ_Q for $\alpha, \beta \in \Phi^+$, then $(\beta, \alpha) = 1$.*

Here ξ is the height function such that $\max\{\, \xi_k \mid k = 1, \ldots, n \,\} = 0$.

After all, the following theorem shows how to find labels of vertices in Γ_Q in an efficient way. In order to introduce the method, we distinguish types of sectional paths in AR quivers.

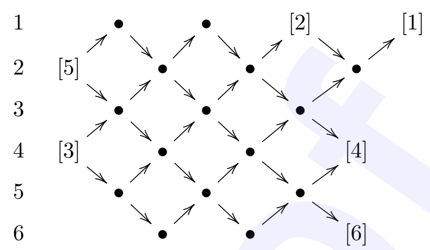
Definition 3.2 (cf. [19, Definition 3.3])**.** In an AR quiver Γ_Q, a sectional path is called *N-sectional* if the path is upwards. On the other hand, if a sectional path is downwards, it is said to be an *S-sectional* path.

Theorem 3.3 ([18])**.** *For a positive root $\alpha = \sum_{j=k_1}^{k_2} \alpha_j$ of type A_n, let us call α_{k_1} the left end and α_{k_2} the right end of α.*

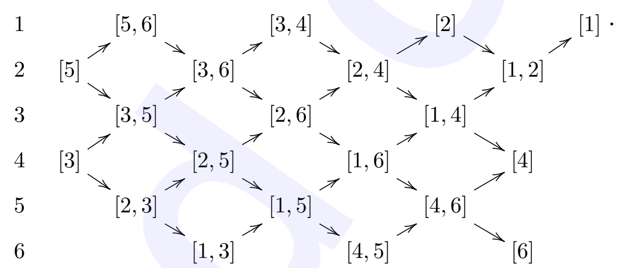
(a) *Every vertex in an N-sectional path in Γ_Q shares its left end.*
(b) *Every vertex in an S-sectional path in Γ_Q shares its right end.*

Now we know how to draw the AR quiver Γ_Q associated to the Dynkin quiver Q of A_n purely combinatorially. We summarize the procedure with the example below.

Example 3.4. For $Q = \underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ} \longleftarrow \underset{4}{\circ} \longrightarrow \underset{5}{\circ} \longleftarrow \underset{6}{\circ}$ of type A_6, Lemma 3.1 tells that Γ_Q can be drawn with partial labels:



Finally, using Theorem 3.3, we can *complete* whole labels of Γ_Q



3.2. Labeling of combinatorial AR-quivers

Now, we generalize the above arguments in Γ_Q. In order to find analogous results for $\Upsilon_{[\widetilde{w}]}$ of any classical finite type, we introduce the notion of *component*:

Definition 3.5. Let $\alpha = \sum_{i \in J} c_i \epsilon_i$ and $\beta = \sum_{i \in J} d_i \epsilon_i$. (Note that J need not to be the same as I.)

(1) If $i \in I$ satisfies $c_i \neq 0$, then ϵ_i is called a component of α.
(2) If $i \in I$ satisfies $c_i > 0$ (resp. $c_i < 0$), then ϵ_i is called a *positive component* (resp. *negative component*) of α.

(3) We say α and β share a component if there is $i \in I$ such that ϵ_i is a positive component to both α and β or a negative component to both α and β.

Remark 3.6. In A_n type, we have $[i,j] = \epsilon_i - \epsilon_{j+1}$. Hence Theorem 3.3 can be restated as follows: An N-sectional (resp. S-sectional) path in Γ_Q shares a positive (resp. negative) component. In short, each sectional path in Γ_Q shares a component.

For type A_n, recall that the action s_i on Φ^+ can be described as follows:

$$[j,k] \mapsto \begin{cases} [j,k-1] & \text{if } j<k=i, \\ [j+1,k] & \text{if } j=i<k, \\ [j,k+1] & \text{if } j<k=i-1, \\ [j-1,k] & \text{if } j=i+1<k, \\ -[i] & \text{if } i=j=k, \\ [j,k] & \text{otherwise.} \end{cases} \tag{3.2}$$

Then the following lemma is an easy consequence induced from the action of simple reflection on Φ^+.

Lemma 3.7. *Let s_t be a simple reflection on W of type A_n and $[i,j]{:=}\sum_{k=i}^{j}\alpha_k$ for $i,j \in I$.*

(1) *If $s_t[i,k]$, $s_t[j,k] \in \Phi^+$, then $s_t[i,k] = [i',k']$ and $s_t[j,k] = [j',k']$ for some $i',j' \le k' \in \{1,2,\dots,n\}$.*
(2) *If $s_t[i,j]$, $s_t[i,k] \in \Phi^+$, then $s_t[i,j] = [i',j']$ and $s_t = [i',k']$ for some $i' \le j',k' \in \{1,2,\dots,n\}$.*

Proposition 3.8. *Let $\widetilde{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_N})$ be a reduced expression of $w \in \mathsf{W}$ of type A_n and $\Upsilon_{[\widetilde{w}]}$ be the combinatorial AR quiver.*

(a) *If there is an arrow from $\beta^{\widetilde{w}}_{k_1}$ of the residue l to $\beta^{\widetilde{w}}_{k_2}$ of the residue $(l-1)$, then the corresponding positive roots $[i_1,j_1]$ and $[i_2,j_2]$ to $\beta^{\widetilde{w}}_{k_1}$ and $\beta^{\widetilde{w}}_{k_2}$ satisfy $i_1 = i_2$.*
(b) *If there is an arrow from $\beta^{\widetilde{w}}_{k_1}$ of the residue l to $\beta^{\widetilde{w}}_{k_2}$ in the residue $(l+1)$, then the corresponding positive roots $[i_1,j_1]$ and $[i_2,j_2]$ to $\beta^{\widetilde{w}}_{k_1}$ and $\beta^{\widetilde{w}}_{k_2}$ satisfy $j_1 = j_2$.*

Proof. (a) The arrow from $\beta^{\widetilde{w}}_{k_1}$ of the residue l to $\beta^{\widetilde{w}}_{k_2}$ of the residue $(l-1)$ implies that $k_1 > k_2$ and

(3.3) the vertices $\{\beta^{\widetilde{w}}_{k} \mid k{=}k_2{+}1,\dots,k_1{-}1\}$ in $\Upsilon_{[\widetilde{w}]}$ are not of the residue l or $(l-1)$.

Denote $\widetilde{w}_{\le k_2-1} = s_{i_1}s_{i_2}\cdots s_{k_2-1}$. Then $[i_1,j_1] = \widetilde{w}_{\le k_2-1}s_{i_{k_2}}s_{i_{k_2}+1}\cdots s_{i_{k_1-1}}$ $(\alpha_{i_{k_1}} = [l])$ and $[i_2,j_2] = \widetilde{w}_{\le k_2-1}$ $(\alpha_{i_{k_2}} = [l-1])$. Using (3.2) and (3.3), we have

$$s_{i_{k_2}}s_{i_{k_2}+1}\cdots s_{i_{k_1-1}}(\alpha_{i_{k_1}}) = [l-1,j]$$

for some $j \geq l$. Then the first assertion follows from Lemma 3.7.

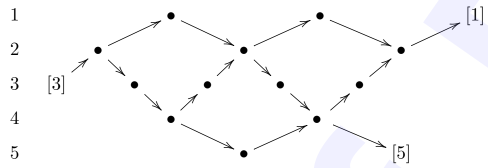
(b) The same argument as that in the proof of (a) works. □

Theorem 3.9. *For any $\Upsilon_{[\widetilde{w}]}$ of type A, if two roots α and β are in an N-sectional (resp. S-sectional) path, then α and β share their positive (resp. negative) components.*

Using Theorem 3.9, we can find labels of combinatorial AR-quivers avoiding large amount of computations.

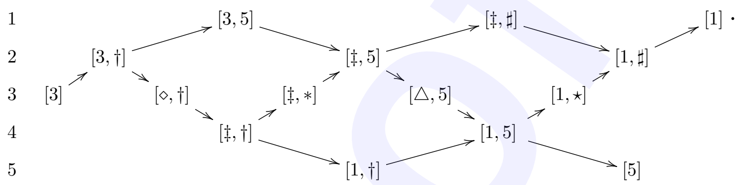
Example 3.10. Let $\widetilde{w}_0 = (s_1, s_2, s_1, s_3, s_5, s_4, s_3, s_2, s_3, s_5, s_4, s_1, s_3, s_2, s_3)$ of A_5. We can easily find that labels of sinks and sources of the quiver $\Upsilon_{[\widetilde{w}_0]}$ are [1], [5] and [3].

(3.4)



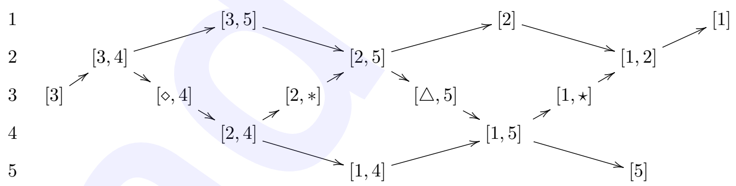
By Proposition 3.8, we can see the labels $\Upsilon_{[\widetilde{w}_0]}$ has the form of

(3.5)



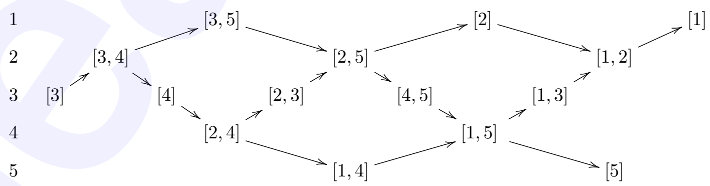
Since (i) there are four different roots with the positive (resp. negative) component $\epsilon_\ddagger$ (resp. $\epsilon_{\dagger+1}$) (ii) $\ddagger \neq 1$ (resp. $\dagger \neq 5$), we have $\ddagger = 2$ (resp. $\dagger = 4$). On the other hand, since $s_1(\alpha_2) = [1, 2]$, $\sharp = 2$.

(3.6)



Now, since $\Phi(w_0) = \Phi^+$, one can see that $\diamond = 4$, $* = 3$, $\triangle = 4$ and $\star = 3$. Hence we complete finding labels of $\Upsilon_{[\widetilde{w}_0]}$.

(3.7)



By applying similar arguments of Lemma 3.7 and Proposition 3.8, we have the following theorem for classical finite types ABCD:

Theorem 3.11. *For any $\Upsilon_{[\widetilde{w}]}$ of classical finite types, a sectional path shares a component; that is, if two roots α and β are in a sectional path, then α and β share one component.*

We can observe the following remark without consideration of types:

Remark 3.12. For α and β in a sectional path in $\Upsilon_{[\widetilde{w}]}$ of any finite type, there exists no set of vertices $\{\gamma_i \mid 1 \le i \le r\} \subset \Phi^+$ in the same sectional path such that

$$\sum_{i=1}^{r} \gamma_i = \alpha + \beta \quad \text{and} \quad \gamma_i \neq \alpha, \beta \quad \text{for all } 1 \le i \le r.$$

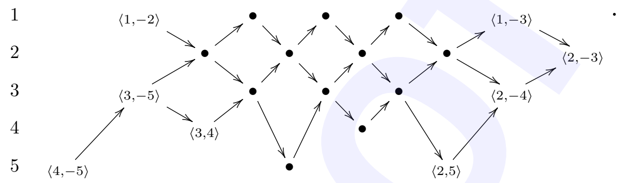
Example 3.13. Recall that the set of positive roots can be expressed as

$$\{\, \epsilon_i \pm \epsilon_j \,|\, 1 \le i < j \le n \,\}.$$

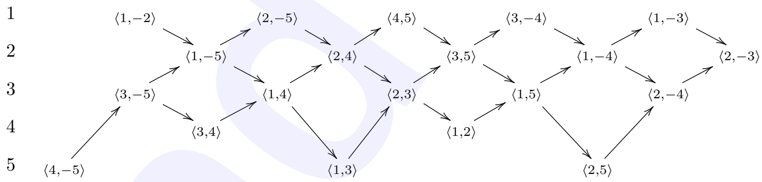
For type D_5, consider the reduced expression

$$\widetilde{w}_0 = (s_2, s_1, s_3, s_2, s_1, s_5, s_3, s_2, s_1, s_4, s_3, s_2, s_1, s_5, s_3, s_2, s_1, s_4, s_3, s_5).$$

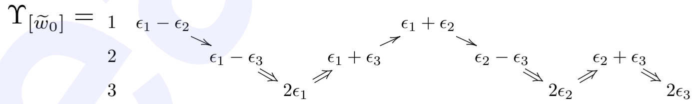
The combinatorial AR quiver $\Upsilon_{[\widetilde{w}_0]}$ has the form of



Here $\epsilon_i \pm \epsilon_j$ is denoted by $\langle i, \pm j \rangle$. Note that the labels filled in the previous quiver are not hard to find by direct computations. Now, by Theorem 3.11, we can complete to find all labels in $\Upsilon_{[\widetilde{w}_0]}$.



Example 3.14. In Example 2.17, $\Upsilon_{[\widetilde{w}_0]}$ of type C_3 can be also labeled in terms of orthonormal basis:



which implies Theorem 3.11. Note that, for any reduced expression of w_0 of type C_n, every positive root of the form $2\epsilon_i$ has residue n and any positive root has residue n is of the form $2\epsilon_i$.

4. Combinatorial reflection functors and r-cluster points

4.1. Reflection maps on $\Upsilon_{[\widetilde{w}_0]}$

The following theorem is a well-known fact about sinks and sources of a Dynkin quiver Q and an AR quiver Γ_Q.

Theorem 4.1. *Let Q be a Dynkin quiver of type A_n, D_n, or E_n and Γ_Q be the associated AR quiver. The followings are equivalent.*

(a) *$i \in I$ is a sink (resp. source) of Q.*
(b) *There are reduced expressions $\widetilde{w}_0$ adapted to Q such that $\widetilde{w}_0$ starts (resp. ends) with s_i (resp. s_{i^*}).*
(c) *α_i is a sink (resp. source) of Γ_Q.*

Let Δ be a Dynkin diagram of simply laced type. On the set of AR quivers $\Gamma_\Delta = \{\, \Gamma_Q \mid Q \text{ is a Dynkin quiver of } \Delta\}$, for $i \in I$, define *right* (*resp. left*) *reflection functor*

$$r_i : \Gamma_\Delta \to \Gamma_\Delta$$

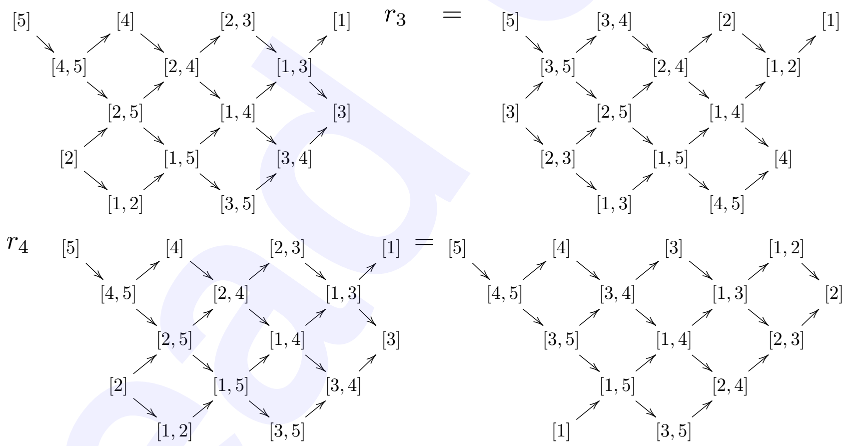
by $\Gamma_Q \mapsto \Gamma_Q r_i$ (resp. $\Gamma_Q \mapsto \Gamma_Q r_i$), where

$$\Gamma_Q r_i = \begin{cases} \Gamma_{s_i(Q)} & \text{if } i \text{ is a sink in } Q, \\ \Gamma_Q & \text{otherwise,} \end{cases} \quad \text{and}$$

(4.1)

$$r_i \Gamma_Q = \begin{cases} \Gamma_{s_{i^*}(Q)} & \text{if } i^* \text{ is a source in } Q, \\ \Gamma_Q & \text{otherwise.} \end{cases}$$

Example 4.2. Let $\widetilde{w}_0 = (s_3, s_1, s_2, s_4, s_1, s_3, s_5, s_2, s_4, s_1, s_3, s_5, s_2, s_1, s_4) \in [Q]$ of A_5. Note that $\widetilde{w}_0$ is adapted. Then α_3 is a sink of Γ_Q and α_2 is a source of Γ_Q.



Let i be a sink (resp. source) in Q. The right (resp. left) reflection functor r_i on Γ_Δ can be described as follows:

(4.2)(i) Delete the sink (resp. source) α_i (resp. α_{i^*}) in Γ_Q.

(ii) Put a new vertex α_i (resp. α_{i^*}) with residue i^* at the beginning (resp. end) of Γ_Q and arrows starting from α_i (resp. ending at α_{i^*}) and ending at the first vertices (resp. starting from the last vertices) with residues j such that $d_\Delta(i^*, j) = 1$.
(iii) Change each label β in $\Phi^+ \setminus \{\alpha_i\}$ (resp. $\Phi^+ \setminus \{\alpha_{i^*}\}$) with $s_i\beta$ (resp. $s_{i^*}\beta$).

Analogously, we can define reflection functors on combinatorial AR quivers. In order to do this, we need notions of source and sink of commutation classes $[\widetilde{w}]$ of W.

Definition 4.3. For a commutation equivalence class $[\widetilde{w}]$, we say that $i \in I$ is a *sink* (resp. *source*) if there is a reduced expression $\widetilde{w}' \in [\widetilde{w}]$ of w starting with s_i (resp. ending with s_i).

The following proposition follows from the construction of the combinatorial AR quiver $\Upsilon_{[\widetilde{w}]}$ and (1.2):

Proposition 4.4.
(a) *i is a sink of $[\widetilde{w}]$ if and only if α_i is a sink in the quiver $\Upsilon_{[\widetilde{w}]}$.*
(b) *i is a source of $[\widetilde{w}]$ if and only if $-w(\alpha_i)$ is a source in the quiver $\Upsilon_{[\widetilde{w}]}$.*

Using sources and sinks of a commutation equivalence class, we shall define a reflection functor on the set of combinatorial AR quivers

$$\Upsilon_{w_0} := \{\, \Upsilon_{[\widetilde{w}_0]} \,|\, \widetilde{w}_0 \text{ is a reduced expression of } w_0\}$$

and divide the set Υ_{w_0} into the orbits $\Upsilon_{[\![\widetilde{w}_0]\!]}$ of reflection functors (see also Definition 4.10 below):

$$\Upsilon_{w_0} = \bigsqcup_{[\![\widetilde{w}_0]\!]} \Upsilon_{[\![\widetilde{w}_0]\!]}$$

Definition 4.5. The right reflection functor r_i on $[\widetilde{w}_0]$ is defined by

$$[\widetilde{w}_0]\, r_i = \begin{cases} [(s_{i_2}, \dots, s_{i_N}, s_{i^*})] & \text{if } i \text{ is a sink and } \widetilde{w}_0' = (s_i, s_{i_2}, \dots, s_{i_N}) \in [\widetilde{w}_0], \\ [\widetilde{w}_0] & \text{if } i \text{ is not a sink of } [\widetilde{w}_0]. \end{cases}$$

On the other hand, the left reflection functor r_i on $[\widetilde{w}_0]$ is defined by

$$r_i\, [\widetilde{w}_0] = \begin{cases} [(s_{i^*}, s_{i_1} \dots, s_{i_{N-1}})] & \text{if } i \text{ is a source and } \widetilde{w}_0' = (s_{i_1}, \dots, s_{i_{N-1}}, s_i) \in [\widetilde{w}_0], \\ [\widetilde{w}_0] & \text{if } i \text{ is not a source of } [\widetilde{w}_0]. \end{cases}$$

The following propositions show that a reflection functor is well-defined on

$$\{[\widetilde{w}_0] \,|\, \widetilde{w}_0 \text{ is a reduced expression of } w_0\}.$$

Proposition 4.6. *Let $\widetilde{w}_0 = (s_{i_1}, \dots, s_{i_{N-1}}, s_{i_N})$ be a reduced expression of w_0.*
(a) *$\widetilde{w}_0' = (s_{i_N^*}, s_{i_1}, \dots, s_{i_{N-1}})$ is a reduced expression of w_0 which is not in $[\widetilde{w}_0]$.*
(b) *$\widetilde{w}_0'' = (s_{i_2}, \dots, s_{i_{N-1}}, s_{i_N}, s_{i_1^*})$ is a reduced expression of w_0 which is not in $[\widetilde{w}_0]$.*

Proof. Remark that $w_0(s_i(\alpha_j)) = -s_{i^*}(\alpha_{j^*})$ for any $i, j \in I$.

(a) We have $s_{i_N^*} w_0 s_{i_N}(\alpha_j) = s_{i_N^*}(-s_{i_N^*}(\alpha_{j^*})) = -\alpha_{j^*}$. Since $s_{i_1}s_{i_2}\cdots s_{i_N} = w_0$, $s_{i_N^*}s_{i_1}s_{i_2}\cdots s_{i_{N-1}} = w_0$. Hence $\widetilde{w}_0'$ is also a reduced expression of w_0. Also, since i_N a source in $\Upsilon_{[\widetilde{w}_0]}$ but is not in $\Upsilon_{\widetilde{w}_0'}$, $[\widetilde{w}_0] \neq [\widetilde{w}_0']$.

(b) By the same argument as (a), we can prove (b). □

Remark 4.7. To the experts, the fact that $\widetilde{w}_0'$ and $\widetilde{w}_0''$ are also reduced expressions of w_0 may be well known (for example, [5, page 7] and [9, page 650]). However, we have had a difficulty finding its proof. Thus we provide a proof by using the system of positive roots.

Proposition 4.8. *Let $\widetilde{w}_0 = (s_{i_1}, \dots, s_{i_N})$ and $\widetilde{w}_0' = (s_{i_1'}, \cdots s_{i_N'})$ be reduced expressions in $[\widetilde{w}_0]$.*

(a) *If $i_1 = i_1'$, then $\widetilde{w}_0^1 = (s_{i_2}, \dots, s_{i_N}, s_{i_1^*})$ and $\widetilde{w}_0^2 = (s_{i_2'}, \dots, s_{i_N'}, s_{i_1^*})$ are in the same commutation equivalence class.*

(b) *If $i_N = i_N'$, then $\widetilde{w}_0^3 = (s_{i_N^*}, s_{i_1}, \dots, s_{i_{N-1}})$ and $\widetilde{w}_0^4 = (s_{i_N^*}, s_{i_1'}, \dots, s_{i_{N-1}'})$ are in the same commutation equivalence class.*

Proof. Since we have $\Upsilon_{[\widetilde{w}_0^1]} = \Upsilon_{[\widetilde{w}_0^2]}$ and $\Upsilon_{[\widetilde{w}_0^3]} = \Upsilon_{[\widetilde{w}_0^4]}$ by (4.2), our assertion follows. □

The reflecting functor on $[\widetilde{w}_0]$ induces the right (resp. left) *reflection functor* r_i for $i \in I$ on Υ_{w_0} as follows:

(4.3) $$\Upsilon_{[\widetilde{w}_0]}\, r_i = \Upsilon_{[\widetilde{w}_0]\, r_i} \qquad (\text{resp. } r_i\, \Upsilon_{[\widetilde{w}_0]} = \Upsilon_{r_i [\widetilde{w}_0]}).$$

Then the right (resp. left) reflection functor on $\Upsilon_{[\widetilde{w}_0]}$ can be described as an analogue of (4.2):

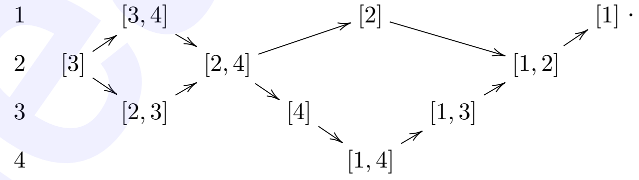
(4.4)(i) Delete the sink (resp. source) α_i (resp. α_{i^*}) with residue i and arrows incident with α_i (resp. α_{i^*}) in $\Upsilon_{[\widetilde{w}_0]}$.

(ii) Put a new vertex α_i (resp. α_{i^*}) in the end (resp. beginning) of $\Upsilon_{[\widetilde{w}_0]}$ and arrows the conditions in Algorithm 2.1.

(iii) Change each label β in $\Phi^+ \setminus \{\alpha_i\}$ (resp. $\Phi^+ \setminus \{\alpha_{i^*}\}$) with $s_i\beta$ (resp. $s_{i^*}\beta$).

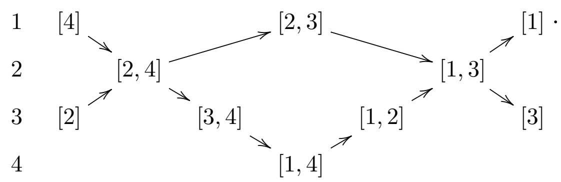
Example 4.9. Let us consider reduced expression $\widetilde{w}_0 = (s_1, s_2, s_1, s_3, s_4, s_3, s_2, s_3, s_1, s_2)$ of A_4 which is not adapted to any Dynkin quiver Q. Then we have

(4.5)



Since 2 is a source of $[\widetilde{w}_0]$, we have $r_2[\widetilde{w}_0] = (s_3, s_1, s_2, s_1, s_3, s_4, s_3, s_2, s_3, s_1)$ and $r_2\Upsilon_{[\widetilde{w}_0]}$ is

(4.6)



Definition 4.10.

(1) Let $[\widetilde{w}_0]$ and $[\widetilde{w}'_0]$ be two commutation equivalence classes. We say $[\widetilde{w}_0]$ and $[\widetilde{w}'_0]$ are in the same *reflection equivalence class* and write $[\widetilde{w}_0] \overset{r}{\sim} [\widetilde{w}'_0]$ if $[\widetilde{w}'_0]$ can be obtained from $[\widetilde{w}_0]$ by a sequence of reflection functors. The family of commutation equivalence classes $[\![\widetilde{w}_0]\!] := \{ [\widetilde{w}_0] \mid [\widetilde{w}_0] \overset{r}{\sim} [\widetilde{w}'_0] \}$ is called an *r-cluster point*.

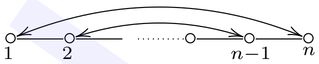
(2) If $[\widetilde{w}_0] \overset{r}{\sim} [\widetilde{w}'_0]$, then we say $\Upsilon_{[\widetilde{w}_0]}$ and $\Upsilon_{[\widetilde{w}'_0]}$ are equivalent via *reflection functors* and write $\Upsilon_{[\widetilde{w}_0]} \overset{r}{\sim} \Upsilon_{[\widetilde{w}'_0]}$. Also, $\Upsilon_{[\![\widetilde{w}_0]\!]} := \{ \Upsilon_{[\widetilde{w}_0]} \mid [\widetilde{w}_0] \overset{r}{\sim} [\widetilde{w}'_0] \}$ is called an *r-cluster point*.

4.2. σ-composition

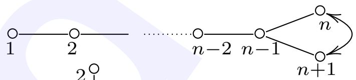
The number of commutation classes for w_0 of a finite simply laced type increases drastically as n increases (see [16, A006245]). Also, in the last subsection, for example (4.4), we showed classes in the same r-cluster point are closely related to each other. Hence, in this section, we introduce a composition shared by classes in the same r-cluster point.

Recall that, for a Dynkin diagram Δ of finite simply-laced type, there exist non-trivial automorphisms σ as follows:

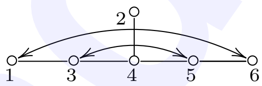
(4.7a) A_n :



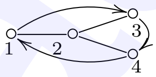
(4.7b) D_{n+1} :



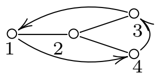
(4.7c) E_6 :



(4.7d) D_4 :



or



Definition 4.11. Let σ be one of Dynkin diagram automorphisms in (4.7a), (4.7b), (4.7c), (4.7d) and k be the number of σ-orbits of the index set I. Take a sequence of σ-orbits $\mathcal{O} = (o_1, o_2, \dots, o_k)$ where $o_i \neq o_j$ for $1 \le i < j \le k$. For a reduced expression $\widetilde{w}_0 = (s_{i_1}, \dots, s_{i_N})$ of w_0, the *σ-composition* of $[\widetilde{w}_0]$ associated to $\mathcal{O}$ is

$$(\mathsf{c}_1, \mathsf{c}_2, \dots, \mathsf{c}_k) \in \mathbb{Z}^k_{\ge 1} \quad \text{where} \quad \mathsf{c}_j = |\{ s_{i_t} \mid i_t \in o_j \quad \text{for some } k \in \mathbb{Z}\}|.$$

The well definedness of σ-composition follows by the fact that if $\widetilde{w}_0 = (s_{i_1}, \ldots, s_{i_N})$ and $\widetilde{w}'_0 = (s_{i'_1}, \ldots, s_{i'_N})$ are in the same commutation class, then

$$\#\{i_k \mid i_k \in o_i\} = \#\{i'_k \mid i'_k \in o_i\} \text{ for any orbit } o_i.$$

Example 4.12. (1) Let us take a Dynkin diagram involution σ of A_4 in (4.7a). Then σ-composition of $[\widetilde{w}_0]$ in Example (4.5) is

$$(4, 6)$$

since there are 4 of s_i's for $i = 1$ or 4 in $\widetilde{w}_0$ and 6 of s_j's for $j = 2$ or 3 in $\widetilde{w}_0$.

(2) Let us take a Dynkin diagram involution σ of D_4 in (4.7b). Then σ-composition of $[\widetilde{w}_0]$ in Example 2.7 is

$$(4, 4, 4).$$

(3) Let us take a Dynkin diagram automorphism σ of D_4 in (4.7d). Then σ-composition of $[\widetilde{w}_0]$ for $\widetilde{w}_0 = (s_1, s_2, s_3, s_2, s_1, s_2, s_4, s_2, s_1, s_2, s_3, s_2)$ is

$$(6, 6).$$

Proposition 4.13. *If two commutation equivalence classes $[\widetilde{w}_0]$ and $[\widetilde{w}'_0]$ of w_0 are in the same r-cluster point, then σ-compositions of $[\widetilde{w}_0]$ and $[\widetilde{w}'_0]$ are the same.*

Proof. Let $\widetilde{w}_0 = (s_{i_1}, \ldots, s_{i_N})$. The only thing we need to show is that σ-compositions of $[\widetilde{w}_0]$, $r_{i_N}[\widetilde{w}_0]$ and $[\widetilde{w}_0]r_{i_1}$ are same. If $r_{i_N}[\widetilde{w}_0] = [\widetilde{w}'_0]$, then $(s_{i^*_N}, s_{i_1}, \ldots, s_{i_{N-1}}) \in [\widetilde{w}'_0]$. Hence σ-compositions of $[\widetilde{w}_0]$ and $[\widetilde{w}'_0]$ are same. Similarly, σ-compositions of $[\widetilde{w}_0]r_{i_1}$ and $[\widetilde{w}_0]$ are same. Hence we proved the proposition. □

Example 4.14.
Let $\widetilde{w}_0$ be a reduced expression of w_0 of A_n adapted to

$$Q = \underset{1}{\circ} \leftarrow \underset{2}{\circ} \leftarrow \cdots\cdots \circ \leftarrow \underset{n-1}{\circ} \leftarrow \underset{n}{\circ}.$$

Let $\sigma = {}^*$. Then the σ-composition of $[\widetilde{w}_0]$ consists of $\lceil \frac{n+1}{2} \rceil$ components such that

$$\begin{cases} (n+1, \ldots, n+1) & \text{if } n \text{ is even,} \\ (n+1, \ldots, n+1, \frac{n+1}{2}) & \text{if } n \text{ is odd.} \end{cases} \tag{4.8}$$

It is well known that all the adapted reduced expressions of w_0 are in this r-cluster point and all of equivalent classes in this r-cluster point are adapted to some Dynkin quiver.

5. Application to KLR algebras and PBW bases

In this section, we apply our results in previous sections to the representation theory of KLR algebras which were introduced by Khovanov-Lauda [10] and Rouquier [23], independently.

5.1. KLR algebra

Let I be an index set. A *symmetrizable Cartan datum* D is a quintuple $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ consisting of (a) an integer-valued matrix $\mathsf{A} = (a_{ij})_{i,j\in I}$, called *the symmetrizable generalized Cartan matrix*, (b) a free abelian group P, called the *weight lattice*, (c) $\Pi = \{\alpha_i \in \mathsf{P} \mid \ i \in I\}$, called the set of *simple roots*, (d) $\mathsf{P}^\vee := \mathrm{Hom}(\mathsf{P}, \mathbb{Z})$, called the *coweight lattice*, (e) $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$, called the set of *simple coroots*, satisfying $\langle h_i, \alpha_j\rangle = a_{ij}$ for all $i, j \in I$ and Π is linearly independent. The free abelian group $\mathsf{Q} := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ is called the *root lattice* and set $\mathsf{Q}^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$.

Let $\mathbf{k}$ be a commutative ring. Take $i, j \in I$ such that $i \neq j$ and a family of polynomials $(Q_{ij})_{i,j\in I}$ in $\mathbf{k}[u, v]$ which satisfy

$$Q_{ij}(u, v) = \delta(i \neq j) \sum_{\substack{(p,q)\in\mathbb{Z}^2_{\geq 0} \\ d_i\times p+d_j\times q=-d_i\times a_{ij}}} t_{i,j;p,q}u^p v^q \tag{5.1}$$

for $t_{i,j;p,q} \in \mathbf{k}$, $t_{i,j;p,q} = t_{j,i;q,p}$ and $t_{i,j;-a_{ij},0} \in \mathbf{k}^\times$. Thus we have $Q_{i,j}(u, v) = Q_{j,i}(v, u)$.

We denote by $\mathfrak{S}_n = \langle \mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1}\rangle$ the symmetric group on n letters, where $\mathfrak{s}_i := (i, i + 1)$ is the transposition of i and $i + 1$. Then $\mathfrak{S}_n$ acts on I^n by place permutations.

For $n \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathsf{Q}^+$ such that $\mathrm{ht}(\beta) = n$, we set

$$I^\beta = \{\nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.$$

Definition 5.1. For $\beta \in \mathsf{Q}^+$ with $|\beta| = n$, the *Khovanov-Lauda-Rouquier* (*KLR*) *algebra* $R(\beta)$ at β associated with a symmetrizable Cartan datum $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ and a matrix $(Q_{ij})_{i,j\in I}$ is the $\mathbb{Z}$-gradable $\mathbf{k}$-algebra generated by the elements $\{e(\nu)\}_{\nu\in I^\beta}$, $\{x_k\}_{1\leq k\leq n}$, $\{\tau_m\}_{1\leq m\leq n-1}$ satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^\beta} e(\nu) = 1, \quad x_k x_m = x_m x_k, \quad x_k e(\nu) = e(\nu)x_k,$$
$$\tau_m e(\nu) = e(\mathfrak{s}_m(\nu))\tau_m, \quad \tau_k\tau_m = \tau_m\tau_k \ \text{ if } |k - m| > 1,$$
$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$
$$(\tau_k x_m - x_{\mathfrak{s}_k(m)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } m = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } m = k + 1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$
$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \delta_{\nu_k,\nu_{k+2}} \frac{Q_{\nu_k,\nu_{k+1}}(x_k,x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2},x_{k+1})}{x_k - x_{k+2}} e(\nu).$$

For $\beta, \gamma \in \mathsf{Q}^+$ with $\mathrm{ht}(\beta) = m$, $\mathrm{ht}(\gamma) = n$, set

$$e(\beta, \gamma) = \sum_{\substack{\nu\in I^{m+n}, \\ (\nu_1,\ldots,\nu_m)\in I^\beta, \ (\nu_{m+1},\ldots,\nu_{m+n})\in I^\gamma}} e(\nu) \in R(\beta + \gamma).$$

Then $e(\beta,\gamma)$ is an idempotent. Let

$$R(\beta)\otimes R(\gamma)\to e(\beta,\gamma)R(\beta+\gamma)e(\beta,\gamma) \tag{5.2}$$

be the **k**-algebra homomorphism given by

$$e(\mu)\otimes e(\nu)\mapsto e(\mu*\nu)\quad(\mu\in I^{\beta}),$$
$$x_k\otimes 1\mapsto x_ke(\beta,\gamma)\quad(1\le k\le m),\qquad 1\otimes x_k\mapsto x_{m+k}e(\beta,\gamma)\quad(1\le k\le n),$$
$$\tau_k\otimes 1\mapsto \tau_ke(\beta,\gamma)\quad(1\le k< m),\qquad 1\otimes \tau_k\mapsto \tau_{m+k}e(\beta,\gamma)\quad(1\le k< n),$$

where $\mu*\nu$ is the concatenation of μ and ν; i.e., $\mu*\nu=(\mu_1,\ldots,\mu_m,\nu_1,\ldots,\nu_n)$.

For a $R(\beta)$-module M and a $R(\gamma)$-module N, we define the *convolution product* $M\circ N$ by

$$M\circ N := R(\beta+\gamma)e(\beta,\gamma)\otimes_{R(\beta)\otimes R(\gamma)}(M\otimes N).$$

and, for a graded $R(\beta)$-module $M=\bigoplus_{k\in\mathbb{Z}}M_k$, we define $qM=\bigoplus_{k\in\mathbb{Z}}(qM)_k$, where

$$(qM)_k = M_{k-1}\ (k\in\mathbb{Z}).$$

We call q the *grading shift functor* on the category of graded $R(\beta)$-modules.

Let $\mathrm{Rep}(R(\beta))$ be the category consisting of finite dimensional graded $R(\beta)$-modules and $[\mathrm{Rep}(R(\beta))]$ be the Grothendieck group of $\mathrm{Rep}(R(\beta))$. Then $[\mathrm{Rep}(R)] := \bigoplus_{\beta\in\mathsf{Q}^+}[\mathrm{Rep}(R(\beta))]$ has a natural $\mathbb{Z}[q,q^{-1}]$-algebra structure induced by the convolution product $\circ$ and the grading shift functor q. In this paper, we usually ignore grading shifts.

For an $R(\beta)$-module M and an $R(\gamma_k)$-module M_k $(1\le k\le n)$, we denote by

$$M^{\circ 0}:=\mathbf{k},\quad M^{\circ r}=\underbrace{M\circ\cdots\circ M}_{r},\quad \mathop{\circ}_{k=1}^{n}M_k=M_1\circ\cdots\circ M_n.$$

Theorem 5.2 ([10, 23])**.** *For a given symmetrizable Cartan datum* D*, let* $U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})^{\vee}$ *the dual of the integral form of the negative part of the quantum group* $U_q(\mathfrak{g})$ *associated with* D *and* R *be the KLR algebra associated with* D *and* $(Q_{ij}(u,v))_{i,j\in I}$*. Then we have*

$$U^-_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})^{\vee}\simeq[\mathrm{Rep}(R)]. \tag{5.3}$$

From now on, we shall deal with the representation theory of KLR algebras which are associated to the Cartan matrix A of finite types.

Convention 5.3. For a reduced expression $\widetilde{w}$ of $w\in\mathsf{W}$, we fix a labeling of $\Phi(w)$ as $\{\beta^{\widetilde{w}}_k \mid 1\le k\le \ell(w)\}$.

(i) We identify a sequence $\underline{m}_{\widetilde{w}}=(m_1,m_2,\ldots,m_{\ell(w)})\in\mathbb{Z}^{\ell(w)}_{\ge 0}$ with
$$(m_1\beta^{\widetilde{w}}_1,m_2\beta^{\widetilde{w}}_2,\ldots,m_{\ell(w)}\beta^{\widetilde{w}}_{\ell(w)})\in(\mathsf{Q}^+)^{\ell(w)}.$$

(ii) For a sequence $\underline{m}_{\widetilde{w}}$ and another reduced expression $\widetilde{w}'$ of w, $\underline{m}_{\widetilde{w}'}$ is a sequence in $\mathbb{Z}^{\ell(w)}_{\ge 0}$ by considering $\underline{m}_{\widetilde{w}}$ as a sequence of positive roots, rearranging with respect to $<_{\widetilde{w}'}$ and applying the convention (i).

(iii) For a sequence $\underline{m}_{\widetilde{w}} \in \mathbb{Z}_{\geq 0}^{\ell(w)}$, a weight $\mathrm{wt}(\underline{m}_{\widetilde{w}})$ of $\underline{m}_{\widetilde{w}}$ is defined by $\sum_{i=1}^{\ell(w)} m_i \beta_i^{\widetilde{w}} \in \mathsf{Q}^+$.

We usually drop the script $\widetilde{w}$ if there is no fear of confusion.

Definition 5.4 ([14,19])**.** For sequences $\underline{m}$, $\underline{m}' \in \mathbb{Z}_{\geq 0}^{\ell(w)}$, we define an order $\leq_{\widetilde{w}}^{\mathtt{b}}$ as follows:
$\underline{m}' = (m'_1, \dots, m'_{\ell(w)}) <_{\widetilde{w}}^{\mathtt{b}} \underline{m} = (m_1, \dots, m_{\ell(w)})$ if and only if $\mathrm{wt}(\underline{m}) = \mathrm{wt}(\underline{m}')$ and there exist integers k, s such that $1 \leq k \leq s \leq \ell(w)$ satisfying

$$m'_t = m_t \text{ if } t < k \text{ or } t > s \text{ and } m'_t < m_t \text{ if } t = s, k.$$

The following order on sequences of positive roots was introduced in [19].

Definition 5.5 ([19])**.** For sequences $\underline{m}$, $\underline{m}' \in \mathbb{Z}_{\geq 0}^{\ell(w)}$, we define an order $\prec_{[\widetilde{w}]}^{\mathtt{b}}$ as follows:

$$\begin{aligned} &\underline{m}' = (m'_1, \dots, m'_{\ell(w)}) \prec_{[\widetilde{w}]}^{\mathtt{b}} \underline{m} = (m_1, \dots, m_{\ell(w)}) \text{ if and only if } \\ &\underline{m}'_{\widetilde{w}'} <_{\widetilde{w}'}^{\mathtt{b}} \underline{m}_{\widetilde{w}'} \text{ for all reduced expression } \widetilde{w}' \in [\widetilde{w}]. \end{aligned} \tag{5.4}$$

Note that $\prec_{[\widetilde{w}]}^{\mathtt{b}}$ is *far coarser* than $<_{\widetilde{w}}^{\mathtt{b}}$.

Definition 5.6. A pair $\underline{m} = (\alpha, \beta) \in (\Phi(w))^2$ is called a *minimal pair of* $\gamma \in \Phi(w)$ *with respect to the convex total order* $\prec_{[\widetilde{w}]}^{\mathtt{b}}$ if $\underline{m}$ is a *cover* of γ. A pair of positive roots is $[\widetilde{w}]$-*simple* if it is minimal with respect to the partial order $\prec_{[\widetilde{w}]}^{\mathtt{b}}$ (see [14, §2.1] and [19]).

Theorem 5.7 ([4,14])**.** *Let R be the KLR algebra corresponding to a Cartan matrix A of finite type. For each positive root $\beta \in \Phi^+$, there exists a simple module $S_{\widetilde{w}_0}(\beta)$ satisfying the following properties*:

(a) *$S_{\widetilde{w}_0}(\beta)^{\circ m}$ is a simple $R(m\beta)$-module.*

(b) *Let $l := \ell(w_0)$ and $\underline{m}_{\widetilde{w}_0} \in \mathbb{Z}_{\geq 0}^{l}$. There exists a non-zero R-module homomorphism*

$$\begin{aligned} \mathbf{r}_{\underline{m}} \colon \overrightarrow{S}_{\widetilde{w}_0}(\underline{m}) &:= S_{\widetilde{w}_0}(\beta_1)^{\circ m_1} \circ \cdots \circ S_{\widetilde{w}_0}(\beta_l)^{\circ m_l} \\ \to \overleftarrow{S}_{\widetilde{w}_0}(\underline{m}) &:= S_{\widetilde{w}_0}(\beta_l)^{\circ m_l} \circ \cdots \circ S_{\widetilde{w}_0}(\beta_1)^{\circ m_1} \end{aligned} \tag{5.5}$$

such that

(i) $\mathrm{Hom}_{R(\mathrm{wt}(\underline{m}))}(\overrightarrow{S}_{\widetilde{w}_0}(\underline{m}), \overleftarrow{S}_{\widetilde{w}_0}(\underline{m})) = \mathbf{k} \cdot \mathbf{r}_{\underline{m}}$,

(ii) $\mathrm{Im}(\mathbf{r}_{\underline{m}}) \simeq \mathrm{hd}\left(\overrightarrow{S}_{\widetilde{w}_0}(\underline{m})\right) \simeq \mathrm{soc}\left(\overleftarrow{S}_{\widetilde{w}_0}(\underline{m})\right)$ *is simple.*

(c) *For any $\underline{m}_{\widetilde{w}_0} \in \mathbb{Z}_{\geq 0}^{\ell(w_0)}$, we have*

$$[\overrightarrow{S}_{\widetilde{w}_0}(\underline{m})] \in [\mathrm{Im}(\mathbf{r}_{\underline{m}})] + \sum_{\underline{m}' <_{\widetilde{w}_0}^{\mathtt{b}} \underline{m}} \mathbb{Z}_{\geq 0}[q^{\pm 1}][\mathrm{Im}(\mathbf{r}_{\underline{m}'})]. \tag{5.6}$$

(d) *For any* $\underline{m}_{\widetilde{w}_0} \in \mathbb{Z}_{\ge 0}^{\ell(w_0)}$, $\overrightarrow{S}_{\widetilde{w}_0}(\underline{m})$ *has a unique simple head* $\mathrm{hd}\left(\overrightarrow{S}_{\widetilde{w}_0}(\underline{m})\right)$ *and* $\mathrm{hd}\left(\overrightarrow{S}_{\widetilde{w}_0}(\underline{m})\right) \not\simeq \mathrm{hd}\left(\overrightarrow{S}_{\widetilde{w}_0}(\underline{m}')\right)$ *if* $\underline{m} \neq \underline{m}'$.

(e) *For every simple R-module M, there exists a unique* $\underline{m} \in \mathbb{Z}_{\ge 0}^{\mathsf{N}}$ *such that* $M \simeq \mathrm{Im}(\mathbf{r}_{\underline{m}}) \simeq \mathrm{hd}\big(\overrightarrow{S}_{\widetilde{w}_0}(\underline{m})\big)$.

(f) *For any minimal pair* $(\beta_k^{\widetilde{w}_0}, \beta_l^{\widetilde{w}_0})$ *of* $\beta_j^{\widetilde{w}_0} = \beta_k^{\widetilde{w}_0} + \beta_l^{\widetilde{w}_0}$ *with respect to* $<_{\widetilde{w}_0}$, *there exists an exact sequence*

$$0 \to S_{\widetilde{w}_0}(\beta_j) \to S_{\widetilde{w}_0}(\beta_k) \circ S_{\widetilde{w}_0}(\beta_l) \xrightarrow{\mathbf{r}_{\underline{m}}} S_{\widetilde{w}_0}(\beta_l) \circ S_{\widetilde{w}_0}(\beta_k) \to S_{\widetilde{w}_0}(\beta_j) \to 0,$$

where $\underline{m}_{\widetilde{w}_0} \in \mathbb{Z}_{\ge 0}^{\ell(w_0)}$ *such that* $m_k = m_l = 1$ *and* $m_i = 0$ *for all* $i \neq k, l$.

Note that the set $\mathrm{Irr}(R)$ of isomorphism classes of all simple R-modules forms a natural basis of $[\mathrm{Rep}(R)]$ and does *not* depend on the choice of reduced expression $\widetilde{w}_0$ of w_0.

We also note that Theorem 5.7 implies that

(i) the subset $\overrightarrow{S}_{\widetilde{w}_0}(R) := \left\{ \big[\overrightarrow{S}_{\widetilde{w}_0}(\underline{m})\big] \mid \underline{m}_{\widetilde{w}_0} \in \mathbb{Z}_{\ge 0}^{\ell(w_0)} \right\}$ of isomorphism classes of R-modules forms another basis of $[\mathrm{Rep}(R)]$,

(ii) $<^{\mathtt{b}}_{\widetilde{w}_0}$ can be interpreted as a unitriangular matrix which plays the role of the transition matrix between $\overrightarrow{S}_{\widetilde{w}_0}(R)$ and $\mathrm{Irr}(R)$ for *any* reduced expression $\widetilde{w}_0$ of w_0.

5.2. Applications of combinatorial AR-quivers

In this subsection, we apply the observations in the previous sections to the representation theory of KLR-algebras and PBW-bases.

Now we shall give an alternative proof of the following theorem:

Theorem 5.8 ([19, Theorem 5.13])**.** *For any* $\widetilde{w}_0$ *of* w_0 *and* $\underline{m}_{\widetilde{w}_0} \in \mathbb{Z}_{\ge 0}^{\ell(w_0)}$, *we can define the module* $\overrightarrow{S}_{[\widetilde{w}_0]}(\underline{m})$; *i.e.,*

$$\overrightarrow{S}_{\widetilde{w}_0}(\underline{m}_{\widetilde{w}_0}) \simeq \overrightarrow{S}_{\widetilde{w}'_0}(\underline{m}_{\widetilde{w}'_0}) \quad \text{for all } \widetilde{w}_0, \widetilde{w}'_0 \in [\widetilde{w}_0].$$

Moreover, we can refine the transition matrix between $\overrightarrow{S}_{[\widetilde{w}_0]}(R) := \{\overrightarrow{S}_{[\widetilde{w}_0]}(\underline{m}) \mid \underline{m} \in \mathbb{Z}_{\ge 0}^{\ell(w_0)}\}$ *and* $\mathrm{Irr}(R)$ *by replacing* $<^{\mathtt{b}}_{\widetilde{w}_0}$ *with the far coarser order* $\prec^{\mathtt{b}}_{[\widetilde{w}_0]}$.

Remark 5.9. For any $\widetilde{w}_0, \widetilde{w}'_0 \in [\widetilde{w}_0]$, Theorem 5.7 tells that

$$S_{\widetilde{w}_0}(\beta) \simeq S_{\widetilde{w}'_0}(\beta) \quad \text{for all } \beta \in \Phi^+.$$

Thus we denote by $S_{[\widetilde{w}_0]}(\beta)$ the simple module $S_{\widetilde{w}'_0}(\beta)$ for any $\widetilde{w}'_0 \in [\widetilde{w}_0]$ and $\beta \in \Phi^+$.

Proposition 5.10. *Let α and β be incomparable positive roots with respect to the order $\prec_{[\widetilde{w}_0]}$. Then (α,β) is $[\widetilde{w}_0]$-simple and we have*

$$S_{[\widetilde{w}_0]}(\alpha)\circ S_{[\widetilde{w}_0]}(\beta)\simeq S_{[\widetilde{w}_0]}(\beta)\circ S_{[\widetilde{w}_0]}(\alpha) \text{ is simple.}$$

Proof. By Lemma 2.18, there exist $\widetilde{w}_0'\in[\widetilde{w}_0]$ and $k\in\mathbb{Z}_{\ge 1}$ such that $\alpha=\beta_k^{\widetilde{w}_0'}$ and $\beta=\beta_{k+1}^{\widetilde{w}_0'}$. Let us denote by (α,β) the sequence $\underline{m}_{\widetilde{w}_0'}$ such that $m_k=m_{k+1}=1$ and $m_i=0$ for all $i\neq k,k+1$. Then there is no $\underline{m}_{\widetilde{w}_0'}$ such that $\underline{m}<^{\mathtt{b}}_{\widetilde{w}_0'}(\alpha,\beta)$. Hence Theorem 5.7(c) tells that the composition series of $S_{[\widetilde{w}_0]}(\alpha)\circ S_{[\widetilde{w}_0]}(\beta)$ consists of $\mathrm{Im}(\mathbf{r}_{(\alpha,\beta)})$. Then our assertion follows from Theorem 5.7(b). □

Remark 5.11. Proposition 5.10 tells that $S_{[\widetilde{w}_0]}(\alpha)$ and $S_{[\widetilde{w}_0]}(\beta)$ *commute* up to grading shift (or *q-commutes*) if α and β are incomparable with respect to $\prec_{[\widetilde{w}_0]}$. However, the converse is not true. As we see in Proposition 5.12 below, when α and β lie in the same sectional path in $\Upsilon_{[\widetilde{w}_0]}$ so that they are comparable, $S_{[\widetilde{w}_0]}(\alpha)$ and $S_{[\widetilde{w}_0]}(\beta)$ commute. This result is a generalization of [19, Proposition 4.2].

Proof of Theorem 5.8. By proposition 5.10, the isomorphism class of the module $\overrightarrow{S}_{\widetilde{w}_0}(\underline{m}_{\widetilde{w}_0})$ and the homomorphism $\mathbf{r}_{\underline{m}_{\widetilde{w}_0}}$ does not depend on the choice of $\widetilde{w}_0\in[\widetilde{w}_0]$. Thus our first assertion follows. By applying the first assertion to (5.6) for all $\widetilde{w}_0'\in[\widetilde{w}_0]$, we have

$$[\overrightarrow{S}_{[\widetilde{w}_0]}(\underline{m})]\in[\mathrm{Im}(\mathbf{r}_{\underline{m}})]+\sum_{\underline{m}'<^{\mathtt{b}}_{\widetilde{w}_0'}\underline{m} \text{ for all } \widetilde{w}_0'\in[\widetilde{w}_0]}\mathbb{Z}_{\ge 0}[q^{\pm1}][\mathrm{Im}(\mathbf{r}_{\underline{m}'})].$$

Thus our second assertion follows from the definition of $\prec^{\mathtt{b}}_{[\widetilde{w}_0]}$; that is,

$$[\overrightarrow{S}_{[\widetilde{w}_0]}(\underline{m})]\in[\mathrm{Im}(\mathbf{r}_{\underline{m}})]+\sum_{\underline{m}'\prec^{\mathtt{b}}_{[\widetilde{w}_0]}\underline{m}}\mathbb{Z}_{\ge 0}[q^{\pm1}][\mathrm{Im}(\mathbf{r}_{\underline{m}'})]. \tag{5.7}$$

□

Proposition 5.12. *Let α and β be in the same sectional path of $\Upsilon_{[\widetilde{w}_0]}$. Then (α,β) is $[\widetilde{w}_0]$-simple and we have*

$$S_{[\widetilde{w}_0]}(\alpha)\circ S_{[\widetilde{w}_0]}(\beta)\simeq S_{[\widetilde{w}_0]}(\beta)\circ S_{[\widetilde{w}_0]}(\alpha) \text{ is simple.}$$

Proof. Proposition 3.12 implies that (α,β) is a simple pair with respect to $\prec_{[\widetilde{w}_0]}$. Thus our assertion follows from Theorem 5.8. □

By Remark 3.12, we have the following corollary from Theorem 5.8.

Corollary 5.13. *Let $\beta_1,\beta_2,\ldots,\beta_p$ be in the same sectional path of $\Upsilon_{[\widetilde{w}_0]}$. Then we have*

$$S_{[\widetilde{w}_0]}(\beta_1)^{\circ m_1}\circ\cdots\circ S_{[\widetilde{w}_0]}(\beta_p)^{\circ m_p} \text{ is simple for any } (m_1,m_2,\ldots,m_p)\in\mathbb{Z}^p_{\ge 0}.$$

Remark 5.14. By the works in [4, 9, 14], $S_{\widetilde{w}_0}(\beta)$'s categorify the dual PBW generators of $\mathfrak{g}$ associated to $\widetilde{w}_0$, which are also elements of the dual canonical basis. Hence our results in this section tell that the dual PBW monomials *depend only on* $[\widetilde{w}_0]$ (up to $q^{\mathbb{Z}}$) and some of them are *q-commutative* under the circumstances we characterized. In particular, when R is *symmetric* and $\mathbf{k}$ is of characteristic 0, simple R-modules categorify the dual canonical basis ([24, 26]). Hence (5.7) provides finer information on transition map between the dual canonical basis and the dual PBW basis associated to $[\widetilde{w}_0]$.

By (4.4), one can observe the following similarity among $\{S_{[\widetilde{w}_0]}(\alpha)\}$ and $\{S_{[\widetilde{w}'_0]}(\alpha')\}$ for $[\widetilde{w}_0], [\widetilde{w}'_0]$ in the same r-cluster point $[\![\widetilde{w}_0]\!]$:

Corollary 5.15. *For a class $[\widetilde{w}_0]$ of reduced expressions of w_0, let $(i_1, i_2, \ldots, i_k)$ be a sequence of indices such that*

$$i_k \text{ is a sink of } [\widetilde{w}_0]\ r_{i_1} \cdots r_{i_{k-1}}.$$

Set $w = s_{i_{k-1}} \cdots s_{i_1}$. For $(\alpha, \beta) \in (\Phi^+)^2$ with $[\widetilde{w}_0]$-simple and $w \cdot \alpha, w \cdot \beta \in \Phi^+$, we have

$$S_{[\widetilde{w}_0]\cdot r_{\widetilde{w}}}(w\cdot\alpha) \circ S_{[\widetilde{w}_0]\cdot r_{\widetilde{w}}}(w\cdot\beta) \simeq S_{[\widetilde{w}_0]\cdot r_{\widetilde{w}}}(w\cdot\beta) \circ S_{[\widetilde{w}_0]\cdot r_{\widetilde{w}}}(w\cdot\alpha) \text{ is simple,}$$

where $r_{\widetilde{w}} := r_{i_1} \cdots r_{i_{k-1}}$.

Appendix A. r-cluster points of A_4

There are 62 commutation classes of w_0 for A_4 (see [2, Table 1] and [16, A006245]). We can check that the 62 commutation classes are classified into 3-cluster points with respect to $\sigma = {}^*$ as follows:

Type 1

$(5, 5)$

A01	1213214321	A02	2132143421	A03	1214342312	A04	3214342341
A05	4342341234	A06	1321434231	A07	2143423412	A08	1434234123

Type 2

$(4, 6)$

B01	2123214321	B02	1232143231	B03	1232124321	B04	1213243212
B05	2132314321	B06	1323124321	B07	1213432312	B08	1323143231
B09	2321243421	B10	2132434212	B11	2124342312	B12	1243421232
B13	3231243421	B14	2321432341	B15	2134323412	B16	2143234312
B17	3212434231	B18	1324342123	B19	1243423123	B20	1432341232
B21	3214323431	B22	1343234123	B23	1432343123	B24	2434212342
B25	3243421234	B26	2434231234	B27	4323412342	B28	4342123423
B29	3432341234	B30	4323431234	B31	4342312343	B32	3231432341

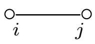
Type 3
(3, 7)

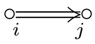
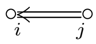
C01	2123243212	C02	2321234321	C03	2132343212	C04	2123432312
C05	3212324321	C06	1232432123	C07	1234321232	C08	3231234321
C09	3212343231	C10	1323432123	C11	1234323123	C12	3234321234
C13	2324321234	C14	2343212342	C15	2432123432	C16	4321234232
C17	3432312343	C18	2343231234	C19	4323123432	C20	3243212343
C21	3432123423	C22	4321234323				

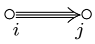
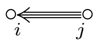
Appendix B. Braid relations and combinatorial AR quivers

By Matsumoto's theorem, for any two reduced expressions $\widetilde{w}$ and $\widetilde{w}'$ of $w \in \mathsf{W}$, $\widetilde{w}$ can be obtained from $\widetilde{w}'$ by commutation relations and braid relations. In Proposition 2.3, we showed if $\widetilde{w}'$ and $\widetilde{w}$ are related by a series of short braid relations, i.e., $[\widetilde{w}] = [\widetilde{w}']$, then $\Upsilon_{[\widetilde{w}']} = \Upsilon_{[\widetilde{w}]}$. In this section, we describe relations between $\Upsilon_{[\widetilde{w}]}$ and $\Upsilon_{[\widetilde{w}'']}$ for $\widetilde{w}''$ which is obtained by a braid relation from $\widetilde{w}$.

Recall that if $d_\Delta(i,j) = 1$, its corresponding braid relation is given as follows:

(Case 1)  implies $s_i s_j s_i = s_j s_i s_j$,

(Case 2)  or  implies $s_i s_j s_i s_j = s_j s_i s_j s_i$,

(Case 3)  or  implies $s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i$.

In Section B.1 and B.2, we shall discuss braid relations on the set of combinatorial AR quivers for (Case 1) and (Case 2). Note that (Case 3) is obvious.

B.1. Case 1

Suppose a Dynkin diagram Δ of type X_n which has the subdiagram in (Case 1) so that $s_i s_j s_i = s_j s_i s_j$.

Proposition B.1. *Let $\widetilde{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_{\ell(w)}})$ and $\widetilde{w}' = (s_{i'_1}, s_{i'_2}, \ldots, s_{i'_{\ell(w)}})$ be reduced expressions of w such that $\widetilde{w}'$ can be obtained by the relation $s_i s_j s_i = s_j s_i s_j$ from $\widetilde{w}$. Equivalently, there exists $2 \le t \le \ell(w) - 1$ such that*

(i) $i_m = i'_m$, *if* $1 \le m \le t-2$ *or* $t+2 \le m \le \ell(w)$,
(ii) $(i_{t-1}, i_t, i_{t+1}) = (i, j, i)$,
(iii) $(i'_{t-1}, i'_t, i'_{t+1}) = (j, i, j)$.

Then we have

(1) $\beta^{\widetilde{w}}_m = \beta^{\widetilde{w}'}_m$, *if* $1 \le m \le t-2$, $t+2 \le m \le \ell(w)$ *or* $m = t$,
(2) $\beta^{\widetilde{w}}_{t-1} = \beta^{\widetilde{w}'}_{t+1}$ *and* $\beta^{\widetilde{w}}_{t+1} = \beta^{\widetilde{w}'}_{t-1}$.

Proof. Our assertion for $1 \le m \le t-2$ is obvious. For $m = t-1, t$ and $t+1$, we have

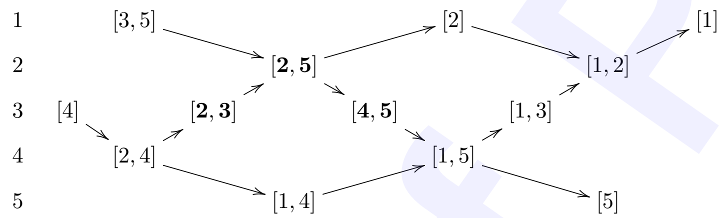
$$\begin{aligned}\beta^{\widetilde{w}}_{t-1} &= s_{i_1} \cdots s_{i_{t-2}}(\alpha_i) = s_{i_1} \cdots s_{i_{t-2}}(s_j s_i(\alpha_j)) \\ &= s_{i'_1} \cdots s_{i'_{t-2}}(s_{i'_{t-1}} s_{i'_t}(\alpha_{i'_{t+1}})) = \beta^{\widetilde{w}'}_{t+1},\end{aligned}$$

$$\begin{aligned}\beta_t^{\widetilde{w}} &= s_{i_1}\cdots s_{i_{t-2}}(s_i(\alpha_j)) = s_{i_1}\cdots s_{i_{t-2}}(s_j(\alpha_i)) \\ &= s_{i'_1}\cdots s_{i'_{t-2}}(s_{i'_{t-1}}(\alpha_{i'_t})) = \beta_t^{\widetilde{w}'}, \\ \beta_{t+1}^{\widetilde{w}} &= s_{i_1}\cdots s_{i_{t-2}}(s_i s_j(\alpha_i)) = s_{i_1}\cdots s_{i_{t-2}}(\alpha_j) \\ &= s_{i'_1}\cdots s_{i'_{t-2}}(\alpha_{i'_{t-1}}) = \beta_{t-1}^{\widetilde{w}'}.\end{aligned}$$

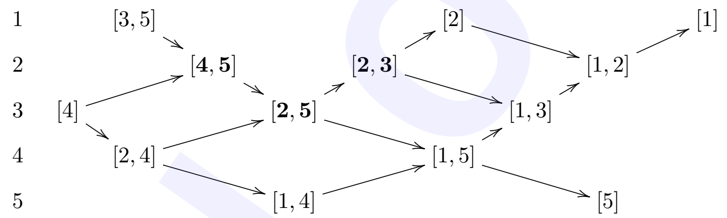
Our assertion for $m \ge t+2$ follow from the fact that

$$s_{i_{t-1}}s_{i_t}s_{i_{t+1}}s_{i_{t+2}}\cdots s_{i_{m-1}} = s_{i'_{t-1}}s_{i'_t}s_{i'_{t+1}}s_{i'_{t+2}}\cdots s_{i'_{m-1}}. \qquad \square$$

Example B.2. Let $\widetilde{w} = (s_1,s_2,s_3,s_5,s_4,s_1,\mathbf{s_3},\mathbf{s_2},\mathbf{s_3},s_5,s_4,s_3,s_1)$ of A_5. The quiver $\Upsilon_{[\widetilde{w}]}$ is drawn as follows:

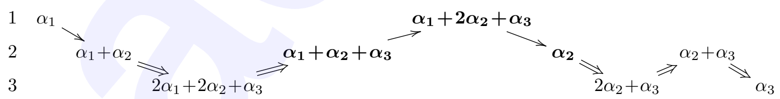
(B.1)


Consider $\widetilde{w}' = (s_1,s_2,s_3,s_5,s_4,s_1,\mathbf{s_2},\mathbf{s_3},\mathbf{s_2},s_5,s_4,s_3,s_1)$ of A_5. The quiver $\Upsilon_{[\widetilde{w}']}$ is drawn as follows:

(B.2)


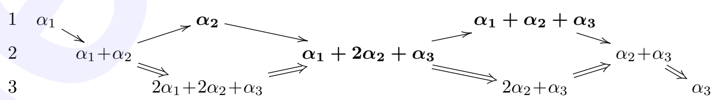
Note that, in $\Upsilon_{[\widetilde{w}'_0]}$, there are arrows from [4] to [4, 5] and from [2, 3] to [1, 3].

Example B.3. In Example 2.17, for $\widetilde{w}_0 = (s_3,s_2,s_3,\mathbf{s_2},\mathbf{s_1},\mathbf{s_2},s_3,s_2,s_1)$ of type C_3,

$\Upsilon_{[\widetilde{w}_0]} =$


.

Let us consider $\widetilde{w}'_0 = (s_3,s_2,s_3,\mathbf{s_1},\mathbf{s_2},\mathbf{s_1},s_3,s_2,s_1)$ of type C_3. Then, by Proposition B.1,

$\Upsilon_{[\widetilde{w}'_0]} =$


.

B.2. Case 2

Suppose Δ of type X_n (X=B,C,F) has the subdiagram in (Case 2), so that $s_is_js_is_j = s_js_is_js_i$. The analogous argument with Proposition B.1, we can see the following proposition.

Proposition B.4. *Let* $\widetilde{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_{\ell(w)}})$ *and* $\widetilde{w}' = (s_{i'_1}, s_{i'_2}, \ldots, s_{i'_{\ell(w)}})$ *be reduced expressions of* w *such that* $\widetilde{w}'$ *can be obtained by the relation* $s_is_js_is_j = s_js_is_js_i$ *from* $\widetilde{w}$. *Equivalently, there exists* $1 \le t \le \ell(w) - 3$ *such that*

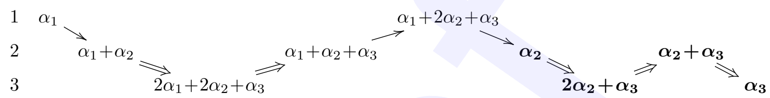
(i) $i_m = i'_m$, *if* $1 \le m < t$ *or* $t + 3 < m \le \ell(w)$,
(ii) $(i_t, i_{t+1}, i_{t+2}, i_{t+3}) = (i, j, i, j)$,
(iii) $(i'_t, i'_{t+1}, i'_{t+2}, i'_{t+3}) = (j, i, j, i)$.

Then we have

(1) $\beta^{\widetilde{w}}_m = \beta^{\widetilde{w}'}_m$ *if* $1 \le m < t$ *or* $t + 3 < m \le \ell(w)$,
(2) $\beta^{\widetilde{w}}_t = \beta^{\widetilde{w}'}_{t+3}$, $\beta^{\widetilde{w}}_{t+1} = \beta^{\widetilde{w}'}_{t+2}$, $\beta^{\widetilde{w}}_{t+2} = \beta^{\widetilde{w}'}_{t+1}$ *and* $\beta^{\widetilde{w}}_{t+3} = \beta^{\widetilde{w}'}_t$.

Example B.5. In Example 2.17, for $\widetilde{w}_0 = (\mathbf{s_3}, \mathbf{s_2}, \mathbf{s_3}, \mathbf{s_2}, s_1, s_2, s_3, s_2, s_1)$ of type C_3,

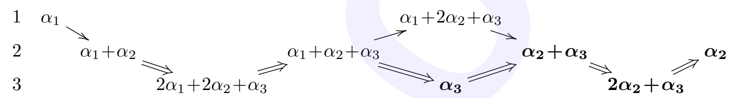
$\Upsilon_{[\widetilde{w}_0]} =$



.

Now, for $\widetilde{w}'_0 = (\mathbf{s_2}, \mathbf{s_3}, \mathbf{s_2}, \mathbf{s_3}, s_1, s_2, s_3, s_2, s_1)$ of type C_3,

$\Upsilon_{[\widetilde{w}'_0]} =$



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