J. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/JKMS.j180151
pISSN: 0304-9914 / eISSN: 2234-3008

THE IDEAL OF WEAKLY p-NUCLEAR OPERATORS AND ITS INJECTIVE AND SURJECTIVE HULLS

Ju Myung Kim

Abstract. We introduce a larger ideal $\mathcal{N}_{wp}$ of the ideal of p-nuclear operators. We obtain isometric representations of the injective and surjective hulls of $\mathcal{N}_{wp}$ and study them.

1. Introduction

Let $1 \le p < \infty$. For a Banach space X, let $\ell_p(X)$ (respectively, $\ell_p^w(X)$) be the Banach space with the norm $\|\cdot\|_p$ (respectively, $\|\cdot\|_p^w$) of all X-valued absolutely (respectively, *weakly*) p-summable sequences. Let $c_0(X)$ (respectively, $c_0^w(X)$) be the Banach space with the norm $\|\cdot\|_\infty$ of all X-valued norm (respectively, *weakly*) null sequences. For the dual space X^* of X, let $c_0^{w^*}(X^*)$ be the Banach space with the norm $\|\cdot\|_\infty$ of all X^*-valued *weak** null sequences.

For $1 \le p \le \infty$, recall the operator ideal $[\mathcal{N}_p, \|\cdot\|_{\mathcal{N}_p}]$ of *p-nuclear operators* (cf. [1, 4, 11–13]). The ideal $\mathcal{N}_p(X,Y)$ is defined as all operators T which have a representation

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \quad \text{and} \quad \|T\|_{\mathcal{N}_p} := \inf \|(x_n^*)_n\|_p \|(y_n)_n\|_{p^*}^w,$$

where $(x_n^*)_n \in \ell_p(X^*)$ ($c_0(X^*)$ when $p=\infty$) and $(y_n)_n \in \ell_{p^*}^w(Y)$ ($c_0(Y)$ when $p=1$). Here $1/p + 1/p^* = 1$ and $x_n^* \underline{\otimes} y_n$ is an operator from X to Y defined by $(x_n^* \underline{\otimes} y_n)(x) = x_n^*(x) y_n$, and the infimum is taken over all such representations. When $\ell_p(\cdot)$ and $\ell_{p^*}^w(\cdot)$ in the above notion are interchanged with each other, we denote the operator ideal consisting of such operators by $[\mathcal{N}^p, \|\cdot\|_{\mathcal{N}^p}]$.

We need an another space of vector valued sequences to introduce a weaker notion of the p-nuclear operator. The closed subspace $\ell_p^u(X)$ of $\ell_p^w(X)$ consists of all sequences $(x_n)_n$ in X satisfying that

$$\|(0, \ldots, 0, x_m, x_{m+1}, \ldots)\|_p^w \longrightarrow 0$$

as $m \to \infty$ (cf. [1, Section 8.2] and [6, 7]). Note that $\ell_\infty^u(X) = c_0(X)$. The ideal of *p-compact operators* (cf. [1, 6, 7, 12]) is denoted by $[\mathfrak{K}_p, \|\cdot\|_{\mathfrak{K}_p}]$. The

Received March 5, 2018; Revised April 24, 2018; Accepted May 3, 2018.
2010 *Mathematics Subject Classification.* 46B28, 46B45, 47L20.
Key words and phrases. Banach operator ideal, tensor norm, nuclear operator.

ideal $\mathfrak{K}_p(X,Y)$ is defined as all operators T which have a representation

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \quad \text{and} \quad \|T\|_{\mathfrak{K}_p} := \inf \|(x_n^*)_n\|_p^w \|(y_n)_n\|_{p^*}^w,$$

where $(x_n^*)_n \in \ell_p^u(X^*)$ and $(y_n)_n \in \ell_{p^*}^u(Y)$, and the infimum is taken over all such representations.

A more general notion of the p-nuclear operator is the *σ-nuclear operator* [12]. It is represented as

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \quad \text{and}$$

$$\|T\|_{\mathcal{N}_\sigma} := \inf \Big\{ \sup \Big\{ \sum_{n=1}^{\infty} |x_n^*(x) y^*(y_n)| : \|x\| \le 1, \|y^*\| \le 1 \Big\} \Big\},$$

where $x_n^* \in X^*$ and $y_n \in Y$ such that $\sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n$ unconditionally converges in the operator norm, and the infimum is taken over all such representations.

Naturally, one may consider the class of operators $T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n$ such that $\sum_{n=1}^{\infty} x_n^*(x) y_n$ unconditionally converges in Y for every $x \in X$. Our main concern in this paper is a special subclass of this class. For $1 \le p \le \infty$, we say that an operator $T : X \to Y$ is *weakly p-nuclear* if it is represented as

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n,$$

where $(x_n^*)_n \in \ell_p^w(X^*)$ ($c_0^{w^*}(X^*)$ when $p = \infty$) and $(y_n)_n \in \ell_{p^*}^w(Y)$ ($c_0^w(Y)$ when $p = 1$). We denote the space of all weakly p-nuclear operators from X to Y by $\mathcal{N}_{wp}(X,Y)$ and define a norm on $\mathcal{N}_{wp}(X,Y)$ by

$$\|T\|_{\mathcal{N}_{wp}} := \inf \|(x_n^*)_n\|_p^w \|(y_n)_n\|_{p^*}^w,$$

where the infimum is taken over all such weakly p-nuclear representations of T. Then $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]$ is a Banach operator ideal (see Theorem 2.1). In this paper, we study the ideal $\mathcal{N}_{wp}$ and its injective and surjective hulls based on the investigation related with the ideals $\mathfrak{K}_p$ and $\mathcal{N}_p$ [3, 5, 8–10].

2. The ideal of weakly p-nuclear operators

Let us recall the definition of a Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ as follows. For each pair (X,Y) of Banach spaces, let $(\mathcal{L}(X,Y), \|\cdot\|)$ be the Banach space of all operators from X to Y. An operator ideal is an association to each pair of Banach spaces X and Y, of a subset, $\mathcal{A}(X,Y)$, of $\mathcal{L}(X,Y)$ such that

(O1) $\mathcal{A}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$ and $(\mathcal{A}(X,Y), \|\cdot\|_{\mathcal{A}})$ is a Banach space.

(O2) $x^* \underline{\otimes} y \in \mathcal{A}(X,Y)$ and $\|x^* \underline{\otimes} y\|_{\mathcal{A}} = \|x^*\|\|y\|$ for every $x^* \in X^*$ and $y \in Y$.

(O3) $STR \in \mathcal{A}(X_0, Y_0)$ and $\|STR\|_{\mathcal{A}} \leq \|S\|\|T\|_{\mathcal{A}}\|R\|$ for every $R \in \mathcal{L}(X_0, X)$, $T \in \mathcal{A}(X, Y)$ and $S \in \mathcal{L}(Y, Y_0)$.

Theorem 2.1. *For $1 \leq p \leq \infty$, $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]$ is a Banach operator ideal.*

Proof. The properties (O2) and (O3) of $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]$ may be easily verified. So we only show that (O1) holds. Let X and Y be Banach spaces.

The case $p = \infty$: It is easily seen that $\alpha T \in \mathcal{N}_{w\infty}(X, Y) \subset \mathcal{L}(X, Y)$, $\|\alpha T\|_{\mathcal{N}_{w\infty}} = |\alpha|\|T\|_{\mathcal{N}_{w\infty}}$ and $\|T\| \leq \|T\|_{\mathcal{N}_{w\infty}}$ for every $T \in \mathcal{N}_{w\infty}(X, Y)$ and scalar α. Let $T, R \in \mathcal{N}_{w\infty}(X, Y)$ and let $\varepsilon > 0$ be given. Let

$$T = \sum_{n=1}^{\infty} x^*_{2n-1} \underline{\otimes} y_{2n-1}, R = \sum_{n=1}^{\infty} x^*_{2n} \underline{\otimes} y_{2n}$$

be weakly ∞-nuclear representations such that

$$\|(y_{2n-1})_n\|_1^w = 1 = \|(y_{2n})_n\|_1^w$$

and

$$\|(x^*_{2n-1})_n\|_\infty \leq (1+\varepsilon)\|T\|_{\mathcal{N}_{w\infty}}, \|(x^*_{2n})_n\|_\infty \leq (1+\varepsilon)\|R\|_{\mathcal{N}_{w\infty}}.$$

Then

$$\begin{aligned} T + R = &\sum_{n=1}^{\infty} \Big(\frac{x^*_{2n-1}}{\|(x^*_{2k-1})_k\|_\infty} \underline{\otimes} \|(x^*_{2k-1})_k\|_\infty y_{2n-1} + \frac{x^*_{2n}}{\|(x^*_{2k})_k\|_\infty} \underline{\otimes} \|(x^*_{2k})_k\|_\infty y_{2n} \Big) \\ &\in \mathcal{N}_{w\infty}(X, Y) \end{aligned}$$

and

$$\begin{aligned} \|T + R\|_{\mathcal{N}_{w\infty}} &\leq \|(\|(x^*_{2k-1})_k\|_\infty y_{2n-1} + \|(x^*_{2k})_k\|_\infty y_{2n})_n\|_1^w \\ &\leq (1+\varepsilon)(\|T\|_{\mathcal{N}_{w\infty}} + \|R\|_{\mathcal{N}_{w\infty}}). \end{aligned}$$

Consequently, $(\mathcal{N}_{w\infty}(X, Y), \|\cdot\|_{\mathcal{N}_{w\infty}})$ is a normed linear subspace of $\mathcal{L}(X, Y)$.

To show that $(\mathcal{N}_{w\infty}(X, Y), \|\cdot\|_{\mathcal{N}_{w\infty}})$ is complete, let $(T_k)_k$ be a sequence in $\mathcal{N}_{w\infty}(X, Y)$ with $\sum_{k=1}^{\infty} \|T_k\|_{\mathcal{N}_{w\infty}} < \infty$. Then $\sum_{k=1}^{\infty} \|T_k\| < \infty$ and so $\sum_{k=1}^{\infty} T_k$ converges in $\mathcal{L}(X, Y)$. We will show that $\sum_{k=1}^{\infty} T_k \in \mathcal{N}_{w\infty}(X, Y)$ and $\|\sum_{k=1}^{\infty} T_k\|_{\mathcal{N}_{w\infty}} \leq \sum_{k=1}^{\infty} \|T_k\|_{\mathcal{N}_{w\infty}}$. Let $\varepsilon > 0$ be given. For each $k \geq 1$, let $(x^*_{kn})_n \in c_0^{w^*}(X^*)$ and $(y_{kn})_n \in \ell_1^w(Y)$ such that $T_k = \sum_{n=1}^{\infty} x^*_{kn} \underline{\otimes} y_{kn}$ and

$$\|(x^*_{kn})_n\|_\infty \leq 1, \ \|(y_{kn})_n\|_1^w \leq \|T_k\|_{\mathcal{N}_{w\infty}} + \frac{\varepsilon}{2^k}.$$

Then, since $\sum_{k=1}^{\infty} \|(y_{kn})_n\|_1^w < \infty$, there exists a sequence $(\beta_k)_k$ of real numbers such that

$$\lim_{k \to \infty} \beta_k = \infty, \ \beta_k > 1, \ \sum_{k=1}^{\infty} \beta_k \|(y_{kn})_n\|_1^w \leq (1+\varepsilon) \sum_{k=1}^{\infty} \|(y_{kn})_n\|_1^w.$$

Now, we consider the following sequence of rectangular array:

$$
\begin{array}{ccccccc}
(1/\beta_1)x_{11}^*\underline{\otimes}\beta_1 y_{11} & \rightarrow & (1/\beta_1)x_{12}^*\underline{\otimes}\beta_1 y_{12} & & (1/\beta_1)x_{13}^*\underline{\otimes}\beta_1 y_{13} & \cdots & (1/\beta_1)x_{1n}^*\underline{\otimes}\beta_1 y_{1n}\cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
(1/\beta_2)x_{21}^*\underline{\otimes}\beta_2 y_{21} & \leftarrow & (1/\beta_2)x_{22}^*\underline{\otimes}\beta_2 y_{22} & & (1/\beta_2)x_{23}^*\underline{\otimes}\beta_2 y_{23} & \cdots & (1/\beta_2)x_{2n}^*\underline{\otimes}\beta_2 y_{2n}\cdots \\
 & & & & \downarrow & & \downarrow \\
(1/\beta_3)x_{31}^*\underline{\otimes}\beta_3 y_{31} & \leftarrow & (1/\beta_3)x_{32}^*\underline{\otimes}\beta_3 y_{32} & \leftarrow & (1/\beta_3)x_{33}^*\underline{\otimes}\beta_3 y_{33} & \cdots & (1/\beta_3)x_{3n}^*\underline{\otimes}\beta_3 y_{3n}\cdots \\
 & & \vdots & & & \vdots & \\
 & & & & & & \downarrow \\
(1/\beta_n)x_{n1}^*\underline{\otimes}\beta_n y_{n1} & \leftarrow & (1/\beta_n)x_{n2}^*\underline{\otimes}\beta_n y_{n2} & \cdots & (1/\beta_n)x_{n(n-1)}^*\underline{\otimes}\beta_n y_{n(n-1)} & \leftarrow & (1/\beta_n)x_{nn}^*\underline{\otimes}\beta_n y_{nn}\cdots \\
 & & \vdots & & & \vdots &
\end{array}
$$

Let $(z_m^*)_m$ and $(z_m)_m$, respectively, be the sequences consisting of the left parts and right parts, respectively, of the above sequence.

Since $\lim_{k\to\infty}\sup_j \|(1/\beta_k)x_{kj}^*\| \leq \lim_{k\to\infty} 1/\beta_k = 0$ and $((1/\beta_k)x_{kj}^*)_j \in c_0^{w^*}(X^*)$ for every k, we see that $(z_m^*)_m \in c_0^{w^*}(X^*)$. Also, since $\sum_{k=1}^{\infty} \|(\beta_k y_{kn})_n\|_1^w < \infty$, $(z_m)_m \in \ell_1^w(Y)$. Consequently, $\sum_{m=1}^{\infty} z_m^*\underline{\otimes} z_m \in \mathcal{N}_{w\infty}(X,Y)$.

Note that if $\sum_{n=1}^{\infty} w_n^*\underline{\otimes} w_n \in \mathcal{N}_{w\infty}(X,Y)$, then for every $x\in X$, $\sum_{n=1}^{\infty} w_n^*(x)w_n$ unconditionally converges in Y. Hence for every $x \in X$,

$$\sum_{k=1}^{\infty} T_k x = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} x_{kn}^*(x)y_{kn} = \sum_{m=1}^{\infty} z_m^*(x)z_m$$

and

$$\Big\|\sum_{k=1}^{\infty} T_k\Big\|_{\mathcal{N}_{w\infty}} \leq \|(z_m^*)_m\|_\infty \|(z_m)_m\|_1^w \leq (1+\varepsilon)\sum_{k=1}^{\infty}\|(y_{kn})_n\|_1^w$$

$$\leq (1+\varepsilon)\sum_{k=1}^{\infty}\Big(\|T_k\|_{\mathcal{N}_{w\infty}} + \frac{\varepsilon}{2^k}\Big).$$

Since $\varepsilon > 0$ was arbitrary, we have the desired conclusion. Also, for each $l \in \mathbb{N}$, we can apply the above argument to $\sum_{k\geq l} T_k$ to complete the proof for the case $p = \infty$.

The case $p = 1$: We can adopt the proof of the case $p = \infty$ by interchanging the roles of the sequences $(x_{kn}^*)_n$ and $(y_{kn})_n$ with each other.

The case $1 < p < \infty$: We can use the proof of [4, Theorem 5.25] or [13, Proposition 8.9] for this case. □

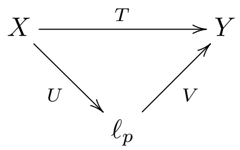
For every n, let e_n^* and e_n, respectively, be the standard unit vector bases in ℓ_{p^*} (c_0 when $p = 1$) and ℓ_p (c_0 when $p = \infty$).

Proposition 2.2. *Let $1 \leq p \leq \infty$ and let $T : X \to Y$ be a linear map. Then $T \in \mathcal{N}_{wp}(X,Y)$ if and only if there exist $R \in \mathcal{L}(X,\ell_p)$ and $S \in \mathcal{L}(\ell_p, Y)$ (ℓ_p is replaced by c_0 if $p = \infty$) such that $T = SR$. In this case, $\|T\|_{\mathcal{N}_{wp}} = \inf \|S\|\|R\|$, where the infimum is taken over all such factorizations.*

Proof. Let $T \in \mathcal{N}_{wp}(X,Y)$. Let $(x_n^*)_n \in \ell_p^w(X^*)$ and $(y_n)_n \in \ell_{p^*}^w(Y)$ such that

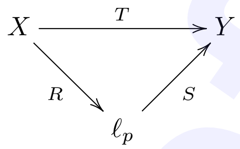
$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n.$$

Consider the maps $U : X \to \ell_p$, $x \mapsto (x_n^*(x))_n$ and $V : \ell_p \to Y$, $(\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n y_n$. Then we see that $\|U\| = \|(x_n^*)_n\|_p^w$ and $\|V\| = \|(y_n)_n\|_{p^*}^w$, and the following diagram is commutative.



Since the weakly p-nuclear representation of T was arbitrary, $\inf \|\cdot\| \|\| \cdot \| \leq \|T\|_{\mathcal{N}_{wp}}$.

Let T have the following factorization.



Consider the sequences $(e_n^* R)_n$ in X^* and $(Se_n)_n$ in Y. Then it is easily seen that $\|(e_n^* R)_n\|_p^w = \|R\|$ and $\|(Se_n)_n\|_{p^*}^w = \|S\|$. It follows that

$$\sum_{n=1}^{\infty} e_n^* R \underline{\otimes} Se_n = SR = T.$$

Since the factorization of T was arbitrary, $\|T\|_{\mathcal{N}_{wp}} \leq \inf \|\cdot\| \|\| \cdot \|$. □

Lemma 2.3. *Let $1 < p \leq \infty$. Then for every Banach space X, $\mathcal{N}_{wp}(X, \ell_p)$ (respectively, $\mathcal{N}_{wp}(\ell_p, X)$) is isometrically equal to $\mathcal{L}(X, \ell_p)$ (respectively, $\mathcal{L}(\ell_p, X)$). Here ℓ_p is replaced by c_0 when $p = \infty$.*

Proof. Let $T \in \mathcal{L}(X, \ell_p)$. Consider the sequences $(e_n^* T)_n$ in X^* and $(e_n)_n$ in ℓ_p. Then we see that $(e_n^* T)_n \in \ell_p^w(X^*)$ and $(e_n)_n \in \ell_{p^*}^w(\ell_p)$. Moreover, $\|(e_n^* T)_n\|_p^w = \|T\|$ and $\|(e_n)_n\|_{p^*}^w = 1$. Hence $T = \sum_n e_n^* T \underline{\otimes} e_n \in \mathcal{N}_{wp}(X, \ell_p)$ and $\|T\|_{\mathcal{N}_{wp}} = \|T\|$.

To show the other part, let $S \in \mathcal{L}(\ell_p, X)$. Consider the sequences $(e_n^*)_n$ in ℓ_{p^*} and $(Se_n)_n$ in X. Then it is easily seen that

$$\|(e_n^*)_n\|_p^w = 1, \|(Se_n)_n\|_{p^*}^w = \|S\|, S = \sum_n e_n^* \underline{\otimes} Se_n.$$

Hence $S \in \mathcal{N}_{wp}(\ell_p, X)$ and $\|S\|_{\mathcal{N}_{wp}} = \|S\|$. □

Remark 2.4. Lemma 2.3 does not hold in general for the case $p = 1$. Indeed, if $\mathcal{N}_{w1}(\ell_1, \ell_1)$ would be equal to $\mathcal{L}(\ell_1, \ell_1)$, then, since weakly 1-nuclear operators are weakly compact, by Schur's property, the identity map on ℓ_1 would be compact. This is a contradiction.

From Proposition 2.2 and Lemma 2.3, we have:

Corollary 2.5. *Let $1 < p \leq \infty$ and let $T : X \to Y$ be a linear map. Then $T \in \mathcal{N}_{wp}(X, Y)$ if and only if there exist $R \in \mathcal{N}_{wp}(X, \ell_p)$ and $S \in \mathcal{N}_{wp}(\ell_p, Y)$ (ℓ_p is replaced by c_0 if $p = \infty$) such that $T = SR$.*

Remark 2.6. For $1 < p \leq \infty$, we see that weakly p-nuclear operator is not always compact considering the fact that $\mathcal{N}_{wp}(\ell_p, \ell_p)$ is equal to $\mathcal{L}(\ell_p, \ell_p)$ by Lemma 2.3. Also weakly 1-nuclear operator is not compact in general. Indeed, let $(z_n)_n$ be a weakly null sequence in a Banach space Z such that the set $\{z_n\}_{n=1}^{\infty}$ is not compact. Consider the weakly 1-nuclear operator $T = \sum_{n=1}^{\infty} e_n \underline{\otimes} z_n$ from ℓ_1 to Z, where each e_n is the standard unit vector in c_0. Then T cannot be compact.

3. The injective and surjective hulls of the ideal of weakly p-nuclear operators

Sinha and Karn [14] introduced a ideal $\mathcal{K}_p$ of 'new' p-compact operators and its weak version. Let $1 \leq p \leq \infty$. A subset K of a Banach space X is called *p-compact* if there exists $(x_n)_n \in \ell_p(X)$ ($c_0(X)$ when $p = \infty$) such that

$$K \subset p\text{-}co(x_n)_n := \Big\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n) \in B_{\ell_{p^*}} \Big\}.$$

We denote the unit ball of X by B_X and replace $B_{\ell_{p^*}}$ with B_{c_0} if $p = 1$. For a linear map $T : Y \to X$, $T \in \mathcal{K}_p(Y, X)$ if $T(B_Y)$ is a p-compact subset of X. Delgado, Piñeiro, and Serrano [2] defined a norm on the space $\mathcal{K}_p(Y, X)$ as follows. For $T \in \mathcal{K}_p(Y, X)$, let

$$\|T\|_{\mathcal{K}_p} := \inf \big\{ \|(x_n)_n\|_p : (x_n)_n \in \ell_p(X) \text{ and } T(B_Y) \subset p\text{-}co(x_n)_n \big\}.$$

Then $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$ is a Banach operator ideal [3]. The ideal $[\mathcal{W}_p, \|\cdot\|_{\mathcal{W}_p}]$ of *weakly p-compact operators* is defined by replacing $\ell_p(X)$ and $c_0(X)$, respectively, by $\ell_p^w(X)$ and $c_0^w(X)$ in the definition of $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$.

The *surjective hull* $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{sur}$ of an operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is defined as follows;

$$\mathcal{A}^{sur}(X, Y) := \{T \in \mathcal{L}(X, Y) : Tq_X \in \mathcal{A}(\ell_1(B_X), Y)\},$$

where $q_X : \ell_1(B_X) \to X$ is the canonical quotient map, and $\|T\|_{\mathcal{A}^{sur}} := \|Tq_X\|_{\mathcal{A}}$ for $T \in \mathcal{A}^{sur}(X, Y)$ (see [1, p. 113] and [12, Section 8.5]). The following lemma is well known (cf. [12, Proposition 8.5.4]).

Lemma 3.1. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal and let X and Y be Banach spaces. A linear map $T \in \mathcal{A}^{sur}(X,Y)$ if and only if there exist a Banach space Z and an $S \in \mathcal{A}(Z,Y)$ such that $T(B_X) \subset S(B_Z)$. In this case,*

$$\|T\|_{\mathcal{A}^{sur}} = \inf \|S\|_{\mathcal{A}},$$

where the infimum is taken over all such inclusions.

Theorem 3.2. *Let $1 \leq p \leq \infty$ and let X and Y be Banach spaces. A linear map $T \in \mathcal{N}_{wp}^{sur}(X,Y)$ if and only if $T \in \mathcal{W}_{p^*}(X,Y)$. In this case, $\|T\|_{\mathcal{N}_{wp}^{sur}} = \|T\|_{\mathcal{W}_{p^*}}$.*

Proof. Let $T \in \mathcal{N}_{wp}^{sur}(X,Y)$. Then $Tq_X \in \mathcal{N}_{wp}(\ell_1(B_X),Y)$. Let

$$Tq_X = \sum_{n=1}^{\infty} (\zeta_x^n)_x \underline{\otimes} y_n$$

be a weakly p-nuclear representation. Then

$$T(B_X) = \Big\{ \sum_{n=1}^{\infty} \zeta_x^n y_n : x \in B_X \Big\} \subset p^*\text{-}co(\|((\zeta_x^k)_x)_k\|_p^w y_n)_n.$$

Hence $T \in \mathcal{W}_{p^*}(X,Y)$ and $\|T\|_{\mathcal{W}_{p^*}} \leq \|((\zeta_x^n)_x)_n\|_p^w \|(y_n)_n\|_{p^*}^w$. Since the representation was arbitrary, $\|T\|_{\mathcal{W}_{p^*}} \leq \|Tq_X\|_{\mathcal{N}_{wp}} = \|T\|_{\mathcal{N}_{wp}^{sur}}$.

In other to show the converse, let $T \in \mathcal{W}_{p^*}(X,Y)$. Let $\varepsilon > 0$ be given. Then there exists $(y_n)_n \in \ell_{p^*}^w(Y)$ with $\|(y_n)_n\|_{p^*}^w \leq \|T\|_{\mathcal{W}_{p^*}} + \varepsilon$ such that

$$T(B_X) \subset p^*\text{-}co(y_n)_n.$$

Consider the map $S : \ell_p \to Y$ ($c_0 \to Y$ when $p = \infty$) defined by

$$S = \sum_{n=1}^{\infty} e_n^* \underline{\otimes} y_n,$$

where $(e_n^*)_n$ is the sequence of standard unit vectors in ℓ_{p^*} ($e_n^* \in c_0 \subset \ell_\infty$ when $p = 1$). Then we see that $S \in \mathcal{N}_{wp}(\ell_p, Y)$ and

$$T(B_X) \subset p^*\text{-}co(y_n)_n = S(B_{\ell_p}).$$

Hence by Lemma 3.1, $T \in \mathcal{N}_{wp}^{sur}(X,Y)$ and

$$\|T\|_{\mathcal{N}_{wp}^{sur}} \leq \|S\|_{\mathcal{N}_{wp}} \leq \|(y_n)_n\|_{p^*}^w \leq \|T\|_{\mathcal{W}_{p^*}} + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we complete the proof. □

It was shown in [3, Proposition 3.11] that $[\mathcal{N}^p, \|\cdot\|_{\mathcal{N}^p}]^{sur} = [\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$. We may use the proof of Theorem 3.2 to show that result. The ideal $[\mathcal{K}_{up}, \|\cdot\|_{\mathcal{K}_{up}}]$ of *unconditionally p-compact operators* is defined by replacing $\ell_p(X)$ by $\ell_p^u(X)$ in the definition of $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$. It was shown in [5, Theorem 4.5] and [10, Proposition 3.1] that $[\mathfrak{K}_p, \|\cdot\|_{\mathfrak{K}_p}]^{sur} = [\mathcal{K}_{up^*}, \|\cdot\|_{\mathcal{K}_{up^*}}]$.

Persson and Pietsch [11] introduced a weaker notion of the p-nuclear operator. For $1 \leq p \leq \infty$, a linear map $T : X \to Y$ is called *quasi p-nuclear* if there exists $(x_n^*)_n \in \ell_p(X^*)$ ($c_0(X^*)$ when $p = \infty$) such that $\|Tx\| \leq \|(x_n^*(x))_n\|_p$ for every $x \in X$. We denote the space of all quasi p-nuclear operators from X to Y by $\mathcal{N}_p^Q(X,Y)$. For $T \in \mathcal{N}_p^Q(X,Y)$, let $\|T\|_{\mathcal{N}_p^Q} := \inf \|(x_n^*)_n\|_p$, where the infimum is taken over all such inequalities. Then $[\mathcal{N}_p^Q, \|\cdot\|_{\mathcal{N}_p^Q}]$ is a Banach operator ideal [11]. The ideal $[\mathcal{N}_{wp}^Q, \|\cdot\|_{\mathcal{N}_{wp}^Q}]$ of *quasi weakly p-nuclear operators* is defined by replacing $\ell_p(X^*)$ and $c_0(X^*)$, respectively, by $\ell_p^w(X^*)$ and $c_0^{w^*}(X^*)$ in the definition of $[\mathcal{N}_p^Q, \|\cdot\|_{\mathcal{N}_p^Q}]$.

The injective hull $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{inj}$ of an operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is defined as follows;

$$\mathcal{A}^{inj}(X,Y) := \{T \in \mathcal{L}(X,Y) : I_Y T \in \mathcal{A}(X, \ell_\infty(B_{Y^*}))\},$$

where $I_Y : Y \to \ell_\infty(B_{Y^*})$ is the canonical isometry, and $\|T\|_{\mathcal{A}^{inj}} := \|I_Y T\|_{\mathcal{A}}$ for $T \in \mathcal{A}^{inj}(X,Y)$ (see [1, p. 112] and [12, Section 8.4]).

Theorem 3.3. *Let $1 \leq p \leq \infty$ and let X and Y be Banach spaces. A linear map $T \in \mathcal{N}_{wp}^{inj}(X,Y)$ if and only if $T \in \mathcal{N}_{wp}^Q(X,Y)$. In this case, $\|T\|_{\mathcal{N}_{wp}^{inj}} = \|T\|_{\mathcal{N}_{wp}^Q}$.*

Recall that a Banach space Z is called *injective* if for every Banach space W and every subspace W_0 of W, any $T \in \mathcal{L}(W_0, Z)$ has an extension $\widehat{T} \in \mathcal{L}(W,Z)$ with $\|T\| = \|\widehat{T}\|$. It is well known that ℓ_∞-spaces are injective.

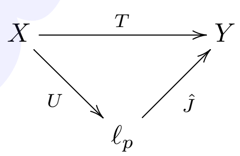
Lemma 3.4. *Let $1 \leq p \leq \infty$. Suppose that Y is injective. If $T \in \mathcal{N}_{wp}^Q(X,Y)$, then $T \in \mathcal{N}_{wp}(X,Y)$ and $\|T\|_{\mathcal{N}_{wp}} = \|T\|_{\mathcal{N}_{wp}^Q}$.*

Proof. Note that if $T \in \mathcal{N}_{wp}(X,Y)$, then $T \in \mathcal{N}_{wp}^Q(X,Y)$ and $\|T\|_{\mathcal{N}_{wp}^Q} \leq \|T\|_{\mathcal{N}_{wp}}$.

Now, let $T \in \mathcal{N}_{wp}^Q(X,Y)$ and let $\varepsilon > 0$ be given. Let $(x_n^*) \in \ell_p^w(X^*)$ be such that for every $x \in X$, $\|Tx\| \leq \|(x_n^*(x))_n\|_p$ and $\|(x_n^*)_n\|_p^w \leq \|T\|_{\mathcal{N}_{wp}^Q} + \varepsilon$. Let us consider the linear subspace

$$Z := \{(x_n^*(x))_n : x \in X\}$$

of ℓ_p (c_0 when $p = \infty$) and the map $J : Z \to Y$ via $(x_n^*(x))_n \mapsto Tx$. Then it follows that J is well defined and linear, and $\|J\| \leq 1$. Since Y is injective, there exists an extension $\hat{J} : \ell_p \to Y$ of J with $\|\hat{J}\| = \|J\|$. Define the operator $U : X \to \ell_p$ by $Ux = (x_n^*(x))_n$. Then the following diagram is commutative.



Hence by Proposition 2.2, $T \in \mathcal{N}_{wp}(X,Y)$ and $\|T\|_{\mathcal{N}_{wp}} \leq \|U\|\|\hat{J}\| \leq \|(x_n^*)_n\|_p^w \leq \|T\|_{\mathcal{N}_{wp}^Q} + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\|T\|_{\mathcal{N}_{wp}} \leq \|T\|_{\mathcal{N}_{wp}^Q}$. □

Proof of Theorem 3.3. If $T \in \mathcal{N}_{wp}^{inj}(X,Y)$, then $I_Y T \in \mathcal{N}_{wp}(X, \ell_\infty(B_{Y^*}))$. We see that $T \in \mathcal{N}_{wp}^{Q}(X,Y)$ and

$$\|T\|_{\mathcal{N}_{wp}^{Q}} \leq \|I_Y T\|_{\mathcal{N}_{wp}} = \|T\|_{\mathcal{N}_{wp}^{inj}}.$$

Conversely, if $T \in \mathcal{N}_{wp}^{Q}(X,Y)$, then $I_Y T \in \mathcal{N}_{wp}^{Q}(X, \ell_\infty(B_{Y^*}))$. By Lemma 3.4, $I_Y T \in \mathcal{N}_{wp}(X, \ell_\infty(B_{Y^*}))$ and

$$\|T\|_{\mathcal{N}_{wp}^{inj}} = \|I_Y T\|_{\mathcal{N}_{wp}} = \|T\|_{\mathcal{N}_{wp}^{Q}}. \qquad \square$$

The above argument also shows that $[\mathcal{N}_p, \|\cdot\|_{\mathcal{N}_p}]^{inj} = [\mathcal{N}_p^{Q}, \|\cdot\|_{\mathcal{N}_p^{Q}}]$. The ideal $[\mathcal{N}_{up}^{Q}, \|\cdot\|_{\mathcal{N}_{up}^{Q}}]$ of *quasi unconditionally p-nuclear operators* is defined by replacing $\ell_p(X^*)$ by $\ell_p^u(X^*)$ in the definition of $[\mathcal{N}_p^{Q}, \|\cdot\|_{\mathcal{N}_p^{Q}}]$. It was shown in [5, Theorem 4.4] that $[\mathfrak{K}_p, \|\cdot\|_{\mathfrak{K}_p}]^{inj} = [\mathcal{N}_{up}^{Q}, \|\cdot\|_{\mathcal{N}_{up}^{Q}}]$.

We now consider the duality relationship between $\mathcal{N}_{wp}^{inj}$ and $\mathcal{N}_{wp}^{sur}$ as in [3, Corollary 3.4].

Lemma 3.5. *For every Banach space X, we have*:

(a) *For $1 \leq p < \infty$, if $(x_n)_n \in \ell_p^w(X)$, then the set p-$co(x_n)_n$ is balanced, convex and weakly compact.*

(b) *If $(x_n)_n \in c_0^w(X)$ (respectively, $(x_n^*)_n \in c_0^{w^*}(X^*)$), then the set $\{\sum_{n=1}^\infty \alpha_n x_n : (\alpha_n) \in B_{\ell_1}\}$ (respectively, $\{\sum_{n=1}^\infty \alpha_n x_n^* : (\alpha_n) \in B_{\ell_1}\}$) is equal to the closed balanced convex hull $\overline{bco}\{x_n\}_{n=1}^\infty$ (respectively, the weak* closed balanced convex hull $\overline{bco}^{w^*}\{x_n^*\}_{n=1}^\infty$) of $\{x_n\}_{n=1}^\infty$ (respectively, $\{x_n^*\}_{n=1}^\infty$).*

Proof. (a) Clearly the set p-$co(x_n)_n$ is balanced and convex. Let

$$\Big(\sum_{n=1}^\infty \alpha_n^k x_n\Big)_{k=1}^\infty$$

be a sequence in $\{\sum_{n=1}^\infty \alpha_n x_n : (\alpha_n) \in B_{\ell_{p^*}}\}$. By the diagonal process, there exists a subsequence $(\sum_{n=1}^\infty \alpha_n^{k_l} x_n)_{l=1}^\infty$ of $(\sum_{n=1}^\infty \alpha_n^k x_n)_{k=1}^\infty$ and $(\beta_n) \in B_{\ell_{p^*}}$ such that for each n, $\alpha_n^{k_l} \longrightarrow \beta_n$ as $l \to \infty$. Then for every $x^* \in X^*$, we have

$$\begin{aligned}
&\Big|x^*\Big(\sum_{n=1}^\infty \alpha_n^{k_l} x_n - \sum_{n=1}^\infty \beta_n x_n\Big)\Big| \\
&\leq \sum_{n\leq N} |\alpha_n^{k_l} - \beta_n||x^*(x_n)| + \sum_{n>N} |\alpha_n^{k_l}||x^*(x_n)| + \sum_{n>N} |\beta_n||x^*(x_n)| \\
&\leq \sum_{n\leq N} |\alpha_n^{k_l} - \beta_n||x^*(x_n)| + 2\Big(\sum_{n>N} |x^*(x_n)|^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

Hence it follows that

$$\sum_{n=1}^\infty \alpha_n^{k_l} x_n \text{ weakly converges to } \sum_{n=1}^\infty \beta_n x_n \in \Big\{\sum_{n=1}^\infty \alpha_n x_n : (\alpha_n) \in B_{\ell_{p^*}}\Big\}$$

as $l \to \infty$.

(b) We only show the case $c_0^{w^*}(X^*)$. Clearly the set $\{\sum_{n=1}^{\infty} \alpha_n x_n^* : (\alpha_n) \in B_{\ell_1}\}$ is contained in $\overline{bco}^{w^*}\{x_n^*\}_{n=1}^{\infty}$, and that set is balanced and convex. So we only need to show that the set is *weak** compact.

Now, let $(\sum_{n=1}^{\infty} \alpha_n^{\gamma} x_n^*)$ be a net in $\{\sum_{n=1}^{\infty} \alpha_n x_n^* : (\alpha_n) \in B_{\ell_1}\}$. By Tychonoff's Theorem, we may assume that for every n, there exists a scalar β_n such that $\lim_{\gamma} \alpha_n^{\gamma} = \beta_n$. Also we see that $(\beta_n)_n \in B_{\ell_1}$. Then for every $x \in X$,

$$\begin{aligned} &\Big|\Big(\sum_{n=1}^{\infty} \alpha_n^{\gamma} x_n^* - \sum_{n=1}^{\infty} \beta_n x_n^*\Big)(x)\Big| \\ \le\ & \sum_{n\le N} |\alpha_n^{\gamma} - \beta_n||x_n^*(x)| + \sum_{n>N} |\alpha_n^{\gamma}||x_n^*(x)| + \sum_{n>N} |\beta_n||x_n^*(x)|. \end{aligned}$$

Since $(x_n^*)_n$ is a *weak** null sequence, $\sum_{n=1}^{\infty} \alpha_n^{\gamma} x_n^*$ *weak** converges to $\sum_n \beta_n x_n^*$ and so we complete the proof. □

By [3, Propositions 3.1 and 3.2] including the case $p = \infty$ and Lemma 3.5, we have:

Corollary 3.6. *Let $1 \le p < \infty$ and let $T : X \to Y$ be a linear map.*

(a) *If $(y_n) \in \ell_p^w(Y)$, then $T(B_X) \subset p$-$co(y_n)_n$ if and only if $\|T^*y^*\| \le \|(y^*(y_n))_n\|_p$ for every $y^* \in Y^*$.*

(b) *If $(y_n) \in c_0^w(Y)$, then $T(B_X) \subset \infty$-$co(y_n)_n$ if and only if $\|T^*y^*\| \le \|(y^*(y_n))_n\|_\infty$ for every $y^* \in Y^*$.*

(c) *If $(x_n^*) \in \ell_p^w(X^*)$, then $T^*(B_{Y^*}) \subset p$-$co(x_n^*)_n$ if and only if $\|Tx\| \le \|(x_n^*(x))_n\|_p$ for every $x \in X$.*

(d) *If $(x_n^*) \in c_0^{w^*}(X^*)$, then $T^*(B_{Y^*}) \subset \infty$-$co(x_n^*)_n$ if and only if $\|Tx\| \le \|(x_n^*(x))_n\|_\infty$ for every $x \in X$.*

From Corollary 3.6, we have:

Theorem 3.7. *Let X and Y be Banach spaces and let $T : X \to Y$ be a linear map.*

(a) *For $1 \le p < \infty$, $T \in \mathcal{N}_{wp}^Q(X,Y)$ if and only if $T^* \in \mathcal{W}_p(Y^*, X^*)$. In this case, $\|T\|_{\mathcal{N}_{wp}^Q} = \|T^*\|_{\mathcal{W}_p}$.*

(b) *If $T^* \in \mathcal{W}_\infty(Y^*, X^*)$, then $T \in \mathcal{N}_{w\infty}^Q(X,Y)$ and $\|T\|_{\mathcal{N}_{w\infty}^Q} \le \|T^*\|_{\mathcal{W}_\infty}$.*

(c) *For $1 \le p \le \infty$, if $T \in \mathcal{W}_p(X,Y)$, then $T^* \in \mathcal{N}_{wp}^Q(Y^*, X^*)$ and $\|T^*\|_{\mathcal{N}_{wp}^Q} \le \|T\|_{\mathcal{W}_p}$.*

4. The maximal hull and minimal kernel of the ideal of weakly p-nuclear operators

A Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is said to *be associated to* a tensor norm α if the canonical map from $\mathcal{A}(M,N)$ to the tensor product $M^* \otimes_\alpha N$ for every finite-dimensional normed spaces M and N is an isometry.

A tensor norm α is called *right-injective* if for every isometry $I : Y \to Z$, the operator

$$id_X \otimes I : X \otimes_\alpha Y \to X \otimes_\alpha Z$$

is an isometry for all Banach spaces X, Y, Z, where id_X is the identity map on X. If the transposed tensor norm α^t of α is right-injective, then the tensor norm α is called *left-injective*. It is well known that there exist the unique largest right-injective tensor norm $\alpha\backslash$ and left-injective tensor norm $/\alpha$ smaller than a tensor norm α (cf. [1, Theorem 20.7]).

Let $1 \le p \le \infty$. For $u \in X \otimes Y$, define

$$w_p(u) = \inf\Big\{\|(x_j)_j\|_p^w \|(y_j)_j\|_{p^*}^w : u = \sum_{j=1}^n x_j \otimes y_j, n \in \mathbb{N}\Big\}.$$

Then w_p is a finitely generated tensor norm (cf. [1, Section 12]) and $\mathfrak{K}_p$ is associated to w_p (cf. [1, Sections 17.12 and 22.3]).

Recall that $\ell_p^w(N) = \ell_p^u(N)$ $(1 \le p < \infty)$ and $c_0^w(N) = c_0(N)$, $c_0^{w^*}(N^*) = c_0(N^*)$ for every finite-dimensional normed space N.

Proposition 4.1. *Let $1 \le p \le \infty$.*

(a) *The ideal $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]$ is associated to the tensor norm w_p.*
(b) *The ideal $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{sur}$ is associated to the tensor norm $/w_p$.*
(c) *The ideal $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{inj}$ is associated to the tensor norm $w_p\backslash$.*

Proof. (a) For every finite-dimensional normed spaces M and N, we have the following isometries;

$$M^* \otimes_{w_p} N \to \mathfrak{K}_p(M, N) \to \mathcal{N}_{wp}(M, N).$$

(b) From Theorem 3.2 and [10, Proposition 3.3], we have the conclusion.

(c) Let β be a finitely generated tensor norm associated to $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{inj}$. Then by [8, Lemma 3.2(b)], β is right injective. We use [1, Proposition 20.9(2), (3)] to show that $\beta = w_p\backslash$. Let M be a finite-dimensional normed space and let $n \in \mathbb{N}$. Then we have the following isometries;

$$M \otimes_\beta \ell_\infty^n \to \mathcal{N}_{wp}^{inj}(M^*, \ell_\infty^n) \to \mathcal{N}_{wp}(M^*, \ell_\infty^n) \to M \otimes_{w_p} \ell_\infty^n.$$

Hence $\beta = w_p\backslash$. □

Let $[\mathcal{A}, \|\cdot\|_\mathcal{A}]$ be a Banach operator ideal. For $T \in \mathcal{L}(X, Y)$, let

$$\|T\|_{\mathcal{A}^{\max}} := \sup\{\|q_L T I_M\|_\mathcal{A} : M \text{ is a finite-dimensional subspace of } X,$$
$$L \text{ is a cofinite-dimensional subspace of } Y\},$$

where $I_M : M \to X$ is the inclusion map and $q_L : Y \to Y/L$ is the quotient map, and let

$$\mathcal{A}^{\max}(X, Y) := \{T \in \mathcal{L}(X, Y) : \|T\|_{\mathcal{A}^{\max}} < \infty\}.$$

Then we call $[\mathcal{A}, \|\cdot\|_\mathcal{A}]^{\max} := [\mathcal{A}^{\max}, \|\cdot\|_{\mathcal{A}^{\max}}]$ the *maximal hull* of $[\mathcal{A}, \|\cdot\|_\mathcal{A}]$. If $[\mathcal{A}, \|\cdot\|_\mathcal{A}] = [\mathcal{A}, \|\cdot\|_\mathcal{A}]^{\max}$, then $[\mathcal{A}, \|\cdot\|_\mathcal{A}]$ is called *maximal*.

The *minimal kernel* of $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is defined by

$$[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{\min} := [\overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}}, \|\cdot\|_{\overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}}}],$$

where $\overline{\mathcal{F}}$ is the ideal of approximable operators. $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is called *minimal* if $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}] = [\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{\min}$.

A Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is called *right-accessible* if for all finite-dimensional normed space M, Banach space Y, $T \in \mathcal{L}(M, Y)$ and $\varepsilon > 0$, there exist a finite-dimensional subspace N of Y and an $S \in \mathcal{L}(M, N)$ such that $T = I_N S$ and $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$.

$[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is called *left-accessible* if for all Banach space X, finite-dimensional normed space N, $T \in \mathcal{L}(X, N)$ and $\varepsilon > 0$, there exist a cofinite-dimensional subspace L of X and an $S \in \mathcal{L}(X/L, N)$ such that $T = Sq_L$ and $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$. A left- and right-accessible Banach operator ideal is called *accessible*.

$[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is called *totally accessible* if for all Banach spaces X and Y, finite rank $T \in \mathcal{L}(X, Y)$ and $\varepsilon > 0$, there exist a cofinite-dimensional subspace L of X, finite-dimensional subspace N of Y and an $S \in \mathcal{L}(X/L, N)$ such that $T = I_N S q_L$ and $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$.

Corollary 4.2. *Let $1 \leq p \leq \infty$.*

(a) $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{\max}$, $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]$ *and* $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{\min}$ *are all accessible.*

(b) $([\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{sur})^{\max}$, $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{sur}$ *and* $([\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{sur})^{\min}$ *are all totally accessible.*

(c) $([\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{inj})^{\max}$, $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{inj}$ *and* $([\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{inj})^{\min}$ *are all totally accessible.*

Proof. (a) By Proposition 4.1(a) and [1, Proposition 21.3, Theorem 21.5(a) and Ex. 21.2(b)].

(b) By Proposition 4.1(b) and [1, the symmetric version of Proposition 21.1(2)].

(c) By Proposition 4.1(c) and [1, Proposition 21.1(2)]. □

We denote the *ideal of p-factorable operators* by $\mathcal{L}_p$ (see [1, Section 18]). Then $\mathcal{L}_p$ is maximal and it is associated to w_p (see [1, Section 17.12]).

Corollary 4.3. *Let $1 \leq p \leq \infty$.*

(a) $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{\max} = [\mathcal{L}_p, \|\cdot\|_{\mathcal{L}_p}]$ *and* $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{\min} = [\mathfrak{K}_p, \|\cdot\|_{\mathfrak{K}_p}]$.

(b) $([\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{sur})^{\max} = [\mathcal{L}_p, \|\cdot\|_{\mathcal{L}_p}]^{sur}$ *and* $(\mathcal{N}_{wp}^{sur})^{\min}(X, Y)$ *is isometric to* $X^* \hat{\otimes}_{/w_p} Y$ *for all Banach spaces X and Y.*

(c) $([\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]^{inj})^{\max} = [\mathcal{L}_p, \|\cdot\|_{\mathcal{L}_p}]^{inj}$ *and* $(\mathcal{N}_{wp}^{inj})^{\min}(X, Y)$ *is isometric to* $X^* \hat{\otimes}_{w_p\backslash} Y$ *for all Banach spaces X and Y.*

Proof. (a) By uniqueness of maximal and minimal operator ideals associated to finitely generated tensor norms.

(b) and (c): By Proposition 4.1(b), (c) and [1, Theorem 20.11(1), (2)], the first parts hold. By [1, Corollary 22.2], the second parts hold. □

Acknowledgement. The author would like to express a sincere gratitude to the referee for valuable comments. This work was supported by NRF-2018R1D1A1B07043566, funded by the Korean Government.

References

[1] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Mathematics Studies, 176, North-Holland Publishing Co., Amsterdam, 1993.

[2] J. M. Delgado, C. Piñeiro, and E. Serrano, *Density of finite rank operators in the Banach space of p-compact operators*, J. Math. Anal. Appl. **370** (2010), no. 2, 498–505.

[3] ———, *Operators whose adjoints are quasi p-nuclear*, Studia Math. **197** (2010), no. 3, 291–304.

[4] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, **43**, Cambridge University Press, Cambridge, 1995.

[5] J. H. Fourie, *Injective and surjective hulls of classical p-compact operators with application to unconditionally p-compact operators*, Studia Math. **240** (2018), no. 2, 147–159.

[6] J. Fourie and J. Swart, *Banach ideals of p-compact operators*, Manuscripta Math. **26** (1978/79), no. 4, 349–362.

[7] ———, *Tensor products and Banach ideals of p-compact operators*, Manuscripta Math. **35** (1981), no. 3, 343–351.

[8] D. Galicer, S. Lassalle, and P. Turco, *The ideal of p-compact operators: a tensor product approach*, Studia Math. **211** (2012), no. 3, 269–286.

[9] J. M. Kim, *Unconditionally p-null sequences and unconditionally p-compact operators*, Studia Math. **224** (2014), no. 2, 133–142.

[10] ———, *The ideal of unconditionally p-compact operators*, Rocky Mountain J. Math. **47** (2017), no. 7, 2277–2293.

[11] A. Persson and A. Pietsch, *p-nukleare une p-integrale Abbildungen in Banachräumen*, Studia Math. **33** (1969), 19–62.

[12] A. Pietsch, *Operator Ideals*, translated from German by the author, North-Holland Mathematical Library, **20**, North-Holland Publishing Co., Amsterdam, 1980.

[13] N. Tomczak-Jaegermann, *Banach-Mazur Distances and Finite-Dimensional Operator Ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, **38**, Longman Scientific & Technical, Harlow, 1989.

[14] D. P. Sinha and A. K. Karn, *Compact operators whose adjoints factor through subspaces of l_p*, Studia Math. **150** (2002), no. 1, 17–33.

Ju Myung Kim
Department of Mathematics
Sejong University
Seoul 05006, Korea
Email address: kjm21@sejong.ac.kr