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Q-MEASURES ON THE DUAL UNIT BALL OF A JB*-TRIPLE

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Abstract. Let A be a JB*-triple with Banach dual space A^* and bi-dual the JBW*-triple A^{**}. Elements x of A^* of norm one may be regarded as normalised 'Q-measures' defined on the complete ortho-lattice $\tilde{\mathcal{U}}(A^{**})$ of tripotents in A^{**}. A Q-measure x possesses a support $e(x)$ in $\tilde{\mathcal{U}}(A^{**})$ and a compact support $e_c(x)$ in the complete atomic lattice $\tilde{\mathcal{U}}_c(A)$ of elements of $\tilde{\mathcal{U}}(A^{**})$ compact relative to A. Necessary and sufficient conditions for an element v of $\tilde{\mathcal{U}}_c(A)$ to be a compact support tripotent $e_c(x)$ are given, one of which is related to the Q-covering numbers of v by families of elements of $\tilde{\mathcal{U}}_c(A)$.

1. Introduction

A complex Banach space A the open unit ball in which is a bounded symmetric domain automatically possesses a unique triple product $(a, b, c) \mapsto \{a\,b\,c\}$ with respect to which A is a JB*-triple, and every JB*-triple arises in this manner. Examples of JB*-triples are C*-algebras, JB*-algebras, the Cartan factor $B(H, K)$ of bounded linear operators from a complex Hilbert space H to a complex Hilbert space K, spin triples, and the exceptional Cartan factors $H_3(\mathbb{O})$ and $M_{1,2}(\mathbb{O})$ of hermitian 3×3 matrices and 1×2 matrices over the complex octonions $\mathbb{O}$.

A JB*-triple B that is the dual of a Banach space B_* is known as a JBW*-triple in which case B_* is unique up to isometric isomorphism and is therefore said to be the *pre-dual* of B. The bi-dual A^{**} of a JB*-triple A is a JBW*-triple with unique pre-dual A^*.

In part, because they provide a link between complex function theory and topological algebra, JB*-triples have been the subject of much investigation over many years. However, their influence has also been substantial because of the connections between the algebraic properties of a JB*-triple A and its

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bi-dual JBW*-triple A^{**} and the geometric properties of the unit balls in A, A^*, and A^{**}.

Briefly, using terms that will be defined precisely in Sections §2-3, these arise in the following manner. Let $\tilde{\mathcal{U}}(A^{**})$ be the complete ortho-lattice of tripotents in A^{**} and let $\tilde{\mathcal{U}}_c(A)$ be the complete atomic lattice of elements of $\tilde{\mathcal{U}}(A^{**})$ compact relative to A. Elements x of A^* of norm one will be referred to as *Q-measures* on $\tilde{\mathcal{U}}(A^{**})$, the *support* of such a Q-measure x being an element $e(x)$ of $\tilde{\mathcal{U}}(A^{**})$ and the *compact support* of which is an element $e_c(x)$ of $\tilde{\mathcal{U}}_c(A)$. The main results provide conditions under which an element v of $\tilde{\mathcal{U}}_c(A)$ is the compact support $e_c(x)$ of a Q-measure x. Whilst the geometric aspect of this problem was discussed in [18], the condition at the centre of this paper relates to the properties of Q-coverings of v by families of elements of $\tilde{\mathcal{U}}(A^{**})$. Incidentally, this result also relates to the question of when a unital C*-algebra possesses a faithful state.

The motivation for the continuing efforts to investigate the properties of JB*-triples not only comes from their central rôle in complex holomorphy, algebra, and geometry, but it is also the case that the pre-dual Banach spaces of JBW*-triples, and, consequently, the dual Banach spaces of JB*-triples, have been proposed as models of state spaces of statistical physical systems. As a consequence, the results obtained in this paper and its many precursors all have physical interpretations. It should be added that these JB*-triple models include as sub-models the usual JBW*-algebra models for quantum theory. The interested reader is referred to [16, 17, 30–33].

The paper is organised as follows. The second section contains a miscellaneous collection of definitions and general information that is needed in order to provide a setting in which the main results can be investigated. The setting itself is described in §3 and the main results are given in §4. The final section is used to illustrate the results by examining in some detail the example in which the JB*-triple A is the unital commutative C*-algebra of all complex-valued continuous functions on a compact Hausdorff space. Much of what appears in §5 can be extracted from [11], to the authors of which a debt of gratitude is owed for providing the authors of this paper with an early sighting of their interesting piece of work.

2. Preliminaries

In this section some, but not all, definitions and properties of those mathematical structures that will be used throughout the paper, are presented, along with references to more detailed accounts and some new approaches.

2.1. Recall that a partially ordered set $\mathcal{P}$ is said to be a *lattice* if, for each pair u and v in $\mathcal{P}$, the supremum $u \vee v$ and the infimum $u \wedge v$ exist with respect to the partial ordering. The partially ordered set $\mathcal{P}$ is said to be a *complete lattice* if, for any subset $\{v_j : j \in \Lambda\}$ of $\mathcal{P}$, the supremum $\vee_{j\in\Lambda} v_j$ and the infimum $\wedge_{j\in\Lambda} v_j$ exist. A complete lattice possesses a greatest element ω and a least element 0. A subset $\mathcal{Q}$ of a lattice $\mathcal{P}$ is said to be a *sub-lattice* if, for all elements u and v in $\mathcal{Q}$, both $u \vee v$ and $u \wedge v$ lie in $\mathcal{Q}$. If $\mathcal{P}$ is a complete lattice and $\mathcal{Q}$ is a sublattice of $\mathcal{P}$ for which the supremum and infimum of arbitrary families of elements of $\mathcal{Q}$ lie in $\mathcal{Q}$, then $\mathcal{Q}$ is said to be a sub-complete lattice of $\mathcal{P}$. A sub-lattice $\mathcal{Q}$ of the lattice $\mathcal{P}$ is said to be an *order ideal* in $\mathcal{P}$ if, for elements u in $\mathcal{Q}$ and v in $\mathcal{P}$ with $v \leq u$, then v also lies in $\mathcal{Q}$, and is said to be an *anti-order ideal* in $\mathcal{P}$ if, for elements u in $\mathcal{Q}$ and v in $\mathcal{P}$ with $u \leq v$, then v also lies $\mathcal{Q}$. A complete lattice $\mathcal{P}$ is said to be *atomic* if, for each element u in $\mathcal{P}$, there exists a non-zero minimal element e such that $e \leq u$. A complete lattice $\mathcal{P}$ along with an anti-order automorphism $u \mapsto u^\perp$ on $\mathcal{P}$ such that, for all elements u and v in $\mathcal{P}$,

$$u \vee u^\perp = 1, \quad u^{\perp\perp} = u,$$

and if $u \leq v$, then

$$v = u \vee (v \wedge u^\perp),$$

is said to be *orthomodular*.

2.2. Let V be a complex (or real) vector space and let C be a convex subset of V. A convex subset F of C is said to be a *face* of C provided that, if x_1 and x_2 lie in C and for some real number t in the open unit interval $(0,1)$ the element $tx_1 + (1-t)x_2$ lies in F, then x_1 and x_2 lie in F. A one-point face $\{x\}$ of C is said to be an *extreme point* of C. Let τ be a locally convex Hausdorff topology on V, let C be a τ-closed convex subset of V, and let $\mathcal{F}_\tau(C)$ denote the set of all τ-closed faces of C. Both $\emptyset$ and C lie in $\mathcal{F}_\tau(C)$ and the intersection of an arbitrary family of elements of $\mathcal{F}_\tau(C)$ also lies in $\mathcal{F}_\tau(C)$. When the infimum of an arbitrary family $\{F_j : j \in \Lambda\}$ of elements of $\mathcal{F}_\tau(C)$ is defined by their intersection and the supremum is defined by

$$\vee_{j\in\Lambda} F_j = \wedge\{F \in \mathcal{F}_\tau(C) : F_j \subseteq F, \ \forall j \in \Lambda\},$$

the set $\mathcal{F}_\tau(C)$ forms a complete lattice. A subset E of C is said to be a *τ-exposed face* of C if there exist a τ-continuous complex linear functional a on V and a real number λ such that, for all elements x in E, the real part $\mathrm{Re}\, a(x)$ of $a(x)$ is equal to λ and, for all elements x in $C \setminus E$,

$$\mathrm{Re}\, a(x) < \lambda.$$

Let $\mathcal{E}_\tau(C)$ denote the family of τ-exposed faces of C and observe that the intersection of a finite number of elements of $\mathcal{E}_\tau(C)$ also lies in $\mathcal{E}_\tau(C)$, that $\emptyset$ and C lie in $\mathcal{E}_\tau(C)$, and that $\mathcal{E}_\tau(C)$ is contained in $\mathcal{F}_\tau(C)$. The intersection of an arbitrary family of elements of $\mathcal{E}_\tau(C)$ is said to be a *τ-semi-exposed* face of C. The intersection of a family of elements of the set $\mathcal{S}_\tau(C)$ of τ-semi-exposed faces of C also lies in $\mathcal{S}_\tau(C)$. Therefore, with respect to the ordering by set inclusion, $\mathcal{S}_\tau(C)$ forms a complete lattice the infimum of a family of elements of $\mathcal{S}_\tau(C)$ coinciding with its infimum in $\mathcal{F}_\tau(C)$.

2.3. Let A be a complex, or real, Banach space with dual space A^* and bi-dual space A^{**}, with closed unit balls A_1, A_1^*, and A_1^{**}, respectively. For each subset

F of A_1 and G of A_1^*, let

$$F' = \{x \in A_1^* : x(a) = 1, \forall a \in F\}, \quad G_{\prime} = \{a \in A_1 : x(a) = 1, \forall x \in G\}.$$

Then, F lies in $\mathcal{S}_{\sigma(A,A^*)}(A_1)$ if and only if F coincides with $(F')_{\prime}$ and G lies in $\mathcal{S}_{\sigma(A^*,A)}(A_1^*)$ if and only if G coincides with $(G_{\prime})'$ and the mappings $F \mapsto F'$ and $G \mapsto G_{\prime}$ are anti-order isomorphisms between the complete lattices $\mathcal{S}_{\sigma(A,A^*)}(A_1)$ and $\mathcal{S}_{\sigma(A^*,A)}(A_1^*)$ and are inverses of each other. Similarly, for subsets G of A_1^* and H of A_1^{**}, let

$$G' = \{a \in A_1^{**} : x(a) = 1, \forall x \in G\}, \quad H_{\prime} = \{x \in A_1^* : x(a) = 1, \forall a \in H\}.$$

Then, G lies in $\mathcal{S}_{\sigma(A^*,A^{**})}(A_1^*)$ if and only if G coincides with $(G')_{\prime}$ and H lies in $\mathcal{S}_{\sigma(A^{**},A^*)}(A_1^{**})$ if and only if H coincides with $(H')_{\prime}$, and, as before, the mappings $G \mapsto G'$ and $H \mapsto H_{\prime}$ are anti-order isomorphisms between the complete lattices $\mathcal{S}_{\sigma(A^*,A^{**})}(A_1^*)$ and $\mathcal{S}_{\sigma(A^{**},A^*)}(A_1^{**})$ and are inverses of each other. For details, the reader is referred to [19, 21].

2.4. Observe that the complete lattice $\mathcal{S}_{\sigma(A^*,A^{**})}(A_1^*)$ contains the complete lattice $\mathcal{S}_{\sigma(A^*,A)}(A_1^*)$ and the infimum of a family of elements of the complete lattice $\mathcal{S}_{\sigma(A^*,A)}(A_1^*)$ coincides with its infimum taken in $\mathcal{S}_{\sigma(A^*,A^{**})}(A_1^*)$. For each element G of $\mathcal{S}_{\sigma(A^*,A^{**})}(A_1^*)$, let G^{c} be the element of $\mathcal{S}_{\sigma(A^*,A)}(A_1^*)$ defined by

$$G^{\mathrm{c}} = (G_{\prime})'. \tag{1}$$

Then, G^{c} is said to be the *Q-closure* of G. Notice that

$$G^{\mathrm{c}} = \wedge_{\{G_1 \in \mathcal{S}_{\sigma(A^*,A)}(A_1^*) : G \leq G_1\}} G_1 = \cap_{\{G_1 \in \mathcal{S}_{\sigma(A^*,A)}(A_1^*) : G \subseteq G_1\}} G_1. \tag{2}$$

An element G of $\mathcal{S}_{\sigma(A^*,A^{**})}(A_1^*)$ is said to be *Q-dense* in an element K of $\mathcal{S}_{\sigma(A^*,A)}(A_1^*)$ when G^{c} coincides with K.

2.5. For each element x of norm one in A_1^* the element $(\{x\}')_{\prime}$ of $\mathcal{S}_{\sigma(A^*,A^{**})}(A_1^*)$ is said to be the *support* $E(x)$ of x and the element $(\{x\}_{\prime})'$ of $\mathcal{S}_{\sigma(A^*,A)}(A_1^*)$ is said to be the *compact support* $E_c(x)$ of x. Notice that $E(x)$ is contained in $E_c(x)$ and that it is not possible that $E(x)$ coincides with A_1^* which would imply that the weak*-continuous complex-valued function x on the weak*-compact unit ball A_1^{**} in A^{**} had a non-compact range. On the other hand it is possible that $E_c(x)$ coincides with A_1^*.

2.6. Elements x and y of A^* are said to be *L-orthogonal*, denoted by $x \Diamond y$, if

$$\|x \pm y\| = \|x\| + \|y\|,$$

and elements a and b in A^{**} are said to be *M-orthogonal*, denoted by $a \,\square\, b$, if

$$\|a \pm b\| = \max\{\|a\|, \|b\|\}.$$

If a and b are of unit norm and M-orthogonal, then the elements $\{a\}_{\prime}$ and $\{b\}_{\prime}$ of $\mathcal{S}_{\sigma(A^*,A^{**})}(A_1^*)$ form L-orthogonal sets. Let x and y be elements of A^* of norm one and let $E(x)$ and $E(y)$ be the corresponding elements of $\mathcal{S}_{\sigma(A^*,A^{**})}(A_1^*)$. Then x and y form an L-orthogonal pair if and only if $E(x)$ and $E(y)$ form an L-orthogonal pair. For details of these and other results of this nature the reader is referred to [25].

2.7. A Jordan*-algebra B which is also a complex Banach space such that, for all elements a and b in B,

$$\|a^*\| = \|a\|, \quad \|a \circ b\| \leq \|a\|\,\|b\|$$

and

$$\|\{a\,a\,a\}\| = \|a\|^3,$$

where

$$\{a\,b\,c\} = a \circ (b^* \circ c) + c \circ (b^* \circ a) - b^* \circ (a \circ c) \tag{3}$$

is the Jordan triple product on B, is said to be a *Jordan C^*-algebra* [52] or *JB^*-algebra* [53]. A Jordan C*-algebra which is the dual of a Banach space is said to be a *Jordan W^*-algebra* [14], or a *JBW^*-algebra* [53]. Examples of JB*-algebras are C*-algebras and examples of JBW*-algebras are W*-algebras in both cases equipped with the Jordan product

$$a \circ b = \tfrac{1}{2}(ab + ba).$$

The self-adjoint parts of JB*-algebras and JBW*-algebras are said to be *JB-algebras* and *JBW-algebras*, respectively. The bidual A^{**} of a JB*-algebra A forms a JBW*-algebra, the bidual $(A_{\mathrm{sa}})^{**}$ of the JB-algebra A_{sa} coinciding with the JBW-algebra A^{**}_{sa}. The set of elements $\{a \circ a : a \in A_{\mathrm{sa}}\}$ in the self-adjoint part A_{sa} of the JB*-algebra A forms a norm-closed generating cone A_+ for A_{sa}. In the case in which B is a JBW*-algebra, B_+ is $\sigma(B, B_*)$-closed. Every JBW*-algebra B possesses a unit element which is an extreme point of the unit ball in B. For the relevant properties of C*-algebras and W*-algebras the reader is referred to [2, 26, 45, 46, 49] and for properties of Jordan algebras to [4, 5, 35, 38, 41–44, 47].

The set $\mathcal{P}(B)$ of self-adjoint idempotents, the *projections*, in a JBW*-algebra B forms a complete orthomodular lattice with respect to the partial ordering defined, for elements p and q in $\mathcal{P}(A)$, by $p \leq q$ if

$$p \circ q = p$$

and the mapping $p \mapsto p^{\perp}$ defined by

$$p^{\perp} = u - p,$$

where u is the unit in B. The ordering of $\mathcal{P}(B)$ coincides with the restriction to $\mathcal{P}(B)$ of the ordering of B_{sa} defined by B_+.

2.8. A complex Banach space A that possesses a triple product $(a, b, c) \mapsto \{a\,b\,c\}$ from $A \times A \times A$ to A that is symmetric and linear in the first and third variables, conjugate linear in the second variable, and, for elements a, b, c and d in A, satisfies the identity

$$[D(a, b), D(c, d)] = D(\{a\,b\,c\}, d) - D(c, \{d\,a\,b\}),$$

where [,] denotes the commutator, and D and Q are the mappings from $A \times A$ to the sets of linear and conjugate linear operators on A, respectively, defined by

$$D(a,b)c = \{a\,b\,c\} = Q(a,c)b,$$

D is norm-continuous from $A \times A$ to the Banach algebra of bounded linear operators on A, and, for each element a in A, $D(a,a)$ is hermitian in the sense of [10], Definition 5.1, with non-negative spectrum, such that

$$\|D(a,a)\| = \|a\|^2,$$

is said to be a *JB***-triple*. A complex Banach space possesses a triple product with respect to which it is a JB*-triple if and only if its open unit ball is a bounded symmetric domain, in which case the triple product is unique. A norm-closed subtriple J of a JB*-triple A is itself a JB*-triple and J is said to be a norm-closed *inner ideal* if the set $\{J\,A\,J\}$ is contained in J. A JB*-triple B that is the dual of a Banach space B_* is said to be a *JBW***-triple*. In this case the *predual* B_* of B is unique up to isometric isomorphism and, for elements a and b in B, the operators $D(a,b)$ and $Q(a,c)$ are $\sigma(B,B_*)$-continuous. The elements a and b of the JB*-triple B are said to be *orthogonal* when $D(a,b)$ is equal to zero. Observe that a pair a and b of elements of B are orthogonal if and only if they are M-orthogonal. In this case $D(b,a)$ is also equal to zero. Examples of JB*-triples are JB*-algebras and examples of JBW*-triples are JBW*-algebras. The bidual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to [7, 8, 12, 13, 34, 37, 39–41, 50, 51].

2.9. An element u in a JBW*-triple B is said to be a *tripotent* if $\{u\,u\,u\}$ is equal to u. The set of tripotents in B is denoted by $\mathcal{U}(B)$. For each tripotent u in the JBW*-triple B and, for j equal to 0, 1, or 2, the $\sigma(B,B_*)$-continuous linear operators, $P_j(u)$, are defined by

$$P_2(u) = Q(u,u)^2,$$

$$P_1(u) = 2(D(u,u) - Q(u,u)^2), \qquad P_0(u) = \mathrm{id}_B - 2D(u,u) + Q(u,u)^2.$$

The results of [7, 8] show that the linear operators $P_j(u)$, are $\sigma(B,B_*)$-continuous contractive projections onto the eigenspaces $B_j(u)$ of $D(u,u)$ corresponding to eigenvalues $j/2$. The corresponding decomposition

$$B = B_0(u) \oplus B_1(u) \oplus B_2(u)$$

is said to be the *Peirce decomposition* of B relative to u. For j, k, l equal to 0, 1, or 2, $B_j(u)$ is a sub-JBW*-triple such that

$$\{B_j(u)\,B_k(u)\,B_l(u)\} \subseteq B_{j-k+l}(u)$$

when $j-k+l$ is equal to 0, 1, or 2, and is equal to $\{0\}$ otherwise. Moreover,

$$\{B_2(u)\,B_0(u)\,B\} = \{B_0(u)\,B_2(u)\,B\} = \{0\}.$$

With respect to the product $(a,b) \mapsto a \circ_u b$ and involution $a \mapsto a^{\dagger_u}$ defined by

$$a \circ_u b = \{a\,u\,b\}, \quad a^{\dagger_u} = \{u\,a\,u\},$$

$B_2(u)$ is a JBW*-algebra with unit u, the Jordan algebra triple product (3) in which coincides with the restriction of that in B to $B_2(u)$.

2.10. Recall that a *GL-space* (*complete base norm space*) W is a real Banach space partially ordered by a norm-closed cone W_+ such that the norm is additive on W_+ and the unit ball W_1 in W coincides with the convex hull $\mathrm{conv}((W_+ \cap W_1) \cup (-W_+ \cap W_1))$ of the set $(W_+ \cap W_1) \cup (-W_+ \cap W_1)$. Then, the set K of elements of W^+ of norm one forms a base for W_+ such that W_1 coincides with $\mathrm{conv}(K \cup (-K))$. A *unital GM-space* (*complete order unit space*) V is a real Banach space partially-ordered by a norm-closed cone V_+ such that the open unit ball in V is upward filtering and possesses a maximal element u in which case the unit ball V_1 coincides with the order interval

$$[-u, u] = (-u + V_+) \cap (u - V_+).$$

The Banach dual space W^* of the GL-space W above endowed with the ordering defined by the dual cone W^*_+ and unit u defined as the element of W^*_+ that takes the value one on the set K, is a unital GM-space. A JBW-algebra B with unit u and positive cone B_+, defined as in §2.7, forms a unital GM-space, its unique pre-dual B_* being a GL-space, the base K of its positive cone being described as the *normal state space* of B. For properties of GL-spaces and GM-spaces the reader is referred to [3, 6, 19, 20].

3. The setting

The previous section mainly contained material of a general nature that will be needed in the description of the subject of main interest of this paper. Some of the approaches that it takes may be of more general interest. This section provides the setting in which the properties of JB*-triples investigated in this paper can be described.

Let A be a JB*-triple with dual A^* and bi-dual JBW*-triple A^{**}, and let $\mathcal{U}(A^{**})$ be the set of tripotents in A^{**}. For elements u and v of $\mathcal{U}(A^{**})$, write $u \le v$ when

$$\{u\,v\,u\} = u.$$

This defines a partial ordering of the set $\mathcal{U}(A^{**})$ which has a least element zero. Observe that the following conditions on elements u and v of $\mathcal{U}(A^{**})$ are equivalent:

$$\{u\,u\,v\} = 0; \quad u \perp v; \quad u \,\square\, v = 0,$$

and the relation is symmetric. Moreover, for elements u and v of $\mathcal{U}(A^{**})$, $v \perp v$ if and only if v is equal to zero, if $u \perp v$, then $u \vee v$ exists and is equal $u + v$, if $u \le v$ there exists uniquely w in $\mathcal{U}(A^{**})$ such that $w \perp u$ and $u \vee w$ is equal to v, and if u, v, and w lie in $\mathcal{U}(A^{**})$ and are such that $u \le v$ and $v \perp w$, then $u \perp w$. For details the reader is referred to [9, 22–24].

Let a be an element of A^{**} of norm one and let (a^{2n-1}) be the sequence of elements of A^{**} defined inductively, for $n = 1, 2, \ldots$ by

$$a^{2n+1} = \{a\,a^{2n-1}\,a\}.$$

Then the sequence (a^{2n+1}) converges in the $\sigma(A^{**}, A^*)$-topology to an element $u(a)$ in $\mathcal{U}(A^{**})$. An element u of $\mathcal{U}(A^{**})$ is said to be *compact* relative to A if, either, u is equal to zero or there exists a family $(a_j)_{j\in\Lambda}$ of elements of A_1 of norm one such that the set $(u(a_j))_{j\in\Lambda}$ of elements of $\mathcal{U}(A^{**})$ is a decreasing net converging in the $\sigma(A^{**}, A^*)$-topology to its infimum u. Let $\mathcal{U}_c(A)$ denote the partially ordered subset of elements of $\mathcal{U}(A^{**})$ that are compact relative to A. Let $\tilde{\mathcal{U}}(A^{**})$ and $\tilde{\mathcal{U}}_c(A)$ respectively denote the sets $\mathcal{U}(A^{**})\cup\{\omega\}$ and $\mathcal{U}_c(A)\cup\{\omega\}$ with partial ordering extended from $\mathcal{U}(A^{**})$ by defining ω to be the largest element. The properties of $\tilde{\mathcal{U}}(A^{**})$ and $\tilde{\mathcal{U}}_c(A)$ and their relationship to the facial structure of A_1^* are listed below. The reader is referred to [15,22,23,25,29] for details.

Theorem 3.1. *Let A be a JB*-triple with dual A^* and bidual JBW*-triple A^{**}, and let A_1, A_1^*, and A_1^{**} be the unit balls in A, A^*, and A^{**}, respectively. Then, the following results hold.*

- (i) (a) *The complete lattices $\mathcal{F}_{\sigma(A^*,A^{**})}(A_1^*)$ of $\sigma(A^*, A^{**})$-closed faces of A_1^* and $\mathcal{S}_{\sigma(A^*,A^{**})}(A_1^*)$ of $\sigma(A^*, A^{**})$-semi-exposed faces of A_1^* coincide.*
  - (b) *The partially ordered set $\tilde{\mathcal{U}}(A^{**})$ is a complete lattice and the mapping $u \mapsto \{u\}_{\prime}$, where $\{\omega\}_{\prime}$ is equal to A_1^*, is an order isomorphism from $\tilde{\mathcal{U}}(A^{**})$ onto the complete lattice $\mathcal{F}_{\sigma(A^*,A^{**})}(A_1^*)$ such that, for elements u and v in $\tilde{\mathcal{U}}(A^{**})$, $u \perp v$ if and only if $\{u\}_{\prime}\Diamond\{v\}_{\prime}$.*
- (ii) (a) *The complete lattices $\mathcal{F}_{\sigma(A^{**},A^*)}(A_1^{**})$ of $\sigma(A^{**}, A^*)$-closed faces of A_1^{**} and $\mathcal{S}_{\sigma(A^{**},A^*)}(A_1^{**})$ of $\sigma(A^{**}, A^*)$-semi-exposed faces of A_1^{**} coincide.*
  - (b) *The mapping $u \mapsto (\{u\}_{\prime})'$ is an anti-order isomorphism from the complete lattice $\tilde{\mathcal{U}}(A^{**})$ onto the complete lattice $\mathcal{F}_{\sigma(A^{**},A^*)}(A_1^{**})$.*
- (iii) (a) *The complete lattices $\mathcal{F}_{\sigma(A^*,A)}(A_1^*)$ of $\sigma(A^*, A)$-closed faces of A_1^* and $\mathcal{S}_{\sigma(A^*,A)}(A_1^*)$ of $\sigma(A^*, A)$-semi-exposed faces of A_1^* coincide.*
  - (b) *The partially ordered set $\tilde{\mathcal{U}}_c(A)$ is a complete lattice and the mapping $u \mapsto \{u\}_{\prime}$, where $\{\omega\}_{\prime}$ is equal to A_1^*, is an order isomorphism from the complete lattice $\tilde{\mathcal{U}}_c(A)$ onto the complete lattice $\mathcal{F}_{\sigma(A^*,A)}(A_1^*)$.*
- (iv) (a) *The complete lattices $\mathcal{F}_{\sigma(A,A^*)}(A_1)$ of $\sigma(A, A^*)$-closed faces of A_1 and $\mathcal{S}_{\sigma(A,A^*)}(A_1)$ of $\sigma(A, A^*)$-semi-exposed faces of A_1 coincide.*
  - (b) *The mapping $u \mapsto (\{u\}_{\prime})_{\prime}$ is an anti-order isomorphism from the complete lattice $\tilde{\mathcal{U}}_c(A)$ onto the complete lattice $\mathcal{F}_{\sigma(A,A^*)}(A_1)$.*

As a consequence of Theorem 3.1(i)(b), $\tilde{\mathcal{U}}(A^{**})$ will be referred to as the *complete ortho-lattice of tripotents* in A^{**}. It should also be said that the results of Theorem 3.1(i)-(ii) hold when A^{**} is replaced by any JBW*-triple B and A^* by the unique pre-dual B_* of B. The same remark applies to the following result, the proof of which can be found in [24].

Corollary 3.2. *Under the conditions of Theorem 3.1 the following results hold.*

(i) *For an increasing net $(v_j)_{j\in\Lambda}$ in $\mathcal{U}(A^{**})$, $\vee_{j\in\Lambda}v_j$ lies in $\mathcal{U}(A^{**})$ and if u is an element of $\mathcal{U}(A^{**})$ such that, for all elements j in Λ, $u \perp v_j$, then $u \perp \vee_{j\in\Lambda}v_j$.*

(ii) *Let $(v_j)_{j\in\Lambda}$ be an orthogonal family in $\mathcal{U}(A^{**})$, let Λ^f be the set of finite subsets of Λ, upward-directed by set inclusion, and for γ in Λ^f, let*

$$v_\gamma = \Sigma_{j\in\gamma}v_j.$$

Then, $\vee_{j\in\Lambda}v_j$ exists in $\mathcal{U}(A^{**})$ and is equal to the $\sigma(A^{**},A^*)$-limit of the $\sigma(A^{**},A^*)$-convergent increasing net $(v_\gamma)_{\gamma\in\Lambda^f}$.

(iii) *For an element u in $\mathcal{U}(A^{**})$ the principal order ideal $\mathcal{I}_u$ of elements v in $\tilde{\mathcal{U}}(A^{**})$ such that $v \leq u$ coincides with the complete ortho-lattice of self-adjoint idempotents in the JBW^*-algebra $A_2^{**}(u)$.*

A further consequence of Theorem 3.1 is that the results of §2.4-2.5 may be related to the properties of $\tilde{\mathcal{U}}(A^{**})$. For each element w of $\tilde{\mathcal{U}}(A^{**})$, let w^{c} be the unique element of $\tilde{\mathcal{U}}_c(A)$ such that

$$\{w^{\mathrm{c}}\}_{\prime} = ((\{w\}_{\prime})_{\prime})'. \tag{4}$$

Then, w^{c} is said to be the *Q-closure* of w. An element w of $\tilde{\mathcal{U}}(A^{**})$ is said to be *Q-dense* in an element v of $\tilde{\mathcal{U}}_c(A)$ if w^{c} coincides with v.

A further consequence of Theorem 3.1, using §2.5, is that, for each element x of A_1^* of norm one, there exist a unique element $e(x)$ of $\tilde{\mathcal{U}}(A^{**})$, known as the *support* of x, such that

$$(\{x\}')_{\prime} = E(x) = \{e(x)\}_{\prime}, \tag{5}$$

and a unique element $e_c(x)$ of $\tilde{\mathcal{U}}_c(A)$, known as the *compact support* of x, such that

$$(\{x\}_{\prime})' = E_c(x) = \{e_c(x)\}_{\prime}. \tag{6}$$

It follows from §2.4-2.5 that, for any element x of A_1^* of norm one, $e(x)^{\mathrm{c}}$ coincides with $e_c(x)$ and, whilst it is possible that $e_c(x)$ is equal to the largest element ω in $\tilde{\mathcal{U}}(A^{**})$ the same does not hold for $e(x)$.

Since, for each element u of $\tilde{\mathcal{U}}_c(A)$ that is not equal to ω or zero, $\{u\}_{\prime}$ is a proper, non-empty $\sigma(A^*,A)$-closed face of A_1^*, the Krein-Milman Theorem shows that $\{u\}_{\prime}$ contains an extreme point x in which case,

$$\{x\} = E(x) = E_c(x) = \{e(x)\}_{\prime} = \{e_c(x)\}_{\prime} \subseteq \{u\}_{\prime}.$$

It follows that the complete lattices $\tilde{\mathcal{U}}_c(A)$ and $\mathcal{F}_{\sigma(A^*,A)}(A_1^*)$ are atomic. Consequently, $\tilde{\mathcal{U}}_c(A)$ will be referred to as the *complete atomic lattice of compact tripotents* in A^{**}. It should be remembered that although it is the case that the infimum of a family of elements of $\tilde{\mathcal{U}}_c(A)$ when taken in $\tilde{\mathcal{U}}_c(A)$ coincides with that taken in the complete lattice $\tilde{\mathcal{U}}(A^{**})$ the same does not, in general, hold for the supremum.

Observe that, from §2.6, a family $(x_j)_{j\in\Lambda}$ of elements of norm one in A_1^* is L-orthogonal if and only the family $(E(x_j))_{j\in\Lambda}$ is L-orthogonal in $\mathcal{F}_{\sigma(A^*,A^{**})}(A_1^*)$

if and only if the family $(e(x_j))_{j\in\Lambda}$ is orthogonal in $\tilde{\mathcal{U}}(A^{**})$. An element u of $\tilde{\mathcal{U}}(A^{**})$ is said to be *σ-finite* if it does not majorize an uncountable orthogonal subset of $\tilde{\mathcal{U}}(A^{**})$. It follows from Theorem 3.1 that, similarly, a $\sigma(A^*, A^{**})$-closed face G of A_1^* may be said to be σ-finite if it does not contain an uncountable family of L-orthogonal $\sigma(A^*, A^{**})$-closed faces of A_1^*. The following corollary of Theorem 3.1 the proof of which can be found in [24] gives two important properties of orthogonal subsets.

Corollary 3.3. *Under the conditions of Theorem 3.1, the following results hold.*

(i) *An element u in $\tilde{\mathcal{U}}(A^{**})$ is σ-finite if and only if there exists an element x in A_1^* such that u coincides with the support tripotent $e(x)$ of x.*
(ii) *For each element u of $\mathcal{U}(A^{**})$ there exists a maximal L-orthogonal subset $\{z_j : j \in \Lambda\}$ of $\{u\}_\prime$ such that u coincides with $\vee_{j\in\Lambda} e(z_j)$.*

4. Main results

Many papers have been written about the properties of a JB*-triple A in some of which an important part is played by the Q-topology discussed above, the classical example of which is the compact Hausdorff topology of the pure state space of a commutative unital C*-algebra. In this paper some answers are given to the question of, when studying the Q-structure related to a JB*-triple A, under what conditions is it the case that a particular Q-closed element v is the compact support $e_c(x)$ of a Q-measure x. Using Theorem 3.1, the question may be interpreted as one about $\sigma(A^*, A)$-closed faces of A_1^* or about compact tripotents in A^{**}. The results depend upon the analogue of 'covering numbers' of families of elements. The connection between this study and the classical problem is described in the final section of the paper.

The first result is concerned with principal order ideals defined by compact tripotents in the spirit of Corollary 3.2(iii).

Lemma 4.1. *Let A be a JB*-triple, with dual Banach space A^* and bidual JBW*-triple A^{**}, and let u be an element of the complete atomic lattice $\tilde{\mathcal{U}}_c(A)$ of compact tripotents in A^{**}. Then, the set*

$$\mathcal{C}_u = \{v \in \tilde{\mathcal{U}}_c(A) : 0 \le v \le u\} \tag{7}$$

is a principal order ideal in the complete atomic lattice $\tilde{\mathcal{U}}_c(A)$ and the mapping $v \mapsto \{v\}_\prime$ is an order isomorphism from $\mathcal{C}_u$ onto the complete atomic lattice $\mathcal{F}_{\sigma(A^*,A)}(\{u\}_\prime)$ of $\sigma(A^*, A)$-closed faces of the $\sigma(A^*, A)$-compact convex set $\{u\}_\prime$.

Proof. Observe that when u is equal to ω then $\mathcal{C}_u$ coincides with $\tilde{\mathcal{U}}_c(A)$ and the result is tautological. When u is not equal to ω then, by Corollary 3.2(iii), $\{u\}_\prime$ can be identified with the normal state space of the JBW*-algebra $A_2^{**}(u)$ or the JBW-algebra $A_2^{**}(u)_{\mathrm{sa}}$. Using §2.10, it follows that the convex hull

$\mathrm{conv}(\{u\}_{\prime} \cup \{-u\}_{\prime})$ is the unit ball in the unique predual $A_2^{**}(u)_{*,\mathrm{sa}}$ of the JBW-algebra $A_2^{**}(u)_{\mathrm{sa}}$. Since u is compact, by Theorem 3.1(iii)(b), $\{u\}_{\prime}$ is $\sigma(A^*, A)$-closed and, therefore, the unit ball in $A_2^{**}(u)_{*,\mathrm{sa}}$ is $\sigma(A^*, A)$-compact. The Krein-Smulian theorem ensures that $A_2^{**}(u)_{*,\mathrm{sa}}$ is $\sigma(A^*, A)$-closed in A^* and that the $\sigma(A^*, A)$-compact convex set $\{u\}_{\prime}$ is regularly embedded in $A_2^{**}(u)_{*,\mathrm{sa}}$. It follows from [3], Propositions II.2-II.2.4, that the $\sigma(A^*, A)$-compact convex set $\{u\}_{\prime}$ can be identified, up to affine homeomorphism, with the state space of the unital GM-space of $\sigma(A^*, A)$-continuous real affine functions on $\{u\}_{\prime}$. The proof is then completed by [18], Theorem 3.1(ii) and Corollary 3.2(i). □

The second result and its proof may be found in [28], Theorem 3.9.

Lemma 4.2. *Let A be a JB*-triple, with dual Banach space A^* and bidual JBW*-triple A^{**}, and let $\{v_j : j = 1, 2, \ldots, n\}$ be a family of pairwise orthogonal elements of the complete atomic lattice $\tilde{\mathcal{U}}_c(A)$ of compact tripotents in A^{**}. Then, $\Sigma_{j=1}^n v_j$ is an element of $\tilde{\mathcal{U}}_c(A)$.*

The next result summarises the properties of the Q-topology of the unit ball A_1^* in A^* the proof of most of which follows immediately from Theorem 3.1 and (4)-(6).

Lemma 4.3. *Let A be a JB*-triple with dual Banach space A^* and bidual JBW*-triple A^{**}, let $\tilde{\mathcal{U}}(A^{**})$ be the complete ortho-lattice of tripotents in A^{**}, and let $\tilde{\mathcal{U}}_c(A)$ be the complete atomic lattice of compact tripotents in A^{**}. Then, the following results hold.*

(i) *For each element w in $\tilde{\mathcal{U}}(A^{**})$, there exists uniquely a smallest element w^{c} in $\tilde{\mathcal{U}}_c(A)$ such that*
$$w \leq w^{\mathrm{c}}$$
given by
$$w^{\mathrm{c}} = \wedge_{\{v \in \tilde{\mathcal{U}}_c(A) : w \leq v\}} v,$$
for which
$$(\{w^{\mathrm{c}}\}_{\prime})_{\prime} = (\{w\}_{\prime})_{\prime}.$$

(ii) *The mapping $w \mapsto w^{\mathrm{c}}$ has the following properties*:
  - (a) $0^{\mathrm{c}} = 0$, $\omega^{\mathrm{c}} = \omega$;
  - (b) *an element w of $\tilde{\mathcal{U}}(A^{**})$ lies in $\tilde{\mathcal{U}}_c(A)$ if and only if w and w^{c} coincide*;
  - (c) *for each family $\{w_j : j \in \Lambda\}$ of elements of $\tilde{\mathcal{U}}(A^{**})$,*
$$(\vee_{j \in \Lambda} w_j)^{\mathrm{c}} = (\vee_{j \in \Lambda} w_j^{\mathrm{c}})^{\mathrm{c}}.$$

(iii) *For each element x in A^* of norm one, let $e(x)$ and $e_c(x)$ be the elements of $\tilde{\mathcal{U}}(A^{**})$ and $\tilde{\mathcal{U}}_c(A)$ defined by*
$$\{e(x)\}_{\prime} = (\{x\}')_{\prime} = E(x), \quad \{e_c(x)\}_{\prime} = (\{x\}_{\prime})' = E_{\mathrm{c}}(x),$$
respectively. Then
$$e(x)^{\mathrm{c}} = e_c(x).$$

Proof. The proofs of (i), (ii)(a)-(ii)(b), and (iii) follow immediately from Theorem 3.1 using §2.3-2.5. In order to prove (ii)(c), notice that $(\vee_{j\in\Lambda}w_j)^{\mathrm{c}}$ exists and may be equal to ω in which case the result holds by (ii)(a). If $(\vee_{j\in\Lambda}w_j)^{\mathrm{c}}$ is not equal to ω, then observe that, for each element k in Λ,

$$w_k \leq \vee_{j\in\Lambda}w_j$$

and, hence,

$$w_k^{\mathrm{c}} \leq (\vee_{j\in\Lambda}w_j)^{\mathrm{c}}.$$

Therefore,

$$\vee_{k\in\Lambda}w_k^{\mathrm{c}} \leq (\vee_{j\in\Lambda}w_j)^{\mathrm{c}},$$

and, hence,

$$(\vee_{k\in\Lambda}w_k^{\mathrm{c}})^{\mathrm{c}} \leq (\vee_{j\in\Lambda}w_j)^{\mathrm{c}}. \tag{8}$$

On the other hand, it is clear that

$$\vee_{j\in\Lambda}w_j \leq (\vee_{k\in\Lambda}w_k^{\mathrm{c}})^{\mathrm{c}}$$

and, hence, that

$$(\vee_{j\in\Lambda}w_j)^{\mathrm{c}} \leq (\vee_{k\in\Lambda}w_k^{\mathrm{c}})^{\mathrm{c}}. \tag{9}$$

The proof is completed by (8)-(9). □

As pointed out in §3, for each element w in $\tilde{\mathcal{U}}(A^{**})$ the element w^{c} of $\tilde{\mathcal{U}}_c(A)$ described in Lemma 4.3 is said to be the *Q-closure* of w. An element w in $\tilde{\mathcal{U}}(A^{**})$ is said to be *Q-dense* in an element v of $\tilde{\mathcal{U}}_c(A)$ if w^{c} coincides with v. Observe that, by Lemma 4.3(ii)(c), the supremum of an arbitrary family of elements of $\tilde{\mathcal{U}}(A^{**})$ each of which is Q-dense in v is also Q-dense in v and, since v is Q-dense in itself, the set of elements that are Q-dense in v is non-empty.

The next result connects the Q-topological concepts described above with the properties of L-orthogonal sets on the surface of the unit ball A_1^* in the dual space A^* of the JB*-triple A that were introduced in Corollary 3.3.

Lemma 4.4. *Let A be a JB*-triple with dual Banach space A^* and bidual JBW*-triple A^{**}, let $\tilde{\mathcal{U}}(A^{**})$ be the complete ortho-lattice of tripotents in A^{**}, and let $\tilde{\mathcal{U}}_c(A)$ be the complete atomic lattice of compact tripotents in A^{**}. Then, the following results hold.*

(i) *Let w be an element of $\mathcal{U}(A^{**})$ and let $\{z_j : j \in \Lambda\}$ be a maximal L-orthogonal subset of $\{w\}_{\prime}$ such that w coincides with $\vee_{j\in\Lambda}e(z_j)$. Then, the Q-closure w^{c} of w is given by*

$$w^{\mathrm{c}} = (\vee_{j\in\Lambda}e_c(z_j))^{\mathrm{c}}. \tag{10}$$

(ii) *Let v be an element of $\mathcal{U}_c(A)$, let w be an element of $\mathcal{U}(A^{**})$ that is majorized by v, and let $\{z_j : j \in \Lambda_1\}$ and $\{z_j : j \in \Lambda_2\}$ be maximal L-orthogonal families of elements of $\{w\}_{\prime}$ and $\{v - w\}_{\prime}$, respectively. Let Λ be the union of Λ_1 and Λ_2, let Λ_1^f, Λ_2^f, and Λ^f be the sets of finite* *subsets of* Λ_1, Λ_2, *and* Λ, *partially ordered by set inclusion, respectively, and for* γ *contained in* Λ_1^f, Λ_2^f, *or* Λ^f, *let*

$$e_\gamma = \sum_{j\in\gamma} e(z_j).$$

Then, $\{e_\gamma : \gamma \in \Lambda_1^f\}$, $\{e_\gamma : \gamma \in \Lambda_2^f\}$, *and* $\{e_\gamma : \gamma \in \Lambda^f\}$ *form increasing nets of elements in* $\tilde{\mathcal{U}}(A^{**})$ *that converge in the* $\sigma(A^{**}, A^*)$*-topology to* w, $v - w$, *and* v, *respectively, and*

$$w = \vee_{j\in\Lambda_1} e(z_j), \quad v - w = \vee_{j\in\Lambda_2} e(z_j), \quad v = \vee_{j\in\Lambda} e(z_j).$$

(iii) *Under the conditions of* (ii), *suppose that* w *is Q-dense in* v. *Then*

$$v = (\vee_{j\in\Lambda_1} e(z_j))^{\mathrm{c}} = (\vee_{j\in\Lambda_1} e_c(z_j))^{\mathrm{c}}. \tag{11}$$

Proof. (i) The formula (10) follows immediately from Corollary 3.2(ii) and Lemma 4.3(ii)(c) and (iii).

(ii) The existence of the maximal L-orthogonal families in $\{w\}_\prime$ and in $\{v - w\}_\prime$ is guaranteed by Corollary 3.2(ii) and that $\{e(z_j) : j \in \Lambda\}$ is an orthogonal family of tripotents dominated by v follows from Corollary 3.2(i). It remains to show that it is maximal. If not, then there exists an element u in $\tilde{\mathcal{U}}(A^{**})$ majorized by v and orthogonal to the set $\{e(z_j) : j \in \Lambda\}$. Then, using Corollary 3.2(i), u is orthogonal to both w and $v - w$, and, therefore, equal to zero. It follows that the orthogonal family $\{e(z_j) : j \in \Lambda\}$ majorized by v is maximal, as required.

(iii) The equation (11) follows from (i) and Lemma 4.3(iii). □

The following result, that is, in part, proved in [18], gives a criterion for a tripotent v in the JBW*-triple A^{**} to be a compact support tripotent $e_c(x)$ of an element x on the surface of the unit ball A_1^*. It was shown in the paper cited above that this is equivalent to the corresponding $\sigma(A, A^*)$-closed face $(\{v\}_\prime)_\prime$ of A_1 being $\sigma(A, A^*)$-exposed.

Theorem 4.5. *Let* A *be a JB*-triple with dual Banach space* A^* *and bidual JBW*-triple* A^{**}, *let* $\tilde{\mathcal{U}}(A^{**})$ *be the complete ortho-lattice of tripotents in* A^{**} *and let* v *be an element of the complete atomic lattice* $\tilde{\mathcal{U}}_c(A)$ *of compact tripotents in* $\tilde{\mathcal{U}}(A^{**})$. *Then, there exists an element* x, *of norm one, in the unit ball* A_1^* *in* A^* *such that* v *coincides with the compact support tripotent* $e_c(x)$ *if and only if there exists a* σ*-finite element* w *of* $\tilde{\mathcal{U}}(A^{**})$ *that is Q-dense in* v.

Proof. Suppose that v is equal to $e_c(x)$ for some element x of norm one in A^*. Let w be equal to $e(x)$, which, by Corollary 3.3(i), is σ-finite. Moreover, by Lemma 4.3(iii), $e(x)$ is Q-dense in $e_c(x)$ and the first part of the proof is complete. Conversely, suppose that the element v of $\mathcal{U}_c(A)$ majorizes a Q-dense σ-finite tripotent w. Then, there exists an element x of norm one in A^* such that w coincides with $e(x)$. It follows that v, which is equal to w^{c}, coincides with $e(x)^{\mathrm{c}}$ which, by Lemma 4.3(iii), coincides with $e_c(x)$. □

It is now possible to investigate an alternative characterisation of the circumstances under which an element v of $\mathcal{U}_c(A)$ is a compact support tripotent. Let u be a further element of $\mathcal{U}_c(A)$ that majorizes v and recall that $\mathcal{C}_u$, defined in (7), denotes the principal order ideal in the complete atomic lattice $\tilde{\mathcal{U}}_c(A)$ corresponding to u. In the remaining part of this section discussions will be restricted to the circumstances that hold in the result above. As a consequence of Corollary 3.2(iii) and Lemma 4.1, it is only necessary to consider the JBW-algebra $A_2^{**}(u)_{\mathrm{sa}}$, its pre-dual $A_2^{**}(u)_{*,\mathrm{sa}}$ and the unital GM-space A_u consisting of all $\sigma(A^*, A)$-continuous real-valued affine functions on the $\sigma(A^*, A)$-compact convex set $\{u\}_\prime$. In this case $A_2^{**}(u)_{\mathrm{sa}}$ is the bidual of the unital GM-space A_u, the ordering, unit, and norm as a unital GM-space agreeing with those as a JBW-algebra. In particular, the order interval $\mathcal{C}_u$, defined in (7), consists of idempotents in the JBW-algebra $A_2^{**}(u)_{\mathrm{sa}}$ compact relative to A_u. For further details, the reader is referred to [3, 6, 18–20].

Lemma 4.6. *Let A be a JB*-triple with dual Banach space A^* and bidual JBW*-triple A^{**}, let $\tilde{\mathcal{U}}(A^{**})$ be the complete ortho-lattice of tripotents in A^{**}, let v be an element of the complete atomic lattice $\tilde{\mathcal{U}}_{\mathrm{c}}(A)$ of compact tripotents in A^{**}, not equal to zero or ω, let*

$$\mathcal{C}_v = \{q \in \tilde{\mathcal{U}}_{\mathrm{c}}(A) : 0 \le q \le v\},$$

let w be an element of $\tilde{\mathcal{U}}(A^{**})$ that is Q-dense in v, and let $\{z_j : j \in \Lambda\}$ be a maximal L-orthogonal family of elements of $\{w\}_\prime$ such that w coincides with $\vee_{j\in\Lambda} e(z_j)$. For each element j in Λ, let

$$\mathcal{D}^{(j)} = \{q \in \mathcal{C}_v \setminus \{v\} : q \wedge e(z_j) \neq 0\}.$$

Then, $\mathcal{D}^{(j)}$ is an anti-order ideal in $\mathcal{C}_v$, closed under finite orthogonal sums, such that

$$\mathcal{C}_v \setminus \{0\} = (\vee_{j\in\Lambda}\mathcal{D}^{(j)})^{\mathrm{c}}. \tag{12}$$

Proof. The existence of the family $\{\mathcal{D}^{(j)} : j \in \Lambda\}$ is a consequence of Lemmas 4.1-4.4. Moreover, it is clear that if q_1 and q_2 are elements of $\mathcal{C}_v$, not equal to v, with q_1 in $\mathcal{D}^{(j)}$ and q_1 majorized by q_2, then

$$0 \neq q_1 \wedge e(z_j) \le q_2 \wedge e(z_j),$$

and q_2 lies in $\mathcal{D}^{(j)}$. Therefore $\mathcal{D}^{(j)}$ is an anti-order ideal in $\mathcal{C}_v$. By Lemma 4.2, finite sums of orthogonal compact tripotents are compact, and, hence, $\mathcal{D}^{(j)}$ is closed under the formation of such sums.

It therefore remains to show that (12) holds. Observe that, for each element j in Λ, the anti-order ideal $\mathcal{D}^{(j)}$ does not contain zero and is contained in $\mathcal{C}_v$. It follows that

$$\vee_{j\in\Lambda}\mathcal{D}^{(j)} \subseteq \mathcal{C}_v.$$

Let q be an element of $\mathcal{C}_v$ and, for j in Λ, let

$$q^{(j)} = q \wedge e_c(z_j).$$

Since the infimum of a family of elements of $\tilde{\mathcal{U}}_c(A)$ also lies in $\tilde{\mathcal{U}}_c(A)$ it follows that $q^{(j)}$ lies in $\mathcal{C}_v$ or is equal to zero. Moreover, if $q^{(j)}$ is non-zero, then

$$q^{(j)} \wedge e(z_j) = q \wedge e_c(z_j) \wedge e(z_j) = q \wedge e(z_j) \neq 0,$$

and it follows that $q^{(j)}$ is contained in $\mathcal{D}^{(j)}$ or is equal to zero.

The remainder of the proof consists of showing that q coincides with $(\vee_{j\in\Lambda} q^{(j)})^{\mathrm{c}}$. As in Lemma 4.4,

$$w = \vee_{j\in\Lambda} e(z_j),$$

where

$$v = w^{\mathrm{c}} = (\vee_{j\in\Lambda} e_c(z_j))^{\mathrm{c}}.$$

Recall that $\{e(z_j)\}_{\prime}$ coincides with $(\{z_j\}')_{\prime}$, $\{e_c(z_j)\}_{\prime}$ coincides with$(\{z_j\}_{\prime})'$ and, since q is compact, $((\{q\}_{\prime})_{\prime})'$ coincides with $\{q\}_{\prime}$. Then,

$$\begin{aligned}\{(q \wedge e(z_j))^{\mathrm{c}}\}_{\prime} &= ((\{q \wedge e(z_j)\}_{\prime})_{\prime})' = ((\{q\}_{\prime} \cap \{e(z_j)\}_{\prime})_{\prime})' \\ &= ((\{q\}_{\prime} \cap (\{z_j\}')_{\prime})_{\prime})' = ((\{q\}_{\prime})_{\prime} \vee ((\{z_j\}')_{\prime})_{\prime})' \\ &= ((\{q\}_{\prime})_{\prime})' \cap (((\{z_j\}')_{\prime})_{\prime})' = \{q\}_{\prime} \cap \{e_c(z_j)\}_{\prime} \\ &= \{q \wedge e_c(z_j)\}_{\prime},\end{aligned}$$

and

$$(q \wedge e(z_j))^{\mathrm{c}} = q \wedge e_c(z_j). \tag{13}$$

For each element k in Λ, using (13),

$$\begin{aligned} q \wedge e(z_k) &\leq q \wedge e_c(z_k) \leq \vee_{j\in\Lambda} q \wedge e_c(z_j) \\ &= \vee_{j\in\Lambda} (q \wedge e(z_j))^{\mathrm{c}} \leq (\vee_{j\in\Lambda} (q \wedge e(z_j))^{\mathrm{c}})^{\mathrm{c}}. \end{aligned} \tag{14}$$

Moreover, using Corollary 3.3(ii) and (14),

$$\begin{aligned} q \wedge w = q \wedge (\vee_{j\in\Lambda} e(z_j)) &= q \wedge \lim_{\sigma(A^{**},A^*)} \{\Sigma_{j\in\gamma} e(z_j) : \gamma \in \Lambda^f\} \\ &= \lim_{\sigma(A^{**},A^*)} \{\Sigma_{j\in\gamma} q \wedge e(z_j) : \gamma \in \Lambda^f\} \leq \vee_{j\in\Lambda} q \wedge e(z_j) \\ &\leq (\vee_{j\in\Lambda} q \wedge e_c(z_j))^{\mathrm{c}}, \end{aligned} \tag{15}$$

and,

$$\begin{aligned} \{(q \wedge w)^{\mathrm{c}}\}_{\prime} &= ((\{q \wedge w\}_{\prime})_{\prime})' = ((\{q\}_{\prime} \cap \{w\}_{\prime})_{\prime})' = ((\{q\}_{\prime})_{\prime} \vee (\{w\}_{\prime})_{\prime})' \\ &= ((\{q\}_{\prime})_{\prime})' \cap ((\{w\}_{\prime})_{\prime})' = \{q\}_{\prime} \cap \{w^{\mathrm{c}}\}_{\prime} = \{q\}_{\prime} \cap \{v\}_{\prime} \\ &= \{q\}_{\prime}. \end{aligned} \tag{16}$$

It follows from (15)-(16) that

$$q = (q \wedge w)^{\mathrm{c}} \leq (\vee_{j\in\Lambda} q \wedge e_c(z_j))^{\mathrm{c}} = (\vee_{j\in\Lambda} q^{(j)})^{\mathrm{c}} \leq q,$$

thereby completing the proof of (12). □

A further property of elements of the family $\{\mathcal{D}^{(j)} : j \in \Lambda\}$ of anti-order ideals of self-adjoint idempotents in the JBW-algebra $A_2^{**}(u)_{\mathrm{sa}}$, compact relative to the unital GM-space A_u, defined above, is given by the following result.

Lemma 4.7. *Under the conditions of Lemma 4.6, for each finite subset S of $\mathcal{D}^{(j)}$ given by*

$$S = \{q_1, q_2, \ldots, q_n\}$$

the covering number $c(S)$ of S, defined by

$$c(S) = \inf_{y \in \{v\}'} \Sigma_{k=1}^{n} n^{-1} y(q_k), \tag{17}$$

is less than one.

Proof. For the set S first suppose that

$$(\vee_{k=1}^{n} q_k)^{\mathrm{c}} \neq v. \tag{18}$$

Then, by [18], Lemma 4.3, there exists an element z in the set $\partial_e\{v\}'$ of extreme points of the $\sigma(A^*, A)$-closed face $\{v\}'$ such that $e(z)$, which coincides with $e_c(z)$, is orthogonal to $(\vee_{k=1}^{n} q_k)^{\mathrm{c}}$ and, therefore, to each element of S, in which case, using §2.6, for k equal to $1, 2, \ldots, n$, $z(q_k)$ is equal to zero. It follows that

$$c(S) \leq \Sigma_{k=1}^{n} n^{-1} z(q_k) = 0,$$

and $c(S)$ is equal to zero. Now suppose that, contrary to (18),

$$(\vee_{k=1}^{n} q_k)^{\mathrm{c}} = v.$$

Consider the set $\{q_1, q_2, \ldots, q_{n-1}\}$ and first suppose that

$$(\vee_{k=1}^{n-1} q_k)^{\mathrm{c}} \neq v.$$

Then, as before, there exists an element z in $\partial_e\{v\}'$ such that, for k equal to $1, 2, \ldots, n-1$, $z(q_k)$ is equal to zero and $z(q_n)$ is equal to a real number between zero and one. It follows that, in this case,

$$c(S) \leq \Sigma_{j=1}^{n} n^{-1} z(q_k) \leq 1/n.$$

If

$$(\vee_{k=1}^{n-1} q_k)^{\mathrm{c}} = v,$$

then the process can be repeated, reaching the conclusion that either $c(S)$ is less than or equal to $2/n$ or that

$$(\vee_{k=1}^{n-2} q_k)^{\mathrm{c}} = v.$$

Consequently, this procedure will terminate when only $\{q_1, q_2\}$ remain, the conclusion being that, either $c(S)$ is less than or equal to $1 - 2/n$ or that $(q_1 \vee q_2)^{\mathrm{c}}$ is equal to v. Since, neither of q_1 or q_2 is equal to v it follows that it is possible to find an element z in $\partial_e\{v\}'$ such that $z(q_1)$ is equal to zero, and for k equal to $2, 3, \ldots, n$, $z(q_k)$ lies between zero and one. It then follows that

$$c(S) \leq 1 - 1/n.$$

Observe that, if S consists of a single element $\{q_1\}$, then q_1 is not equal to v and it is possible to find an element z of $\{v\}_{\prime}$ such that $z(q_1)$ is equal to zero implying that $c(S)$ is equal to zero. This completes the proof of the lemma. □

The previous results may now be combined in order to present a new criterion for determining when an element v of the of the complete atomic lattice $\tilde{\mathcal{U}}_c(A)$ of tripotents in A^{**} compact relative to A, not equal to the adjoined largest element ω, is a compact support tripotent of an element x of A^* of norm one.

Theorem 4.8. *Let A be a JB*-triple with dual A^* and bidual JBW*-triple A^{**}, let u and v be tripotents in A^{**}, compact relative to A, with v majorized by u, let $\mathcal{C}_v$ be the complete atomic lattice of self-adjoint idempotents in the JBW*-algebra $A_2^{**}(v)$ compact relative to the unital GM-space A_v of $\sigma(A^*, A)$-continuous real-valued affine functions on $\{v\}_{\prime}$, and let $\{\mathcal{D}^{(j)} : j \in \Lambda\}$ be the family of anti-order ideals in $\mathcal{C}_v$ constructed in Lemma 4.6. Then, there exists an element x in A^* of norm one such that v is the compact support tripotent $e_c(x)$ of x if and only if the family $\{\mathcal{D}^{(j)} : j \in \Lambda\}$ is countable.*

Proof. The result follows immediately from Theorem 4.5 using Lemmas 4.1, 4.6, and 4.7. □

Under the conditions of Theorem 4.8, let v be an element of $\mathcal{C}_u$ and let A_v be the unital GM-space of $\sigma(A^*, A)$-continuous real affine functions on $\{v\}_{\prime}$. The unit in A_v is the $\sigma(A^*, A)$-continuous real-valued affine function taking the constant value one on $\{v\}_{\prime}$ and, therefore, can be identified with v, in which case $\{v\}_{\prime}$ and $\{v\}'$ may also be identified. Elements of $\{v\}_{\prime}$ are said to be *states* of A_v and a state x is said to be *faithful* if, for an element a in the positive cone $A_{v,+}$ of the unital GM-space A_v, $x(a)$ is equal to zero then a is equal to zero.

A particular case of the situation described above occurs when A is a unital C*-algebra in which case the self-adjoint part A_{sa} is a unital GM-space. It was shown in [48] that a separable unital C*-algebra always possesses a faithful state as does a C*-algebra of operators on a separable Hilbert space. The following result is certainly well known and its proof, given below, can be reconstructed from those of related results in [1] and [27].

Lemma 4.9. *Let A be a unital C*-algebra with unit v and self-adjoint part the unital JB-algebra and unital GM-space A_v, and let x be an element of the state space $\{v\}'$ of A. Then, x is a faithful state of A_v if and only if the compact support tripotent $e_c(x)$ of x coincides with v.*

Proof. Observe that, since x is contained in $\{v\}'$, $e_c(x)$ is majorized by v and is, therefore, a projection in the JBW-algebra A_{sa}^{**}.

First suppose that v and $e_c(x)$ coincide and let a be an element of $A_{e_c(x),+}$ such that $x(a)$ is equal to zero. Then, a is contained in the order interval

$$[0, \|a\|e_c(x)] = \{b \in A_{e_c(x)} : 0 \leq b \leq \|a\|e_c(x)\}$$

and

(19) $$x(\|a\|e_c(x) - a) = \|a\|x(e_c(x)) - x(a) = \|a\|.$$

Using (19),

$$x(e_c(x) - a/\|a\|) = 1,$$

and $e_c(x) - a/\|a\|$ lies in $\{x\}_{\prime}$, which coincides with $\{e_c(x)\}$. It follows that a is equal to zero and x is faithful, as required.

Conversely, if x is a faithful state of A_v, then $e_c(x)$ is majorized by v, and suppose that $e_c(x)$ is not equal to v. As in the proof of [18], Lemma 4.3, there exists an element z in the set $\partial_{\mathrm{e}}\{v\}'$ of extreme points of the $\sigma(A^*, A)$-compact state space $\{v\}'$ such that $e_c(z)$, which coincides with $e(z)$, is orthogonal to $e_c(x)$ and

$$e_c(x) + e_c(z) \leq v.$$

Observe that $e_c(x)$ and $e_c(z)$ are orthogonal compact projections in the W*-algebra A^{**}. Using [27], Theorem 1.4, there exist orthogonal positive elements a and c of norm one in $(\{e_c(x)\}_{\prime})_{\prime}$ and $(\{e_c(z)\}_{\prime})_{\prime}$, respectively, such that

$$0 \leq a + c \leq v$$

which implies that

$$0 \leq x(a) + x(c) \leq x(v)$$

and, since $x(a)$ and $x(v)$ are both equal to one, $x(c)$ is equal to zero. Since x is faithful it follows that c is equal to zero, yielding the required contradiction, which implies that $e_c(x)$ and v coincide. □

The following corollary of Theorems 4.5 and 4.8 connects the results above with the question of when a unital C*-algebra possesses a faithful state.

Corollary 4.10. *Let A be a unital C^*-algebra with unit v and bidual W^*-algebra A^{**}. Then, the following are equivalent*:

(i) *A possesses a faithful state.*
(ii) *There exists a σ-finite projection w in A^{**} that is Q-dense in v.*
(iii) *There exists a projection w in A^{**} such that the family $\{\mathcal{D}^{(j)} : j \in \Lambda\}$ of anti-order ideals of the complete atomic lattice $\mathcal{C}_v$ of compact projections in A^{**}, constructed as in Lemma 4.6, is countable.*

5. The classical example

The results of the preceding section are best illustrated by examining the situation in which A is a unital commutative C*-algebra. Much of what appears below can be extracted from [11].

Let (Ω, ψ) be a compact Hausdorff space and let $C^{\psi}(\Omega)$ denote the JB*-triple of ψ-continuous complex-valued functions on Ω. A positive regular Borel measure x on (Ω, ψ) identified, by the Riesz representation, with a positive linear functional on $C^{\psi}(\Omega)$ is said to be *normal* if for each bounded increasing net (a_j) of elements of $C^{\psi}(\Omega)$ with supremum a, the net $(x(a_j))$ converges to

$x(a)$. The space (Ω, ψ) is said to be *hyper-Stonean* if it has the two properties that the closure of each element of ψ is itself contained in ψ and that the set of normal positive regular Borel measures on (Ω, ψ) separates points in $C^{\psi}(\Omega)$. The proof of the following result may be found in [46].

Lemma 5.1. *The compact Hausdorff space (Ω, ψ) is hyper-Stonean if and only if the JB^*-triple $C^{\psi}(\Omega)$ is a JBW^*-triple.*

Observe that, in this case, the complex linear span $N^{\psi}(\Omega)$ of the cone of normal positive regular Borel measures on (Ω, ψ) is a norm-closed subspace of the complex Banach space $M^{\psi}(\Omega)$ of all complex regular Borel measures on (Ω, ψ), which, by the Riesz representation, may be identified with the dual space $C^{\psi}(\Omega)^*$ of $C^{\psi}(\Omega)$. Moreover, $N^{\psi}(\Omega)$ is the unique predual of $C^{\psi}(\Omega)$ and the identification of $N^{\psi}(\Omega)$ as a closed subspace of $M^{\psi}(\Omega)$ is merely the isometric embedding of the complex Banach space $N^{\psi}(\Omega)$ into its bidual.

Lemma 5.2. *Under the conditions of Lemma 5.1, the following results hold.*

(i) *Let u be a non-zero element of the set $\mathcal{U}(C^{\psi}(\Omega))$ of tripotents in $C^{\psi}(\Omega)$. Then, there exists a unique ψ-clopen subset E_u of Ω given by*

$$E_u = \{t \in \Omega : |u(t)| = 1\}$$

such that

$$u = \chi_{E_u} u \tag{20}$$

in which case

$$E_u = \Omega \setminus \{t \in \Omega : u(t) = 0\}. \tag{21}$$

(ii) *For non-zero elements u and v of $\mathcal{U}(C^{\psi}(\Omega))$, $u \leq v$ if and only if $E_u \subseteq E_v$ and*

$$u = \chi_{E_u} v, \tag{22}$$

and $u \perp v$ if and only if

$$E_u \cap E_v = \emptyset.$$

(iii) *Let $(u_j)_{j\in\Lambda}$ be a decreasing net of non-zero elements of $\mathcal{U}(C^{\psi}(\Omega))$. Then, the net has a non-zero infimum u such that E_u coincides with the interior $(\cap_{j\in\Lambda} E_{u_j})^{\circ\psi}$ of the ψ-closed set $\cap_{j\in\Lambda} E_{u_j}$.*

Proof. The proofs of (i) and (ii) are straightforward calculations.

In order to prove (iii) observe that $\{E_{u_j} : j \in \Lambda, j \geq j_0\}$ is a decreasing family of non-empty ψ-clopen sets in Ω such that, for $j \geq j_0$,

$$u_j = \chi_{E_{u_j}} u_{j_0}.$$

Since, for each element j in Λ, the set E_{u_j} is a non-empty compact subset of a compact space it follows that the set $\cap_{\{j\in\Lambda : j\geq j_0\}} E_{u_j}$ is non-empty and compact and equal to $\cap_{j\in\Lambda} E_{u_j}$. Observe that the infimum u of the net $(u_j)_{j\in\Lambda}$ is equal to $\chi_{E_u} u_{j_0}$. It follows that E_u is the largest ψ-clopen set contained in $\cap_{j\in\Lambda} E_{u_j}$ which, from the definition of a hyper-Stonean space, is its interior. □

Before continuing the investigation, for future reference, the following easily demonstrated results concerning the support of elements of norm one in $C^{\psi}(\Omega)$ are included.

Lemma 5.3. *Under the conditions of Lemma 5.2, let a be an element of norm one in $C^{\psi}(\Omega)$. Then, the sequence $(|a|^{2j}a)$ in $C^{\psi}(\Omega)$ converges in the $\sigma(C^{\psi}(\Omega), C^{\psi}(\Omega)_*)$-topology to the non-zero tripotent $u(a)$ in $C^{\psi}(\Omega)$ defined by*

$$u(a) = \chi_{E_{u(a)}} a,$$

where

$$E_{u(a)} = \{t \in \Omega : |a(t)| = 1\}.$$

Lemma 5.4. *Under the conditions of Lemma 5.2, let x be a normal measure on (Ω, ψ), let $\tilde{\mathcal{U}}(C^{\psi}(\Omega))$ be the complete ortho-lattice of tripotents in $C^{\psi}(\Omega)$, and let*

$$e(x) = \wedge\{u \in \tilde{\mathcal{U}}(C^{\psi}(\Omega)) : P_2(u)_* x = x\}.$$

Then

$$E_{e(x)} = \cap\{E_u : u \in \tilde{\mathcal{U}}(C^{\psi}(\Omega)), P_2(u)_* x = x\}$$

Let x be a positive regular Borel measure on (Ω, ψ). Then, the *support* of x is defined to be the ψ-closed subset of Ω given by

$$\mathrm{supp}_{(\Omega,\psi)} x = \Omega \setminus (\cup\{U \in \psi : x(\chi_U) = 0\}).$$

The support of any complex regular Borel measure on (Ω, ψ) is defined to be the support of its modulus. It is now possible to give a proof of the result that identifies the support of x in terms of those in §3.

Theorem 5.5. *Let x be a normal positive measure on the hyper-Stonean space (Ω, ψ) regarded as an element of $C^{\psi}(\Omega)_*$ and let $e(x)$ be the tripotent defined in Lemma 5.4. Then the support of x coincides with $E_{e(x)}$.*

Proof. Observe that

$$\Omega \setminus \mathrm{supp}_{(\Omega,\psi)} x = \cup\{U \in \psi : P_2(\chi_U)_* x = 0\}.$$

Suppose that U lies in ψ and is contained in $\Omega \setminus \mathrm{supp}_{(\Omega,\psi)} x$. Then, $\mathrm{supp}_{(\Omega,\psi)} x \cap U$ is empty, with $\mathrm{supp}_{(\Omega,\psi)} x$ closed and U open. Since (Ω, ψ) is hyper-Stonean and, hence, extremally disconnected, the set $\overline{\mathrm{supp}_{(\Omega,\psi)} x}^{\psi} \cap \overline{U}^{\psi}$ is also empty and it follows that the ψ-clopen set $\overline{U}^{\psi}$ also satisfies the condition that $P_2(\chi_{\overline{U}^{\psi}})_* x$ is equal to zero. It follows that

$$\begin{aligned} \Omega \setminus \mathrm{supp}_{(\Omega,\psi)} x &= \cup\{U : U, \Omega \setminus U \in \psi, P_2(\chi_U)_* x = 0\} \\ &= \cup\{U : U, \Omega \setminus U \in \psi, x(\chi_U) = 0\}. \end{aligned}$$

Hence, by Lemma 5.4,

$$\mathrm{supp}_{(\Omega,\psi)} x = \cap\{V : V, \Omega \setminus V \in \psi, P_2(\chi_V)_* x = x\} = E_{e(x)},$$

as required. □

It is now possible to turn to the properties of compact tripotents. Let (Ω, τ) be a compact Hausdorff space and let $M^\tau(\Omega)$ be the complex Banach space of regular Borel measures on (Ω, τ) which, by means of the Riesz representation, will be identified with the dual space $C^\tau(\Omega)^*$ of the JB*-triple $C^\tau(\Omega)$. Observe that every complex-valued non-zero homomorphism (or character) on $C^\tau(\Omega)$ is of the form δ_t for some unique element t of Ω defined, for each element a of $C^\tau(\Omega)$, by

$$\delta_t(a) = a(t). \tag{23}$$

Moreover, the mapping $t \mapsto \delta_t$ is a homeomorphism from (Ω, τ) onto the set $\partial_{\mathrm{e}}(M^\tau(\Omega)_{+,1})$ of extreme points of the set of positive regular Borel measures on (Ω, τ) of norm one endowed with the relative weak*-topology $\sigma(M^\tau(\Omega), C^\tau(\Omega))$. The bidual $C^\tau(\Omega)^{**}$ is a JBW*-triple, the canonical linear isometry from $C^\tau(\Omega)$ into $C^\tau(\Omega)^{**}$ being an algebraic triple isomorphism. The unique predual $C^\tau(\Omega)^*$ of $C^\tau(\Omega)^{**}$ can be identified with $M^\tau(\Omega)$. The compact Hausdorff space $(\hat{\Omega}, \psi)$ of characters of $C^\tau(\Omega)^{**}$ endowed with the relative $\sigma(C^\tau(\Omega)^{***}, C^\tau(\Omega)^{**})$-topology, denoted by ψ, is such that the JBW*-triple $C^\tau(\Omega)^{**}$ can be identified with $C^\psi(\hat{\Omega})$ the unique predual of which can be identified with any of $C^\tau(\Omega)^*$, $M^\tau(\Omega)$, and $N^\psi(\hat{\Omega})$. The hyper-Stonean space $(\hat{\Omega}, \psi)$ is said to be the *hyper-Stonean envelope* of (Ω, τ).

For t in $\hat{\Omega}$, the linear functional δ_t on $C^\psi(\hat{\Omega})$ defined as in (23), is a character of $C^\psi(\hat{\Omega})$, the restriction of which to $C^\tau(\Omega)$ is clearly also a character. Hence, there exists uniquely an element πt of Ω such that

$$\delta_{\pi t} = \delta_t|_{C^\tau(\Omega)}. \tag{24}$$

Suppose that (t_j) is a net in $\hat{\Omega}$ converging in the ψ-topology to t. Then, for each element a of $C^\tau(\Omega)^{**}$, the net $(a(t_j))$ converges to $a(t)$. In particular, this holds for all elements a in $C^\tau(\Omega)$ which implies that the net (πt_j) in Ω converges in the τ-topology to πt. It follows from (23) and (24) that π is a continuous mapping from $(\hat{\Omega}, \psi)$ to (Ω, τ). Let s be an element of Ω and let ιs be the element of $\hat{\Omega}$ such that, for all elements a of $C^\psi(\hat{\Omega})$,

$$\delta_{\iota s}(a) = a(\delta_s) = a(s), \tag{25}$$

where δ_s is the character in $\partial_{\mathrm{e}} M^\tau(\Omega)_{+,1}$ corresponding to s or, equivalently, the point measure at s of norm one. Observe that, by (25), for s_1 and s_2 in Ω if $\delta_{\iota s_1}$ and $\delta_{\iota s_2}$ coincide, then, for all elements a in $C^\psi(\hat{\Omega})$, and, hence, for all elements a in $C^\tau(\Omega)$, $a(s_1)$ and $a(s_2)$ coincide, which, by the Stone-Weierstrass theorem, implies that s_1 and s_2 coincide. Hence ι is an injection from Ω into $\hat{\Omega}$. Furthermore, for all elements s in Ω and a in $C^\tau(\Omega)$, using (24) and (25),

$$\delta_{\pi\iota s}(a) = \delta_{\iota s}|_{C^\tau(\Omega)}(a) = a(s) = \delta_s(a),$$

and, hence, $\pi\iota s$ coincides with s. It follows that π is a continuous surjection from $(\hat{\Omega}, \psi)$ onto (Ω, τ) and that Ω may be regarded as a subset of $\hat{\Omega}$ possessing two topologies, τ and $\psi|_\Omega$. It can be seen that, for each element s of Ω, the mapping $x \mapsto x(\{s\})$ is a bounded linear functional λ_s on $M^\tau(\Omega)$ and, therefore, an element of $C^\psi(\hat{\Omega})$. It follows that, for all elements s in Ω, when restricted to Ω the function χ_s is ψ-continuous and, hence, that the topology ψ restricted to Ω is discrete.

For each element s of Ω, observe that, since $\{s\}$ is τ-closed and π is continuous, the fibre $\pi^{-1}(\{s\})$ is ψ-closed. Moreover,

$$\hat{\Omega} = \cup\{\pi^{-1}(\{s\}) : s \in \Omega\}.$$

Furthermore, the canonical image of $C^\tau(\Omega)$ in $C^\psi(\hat{\Omega})$ consists of functions that are constant on each of the fibres $\pi^{-1}(\{s\})$. It can be shown that $\pi^{-1}(\Omega)$ is an element of ψ such that $\overline{\pi^{-1}(\Omega)}^{\psi}$ coincides with $\hat{\Omega}$.

It is now possible to identify the tripotents in $C^\psi(\hat{\Omega})$ that are compact relative to $C^\tau(\Omega)$.

Theorem 5.6. *Let (Ω, τ) be a compact Hausdorff space and let $(\hat{\Omega}, \psi)$ be its hyper-Stonean envelope. For each non-zero tripotent u in $C^\psi(\hat{\Omega})$, let*

$$u = \chi_{E_u} u$$

be the unique decomposition of u described in Lemma 5.2, and, for u equal to zero, let E_u be the empty set. Then, the mapping $u \mapsto E_u \cap \Omega$ is an order isomorphism from the complete atomic lattice $\tilde{\mathcal{U}}_c(C^\tau(\Omega))$ of tripotents u in $C^\psi(\hat{\Omega})$ compact relative to $C^\tau(\Omega)$ onto the complete atomic lattice of τ-compact subsets of Ω.

Proof. Suppose that u is compact relative to $C^\tau(\Omega)$. If u is equal to zero, then the result is immediate. If not, then there exists a set $(a_j)_{j\in\Lambda}$ of elements of $C^\tau(\Omega)$ of norm one such that $(u(a_j))_{j\in\Lambda}$ forms a decreasing net in $\mathcal{U}(C^\psi(\hat{\Omega}))$ with infimum u to which it converges in the weak*-topology $\sigma(C^\psi(\hat{\Omega}), M^\tau(\Omega))$. It follows from Lemmas 5.2 and 5.3 that, for j_0 and j in Λ with $j_0 \le j$,

$$u(a_j) = \chi_{E_{u(a_j)}} a_{j_0}, \quad u = \chi_{E_u} a_{j_0},$$

and

$$E_{u(a_j)} \subseteq E_{u(a_{j_0})},$$

where

$$u = \wedge_{j\in\Lambda} u(a_j), \quad E_u = (\cap_{j\in\Lambda} E_{u(a_j)})^{\circ\psi},$$

and, being the intersection of a decreasing family of non-empty ψ-compact sets, $\cap_{j\in\Lambda} E_{u(a_j)}$ is ψ-compact and non-empty. Since the net $(\chi_{E_{u(a_j)}} a_{j_0})_{j\ge j_0}$ converges in the weak*-topology, $\sigma(C^\psi(\hat{\Omega}), M^\tau(\Omega))$ to $\chi_{E_u} a_{j_0}$, and, for all elements s in Ω, the point measures δ_s lie in $M^\tau(\Omega)$, it can be seen that, for all elements s in Ω, the net $(\chi_{E_{u(a_j)}}(s) a_{j_0}(s))_{j\ge j_0}$ converges to $\chi_{E_u}(s) a_{j_0}(s)$. It follows that

$$E_u \cap \Omega = (\cap_{j\in\Lambda} E_{u(a_j)}) \cap \Omega = \cap_{j\in\Lambda}(E_{u(a_j)} \cap \Omega).$$

Notice that $E_{u(a_j)} \cap \Omega$ is the set of points in Ω at which the modulus of the continuous function a_j of norm one attains its maximum value, which is a τ-compact G_δ subset of Ω. Then, since $E_u \cap \Omega$ is the intersection of a decreasing family of such subsets, $E_u \cap \Omega$ is a τ-compact subset of Ω, as required.

Conversely, let F be a non-empty τ-compact subset of Ω. Then, using Urysohn's lemma, there exists a decreasing net $(E_{u(a_j)} \cap \Omega)_{j \in \Lambda}$ of non-empty τ-compact G_δ subsets of Ω such that

$$F = \cap_{j \in \Lambda} E_{u(a_j)} \cap \Omega.$$

Then, the decreasing net $(u(a_j))_{j \in \Lambda}$ is $\sigma(C^\psi(\hat{\Omega}), M^\tau(\Omega))$-convergent to its infimum u which, by definition, lies in $\tilde{\mathcal{U}}_c(C^\tau(\Omega))$. As in the first part of the proof,

$$E_u = (\cap_{j \in \Lambda} E_{u(a_j)})^{\circ\psi},$$

and

$$E_u \cap \Omega = \cap_{j \in \Lambda} E_{u(a_j)} \cap \Omega = F.$$

Observe that, if u_1 and u_2 are compact tripotents such that $E_{u_1} \cap \Omega$ and $E_{u_2} \cap \Omega$ coincide, then

$$E_{u_1} = \pi\iota(E_{u_1} \cap \Omega) = \pi\iota(E_{u_2} \cap \Omega) = E_{u_2},$$

and the mapping is a bijection. This completes the proof of the theorem. □

Let x be a positive regular Borel measure on (Ω, τ) and, therefore, a normal measure on $(\hat{\Omega}, \psi)$. Then, Theorem 5.5 shows that the element $e(x)$ of $\tilde{\mathcal{U}}(C^\psi(\hat{\Omega}))$ defined by

$$e(x) = \wedge\{u \in \tilde{\mathcal{U}}(C^\psi(\hat{\Omega})) : P_2(u)_* x = x\}$$

is such that

$$E_{e(x)} = \hat{\Omega} \setminus \cup\{U \in \psi : x(\chi_U) = 0\} = \mathrm{supp}_{(\hat{\Omega}, \psi)} x.$$

For a regular Borel measure x on (Ω, τ) define

$$e_c(x) = \wedge\{u \in \tilde{\mathcal{U}}_c(C^\tau(\Omega)) : P_2(u)_* x = x\}.$$

Then, $e(x)$ is majorized by $e_c(x)$ which lies in $\tilde{\mathcal{U}}_c(C^\tau(\Omega))$. The next result explains why $e_c(x)$ is defined to be the compact support tripotent of x.

Theorem 5.7. *Let x be a regular Borel probability measure on the compact Hausdorff space (Ω, τ), with compact support tripotent $e_c(x)$, and let*

$$\mathrm{supp}_{(\Omega, \tau)} x = \Omega \setminus (\cup\{U \in \tau : x(\chi_U) = 0\})$$

be the support of x. Then

$$\mathrm{supp}_{(\Omega, \tau)} x = E_{e_c(x)} \cap \Omega.$$

Proof. Notice that

$$E_{e_c(x)} = (\cap\{E_u : E_u, \hat{\Omega} \setminus E_u \in \psi, \Omega \setminus (E_u \cap \Omega) \in \tau, P_2(u)_* x = x\})^{\circ\psi}.$$

However, $P_2(e_c(x))_* x$ and x coincide and, hence,

$$\cap\{E_u : E_u, \hat{\Omega} \setminus E_u \in \psi, \Omega \setminus (E_u \cap \Omega) \in \tau, P_2(u)_* x = x\} \subseteq E_{e_c(x)},$$

which implies that

$$E_{e_c(x)} = \cap\{E_u : E_u, \hat{\Omega} \setminus E_u \in \psi, \Omega \setminus (E_u \cap \Omega) \in \tau, P_2(u)_* x = x\}.$$

Therefore, observing that, since the relative ψ-topology on Ω is discrete, every subset of Ω is the intersection of a ψ-clopen set in $\hat{\Omega}$ with Ω,

$$\begin{aligned} E_{e_c(x)} \cap \Omega &= \cap\{E_u \cap \Omega : E_u, \hat{\Omega} \setminus E_u \in \psi, \Omega \setminus (E_u \cap \Omega) \in \tau, P_2(u)_* x = x\} \\ &= \cap\{F : \Omega \setminus F \in \tau, x(\chi_F) = \|x\|\} \\ &= \Omega \setminus \cup\{G : G \in \tau, x(\chi_G) = 0\} \\ &= \mathrm{supp}_{(\Omega,\tau)} x, \end{aligned}$$

as required. □

It is now possible to apply the results of §4 to the classical example. Let (Ω, τ) be a compact Hausdorff space and let $(\hat{\Omega}, \psi)$ be its hyper-Stonean envelope. By Lemma 5.2 there is associated with each tripotent v of the complete lattice $\tilde{\mathcal{U}}(C^\psi(\hat{\Omega}))$ a unique ψ-clopen subset E_v of $\hat{\Omega}$ such that, for each element t in E_v, $|v(t)|$ is equal to one and v coincides with $\chi_{E_v} v$. Moreover, for u and v in $\tilde{\mathcal{U}}(C^\psi(\hat{\Omega}))$, $v \leq u$ if and only if E_v is contained in E_u and u and v agree on E_v, and u is orthogonal to v if and only if E_u and E_v have empty intersection in which case $E_{v\vee u}$ coincides with both E_{v+u} and $E_u \cup E_v$. Furthermore, for each family $\{v_j : j \in \Lambda\}$ of tripotents the ψ-clopen set $(\cap_{j\in\Lambda} E_{v_j})^{\circ\psi}$ is equal to $E_{\wedge_{j\in\Lambda} v_j}$ and, if the family is an increasing net then $E_{\vee_{j\in\Lambda} v_j}$ coincides with the ψ-closure $\overline{\cup_{j\in\Lambda} E_{v_j}}^{\psi}$, which is ψ-clopen. This result applies, in particular, to any orthogonal family $\{v_j : j \in \Gamma\}$ when Λ is equal to the set Γ^f of finite subsets of Γ.

As was remarked above, the space Ω is naturally embedded in $\hat{\Omega}$ and, therefore, possesses two topologies, its own topology τ and the restriction of ψ to Ω, which is discrete. In this situation Lemma 4.3 may be interpreted as follows. Let E_w be a ψ-clopen subset corresponding to the element w of $\tilde{\mathcal{U}}(C^\psi(\hat{\Omega}))$, having Q-closure w^{c}. Then ,

$$E_{w^{\mathrm{c}}} = (\cap\{E_v \subseteq \hat{\Omega} : \Omega \setminus (E_v \cap \Omega) \in \tau, E_w \subseteq E_v\})^{\circ\psi}.$$

Therefore, $E_{w^{\mathrm{c}}}$ is the ψ-closure of E_w and $E_{w^{\mathrm{c}}} \cap \Omega$ is the τ-closure of $E_w \cap \Omega$.

When x is a positive regular Borel probability measure on the compact Hausdorff space (Ω, τ) and, therefore, a normal measure on its hyper-Stonean envelope $(\hat{\Omega}, \psi)$, it follows from Theorems 5.5 and 5.7 that $E_{e(x)}$ is the support of x in $(\hat{\Omega}, \psi)$ and $E_{e_c(x)} \cap \Omega$ is the support of x in (Ω, τ). In this example, Lemma 4.4 and Theorem 4.5 lead to the following result.

Corollary 5.8. *Let E_v be a ψ-clopen subset of $\hat{\Omega}$ such that $E_v \cap \Omega$ is τ-closed, and let w be Q-dense in v, in which case $E_v \cap \Omega$ is the τ-closure of $E_w \cap \Omega$. Then, the following results hold.*

(i) *There exists a maximal set $\{z_j : j \in \Lambda\}$ of normal L-orthogonal regular Borel probability measures on $(\hat{\Omega}, \psi)$ with pairwise disjoint supports $\{E_{e(z_j)} : j \in \Lambda\}$ such that*
$$E_w = \overline{\cup_{j\in\Lambda} E_{e(z_j)}}^{\psi},$$
and
$$E_v \cap \Omega = \overline{\cup_{j\in\Lambda}(E_{e_c(z_j)} \cap \Omega)}^{\tau}.$$

(ii) *There exists a regular Borel probability measure x on (Ω, τ) with support $E_v \cap \Omega$ if and only if Λ is countable.*

The results of Lemmas 4.6 and 4.7, and Theorem 4.8 can now be interpreted in this example.

Corollary 5.9. *Under the conditions of Corollary 5.8, let $\mathcal{C}_v$ be the set of ψ-clopen subsets of $\hat{\Omega}$ the intersections with $E_v \cap \Omega$ of which are τ-closed, and, for each element j in Λ, let*
$$\mathcal{D}^{(j)} = \{E_q \in \mathcal{C}_v \setminus \{E_v\} : E_q \cap E_{e(z_j)} \cap \Omega \neq \emptyset\}.$$
Then, the set of such subsets of $\mathcal{C}_v$ has the following properties.

(i) *For each pair E_{q_1} and E_{q_2} of elements of $\mathcal{C}_v$ with $E_{q_1} \subseteq E_{q_2}$, if E_{q_1} is contained in $\mathcal{D}^{(j)}$ then E_{q_2} is contained in $\mathcal{D}^{(j)}$. ($\mathcal{D}^{(j)}$ is an anti-order ideal in $\mathcal{C}_v$).*

(ii) *The union E of each finite set $E_{q_1}, E_{q_2}, \ldots, E_{q_n}$ of disjoint elements of $\mathcal{D}^{(j)}$ such that $E \neq E_v$ is contained in $\mathcal{D}^{(j)}$.*

(iii) *For each element E_q of $\mathcal{C}_v \setminus \{\emptyset\}$, and each element j in Λ, there exist elements*
$$E_{q^{(j)}} = E_q \cap E_{e_c(z_j)}$$
in $\mathcal{D}^{(j)}$ such that
$$E_q \cap \Omega = \overline{\cup_{j\in\Lambda}(E_{q^{(j)}} \cap \Omega)}^{\tau}.$$

(iv) *For each finite subset $E_{q_1}, E_{q_2}, \ldots, E_{q_n}$ of elements of $\mathcal{D}^{(j)}$, the union E of which is a proper subset of E_q, the covering number $c(S)$ of S, defined by*
$$c(S) = \inf_{t\in E_v\cap\Omega} \Sigma_{k=1}^{n} n^{-1} \chi_{E_{q_k}\cap\Omega}(t), \tag{26}$$
is less than one.

It should be recognised that the set $E_v \cap \Omega$ over which the infimum in Corollary 5.8 is taken can be identified with the set of extreme points of the $\sigma(A^*, A)$-compact convex set $\{v\}$, in A_1^* and the Krein-Milman theorem may be applied to show that the formulae in (17) and (26) are equivalent.

In the classical example Theorem 4.8 reduces to the following final result.

Corollary 5.10. *Let (Ω, τ) be a compact Hausdorff space with hyper-Stonean envelope $(\hat{\Omega}, \psi)$ and let E be a ψ-clopen subset of $\hat{\Omega}$ such that $E \cap \Omega$ is τ-closed. Then, the following conditions are equivalent.*

(i) *There exists a regular Borel probability measure x on (Ω, τ) with support $E \cap \Omega$.*

(ii) *The family $\{\mathcal{D}^{(j)} : j \in \Lambda\}$, as defined in Corollary 5.9, consisting of sets of ψ-clopen subsets of $\hat{\Omega}$ the intersections of which with Ω are τ-closed, is countable.*

A similar result was proved directly by Hebert and Lacey in [36], Theorem 1.6.

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