

## ON SOME TYPE ELEMENTS OF ZERO-SYMMETRIC NEAR-RING OF POLYNOMIALS

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ABSTRACT. Let  $R$  be a commutative ring with unity. In this paper, we characterize the unit elements, the regular elements, the  $\pi$ -regular elements and the clean elements of zero-symmetric near-ring of polynomials  $R_0[x]$ , when  $\text{nil}(R)^2 = 0$ . Moreover, it is shown that the set of  $\pi$ -regular elements of  $R_0[x]$  forms a semigroup. These results are somewhat surprising since, in contrast to the polynomial ring case, the near-ring of polynomials has substitution for its “multiplication” operation.

### 1. Introduction and preliminary definitions

Through this paper, all rings are commutative with unity and all near-rings are abelian left near-ring with unity. A set  $N$  together with two binary operations “+” and “ $\cdot$ ” is called left near-ring if  $(N, +)$  is a group,  $(N, \cdot)$  is a semigroup and  $a \cdot (b + c) = a \cdot b + a \cdot c$  for each  $a, b, c \in N$ . If  $(N, +)$  is abelian, then we call  $N$  *abelian*.

For a near-ring  $N$ ,  $N_0 = \{a \in N \mid 0 \cdot a = 0\}$  is called the zero-symmetric part of  $N$ ,  $N_c = \{a \in N \mid 0 \cdot a = a\}$  is called the *constant part* of  $N$ . A near-ring  $N$  is called *zero-symmetric* if  $N = N_0$ . A near-ring  $N$  is called *constant near-ring* if  $N_c = N$ . Also, a subgroup  $M$  of a near-ring  $N$  with  $MM \subseteq M$  is called a *subnear-ring* of  $N$ . Thus  $N_0$  and  $N_c$  are subnear-rings of  $N$ . The most general class of examples of zero-symmetric near-rings comes from the following construction: Let  $(G, +)$  be a not necessarily abelian group. Then the set  $M_0(G)$  of all functions  $f : G \rightarrow G$  with  $f(0) = 0$  under pointwise addition  $+$  and function composition  $\circ$  determines a zero-symmetric near-ring  $(M_0(G), +, \circ)$ . Evidently, also each ring is a zero-symmetric (left) near-ring and so we may view near-rings as generalized rings. For basic definitions and comprehensive discussion on near-rings, we refer the reader to [11].

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Recall that, a near-ring  $N$  is a *near-field*, if every nonzero element  $a \in N$  has multiplicatively inverse  $a^{-1}$ . Thus the nonzero elements of  $N$  form a group under multiplication.

A subgroup  $M$  of  $(N, +)$  is called  *$N$ -subgroup*, if  $MN \subseteq M$ . It is proved that  $N$  is a zero-symmetric near-ring if and only if each right ideal of  $N$  is an  $N$ -subgroup of  $N$  by [11, Proposition 1.34]. A zero-symmetric near-ring  $N$  is called *local* if  $L = \{k \in N \mid kN \neq N\}$  is an  $N$ -subgroup. Near-fields are local near-rings with  $L = 0$ . Maxson in [9, Theorem 4.2], proved that if  $N$  is a local near-ring, then  $N$  contains no idempotent other than 0 and 1. A near-ring  $N$  is called *integral*, if  $N$  has no nonzero zero divisor.

For a near-ring  $N$ ,  $\text{nil}(N)$ ,  $\text{idem}(N)$  and  $U(N)$  denote the set of all nilpotent elements of  $N$ , the set of all idempotent elements of  $N$  and the set of all units of  $N$ , respectively. Given a ring or near-ring  $N$ , we say that it is *reduced* if it has no nonzero nilpotent element. Also, we write  $Z_\ell(N)$ ,  $Z_r(N)$  and  $Z(N)$  for the set of all left zero divisors of  $N$ , the set of all right zero divisors and the set  $Z_\ell(N) \cup Z_r(N)$ , respectively.

An element  $a$  of a near-ring  $N$  is called *regular* if there exists  $b \in N$  such that  $a = aba$ . The set of all regular elements of  $N$  is denoted by  $\text{vnr}(N)$ . A near-ring  $N$  is called *regular*, whenever  $\text{vnr}(N) = N$ . For example, every constant near-ring is regular. Further, Beidleman in [2], proved that the near-rings  $M(G)$  and  $M_0(G)$  are regular. Also, he showed that a regular near-ring with identity contains no nonzero nil  $N$ -subgroup. In [4], Chao proved that if  $N$  is a reduced zero-symmetric near-ring with unity, then  $N$  is regular if and only if  $aN$  is a direct summand of  $N$  for each  $a \in N$ . According to [11, p. 347], a regular near-ring with identity is integral if and only if it is a near-field. Properties of regular near-rings have been studied by Ghoudhari, Goyal, Heatherly, Hongan, Ligh, Mason and Murty. Their main results are suggested in the book [11].

A near-ring  $N$  is said to be  *$\pi$ -regular* if for each element  $a \in N$ , there exists a positive integer  $n$  such that  $a^n$  is a regular element, that is,  $a^n = a^n b a^n$  for some  $b \in N$ . Such an element  $a$  is called  *$\pi$ -regular*. The set of all  $\pi$ -regular elements of  $N$  is denoted by  $\pi - r(N)$ . Clearly every regular near-ring is  $\pi$ -regular, but Cho in [5] gives an example of a  $\pi$ -regular near-ring which is not regular. As in [10] for a ring, we say that an element  $a$  of a near-ring  $N$  is *clean* if  $a$  is the sum of a unit and an idempotent of  $R$ . The set of all clean elements of  $N$  is denoted by  $\text{cln}(N)$ . Moreover,  $N$  is said to be a clean near-ring if  $\text{cln}(N) = N$ .

We say that a subset  $S$  of a ring or near-ring is *locally nilpotent* if for any finite subset  $\{s_1, s_2, \dots, s_n\} \subseteq S$ , there exists an integer  $k$  such that any product of  $k$  elements from  $\{s_1, s_2, \dots, s_n\}$  is zero. In other words,  $S$  is locally nilpotent if any subring without identity generated by a finite number of elements in  $S$  is nilpotent.

Let  $R$  be a ring. Since  $R[x]$  is an abelian near-ring under addition and substitution, it is natural to investigate the near-ring of polynomials  $(R[x], +, \circ)$ . The binary operation of substitution, denoted by " $\circ$ ", of one polynomial into

another is both natural and important in the theory of polynomials. We adopt the convention that for polynomials  $(x)f = \sum_{i=0}^m a_i x^i$  and  $(x)g \in R[x]$ ,

$$(x)g \circ (x)f = \sum_{i=0}^m a_i ((x)g)^i.$$

For example,  $(a_0 + a_1x) \circ x^2 = (a_0 + a_1x)^2 = a_0^2 + (a_0a_1 + a_1a_0)x + a_1^2x^2$ . However, the operation  $\circ$ , left distributes but does not right distribute over addition. Thus  $(R[x], +, \circ)$  forms a left near-ring but not a ring. We use  $R[x]$  to denote the left near-ring  $(R[x], +, \circ)$  with coefficients from  $R$  and  $R_0[x] = \{(x)f \mid (x)f \text{ has zero constant term}\}$  is the zero-symmetric left near-ring of polynomials with coefficients in  $R$ . Also, for each  $(x)f = \sum_{i=0}^m a_i x^i$  and  $(x)g = \sum_{j=0}^n b_j x^j \in R[x]$ , we write  $(x)f(x)g = \sum_{k=0}^{n+m} (\sum_{i+j=k} a_i b_j) x^k$ .

In this paper, we characterize all of the unit elements, the regular elements, the  $\pi$ -regular elements and the clean elements of the zero-symmetric near-ring  $R_0[x]$ , when  $R$  is a commutative ring with  $\text{nil}(R)^2 = 0$ . Also, we prove that  $\text{vnr}(R_0[x])$  is a subnear-ring of  $R_0[x]$  if and only if  $\text{vnr}(R)$  is a subring of  $R$ . Moreover, it is shown that the set of  $\pi$ -regular elements of  $R_0[x]$  is multiplicatively closed. These results are somewhat surprising since, in contrast to the polynomial ring case, the near-ring of polynomials has substitution for its “multiplication” operation.

## 2. Regular elements

In this section we investigate regular elements of the near-ring  $R_0[x]$ , when  $R$  is a commutative ring with  $\text{nil}(R)^2 = 0$ .

**Theorem 2.1.** *Let  $N$  be a near-ring with central idempotents.*

- (1) *Let  $a \in N$ . If  $aba = a$  for some  $b \in N$ , then  $ab = ba$  is an idempotent of  $N$ .*
- (2)  *$\text{vnr}(N)$  is multiplicatively closed.*
- (3)  *$\text{vnr}(N) \cap \text{nil}(N) = \{0\}$ .*
- (4)  *$U(N) \cup \text{Idem}(N) \subseteq \text{vnr}(N) \subseteq U(N) \cup Z(N)$ .*
- (5)  *$\text{vnr}(N) = U(N) \cup \{0\}$  if and only if  $\text{Idem}(N) = \{0, 1\}$ . In particular,  $\text{vnr}(N) = U(N) \cup \{0\}$  if  $N$  is either integral or local.*
- (6)  *$\text{vnr}(N)$  contains a nonzero nonunit if and only if  $\text{Idem}(N) \neq \{0, 1\}$ .*

*Proof.* (1) Let  $a \in \text{vnr}(N)$ . Then  $a = aba$  for some  $b \in N$ . Hence  $ab = (ab)^2 = abab = a(ba)b = (ba)ab = b(ab)a = (ba)^2 = ba$ , since  $ab$  and  $ba$  are central idempotents.

(2) Let  $a, a' \in \text{vnr}(N)$ . Then  $a = aba$  and  $a' = a'ca'$  for some  $b, c \in R$ . Since idempotent elements of  $N$  are central, it follows that  $aa' = (aba)(a'ca') = aa'(cb)aa'$  by (1).

By a similar argument one can prove the other statements.  $\square$

**Proposition 2.2.** *Let  $N$  be a near-ring which whose idempotents are central. If  $a \in \text{vnr}(N)$ , then there exists a unique  $b \in N$  with  $aba = a$  and  $bab = b$ .*

*Proof.* Suppose that  $a \in \text{vnr}(N)$ . Then  $a = aca$  for some  $c \in N$ . Let  $b = cac$ , hence  $ca = ac \in \text{Idem}(N)$  by Theorem 2.1. Thus  $aba = a$  and  $bab = b$ . Now assume that there exists  $b_1 \in N$  such that  $ab_1a = a$  and  $b_1ab_1 = b_1$ . Thus  $b_1a = ab_1 \in \text{Idem}(N)$  by Theorem 2.1. So we have  $b_1 = b_1ab_1 = b_1(aba)b_1 = b_1(ab_1a)b = b_1ab = b_1(aba)b = bab_1ab = b$ . Therefore  $b$  is unique.  $\square$

Since every idempotent is central in each commutative ring, then by [7, Lemma 2.1], we have the following result.

**Lemma 2.3.** *Let  $R$  be a commutative ring and  $(x)f \in R_0[x]$ . Then  $(x)f$  is an idempotent element of the near-ring  $R_0[x]$  if and only if  $(x)f = e_1x$ , where  $e_1$  is an idempotent of  $R$ . In particular, the idempotent elements of  $R_0[x]$  are central.*

For each  $(x)f \in R_0[x]$  and positive integer  $n$ , we write

$$((x)f)^{(n)} = \underbrace{(x)f \circ (x)f \circ \cdots \circ (x)f}_n.$$

**Lemma 2.4.** *Let  $R$  be a reduced commutative ring and  $(x)f = \sum_{i=1}^m a_i x^i$ ,  $(x)g = \sum_{j=1}^n b_j x^j \in R_0[x]$ . If  $(x)g \circ (x)f = cx$ , then  $a_1 b_1 = c$  and  $a_i b_j = 0$  for  $i + j \neq 2$ .*

*Proof.* Let  $n = 1$ . Then  $(x)g \circ (x)f = a_1(b_1x) + \cdots + a_m(b_1x)^m = cx$ . Hence  $a_1 b_1 = c$  and  $a_i b_1 = 0$  for  $i = 2, \dots, m$ , since  $a_i b_1^i = 0$  and  $R$  is reduced. Now assume that  $n > 1$ . Then we have

$$(2.1) \quad (x)g \circ (x)f = a_1((x)g) + a_2((x)g)^2 + \cdots + a_m((x)g)^m = cx,$$

which implies that  $a_1 b_1 = c$  and  $a_m b_n^m = 0$ , since it is the leading coefficient of Eq. (2.1). Thus  $a_m b_n = b_n a_m = 0$ , since  $R$  is reduced. By multiplying  $b_n$  to Eq. (2.1), we obtain

$$(2.2) \quad b_n a_1((x)g) + b_n a_2((x)g)^2 + \cdots + b_n a_{m-1}((x)g)^{m-1} = b_n cx.$$

Hence  $b_n a_{m-1} (b_n)^{m-1} = 0$ , since it is the leading coefficient of Eq. (2.2). Therefore  $b_n a_{m-1} = a_{m-1} b_n = 0$ , since  $R$  is reduced. Inductively, we have  $b_n a_i = a_i b_n = 0$  for  $i = 1, \dots, m$ . Hence from Eq. (2.1) we have  $(\sum_{j=1}^{n-1} b_j x^j) \circ (\sum_{i=1}^m a_i x^i) = cx$ . Continuing this process, one can prove that  $b_j a_i = a_i b_j = 0$  for  $i + j \neq 2$ .  $\square$

It is well known that if  $R$  is a commutative ring, then  $(x)f = \sum_{i=0}^m a_i x^i$  is a unit element of the polynomial ring  $R[x]$  if and only if  $a_0 \in U(R)$  and  $a_1, \dots, a_m \in \text{nil}(R)$ . In the next theorem, we determine unit elements of the near-ring  $R_0[x]$ , when  $R$  is a commutative ring with  $\text{nil}(R)^2 = 0$ .

**Theorem 2.5.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$ . Then  $(x)f = \sum_{i=1}^m a_i x^i \in U(R_0[x])$  if and only if  $a_1 \in U(R)$  and  $a_2, \dots, a_m \in \text{nil}(R)$ .*

*Proof.* Suppose that  $(x)f \in U(R_0[x])$ . Then  $(x)f \circ (x)g = (x)g \circ (x)f = x$  for some  $(x)g = \sum_{j=1}^n b_j x^j \in R_0[x]$ . Since  $\text{nil}(R)$  is an ideal of  $R$ , it follows that  $\bar{R} = R/\text{nil}(R)$  is reduced and so  $(x)\bar{f} \circ (x)\bar{g} = (x)\bar{g} \circ (x)\bar{f} = \bar{1}x = (1 + \text{nil}(R))x$ , where  $(x)\bar{f} = \sum_{i=1}^m (a_i + \text{nil}(R))x^i$  and  $(x)\bar{g} = \sum_{j=1}^n (b_j + \text{nil}(R))x^j$ . By Lemma 2.4,  $\bar{a}_1 \bar{b}_1 = \bar{b}_1 \bar{a}_1 = \bar{1}$  and  $\bar{b}_1 \bar{a}_i = \bar{0}$  for  $i = 2, \dots, m$ , which implies that  $\bar{a}_i = \bar{0}$  for  $i = 2, \dots, m$ . Since  $\text{nil}(R) \subseteq J(R)$ , it follows that  $a_1 \in U(R)$  and  $a_i \in \text{nil}(R)$  for  $i = 2, \dots, m$ .

Conversely, let  $(x)f = a_0 x + a_1 x^2 + \dots + a_n x^{n+1}$ , where  $a_0 \in U(R)$  and  $a_1, a_2, \dots, a_n \in \text{nil}(R)$ . We show that  $(x)f$  has right and left inverse. Since  $R$  is commutative, then  $(x)f_1 = a_0 + a_1 x + \dots + a_n x^n$  is a unit element of the polynomial ring  $R[x]$ . Thus there exists  $(x)g = b_0 + b_1 x + \dots + b_m x^m$  of  $R[x]$  such that  $(x)f_1(x)g = (x)g(x)f_1 = 1$ . Hence  $b_0 \in U(R)$  and  $b_1, \dots, b_m \in \text{nil}(R)$ . Since  $\text{nil}(R[x]) = \text{nil}(R)[x]$ , it follows that  $(x)g_1 = b_1 x + \dots + b_m x^m$  is a nilpotent element of the polynomial ring  $R[x]$  and so there is a non-negative integer  $k$  such that  $((x)g_1)^k = 0$ , which implies that  $\deg[(x)g^t] \leq (k-1)m$  for each  $t \geq k$ . Put  $r = (k-1)m$ . We have to find  $(x)h = h_1 x + h_2 x^2 + \dots + h_{r+1} x^{r+1} \in R_0[x]$  such that  $(x)f \circ (x)h = x$ . Then we have

$$\begin{aligned}
& (x)f \circ (x)h = x \\
& \Leftrightarrow h_1((x)f) + h_2((x)f)^2 + \dots + h_{r+1}((x)f)^{r+1} = x \\
& \Leftrightarrow [h_1 + h_2((x)f) + \dots + h_{r+1}((x)f)^r](x)f = x \\
& \Leftrightarrow [h_1 + h_2((x)f) + \dots + h_{r+1}((x)f)^r](x)f_1 = 1 \\
& \Leftrightarrow [h_1 + h_2((x)f) + \dots + h_{r+1}((x)f)^r] = (x)g \\
& \Leftrightarrow [h_2 x((x)f_1) + \dots + h_{r+1} x^r ((x)f_1)^r] = (x)g - h_1 \\
& \Leftrightarrow [h_2 x + \dots + h_{r+1} x^r ((x)f_1)^{r-1}](x)f_1 = (x)g - h_1 \\
& \Leftrightarrow [h_2 x + \dots + h_{r+1} x^r ((x)f_1)^{r-1}] = ((x)g - h_1)(x)g \\
& \Leftrightarrow [h_3 x^2((x)f_1) + \dots + h_{r+1} x^r ((x)f_1)^{r-1}] = ((x)g)^2 - h_1((x)g) - h_2 x \\
& \Leftrightarrow [h_3 x^2 + \dots + h_{r+1} x^r ((x)f_1)^{r-2}](x)f_1 = ((x)g)^2 - h_1((x)g) - h_2 x \\
& \Leftrightarrow [h_3 x^2 + \dots + h_{r+1} x^r ((x)f_1)^{r-2}] = ((x)g)^3 - h_1((x)g)^2 - h_2 x((x)g) \\
& \vdots \\
& \Leftrightarrow ((x)g)^{r+1} - h_1((x)g)^r - \dots - h_r x^{r-1}(x)g - h_{r+1} x^r = 0 \\
& \Leftrightarrow h_1 = b_0, \quad h_2 = b_0 b_1, \quad h_3 = b_0^2 b_2 + b_0 b_1^2, \quad \dots, \\
& \quad h_{r+1} = \sum_{i_1 + \dots + i_{r+1} = r} b_{i_1} \dots b_{i_{r+1}} - h_1 \sum_{i_1 + \dots + i_r = r} b_{i_1} \dots b_{i_r} - \dots - h_r b_1,
\end{aligned}$$

where  $b_{i_j} \in \{b_0, b_1, \dots, b_m\}$  for  $j = 1, \dots, r+1$ . Hence  $(x)h$  is a right inverse for  $(x)f$ .

Since  $b_0 \in U(R)$  and  $\{b_1, \dots, b_m\} \subseteq \text{nil}(R)$ , hence  $h_1 \in U(R)$  and  $\{h_2, \dots, h_{r+1}\} \subseteq \text{nil}(R)$ . Thus with a similar argument as used in the previous paragraph, one can find  $(x)k \in R_0[x]$  such that  $(x)h \circ (x)k = x$ . Hence  $(x)h \in U(R_0[x])$ , which implies that  $(x)f \in U(R_0[x])$ .  $\square$

**Corollary 2.6.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$ . Then  $U(R_0[x]) = U(R)x + \text{nil}(R_0[x])$ . In particular, if  $R$  is reduced, then  $U(R_0[x]) = \{ux \mid u \in U(R)\}$ .*

**Corollary 2.7.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$  and  $(x)f \in R_0[x]$ . If  $(x)f$  has right or left inverse, then  $(x)f$  is invertible in  $R_0[x]$ .*

*Proof.* It follows from the proof of Theorem 2.5.  $\square$

Let  $R$  be a commutative ring and  $a \in R$ . Anderson and Badawi [1, Theorem 2.2], proved that  $a \in \text{vnr}(R)$  if and only if  $a = ue$  for some  $u \in U(R)$  and  $e \in \text{Idem}(R)$ . In the next proposition, we extend this result to the near-ring  $R_0[x]$ .

**Proposition 2.8.** *Let  $R$  be a commutative ring and  $(x)f \in R_0[x]$ . Then the following statements are equivalent:*

- (1)  $(x)f \in \text{vnr}(R_0[x])$ .
- (2)  $(x)f = (x)f \circ (x)u \circ (x)f$  for some  $(x)u \in U(R_0[x])$ .
- (3)  $(x)f = (x)u \circ (x)h$  for some  $(x)h \in \text{Idem}(R_0[x])$  and  $(x)u \in U(R_0[x])$ .

*Proof.* (1)  $\Rightarrow$  (2) Let  $(x)f \in \text{vnr}(R_0[x])$ . Then  $(x)f = (x)f \circ (x)g \circ (x)f$  for some  $(x)g \in R_0[x]$  and so we have  $(x)f \circ (x)g = (x)g \circ (x)f \in \text{Idem}(R_0[x])$  by Theorem 2.1. Thus  $(x)f \circ (x)g = ex$  for some  $e \in \text{Idem}(R)$  by Lemma 2.3. Clearly,  $1 - e$  is an idempotent of  $R$ . Let  $(x)u = ex \circ (x)g + (1 - e)x$ . Then by using Lemma 2.3, we have

$$\begin{aligned}
& (x)u \circ [(x)f + (1 - e)x] \\
&= (x)u \circ (x)f + (x)u \circ (1 - e)x \\
&= [ex \circ (x)g + (1 - e)x] \circ ex \circ (x)f + [ex \circ (x)g + (1 - e)x] \circ (1 - e)x \\
&= ex \circ [ex \circ (x)g + (1 - e)x] \circ (x)f + [ex \circ (x)g + (1 - e)x] \circ (1 - e)x \\
&= ex \circ (x)g \circ (x)f + (1 - e)x \\
&= ex + (1 - e)x \\
&= x
\end{aligned}$$

and so  $(x)u$  is invertible in  $R_0[x]$  by Corollary 2.7. Further,  $(1 - e)x \circ (x)f = (x)f \circ (1 - e)x = (x)f - (x)f \circ ex = (x)f - (x)f \circ (x)g \circ (x)f = 0$  by Lemma 2.3. Hence  $(x)f \circ (x)u \circ (x)f = (x)f \circ [ex \circ (x)g + (1 - e)x] \circ (x)f = [(x)f \circ ex] \circ (x)g + (x)f \circ (1 - e)x \circ (x)f = (x)f \circ (x)g \circ (x)f = (x)f$ .

(2)  $\Rightarrow$  (3) Assume that  $(x)f = (x)f \circ (x)v \circ (x)f$  for some  $(x)v \in U(R_0[x])$  and let  $u(x) = (x)v^{-1} \in U(R_0[x])$ . Since  $(x)h = (x)v \circ (x)f \in \text{Idem}(R_0[x])$ , it follows that  $(x)u \circ (x)h = (x)v^{-1} \circ (x)v \circ (x)f = (x)f$ .

(3)  $\Rightarrow$  (1) Suppose that  $(x)f = (x)u \circ (x)h$ , where  $(x)u \in U(R_0[x])$  and  $(x)h \in \text{Idem}(R_0[x])$ . Hence by Lemma 2.3,  $(x)h = ex$  for some  $e \in \text{Idem}(R)$ . So  $(x)f = (x)u \circ ex = ex \circ (x)u$ , since  $ex$  is central. Therefore  $(x)f \circ (x)u^{-1} \circ (x)f = (ex \circ (x)u) \circ (x)u^{-1} \circ (x)f = ex \circ (x)f = ex \circ (x)u \circ ex = (x)f$ , since idempotents of  $R_0[x]$  are central.  $\square$

Now we give a characterization of regular elements of  $R_0[x]$ , when  $R$  is a commutative ring with  $\text{nil}(R)^2 = 0$ .

**Theorem 2.9.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$ . Then  $\text{vnr}(R_0[x]) = \{ \sum_{i=1}^n a_i x^i \in R_0[x] \mid n \geq 1, a_1 = ue \text{ and } a_i \in e(\text{nil}(R)) \text{ for each } i \geq 2, \text{ where } u \in U(R) \text{ and } e \in \text{Idem}(R) \}$ .*

*Proof.* It follows directly from Proposition 2.8, Theorem 2.5 and Lemma 2.3.  $\square$

**Corollary 2.10.**  *$R$  be a commutative ring with  $\text{nil}(R)^2 = 0$ . If  $R$  is reduced, then  $\text{vnr}(R_0[x]) = (\text{vnr}(R))x$ . In particular, if  $\text{vnr}(R)$  is a subring of  $R$ , then  $\text{vnr}(R_0[x]) = (\text{vnr}(R))x$ .*

*Proof.* If  $\text{nil}(R) = 0$ , then  $\text{vnr}(R_0[x]) = (\text{vnr}(R))x$  by Theorem 2.9. Now, assume that  $\text{vnr}(R)$  be a subring of  $R$ . Then by [1, Theorem 2.9],  $R$  is reduced and so the result follows.  $\square$

**Theorem 2.11.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$ . If  $\text{vnr}(R_0[x])$  is a subnear-ring of  $R_0[x]$ , then  $R$  is reduced and  $\text{vnr}(R_0[x]) = (\text{vnr}(R))x$ .*

*Proof.* Let  $(x)f$  be a nilpotent element of  $R_0[x]$ . Then by Theorem 2.5,  $x + (x)f \in U(R_0[x]) \subseteq \text{vnr}(R_0[x])$ . Since  $\text{vnr}(R_0[x])$  is a subnear-ring of  $R_0[x]$ , we have  $(x)f = -x + (x + (x)f) \in \text{vnr}(R_0[x])$ , which implies that  $(x)f \in \text{vnr}(R_0[x]) \cap \text{nil}(R_0[x]) = \{0\}$  by Theorem 2.1. Therefore  $\text{nil}(R_0[x]) = \{0\}$  and  $R$  is reduced by [3, Proposition 3.1]. Also,  $\text{vnr}(R_0[x]) = (\text{vnr}(R))x$  by Corollary 2.10.  $\square$

Let  $R$  be a commutative ring. Anderson and Badawi [1, Theorem 2.1], proved that the set of regular elements of  $R$ , is multiplicatively closed. Thus we have the following result.

**Corollary 2.12.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$ . Then  $\text{vnr}(R_0[x])$  is a subnear-ring of  $R_0[x]$  if and only if  $\text{vnr}(R)$  is a subring of  $R$ .*

*Proof.* If  $\text{vnr}(R_0[x])$  is a subnear-ring of  $R_0[x]$ , then  $\text{vnr}(R_0[x]) = (\text{vnr}(R))x$  by Theorem 2.11. Hence  $(\text{vnr}(R))x$  is a subgroup of  $(R_0[x], +)$ , which implies that  $\text{vnr}(R)$  is a subring of  $R$  by [1, Theorem 2.1].

Conversely, assume that  $\text{vnr}(R)$  is a subring of  $R$ . Thus  $\text{vnr}(R_0[x]) = (\text{vnr}(R))x$  by Corollary 2.10. Then  $\text{vnr}(R_0[x])$  is a subgroup of  $(R_0[x], +)$ , and so the result follows from Theorem 2.1.  $\square$

**Theorem 2.13.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$  and  $2 \in U(R)$ . Then every  $(x)f \in \text{vnr}(R_0[x])$  is the sum of two units of  $R_0[x]$ .*

*Proof.* Let  $(x)f = \sum_{i=1}^m a_i x^i$  be a regular element of  $R_0[x]$ . Then  $a_1 = ue$  and  $a_i \in e(\text{nil}(R))$  for some  $u \in U(R)$  and  $e \in \text{Idem}(R)$  by Theorem 2.9. Hence  $a_1 \in \text{vnr}(R)$  by [1, Theorem 2.2]. Since  $2 \in U(R)$ , it follows that  $a_1 = u' + v'$  for some  $u', v' \in U(R)$  by [1, Theorem 2.10]. Let  $(x)g = u'x$  and  $(x)h = v'x + a_2 x^2 + \cdots + a_m x^m$ . Then  $(x)g, (x)h \in U(R_0[x])$  by Theorem 2.5. Hence  $(x)f = (x)g + (x)h$  is the sum of two units of  $R_0[x]$ .  $\square$

**Theorem 2.14.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$  and  $2 \in U(R)$ . Then the following statements are equivalent.*

- (1)  $\text{vnr}(R_0[x])$  is a subnear-ring of  $R_0[x]$ .
- (2) The sum of any four units of  $R_0[x]$  is a regular element of  $R_0[x]$ .

*Proof.* (1)  $\Rightarrow$  (2) It is clear since  $U(R_0[x]) \subseteq \text{vnr}(R_0[x])$  by Theorem 2.1.

(2)  $\Rightarrow$  (1) By Theorem 2.1,  $\text{vnr}(R_0[x])$  is multiplicatively closed. Now, let  $(x)f, (x)g \in \text{vnr}(R_0[x])$ . Hence there exist  $(x)u_1, (x)u_2, (x)v_1, (x)v_2 \in U(R_0[x])$  such that  $(x)f = (x)u_1 + (x)u_2$  and  $(x)g = (x)v_1 + (x)v_2$  by Theorem 2.13. Thus  $(x)f + (x)g$  is the sum of four units of  $R_0[x]$ , which implies that  $(x)f + (x)g \in \text{vnr}(R_0[x])$  by hypothesis.  $\square$

**Corollary 2.15.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$  and  $2 \in U(R)$ . If the sum of any four units of  $R_0[x]$  is a regular element of  $R_0[x]$ , then  $\text{vnr}(R_0[x]) = (\text{vnr}(R))x$ .*

*Proof.* It follows from Theorem 2.14 and Corollaries 2.12 and 2.10.  $\square$

### 3. $\pi$ -regular elements and clean elements of $R_0[x]$

In this section, we investigate  $\pi$ -regular and clean elements of  $R_0[x]$  when  $R$  is a commutative ring with  $\text{nil}(R)^2 = 0$ .

**Theorem 3.1.** *Let  $N$  be a near-ring with central idempotents. Then*

- (1)  $\text{vnr}(N) \subseteq \pi - r(N)$ . In particular, each regular near-ring is  $\pi$ -regular near-ring.
- (2)  $\text{vnr}(N) \cup \text{nil}(N) \subseteq \pi - r(N) \subseteq U(N) \cup Z(N)$ .
- (3)  $\pi - r(N) = U(N) \cup \text{nil}(N)$  if and only if  $\text{Idem}(N) = \{0, 1\}$ . In particular,  $\pi - r(N) = U(N) \cup \text{nil}(N)$  if  $N$  is either integral or local.
- (4)  $\pi - r(N)$  contains a non-nilpotent nonunit if and only if  $\text{Idem}(N) \neq \{0, 1\}$ .

*Proof.* By a similar way as used in the proof of [1, Theorem 4.1], one can prove it.  $\square$

**Theorem 3.2.** *Let  $R$  be a commutative ring and  $(x)f \in R_0[x]$ . Then  $(x)f$  is  $\pi$ -regular if and only if there exists  $(x)g \in \text{Idem}(R_0[x])$  such that  $(x)g \circ (x)f$  is regular and  $(x - (x)g) \circ (x)f \in \text{nil}(R_0[x])$ .*



*Proof.* Since  $(x)f$  is  $\pi$ -regular, then  $((x)f)^{(n)}$  is regular for some  $n \geq 1$ . Hence  $((x)f)^{(n)} = (x)u \circ (x)g$  for some  $(x)u \in U(R_0[x])$  and  $(x)g \in \text{Idem}(R_0[x])$  by Proposition 2.8. By Lemma 2.3, there exists  $e \in \text{Idem}(R)$  such that  $(x)g = ex$ . First we show that  $ex \circ (x)f$  is regular. Since idempotents of  $R_0[x]$  are central, we have  $ex \circ (x)f \circ [((x)f)^{(n-1)} \circ (x)u^{-1}] \circ ex \circ (x)f = [ex \circ ((x)f)^{(n)} \circ (x)u^{-1}] \circ ex \circ (x)f = [ex \circ (x)u \circ ex \circ (x)u^{-1}] \circ ex \circ (x)f = [ex \circ (x)u \circ (x)u^{-1}] \circ ex \circ (x)f = ex \circ (x)f$ , which implies that  $ex \circ (x)f \in \text{vnr}(R_0[x])$ . Also  $((1-e)x \circ (x)f)^{(n)} = (1-e)x \circ ((x)f)^{(n)} = (1-e)x \circ (x)u \circ ex = 0$ , since  $(1-e)x \in \text{Idem}(R_0[x])$ . Hence  $(1-e)x \circ (x)f \in \text{nil}(R_0[x])$ .

Conversely, suppose that for some  $e \in \text{Idem}(R)$ ,  $ex \circ (x)f \in \text{vnr}(R_0[x])$  and  $(1-e)x \circ (x)f \in \text{nil}(R_0[x])$ . Then for some  $n \geq 1$ ,  $0 = ((1-e)x \circ (x)f)^{(n)} = (1-e)x \circ ((x)f)^{(n)} = ((x)f)^{(n)} \circ (1-e)x$ , since  $(1-e)x$  is a central idempotent of  $R_0[x]$ . Hence

$$(3.1) \quad ((x)f)^{(n)} = ex \circ ((x)f)^{(n)}.$$

Since  $ex \circ (x)f$  is regular,  $ex \circ (x)f = (x)u \circ cx$  for some  $(x)u \in U(R_0[x])$  and  $c \in \text{Idem}(R)$  by Proposition 2.8 and Lemma 2.3. Thus  $(ex \circ (x)f)^{(n)} = ((x)u \circ cx)^{(n)} = cx \circ ((x)u)^{(n)}$ . But  $(ex \circ (x)f)^{(n)} = ex \circ ((x)f)^{(n)} = ((x)f)^{(n)}$  by Eq. (3.1). Hence  $((x)f)^{(n)} = cx \circ ((x)u)^{(n)}$ . Let  $(x)g = cx \circ ((x)u^{-1})^{(n)}$ . Then  $((x)f)^{(n)} \circ (x)g \circ ((x)f)^{(n)} = ((x)f)^{(n)} \circ cx \circ ((x)u^{-1})^{(n)} \circ ((x)f)^{(n)} = cx \circ ((x)u)^{(n)} = ((x)f)^{(n)}$ , since idempotents of the near-ring  $R_0[x]$  are central. Therefore  $(x)f$  is  $\pi$ -regular.  $\square$

**Lemma 3.3.** *Let  $R$  be a commutative ring and  $(x)f$  be a  $\pi$ -regular element of the near-ring  $R_0[x]$ . Then for some  $(x)g \in \text{Idem}(R_0[x])$  and  $(x)u \in U(R_0[x])$  we have  $(x)g \circ (x)f = (x)g \circ (x)u$ .*

*Proof.* Since  $(x)f$  is  $\pi$ -regular, by Proposition 2.8, we have  $((x)f)^{(n)} = (x)u \circ (x)g$  for some  $(x)g \in \text{Idem}(R_0[x])$ ,  $(x)u \in U(R_0[x])$  and  $n \geq 1$ . By Lemma 2.3,  $(x)g = ex$  for some  $e \in \text{Idem}(R)$ . As shown in the proof of Theorem 3.2,  $ex \circ (x)f$  is regular. Hence  $ex \circ (x)f = cx \circ (x)v$  for some  $c \in \text{Idem}(R)$  and  $(x)v \in U(R_0[x])$  by Proposition 2.8 and Lemma 2.3. Now we show that  $e = c$ . Since  $ex \circ (x)f = ex \circ (ex \circ (x)f) = ex \circ (cx \circ (x)v)$ , we have  $ecx \circ (x)v = cx \circ (x)v$  and therefore  $ec = c$ . Since  $ex$  and  $cx$  are central,  $(ex \circ (x)f)^{(n)} = ex \circ ((x)f)^{(n)} = cx \circ ((x)v)^{(n)}$ . Thus  $ex \circ ((x)f)^{(n)} = ex \circ (x)u = cx \circ ((x)v)^{(n)}$ , since  $((x)f)^{(n)} = (x)u \circ ex$ . Hence  $ex = cx \circ ((x)v)^{(n)} \circ (x)u^{-1}$ . Thus  $ecx = ex \circ cx = cx \circ ((x)v)^{(n)} \circ (x)u^{-1} \circ cx = cx \circ ((x)v)^{(n)} \circ (x)u^{-1}$ , which implies that  $ec = e$ . Thus  $e = c$ , since  $ec = c$ . Therefore  $(x)g \circ (x)f = (x)g \circ (x)v$ .  $\square$

**Lemma 3.4** ([8, Theorem 21.28]). *Let  $R$  be a ring with unity and  $I$  a two-sided nil ideal of  $R$ . If  $c + I \in \text{Idem}(R/I)$ , then there is  $e \in \text{Idem}(R)$  such that  $c + I = e + I$  in  $R/I$ .*

Let  $R$  be a commutative ring. Then  $\text{nil}(R)$  is a locally nilpotent ideal of  $R$ , and so  $\text{nil}(R[x]) = \text{nil}(R)_0[x]$  is a right ideal of the near-ring  $R[x]$  by [6, Theorem 3 and Proposition 8]. Since  $\text{nil}(R[x]) = \text{nil}(R_0[x])$ , then  $\text{nil}(R_0[x])$  is a

right ideal of  $R_0[x]$ . Let  $(x)f = \sum_{i=1}^m a_i x^i \in \text{nil}(R_0[x])$  and  $(x)g = \sum_{j=1}^n b_j x^j \in R_0[x]$ . Hence  $(x)g \circ (x)f = a_1((x)g) + \cdots + a_m((x)g)^m \in \text{nil}(R_0[x]) = \text{nil}(R_0[x])$ , since  $a_i \in \text{nil}(R)$ . Therefore  $\text{nil}(R_0[x])$  is a two-sided ideal of the near-ring  $R_0[x]$ . One can easily show that the map  $\varphi : R_0[x] \rightarrow (R/\text{nil}(R))_0[x]$  with  $\varphi(\sum_{i=1}^n a_i x^i) = \sum_{i=1}^n \bar{a}_i x^i$ , where  $\bar{a}_i = a_i + \text{nil}(R)$  is a near-ring epimorphism. Hence  $R_0[x]/\text{nil}(R_0[x]) \cong (R/\text{nil}(R))_0[x]$ .

**Theorem 3.5.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$  and  $(x)f \in R_0[x]$ . Then  $(x)f$  is  $\pi$ -regular if and only if  $(x)f + \text{nil}(R_0[x])$  is regular.*

*Proof.* Suppose that  $(x)f$  is  $\pi$ -regular and  $(x)\bar{f} = (x)f + \text{nil}(R_0[x])$ . Then  $((x)f)^{(n)} = ((x)f)^{(n)} \circ (x)g \circ ((x)f)^{(n)}$  for some  $(x)g \in R_0[x]$  and  $n \geq 1$ . Hence  $((x)f)^{(n)} \circ (x)g \in \text{Idem}(R_0[x])$ . Thus by Lemma 2.3,  $((x)f)^{(n)} \circ (x)g = ex$ , for some  $e \in \text{Idem}(R)$ . Therefore  $((1-e)x \circ (x)f)^{(n)} = (1-e)x \circ ((x)f)^{(n)} = (1-e)x \circ ex \circ ((x)f)^{(n)} = 0$ , since idempotents of  $R_0[x]$  are central. Hence  $[x - ((x)f)^{(n)} \circ (x)g] \circ (x)f = (1-e)x \circ (x)f \in \text{nil}(R_0[x])$ . Since  $x - ((x)f)^{(n)} \circ (x)g$  is idempotent, hence we have

$$\begin{aligned} & (x)f - (x)f \circ [((x)f)^{(n-1)} \circ (x)g] \circ (x)f \\ &= (x)f - ((x)f)^{(n)} \circ (x)g \circ (x)f \\ &= (x)f - (x)f \circ ((x)f)^{(n)} \circ (x)g \\ &= (x)f \circ [x - ((x)f)^{(n)} \circ (x)g] \\ &= [x - ((x)f)^{(n)} \circ (x)g] \circ (x)f \in \text{nil}(R_0[x]) \end{aligned}$$

which implies that  $(x)f + \text{nil}(R_0[x]) = (x)f \circ [((x)f)^{(n-1)} \circ (x)g] \circ (x)f + \text{nil}(R_0[x])$ . Hence  $(x)\bar{f}$  is regular.

Conversely, assume that

$$(x)\bar{f} = (x)f + \text{nil}(R_0[x])$$

is regular in  $R_0[x]/\text{nil}(R_0[x])$ , where  $(x)f = \sum_{i=1}^m a_i x^i$ . Then  $(x)\bar{f} = (x)\bar{u} \circ (x)\bar{c}$  for some  $(x)\bar{u} \in U(R_0[x]/\text{nil}(R_0[x]))$  and  $\bar{c} \in \text{Idem}(R_0[x]/\text{nil}(R_0[x]))$  by Proposition 2.8. Since  $R_0[x]/\text{nil}(R_0[x]) \cong (R/\text{nil}(R))_0[x]$ , we have  $(x)\bar{u} \in U((R/\text{nil}(R))_0[x])$  and  $(x)\bar{c} \in \text{Idem}((R/\text{nil}(R))_0[x])$ . Hence by Corollary 2.6,  $(x)\bar{u} = \bar{v}x$  for some  $\bar{v} \in U(R/\text{nil}(R))$ . Since  $\text{nil}(R) \subseteq J(R)$ ,  $(x)\bar{u} = \bar{v}'x$  for some  $v' \in U(R)$ . Furthermore, by Lemmas 2.3 and 3.4,  $(x)\bar{c} = \bar{e}x = (e + \text{nil}(R))x$  for some  $e \in \text{Idem}(R)$ . Thus  $(x)\bar{f} = \bar{v}'x \circ \bar{e}x = \bar{v}'\bar{e}x = \bar{v}'ex$ . Therefore  $(x)\bar{f} = \sum_{i=1}^m \bar{a}_i x^i = \bar{v}'ex$ , which implies that  $a_1 - v'e, a_i \in \text{nil}(R)$  for each  $i \geq 2$ . Then  $a_1 = v'e + b$  for some  $b \in \text{nil}(R)$ . Hence  $(x)w = bx + a_2 x^2 + \cdots + a_m x^m \in \text{nil}(R_0[x]) = \text{nil}(R_0[x])$  and  $a_1$  is  $\pi$ -regular by [1, Theorem 4.2]. Therefore  $(x)f = v'x \circ ex + (x)w$ . By Theorem 2.5,  $v'x + (x)w \in U(R_0[x])$ , hence  $ex \circ (x)f = ex \circ (ex \circ v'x + (x)w) = ex \circ (v'x + (x)w)$  is regular by Proposition 2.8. Further,  $(1-e)x \circ (x)f = (x)f - (x)f \circ ex = (v'x \circ ex + (x)w) - (v'x \circ ex + (x)w) \circ ex = (x)w - ex \circ (x)w \in \text{nil}(R_0[x])$ , since idempotents of  $R_0[x]$

are central and  $\text{nil}(R_0[x])$  is an ideal of  $R_0[x]$ . Therefore  $(x)f$  is  $\pi$ -regular by Theorem 3.2.  $\square$

From Theorem 3.5 we conclude that  $R_0[x]$  is not  $\pi$ -regular. Now we give a characterization of  $\pi$ -regular elements of  $R_0[x]$ , when  $R$  is a commutative ring with  $\text{nil}(R)^2 = 0$ .

**Theorem 3.6.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$  and  $(x)f \in R_0[x]$ . Then the following statements are equivalent:*

- (1)  $(x)f \in \pi - r(R_0[x])$ .
- (2)  $((x)f)^{(n)} \in \text{vnr}(R_0[x])$  for some  $n \geq 1$ .
- (3)  $((x)f)^{(n)} = (x)u \circ (x)h$  for some  $(x)u \in U(R_0[x])$  and  $(x)h \in \text{Idem}(R_0[x])$ .
- (4)  $(x)f = (x)g + (x)w$  for some  $(x)g \in \text{vnr}(R_0[x])$  and  $(x)w \in \text{nil}(R_0[x])$ .
- (5)  $(x)f = (x)u \circ (x)h + (x)w$  for some  $(x)u \in U(R_0[x])$ ,  $(x)h \in \text{Idem}(R_0[x])$  and  $(x)w \in \text{nil}(R_0[x])$ .
- (6)  $(x)f + \text{nil}(R_0[x]) \in \text{vnr}(R_0[x]/\text{nil}(R_0[x]))$ .

*Proof.* (1)  $\Leftrightarrow$  (2) It is clear.

(2)  $\Leftrightarrow$  (3) and (4)  $\Leftrightarrow$  (5) It follows from Proposition 2.8.

(1)  $\Rightarrow$  (5) It follows from Theorem 3.5.

(4)  $\Rightarrow$  (6) It is clear.

(6)  $\Rightarrow$  (1) It follows from Theorem 3.5.  $\square$

**Corollary 3.7.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$ . Then we have:*

- (1)  $\pi - r(R_0[x]) = \text{vnr}(R_0[x]) + \text{nil}(R_0[x])$ .
- (2)  $\pi - r(R_0[x])/\text{nil}(R_0[x]) = \text{vnr}(R_0[x]/\text{nil}(R_0[x]))$ .
- (3)  $\pi - r(R_0[x]) = \text{vnr}(R_0[x])$  if and only if  $R$  is reduced.
- (4) If  $2 \in U(R)$ , then every  $(x)f \in \pi - r(R_0[x])$  is the sum of two units of  $R_0[x]$ .

*Proof.* (1) This follows from the equivalence of (1) and (4) in Theorem 3.6.

(2) This follows from the equivalence of (1) and (6) in Theorem 3.6.

(3) Since by Theorem 2.1,  $\text{nil}(R_0[x]) \cap \text{vnr}(R_0[x]) = \{0\}$ , the result follows from (1).

(4) By (1),  $(x)f = (x)g + (x)w$  with  $(x)g \in \text{vnr}(R_0[x])$  and  $(x)w \in \text{nil}(R_0[x])$ . Then  $(x)g = (x)u + (x)v$  for some  $(x)u, (x)v \in U(R_0[x])$  by Theorem 2.13. Thus  $(x)u' = (x)v + (x)w \in U(R_0[x])$  by Theorem 2.5. Hence  $(x)f = (x)u + (x)u'$  is the sum of two units of  $R_0[x]$ .  $\square$

**Proposition 3.8.** *If  $R$  is a commutative ring with  $\text{nil}(R)^2 = 0$ , then  $\pi - r(R_0[x])$  is multiplicatively closed.*

*Proof.* Let  $(x)f_1, (x)f_2 \in \pi - r(R_0[x])$ . Thus  $(x)f_1 = u_1e_1x + (x)h_1$  and  $(x)f_2 = u_2e_2x + (x)h_2$  for some  $u_1, u_2 \in U(R)$ ,  $e_1, e_2 \in \text{Idem}(R)$  and  $(x)h_1, (x)h_2 \in \text{nil}(R_0[x])$  by Corollary 3.7. Thus  $(x)w_1 = u_2e_2((x)h_1)$  and  $(x)w_2 = (x)f_1 \circ$

$(x)h_2$  are nilpotent elements of  $R_0[x]$ , since  $\text{nil}(R_0[x])$  is an ideal of  $R_0[x]$ . Hence

$$\begin{aligned} (x)f_1 \circ (x)f_2 &= (u_1e_1x + (x)h_1) \circ (u_2e_2x + (x)h_2) \\ &= (u_1e_1x + (x)h_1) \circ u_2e_2x + (u_1e_1x + (x)h_1) \circ (x)h_2 \\ &= u_2e_2(u_1e_1x + (x)h_1) + (x)w_2 \\ &= u_2e_2u_1e_1x + (x)w_1 + (x)w_2. \end{aligned}$$

Then by [1, Theorem 2.1],  $u_2e_2u_1e_1 \in \text{vnr}(R)$ . Also,  $(x)w_1 + (x)w_2 \in \text{nil}(R_0[x])$ , since  $\text{nil}(R_0[x])$  is an ideal of  $R_0[x]$ . Therefore  $(x)f_1 \circ (x)f_2 \in \pi - r(R_0[x])$  by Corollary 3.7.  $\square$

**Theorem 3.9.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$ . Then  $\pi - r(R_0[x]) = \text{vnr}(R_0[x]) \cup \text{nil}(R_0[x])$  if and only if either  $\text{Idem}(R) = \{0, 1\}$  or  $R$  is reduced.*

*Proof.* Suppose that  $\pi - r(R_0[x]) = \text{vnr}(R_0[x]) \cup \text{nil}(R_0[x])$  and there exists  $e \in \text{Idem}(R) \setminus \{0, 1\}$ . Thus  $\text{Idem}(R_0[x]) \neq \{0, x\}$  by Lemma 2.3. Let  $(x)f \in \text{nil}(R_0[x])$ . Then  $ex + (x)f \in \text{vnr}(R_0[x]) + \text{nil}(R_0[x]) = \pi - r(R_0[x]) = \text{vnr}(R_0[x]) \cup \text{nil}(R_0[x])$  by Corollary 3.7 and hypothesis. Thus  $ex + (x)f \in \text{vnr}(R_0[x])$ , since  $e \neq 0$ . Hence by Theorem 2.1,  $(x)f - ex \circ (x)f = (1 - e)x \circ (x)f = (1 - e)x \circ (ex + (x)f) \in \text{vnr}(R_0[x])$ , since idempotents of  $R_0[x]$  are central. Also,  $(x)f - ex \circ (x)f = (1 - e)x \circ (x)f \in \text{nil}(R_0[x])$ , since  $\text{nil}(R_0[x])$  is an ideal of  $R_0[x]$ . Hence by Theorem 2.1,  $(x)f - ex \circ (x)f = 0$ . By replacing  $ex$  with  $(1 - e)x$ , a similar argument yields that  $ex \circ (x)f = 0$ , and so  $(x)f = 0$ . Therefore  $\text{nil}(R) = \{0\}$  by [3, Proposition 3.1].

Conversely, if  $\text{Idem}(R) = \{0, 1\}$ , then  $\text{Idem}(R_0[x]) = \{0, x\}$  by Lemma 2.3. Hence by Theorem 2.1,  $\text{vnr}(R_0[x]) = U(R_0[x]) \cup \{0\}$ . Thus  $\pi - r(R_0[x]) = U(R_0[x]) + \text{nil}(R_0[x]) = U(R_0[x])$  by Corollaries 2.6 and 3.7. Also, if  $\text{nil}(R) = \{0\}$ , then  $\text{nil}(R_0[x]) = \text{nil}(R)_0[x] = \{0\}$ . Therefore by Corollary 3.7,  $\pi - r(R_0[x]) = \text{vnr}(R_0[x])$ . Hence  $\pi - r(R_0[x]) = \text{vnr}(R_0[x]) \cup \text{nil}(R_0[x])$ .  $\square$

**Theorem 3.10.** *Let  $R$  be a commutative ring with  $\text{nil}(R)^2 = 0$ . Then*

- (1)  $\text{cln}(R_0[x]) = (\text{cln}(R))x + (\text{nil}(R_0[x]))x$   
 $= \left\{ \sum_{i=1}^n a_i x^i \mid a_1 \in \text{cln}(R), a_i \in \text{nil}(R) \text{ for every } i \geq 2 \right\}.$
- (2)  $R_0[x]$  is never a clean near-ring.

*Proof.* (1) By Theorem 2.5 and Lemma 2.3, we have  $\text{cln}(R_0[x]) = U(R_0[x]) + \text{Idem}(R_0[x]) = \left\{ \sum_{i=1}^n a_i x^i \mid a_1 = u + e \text{ for some } u \in U(R), e \in \text{Idem}(R) \text{ and } a_i \in \text{nil}(R) \text{ for every } i \geq 2 \right\} = \left\{ \sum_{i=1}^n a_i x^i \mid a_1 \in \text{cln}(R), a_i \in \text{nil}(R) \text{ for every } i \geq 2 \right\}.$

(2) It follows from (1), since  $x^2 \notin \text{cln}(R_0[x])$ .  $\square$

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