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ON SOME TYPE ELEMENTS OF ZERO-SYMMETRIC NEAR-RING OF POLYNOMIALS

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Abstract. Let R be a commutative ring with unity. In this paper, we characterize the unit elements, the regular elements, the π-regular elements and the clean elements of zero-symmetric near-ring of polynomials $R_0[x]$, when $\mathrm{nil}(R)^2 = 0$. Moreover, it is shown that the set of π-regular elements of $R_0[x]$ forms a semigroup. These results are somewhat surprising since, in contrast to the polynomial ring case, the near-ring of polynomials has substitution for its "multiplication" operation.

1. Introduction and preliminary definitions

Through this paper, all rings are commutative with unity and all near-rings are abelian left near-ring with unity. A set N together with two binary operations " $+$ " and " $\cdot$ " is called left near-ring if $(N, +)$ is a group, $(N, \cdot)$ is a semigroup and $a \cdot (b + c) = a \cdot b + a \cdot c$ for each $a, b, c \in N$. If $(N, +)$ is abelian, then we call N *abelian*.

For a near-ring N, $N_0 = \{a \in N \mid 0 \cdot a = 0\}$ is called the zero-symmetric part of N, $N_c = \{a \in N \mid 0 \cdot a = a\}$ is called the *constant part* of N. A near-ring N is called *zero-symmetric* if $N = N_0$. A near-ring N is called constant *near-ring* if $N_c = N$. Also, a subgroup M of a near-ring N with $MM \subseteq M$ is called a *subnear-ring* of N. Thus N_0 and N_c are subnear-rings of N. The most general class of examples of zero-symmetric near-rings comes from the following construction: Let $(G, +)$ be a not necessarily abelian group. Then the set $M_0(G)$ of all functions $f : G \to G$ with $f(0) = 0$ under pointwise addition $+$ and function composition $\circ$ determines a zero-symmetric near-ring $(M_0(G), +, \circ)$. Evidently, also each ring is a zero-symmetric (left) near-ring and so we may view near-rings as generalized rings. For basic definitions and comprehensive discussion on near-rings, we refer the reader to [11].

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Recall that, a near-ring N is a *near-field*, if every nonzero element $a \in N$ has multiplicatively inverse a^{-1}. Thus the nonzero elements of N form a group under multiplication.

A subgroup M of $(N, +)$ is called N-*subgroup*, if $MN \subseteq M$. It is proved that N is a zero-symmetric near-ring if and only if each right ideal of N is an N-subgroup of N by [11, Proposition 1.34]. A zero-symmetric near-ring N is called *local* if $L = \{k \in N \mid kN \neq N\}$ is an N-subgroup. Near-fields are local near-rings with $L = 0$. Maxson in [9, Theorem 4.2], proved that if N is a local near-ring, then N contains no idempotent other than 0 and 1. A near-ring N is called *integral*, if N has no nonzero zero divisor.

For a near-ring N, $\mathrm{nil}(N)$, $\mathrm{idem}(N)$ and $U(N)$ denote the set of all nilpotent elements of N, the set of all idempotent elements of N and the set of all units of N, respectively. Given a ring or near-ring N, we say that it is *reduced* if it has no nonzero nilpotent element. Also, we write $Z_\ell(N)$, $Z_r(N)$ and $Z(N)$ for the set of all left zero divisors of N, the set of all right zero divisors and the set $Z_\ell(N) \cup Z_r(N)$, respectively.

An element a of a near-ring N is called *regular* if there exists $b \in N$ such that $a = aba$. The set of all regular elements of N is denoted by $\mathrm{vnr}(N)$. A near-ring N is called *regular*, whenever $\mathrm{vnr}(N) = N$. For example, every constant near-ring is regular. Further, Beidleman in [2], proved that the near-rings $M(G)$ and $M_0(G)$ are regular. Also, he showed that a regular near-ring with identity contains no nonzero nil N-subgroup. In [4], Chao proved that if N is a reduced zero-symmetric near-ring with unity, then N is regular if and only if aN is a direct summand of N for each $a \in N$. According to [11, p. 347], a regular near-ring with identity is integral if and only if it is a near-field. Properties of regular near-rings have been studied by Ghoudhari, Goyal, Heatherly, Hongan, Ligh, Mason and Murty. Their main results are suggested in the book [11].

A near-ring N is said to be π-*regular* if for each element $a \in N$, there exists a positive integer n such that a^n is a regular element, that is, $a^n = a^n b a^n$ for some $b \in N$. Such an element a is called π-*regular*. The set of all π-regular elements of N is denoted by $\pi - r(N)$. Clearly every regular near-ring is π-regular, but Cho in [5] gives an example of a π-regular near-ring which is not regular. As in [10] for a ring, we say that an element a of a near-ring N is *clean* if a is the sum of a unit and an idempotent of R. The set of all clean elements of N is denoted by $\mathrm{cln}(N)$. Moreover, N is said to be a clean near-ring if $\mathrm{cln}(N) = N$.

We say that a subset S of a ring or near-ring is *locally nilpotent* if for any finite subset $\{s_1, s_2, \ldots, s_n\} \subseteq S$, there exists an integer k such that any product of k elements from $\{s_1, s_2, \ldots, s_n\}$ is zero. In other words, S is locally nilpotent if any subring without identity generated by a finite number of elements in S is nilpotent.

Let R be a ring. Since $R[x]$ is an abelian near-ring under addition and substitution, it is natural to investigate the near-ring of polynomials $(R[x], +, \circ)$. The binary operation of substitution, denoted by " $\circ$ ", of one polynomial into another is both natural and important in the theory of polynomials. We adopt the convention that for polynomials $(x)f = \sum_{i=0}^{m} a_i x^i$ and $(x)g \in R[x]$,

$$(x)g \circ (x)f = \sum_{i=0}^{m} a_i((x)g)^i.$$

For example, $(a_0 + a_1x) \circ x^2 = (a_0 + a_1x)^2 = a_0^2 + (a_0a_1 + a_1a_0)x + a_1^2x^2$. However, the operation $\circ$, left distributes but does not right distribute over addition. Thus $(R[x], +, \circ)$ forms a left near-ring but not a ring. We use $R[x]$ to denote the left near-ring $(R[x], +, \circ)$ with coefficients from R and $R_0[x] = \{(x)f \mid (x)f$ has zero constant term$\}$ is the zero-symmetric left near-ring of polynomials with coefficients in R. Also, for each $(x)f = \sum_{i=0}^{m} a_ix^i$ and $(x)g = \sum_{j=0}^{n} b_jx^j \in R[x]$, we write $(x)f(x)g = \sum_{k=0}^{n+m}(\sum_{i+j=k} a_ib_j)x^k$.

In this paper, we characterize all of the unit elements, the regular elements, the π-regular elements and the clean elements of the zero-symmetric near-ring $R_0[x]$, when R is a commutative ring with $\mathrm{nil}(R)^2 = 0$. Also, we prove that $\mathrm{vnr}(R_0[x])$ is a subnear-ring of $R_0[x]$ if and only if $\mathrm{vnr}(R)$ is a subring of R. Moreover, it is shown that the set of π-regular elements of $R_0[x]$ is multiplicatively closed. These results are somewhat surprising since, in contrast to the polynomial ring case, the near-ring of polynomials has substitution for its "multiplication" operation.

2. Regular elements

In this section we investigate regular elements of the near-ring $R_0[x]$, when R is a commutative ring with $\mathrm{nil}(R)^2 = 0$.

Theorem 2.1. *Let N be a near-ring with central idempotents.*

1. *Let $a \in N$. If $aba = a$ for some $b \in N$, then $ab = ba$ is an idempotent of N.*
2. $\mathrm{vnr}(N)$*is multiplicatively closed.*
3. $\mathrm{vnr}(N) \cap \mathrm{nil}(N) = \{0\}$.
4. $U(N) \cup \mathrm{Idem}(N) \subseteq \mathrm{vnr}(N) \subseteq U(N) \cup \mathrm{Z}(N)$.
5. $\mathrm{vnr}(N) = U(N) \cup \{0\}$ *if and only if* $\mathrm{Idem}(N) = \{0, 1\}$. *In particular,* $\mathrm{vnr}(N) = U(N) \cup \{0\}$ *if N is either integral or local.*
6. $\mathrm{vnr}(N)$ *contains a nonzero nonunit if and only if* $\mathrm{Idem}(N) \neq \{0, 1\}$.

Proof. (1) Let $a \in \mathrm{vnr}(N)$. Then $a = aba$ for some $b \in N$. Hence $ab = (ab)^2 = abab = a(ba)b = (ba)ab = b(ab)a = (ba)^2 = ba$, since ab and ba are central idempotents.

(2) Let $a, a' \in \mathrm{vnr}(N)$. Then $a = aba$ and $a' = a'ca'$ for some $b, c \in R$. Since idempotent elements of N are central, it follows that $aa' = (aba)(a'ca') = aa'(cb)aa'$ by (1).

By a similar argument one can prove the other statements. □

Proposition 2.2. *Let N be a near-ring which whose idempotents are central. If $a \in \mathrm{vnr}(N)$, then there exists a unique $b \in N$ with $aba = a$ and $bab = b$.*

Proof. Suppose that $a \in \mathrm{vnr}(N)$. Then $a = aca$ for some $c \in N$. Let $b = cac$, hence $ca = ac \in \mathrm{Idem}(N)$ by Theorem 2.1. Thus $aba = a$ and $bab = b$. Now assume that there exists $b_1 \in N$ such that $ab_1a = a$ and $b_1ab_1 = b_1$. Thus $b_1a = ab_1 \in \mathrm{Idem}(N)$ by Theorem 2.1. So we have $b_1 = b_1ab_1 = b_1(aba)b_1 = b_1(ab_1a)b = b_1ab = b_1(aba)b = bab_1ab = b$. Therefore b is unique. □

Since every idempotent is central in each commutative ring, then by [7, Lemma 2.1], we have the following result.

Lemma 2.3. *Let R be a commutative ring and $(x)f \in R_0[x]$. Then $(x)f$ is an idempotent element of the near-ring $R_0[x]$ if and only if $(x)f = e_1x$, where e_1 is an idempotent of R. In particular, the idempotent elements of $R_0[x]$ are central.*

For each $(x)f \in R_0[x]$ and positive integer n, we write

$$((x)f)^{(n)} = \underbrace{(x)f \circ (x)f \circ \cdots \circ (x)f}_{n}.$$

Lemma 2.4. *Let R be a reduced commutative ring and $(x)f = \sum_{i=1}^{m} a_ix^i$, $(x)g = \sum_{j=1}^{n} b_jx^j \in R_0[x]$. If $(x)g \circ (x)f = cx$, then $a_1b_1 = c$ and $a_ib_j = 0$ for $i + j \neq 2$.*

Proof. Let $n = 1$. Then $(x)g \circ (x)f = a_1(b_1x) + \cdots + a_m(b_1x)^m = cx$. Hence $a_1b_1 = c$ and $a_ib_1 = 0$ for $i = 2, \ldots, m$, since $a_ib_1^i = 0$ and R is reduced. Now assume that $n > 1$. Then we have

$$(x)g \circ (x)f = a_1((x)g) + a_2((x)g)^2 + \cdots + a_m((x)g)^m = cx, \tag{2.1}$$

which implies that $a_1b_1 = c$ and $a_mb_n^m = 0$, since it is the leading coefficient of Eq. (2.1). Thus $a_mb_n = b_na_m = 0$, since R is reduced. By multiplying b_n to Eq. (2.1), we obtain

$$b_na_1((x)g) + b_na_2((x)g)^2 + \cdots + b_na_{m-1}((x)g)^{m-1} = b_ncx. \tag{2.2}$$

Hence $b_na_{m-1}(b_n)^{m-1} = 0$, since it is the leading coefficient of Eq. (2.2). Therefore $b_na_{m-1} = a_{m-1}b_n = 0$, since R is reduced. Inductively, we have $b_na_i = a_ib_n = 0$ for $i = 1, \ldots, m$. Hence from Eq. (2.1) we have $(\sum_{j=1}^{n-1} b_jx^j) \circ (\sum_{i=1}^{m} a_ix^i) = cx$. Continuing this process, one can prove that $b_ja_i = a_ib_j = 0$ for $i + j \neq 2$. □

It is well known that if R is a commutative ring, then $(x)f = \sum_{i=0}^{m} a_ix^i$ is a unit element of the polynomial ring $R[x]$ if and only if $a_0 \in U(R)$ and $a_1, \ldots, a_m \in \mathrm{nil}(R)$. In the next theorem, we determine unit elements of the near-ring $R_0[x]$, when R is a commutative ring with $\mathrm{nil}(R)^2 = 0$.

Theorem 2.5. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$. Then $(x)f = \sum_{i=1}^{m} a_ix^i \in U(R_0[x])$ if and only if $a_1 \in U(R)$ and $a_2, \ldots, a_m \in \mathrm{nil}(R)$.*

Proof. Suppose that $(x)f \in U(R_0[x])$. Then $(x)f \circ (x)g = (x)g \circ (x)f = x$ for some $(x)g = \sum_{j=1}^{n} b_j x^j \in R_0[x]$. Since $\mathrm{nil}(R)$ is an ideal of R, it follows that $\overline{R} = R/\mathrm{nil}(R)$ is reduced and so $(x)\overline{f} \circ (x)\overline{g} = (x)\overline{g} \circ (x)\overline{f} = \overline{1}x = (1+\mathrm{nil}(R))x$, where $(x)\overline{f} = \sum_{i=1}^{m}(a_i+\mathrm{nil}(R))x^i$ and $(x)\overline{g} = \sum_{j=1}^{n}(b_j+\mathrm{nil}(R))x^j$. By Lemma 2.4, $\overline{a}_1\overline{b}_1 = \overline{b}_1\overline{a}_1 = \overline{1}$ and $\overline{b}_1\overline{a}_i = \overline{0}$ for $i = 2, \dots, m$, which implies that $\overline{a}_i = \overline{0}$ for $i = 2, \dots, m$. Since $\mathrm{nil}(R) \subseteq \mathrm{J}(R)$, it follows that $a_1 \in U(R)$ and $a_i \in \mathrm{nil}(R)$ for $i = 2, \dots, m$.

Conversely, let $(x)f = a_0x + a_1x^2 + \cdots + a_nx^{n+1}$, where $a_0 \in U(R)$ and $a_1, a_2, \dots, a_n \in \mathrm{nil}(R)$. We show that $(x)f$ has right and left inverse. Since R is commutative, then $(x)f_1 = a_0 + a_1x + \cdots + a_nx^n$ is a unit element of the polynomial ring $R[x]$. Thus there exists $(x)g = b_0+b_1x+\cdots+b_mx^m$ of $R[x]$ such that $(x)f_1(x)g = (x)g(x)f_1 = 1$. Hence $b_0 \in U(R)$ and $b_1, \dots, b_m \in \mathrm{nil}(R)$. Since $\mathrm{nil}(R[x]) = \mathrm{nil}(R)[x]$, it follows that $(x)g_1 = b_1x+\cdots+b_mx^m$ is a nilpotent element of the polynomial ring $R[x]$ and so there is a non-negative integer k such that $((x)g_1)^k = 0$, which implies that $\deg[((x)g)^t] \le (k-1)m$ for each $t \ge k$. Put $r = (k-1)m$. We have to find $(x)h = h_1x + h_2x^2 + \cdots + h_{r+1}x^{r+1} \in R_0[x]$ such that $(x)f \circ (x)h = x$. Then we have

$$
\begin{aligned}
& (x)f \circ (x)h = x \\
\Leftrightarrow\ & h_1((x)f) + h_2((x)f)^2 + \cdots + h_{r+1}((x)f)^{r+1} = x \\
\Leftrightarrow\ & [h_1 + h_2((x)f) + \cdots + h_{r+1}((x)f)^r](x)f = x \\
\Leftrightarrow\ & [h_1 + h_2((x)f) + \cdots + h_{r+1}((x)f)^r](x)f_1 = 1 \\
\Leftrightarrow\ & [h_1 + h_2((x)f) + \cdots + h_{r+1}((x)f)^r] = (x)g \\
\Leftrightarrow\ & [h_2x((x)f_1) + \cdots + h_{r+1}x^r((x)f_1)^r] = (x)g - h_1 \\
\Leftrightarrow\ & [h_2x + \cdots + h_{r+1}x^r((x)f_1)^{r-1}]((x)f_1) = (x)g - h_1 \\
\Leftrightarrow\ & [h_2x + \cdots + h_{r+1}x^r((x)f_1)^{r-1}] = ((x)g - h_1)(x)g \\
\Leftrightarrow\ & [h_3x^2((x)f_1)+\cdots+h_{r+1}x^r((x)f_1)^{r-1}] = ((x)g)^2 - h_1((x)g) - h_2x \\
\Leftrightarrow\ & [h_3x^2+\cdots+h_{r+1}x^r((x)f_1)^{r-2}]((x)f_1) = ((x)g)^2 - h_1((x)g) - h_2x \\
\Leftrightarrow\ & [h_3x^2+\cdots+h_{r+1}x^r((x)f_1)^{r-2}] = ((x)g)^3 - h_1((x)g)^2 - h_2x((x)g) \\
& \vdots \\
\Leftrightarrow\ & ((x)g)^{r+1} - h_1((x)g)^r - \cdots - h_rx^{r-1}(x)g - h_{r+1}x^r = 0 \\
\Leftrightarrow\ & h_1 = b_0,\ h_2 = b_0b_1,\ h_3 = b_0^2b_2 + b_0b_1^2,\ \dots, \\
& h_{r+1} = \sum_{i_1+\cdots+i_{r+1}=r} b_{i_1}\dots b_{i_{r+1}} - h_1 \sum_{i_1+\cdots+i_r=r} b_{i_1}\dots b_{i_r} - \cdots - h_rb_1,
\end{aligned}
$$

where $b_{i_j} \in \{b_0, b_1, \dots, b_m\}$ for $j = 1, \dots, r+1$. Hence $(x)h$ is a right inverse for $(x)f$.

Since $b_0 \in U(R)$ and $\{b_1, \ldots, b_m\} \subseteq nil(R)$, hence $h_1 \in U(R)$ and $\{h_2, \ldots, h_{r+1}\} \subseteq \mathrm{nil}(R)$. Thus with a similar argument as used in the previous paragraph, one can find $(x)k \in R_0[x]$ such that $(x)h \circ (x)k = x$. Hence $(x)h \in U(R_0[x])$, which implies that $(x)f \in U(R_0[x])$. $\square$

Corollary 2.6. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$. Then $U(R_0[x]) = U(R)x + \mathrm{nil}(R_0[x])$. In particular, if R is reduced, then $U(R_0[x]) = \{ux \mid u \in U(R)\}$.*

Corollary 2.7. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$ and $(x)f \in R_0[x]$. If $(x)f$ has right or left inverse, then $(x)f$ is invertible in $R_0[x]$.*

Proof. It follows from the proof of Theorem 2.5. $\square$

Let R be a commutative ring and $a \in R$. Anderson and Badawi [1, Theorem 2.2], proved that $a \in \mathrm{vnr}(R)$ if and only if $a = ue$ for some $u \in U(R)$ and $e \in \mathrm{Idem}(R)$. In the next proposition, we extend this result to the near-ring $R_0[x]$.

Proposition 2.8. *Let R be a commutative ring and $(x)f \in R_0[x]$. Then the following statements are equivalent:*

(1) $(x)f \in \mathrm{vnr}(R_0[x])$.
(2) $(x)f = (x)f \circ (x)u \circ (x)f$ *for some* $(x)u \in U(R_0[x])$.
(3) $(x)f = (x)u \circ (x)h$ *for some* $(x)h \in \mathrm{Idem}(R_0[x])$ *and* $(x)u \in U(R_0[x])$.

Proof. (1) $\Rightarrow$ (2) Let $(x)f \in \mathrm{vnr}(R_0[x])$. Then $(x)f = (x)f \circ (x)g \circ (x)f$ for some $(x)g \in R_0[x]$ and so we have $(x)f \circ (x)g = (x)g \circ (x)f \in \mathrm{Idem}(R_0[x])$ by Theorem 2.1. Thus $(x)f \circ (x)g = ex$ for some $e \in \mathrm{Idem}(R)$ by Lemma 2.3. Clearly, $1 - e$ is an idempotent of R. Let $(x)u = ex \circ (x)g + (1 - e)x$. Then by using Lemma 2.3, we have

$$\begin{aligned}
&(x)u \circ [(x)f + (1-e)x] \\
&= (x)u \circ (x)f + (x)u \circ (1-e)x \\
&= [ex \circ (x)g + (1-e)x] \circ ex \circ (x)f + [ex \circ (x)g + (1-e)x] \circ (1-e)x \\
&= ex \circ [ex \circ (x)g + (1-e)x] \circ (x)f + [ex \circ (x)g + (1-e)x] \circ (1-e)x \\
&= ex \circ (x)g \circ (x)f + (1-e)x \\
&= ex + (1-e)x \\
&= x
\end{aligned}$$

and so $(x)u$ is invertible in $R_0[x]$ by Corollary 2.7. Further, $(1 - e)x \circ (x)f = (x)f \circ (1 - e)x = (x)f - (x)f \circ ex = (x)f - (x)f \circ (x)g \circ (x)f = 0$ by Lemma 2.3. Hence $(x)f \circ (x)u \circ (x)f = (x)f \circ [ex \circ (x)g + (1 - e)x)] \circ (x)f = [((x)f \circ ex) \circ (x)g + (x)f \circ (1 - e)x] \circ (x)f = (x)f \circ (x)g \circ (x)f = (x)f$.

(2) $\Rightarrow$ (3) Assume that $(x)f = (x)f \circ (x)v \circ (x)f$ for some $(x)v \in U(R_0[x])$ and let $u(x) = (x)v^{-1} \in U(R_0[x])$. Since $(x)h = (x)v \circ (x)f \in \mathrm{Idem}(R_0[x])$, it follows that $(x)u \circ (x)h = (x)v^{-1} \circ (x)v \circ (x)f = (x)f$.

(3) $\Rightarrow$ (1) Suppose that $(x)f = (x)u \circ (x)h$, where $(x)u \in U(R_0[x])$ and $(x)h \in \mathrm{Idem}(R_0[x])$. Hence by Lemma 2.3, $(x)h = ex$ for some $e \in \mathrm{Idem}(R)$. So $(x)f = (x)u \circ ex = ex \circ (x)u$, since ex is central. Therefore $(x)f \circ (x)u^{-1} \circ (x)f = (ex \circ (x)u) \circ (x)u^{-1} \circ (x)f = ex \circ (x)f = ex \circ (x)u \circ ex = (x)f$, since idempotents of $R_0[x]$ are central. $\square$

Now we give a characterization of regular elements of $R_0[x]$, when R is a commutative ring with $\mathrm{nil}(R)^2 = 0$.

Theorem 2.9. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$. Then* $\mathrm{vnr}(R_0[x]) = \big\{ \sum_{i=1}^{n} a_i x^i \in R_0[x] \mid n \geq 1,\ a_1 = ue$ *and* $a_i \in e(\mathrm{nil}(R))$ *for each* $i \geq 2$, *where* $u \in U(R)$ *and* $e \in \mathrm{Idem}(R)\big\}$.

Proof. It follows directly from Proposition 2.8, Theorem 2.5 and Lemma 2.3. $\square$

Corollary 2.10. *R be a commutative ring with $\mathrm{nil}(R)^2 = 0$. If R is reduced, then $\mathrm{vnr}(R_0[x]) = (\mathrm{vnr}(R))x$. In particular, if $\mathrm{vnr}(R)$ is a subring of R, then $\mathrm{vnr}(R_0[x]) = (\mathrm{vnr}(R))x$.*

Proof. If $\mathrm{nil}(R) = 0$, then $\mathrm{vnr}(R_0[x]) = (\mathrm{vnr}(R))x$ by Theorem 2.9. Now, assume that $\mathrm{vnr}(R)$ be a subring of R. Then by [1, Theorem 2.9], R is reduced and so the result follows. $\square$

Theorem 2.11. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$. If $\mathrm{vnr}(R_0[x])$ is a subnear-ring of $R_0[x]$, then R is reduced and $\mathrm{vnr}(R_0[x]) = (\mathrm{vnr}(R))x$.*

Proof. Let $(x)f$ be a nilpotent element of $R_0[x]$. Then by Theorem 2.5, $x + (x)f \in U(R_0[x]) \subseteq \mathrm{vnr}(R_0[x])$. Since $\mathrm{vnr}(R_0[x])$ is a subnear-ring of $R_0[x]$, we have $(x)f = -x + (x + (x)f) \in \mathrm{vnr}(R_0[x])$, which implies that $(x)f \in \mathrm{vnr}(R_0[x]) \cap \mathrm{nil}(R_0[x]) = \{0\}$ by Theorem 2.1. Therefore $\mathrm{nil}(R_0[x]) = \{0\}$ and R is reduced by [3, Proposition 3.1]. Also, $\mathrm{vnr}(R_0[x]) = (\mathrm{vnr}(R))x$ by Corollary 2.10. $\square$

Let R be a commutative ring. Anderson and Badawi [1, Theorem 2.1], proved that the set of regular elements of R, is multiplicatively closed. Thus we have the following result.

Corollary 2.12. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$. Then $\mathrm{vnr}(R_0[x])$ is a subnear-ring of $R_0[x]$ if and only if $\mathrm{vnr}(R)$ is a subring of R.*

Proof. If $\mathrm{vnr}(R_0[x])$ is a subnear-ring of $R_0[x]$, then $\mathrm{vnr}(R_0[x]) = (\mathrm{vnr}(R))x$ by Theorem 2.11. Hence $(\mathrm{vnr}(R))x$ is a subgroup of $(R_0[x], +)$, which implies that $\mathrm{vnr}(R)$ is a subring of R by [1, Theorem 2.1].

Conversely, assume that $\mathrm{vnr}(R)$ is a subring of R. Thus $\mathrm{vnr}(R_0[x]) = (\mathrm{vnr}(R))x$ by Corollary 2.10. Then $\mathrm{vnr}(R_0[x])$ is a subgroup of $(R_0[x], +)$, and so the result follows from Theorem 2.1. $\square$

Theorem 2.13. *Let R be a commutative ring with* $\mathrm{nil}(R)^2 = 0$ *and* $2 \in U(R)$. *Then every* $(x)f \in \mathrm{vnr}(R_0[x])$ *is the sum of two units of* $R_0[x]$.

Proof. Let $(x)f = \sum_{i=1}^{m} a_i x^i$ be a regular element of $R_0[x]$. Then $a_1 = ue$ and $a_i \in e(\mathrm{nil}(R))$ for some $u \in U(R)$ and $e \in \mathrm{Idem}(R)$ by Theorem 2.9. Hence $a_1 \in \mathrm{vnr}(R)$ by [1, Theorem 2.2]. Since $2 \in U(R)$, it follows that $a_1 = u' + v'$ for some $u', v' \in U(R)$ by [1, Theorem 2.10]. Let $(x)g = u'x$ and $(x)h = v'x + a_2x^2 + \cdots + a_mx^m$. Then $(x)g, (x)h \in U(R_0[x])$ by Theorem 2.5. Hence $(x)f = (x)g + (x)h$ is the sum of two units of $R_0[x]$. $\square$

Theorem 2.14. *Let R be a commutative ring with* $\mathrm{nil}(R)^2 = 0$ *and* $2 \in U(R)$. *Then the following statements are equivalent.*

(1) $\mathrm{vnr}(R_0[x])$ *is a subnear-ring of* $R_0[x]$.
(2) *The sum of any four units of* $R_0[x]$ *is a regular element of* $R_0[x]$.

Proof. (1) $\Rightarrow$ (2) It is clear since $U(R_0[x]) \subseteq \mathrm{vnr}(R_0[x])$ by Theorem 2.1.

(2) $\Rightarrow$ (1) By Theorem 2.1, $\mathrm{vnr}(R_0[x])$ is multiplicatively closed. Now, let $(x)f, (x)g \in \mathrm{vnr}(R_0[x])$. Hence there exist $(x)u_1, (x)u_2, (x)v_1, (x)v_2 \in U(R_0[x])$ such that $(x)f = (x)u_1 + (x)u_2$ and $(x)g = (x)v_1 + (x)v_2$ by Theorem 2.13. Thus $(x)f + (x)g$ is the sum of four units of $R_0[x]$, which implies that $(x)f + (x)g \in \mathrm{vnr}(R_0[x])$ by hypothesis. $\square$

Corollary 2.15. *Let R be a commutative ring with* $\mathrm{nil}(R)^2 = 0$ *and* $2 \in U(R)$. *If the sum of any four units of* $R_0[x]$ *is a regular element of* $R_0[x]$, *then* $\mathrm{vnr}(R_0[x]) = (\mathrm{vnr}(R))x$.

Proof. It follows from Theorem 2.14 and Corollaries 2.12 and 2.10. $\square$

3. π-regular elements and clean elements of $R_0[x]$

In this section, we investigate π-regular and clean elements of $R_0[x]$ when R is a commutative ring with $\mathrm{nil}(R)^2 = 0$.

Theorem 3.1. *Let N be a near-ring with central idempotents. Then*

(1) $\mathrm{vnr}(N) \subseteq \pi - r(N)$. *In particular, each regular near-ring is π-regular near-ring.*
(2) $\mathrm{vnr}(N) \cup \mathrm{nil}(N) \subseteq \pi - r(N) \subseteq U(N) \cup \mathrm{Z}(N)$.
(3) $\pi - r(N) = U(N) \cup \mathrm{nil}(N)$ *if and only if* $\mathrm{Idem}(N) = \{0, 1\}$. *In particular,* $\pi - r(N) = U(N) \cup \mathrm{nil}(N)$ *if N is either integral or local.*
(4) $\pi - r(N)$ *contains a non-nilpotent nonunit if and only if* $\mathrm{Idem}(N) \neq \{0, 1\}$.

Proof. By a similar way as used in the proof of [1, Theorem 4.1], one can prove it. $\square$

Theorem 3.2. *Let R be a commutative ring and* $(x)f \in R_0[x]$. *Then* $(x)f$ *is π-regular if and only if there exists* $(x)g \in \mathrm{Idem}(R_0[x])$ *such that* $(x)g \circ (x)f$ *is regular and* $(x - (x)g) \circ (x)f \in \mathrm{nil}(R_0[x])$.

Proof. Since $(x)f$ is π-regular, then $((x)f)^{(n)}$ is regular for some $n \geq 1$. Hence $((x)f)^{(n)} = (x)u \circ (x)g$ for some $(x)u \in U(R_0[x])$ and $(x)g \in \mathrm{Idem}(R_0[x])$ by Proposition 2.8. By Lemma 2.3, there exists $e \in \mathrm{Idem}(R)$ such that $(x)g = ex$. First we show that $ex \circ (x)f$ is regular. Since idempotents of $R_0[x]$ are central, we have $ex \circ (x)f \circ [((x)f)^{(n-1)} \circ (x)u^{-1}] \circ ex \circ (x)f = [ex \circ ((x)f)^{(n)} \circ (x)u^{-1}] \circ ex \circ (x)f = [ex \circ (x)u \circ ex \circ (x)u^{-1}] \circ ex \circ (x)f = [ex \circ (x)u \circ (x)u^{-1}] \circ ex \circ (x)f = ex \circ (x)f$, which implies that $ex \circ (x)f \in \mathrm{vnr}(R_0[x])$. Also $((1-e)x \circ (x)f)^{(n)} = (1-e)x \circ ((x)f)^{(n)} = (1-e)x \circ (x)u \circ ex = 0$, since $(1-e)x \in \mathrm{Idem}(R_0[x])$. Hence $(1-e)x \circ (x)f \in \mathrm{nil}(R_0[x])$.

Conversely, suppose that for some $e \in \mathrm{Idem}(R)$, $ex \circ (x)f \in \mathrm{vnr}(R_0[x])$ and $(1-e)x \circ (x)f \in \mathrm{nil}(R_0[x])$. Then for some $n \geq 1$, $0 = ((1-e)x \circ (x)f)^{(n)} = (1-e)x \circ ((x)f)^{(n)} = ((x)f)^{(n)} \circ (1-e)x$, since $(1-e)x$ is a central idempotent of $R_0[x]$. Hence

$$((x)f)^{(n)} = ex \circ ((x)f)^{(n)}. \tag{3.1}$$

Since $ex \circ (x)f$ is regular, $ex \circ (x)f = (x)u \circ cx$ for some $(x)u \in U(R_0[x])$ and $c \in \mathrm{Idem}(R)$ by Proposition 2.8 and Lemma 2.3. Thus $(ex \circ (x)f)^{(n)} = ((x)u \circ cx)^{(n)} = cx \circ ((x)u)^{(n)}$. But $(ex \circ (x)f)^{(n)} = ex \circ ((x)f)^{(n)} = ((x)f)^{(n)}$ by Eq. (3.1). Hence $((x)f)^{(n)} = cx \circ ((x)u)^{(n)}$. Let $(x)g = cx \circ ((x)u^{-1})^{(n)}$. Then $((x)f)^{(n)} \circ (x)g \circ ((x)f)^{(n)} = ((x)f)^{(n)} \circ cx \circ ((x)u^{-1})^{(n)} \circ ((x)f)^{(n)} = cx \circ ((x)u)^{(n)} = ((x)f)^{(n)}$, since idempotents of the near-ring $R_0[x]$ are central. Therefore $(x)f$ is π-regular. □

Lemma 3.3. *Let R be a commutative ring and $(x)f$ be a π-regular element of the near-ring $R_0[x]$. Then for some $(x)g \in \mathrm{Idem}(R_0[x])$ and $(x)u \in U(R_0[x])$ we have $(x)g \circ (x)f = (x)g \circ (x)u$.*

Proof. Since $(x)f$ is π-regular, by Proposition 2.8, we have $((x)f)^{(n)} = (x)u \circ (x)g$ for some $(x)g \in \mathrm{Idem}(R_0[x])$, $(x)u \in U(R_0[x])$ and $n \geq 1$. By Lemma 2.3, $(x)g = ex$ for some $e \in \mathrm{Idem}(R)$. As shown in the proof of Theorem 3.2, $ex \circ (x)f$ is regular. Hence $ex \circ (x)f = cx \circ (x)v$ for some $c \in \mathrm{Idem}(R)$ and $(x)v \in U(R_0[x])$ by Proposition 2.8 and Lemma 2.3. Now we show that $e = c$. Since $ex \circ (x)f = ex \circ (ex \circ (x)f) = ex \circ (cx \circ (x)v)$, we have $ecx \circ (x)v = cx \circ (x)v$ and therefore $ec = c$. Since ex and cx are central, $(ex \circ (x)f)^{(n)} = ex \circ ((x)f)^{(n)} = cx \circ ((x)v)^{(n)}$. Thus $ex \circ ((x)f)^{(n)} = ex \circ (x)u = cx \circ ((x)v)^{(n)}$, since $((x)f)^{(n)} = (x)u \circ ex$. Hence $ex = cx \circ ((x)v)^{(n)} \circ (x)u^{-1}$. Thus $ecx = ex \circ cx = cx \circ ((x)v)^{(n)} \circ (x)u^{-1} \circ cx = cx \circ ((x)v)^{(n)} \circ (x)u^{-1}$, which implies that $ec = e$. Thus $e = c$, since $ec = c$. Therefore $(x)g \circ (x)f = (x)g \circ (x)v$. □

Lemma 3.4 ([8, Theorem 21.28])**.** *Let R be a ring with unity and I a two-sided nil ideal of R. If $c + I \in \mathrm{Idem}(R/I)$, then there is $e \in \mathrm{Idem}(R)$ such that $c + I = e + I$ in R/I.*

Let R be a commutative ring. Then $\mathrm{nil}(R)$ is a locally nilpotent ideal of R, and so $\mathrm{nil}(R[x]) = \mathrm{nil}(R)_0[x]$ is a right ideal of the near-ring $R[x]$ by [6, Theorem 3 and Proposition 8]. Since $\mathrm{nil}(R[x]) = \mathrm{nil}(R_0[x])$, then $\mathrm{nil}(R_0[x])$ is a right ideal of $R_0[x]$. Let $(x)f = \sum_{i=1}^{m} a_i x^i \in \mathrm{nil}(R_0[x])$ and $(x)g = \sum_{j=1}^{n} b_j x^j \in R_0[x]$. Hence $(x)g \circ (x)f = a_1((x)g) + \cdots + a_m((x)g)^m \in \mathrm{nil}(R)_0[x] = \mathrm{nil}(R_0[x])$, since $a_i \in \mathrm{nil}(R)$. Therefore $\mathrm{nil}(R_0[x])$ is a two-sided ideal of the near-ring $R_0[x]$. One can easy show that the map $\varphi : R_0[x] \longrightarrow (R/\mathrm{nil}(R))_0[x]$ with $\varphi(\sum_{i=1}^{n} a_i x^i) = \sum_{i=1}^{n} \overline{a_i} x^i$, where $\overline{a_i} = a_i + \mathrm{nil}(R)$ is a near-ring epimomorphism. Hence $R_0[x]/\mathrm{nil}(R_0[x]) \cong (R/\mathrm{nil}(R))_0[x]$.

Theorem 3.5. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$ and $(x)f \in R_0[x]$. Then $(x)f$ is π-regular if and only if $(x)f + \mathrm{nil}(R_0[x])$ is regular.*

Proof. Suppose that $(x)f$ is π-regular and $(x)\overline{f} = (x)f + \mathrm{nil}(R_0[x])$. Then $((x)f)^{(n)} = ((x)f)^{(n)} \circ (x)g \circ ((x)f)^{(n)}$ for some $(x)g \in R_0[x]$ and $n \geq 1$. Hence $((x)f)^{(n)} \circ (x)g \in \mathrm{Idem}(R_0[x])$. Thus by Lemma 2.3, $((x)f)^{(n)} \circ (x)g = ex$, for some $e \in \mathrm{Idem}(R)$. Therefore $((1-e)x \circ (x)f)^{(n)} = (1-e)x \circ ((x)f)^{(n)} = (1-e)x \circ ex \circ ((x)f)^{(n)} = 0$, since idempotents of $R_0[x]$ are central. Hence $[x - ((x)f)^{(n)} \circ (x)g] \circ (x)f = (1-e)x \circ (x)f \in \mathrm{nil}(R_0[x])$. Since $x - ((x)f)^{(n)} \circ (x)g$ is idempotent, hence we have

$$\begin{aligned} & (x)f - (x)f \circ [((x)f)^{(n-1)} \circ (x)g] \circ (x)f \\ = \; & (x)f - ((x)f)^{(n)} \circ (x)g \circ (x)f \\ = \; & (x)f - (x)f \circ ((x)f)^{(n)} \circ (x)g \\ = \; & (x)f \circ [x - ((x)f)^{(n)} \circ (x)g] \\ = \; & [x - ((x)f)^{(n)} \circ (x)g] \circ (x)f \in \mathrm{nil}(R_0[x]) \end{aligned}$$

which implies that $(x)f + \mathrm{nil}(R_0[x]) = (x)f \circ [((x)f)^{(n-1)} \circ (x)g] \circ (x)f + \mathrm{nil}(R_0[x])$. Hence $(x)\overline{f}$ is regular.

Conversely, assume that

$$(x)\overline{f} = (x)f + \mathrm{nil}(R_0[x])$$

is regular in $R_0[x]/\mathrm{nil}(R_0[x])$, where $(x)f = \sum_{i=1}^{m} a_i x^i$. Then $(x)\overline{f} = (x)\overline{u} \circ (x)\overline{c}$ for some $(x)\overline{u} \in U(R_0[x]/\mathrm{nil}(R_0[x]))$ and $\overline{c} \in \mathrm{Idem}(R_0[x]/\mathrm{nil}(R_0[x]))$ by Proposition 2.8. Since $R_0[x]/\mathrm{nil}(R_0[x]) \cong (R/\mathrm{nil}(R))_0[x]$, we have $(x)\overline{u} \in U((R/\mathrm{nil}(R))_0[x])$ and $(x)\overline{c} \in \mathrm{Idem}((R/\mathrm{nil}(R))_0[x])$. Hence by Corollary 2.6, $(x)\overline{u} = \overline{v}x$ for some $\overline{v} \in U(R/\mathrm{nil}(R))$. Since $\mathrm{nil}(R) \subseteq J(R)$, $(x)\overline{u} = \overline{v'}x$ for some $v' \in U(R)$. Furthermore, by Lemmas 2.3 and 3.4, $(x)\overline{c} = \overline{e}x = (e + \mathrm{nil}(R))x$ for some $e \in \mathrm{Idem}(R)$. Thus $(x)\overline{f} = \overline{v'}x \circ \overline{e}x = \overline{v'}\overline{e}x = \overline{v'e}x$. Therefore $(x)\overline{f} = \sum_{i=1}^{m} \overline{a_i} x^i = \overline{v'e}x$, which implies that $a_1 - v'e, a_i \in \mathrm{nil}(R)$ for each $i \geq 2$. Then $a_1 = v'e + b$ for some $b \in \mathrm{nil}(R)$. Hence $(x)w = bx + a_2x^2 + \cdots + a_m x^m \in \mathrm{nil}(R)_0[x] = \mathrm{nil}(R_0[x])$ and a_1 is π-regular by [1, Theorem 4.2]. Therefore $(x)f = v'x \circ ex + (x)w$. By Theorem 2.5, $v'x + (x)w \in U(R_0[x])$, hence $ex \circ (x)f = ex \circ (ex \circ v'x + (x)w) = ex \circ (v'x + (x)w)$ is regular by Proposition 2.8. Further, $(1-e)x \circ (x)f = (x)f - (x)f \circ ex = (v'x \circ ex + (x)w) - (v'x \circ ex + (x)w) \circ ex = (x)w - ex \circ (x)w \in \mathrm{nil}(R_0[x])$, since idempotents of $R_0[x]$

are central and $\mathrm{nil}(R_0[x])$ is an ideal of $R_0[x]$. Therefore $(x)f$ is π-regular by Theorem 3.2. □

From Theorem 3.5 we conclude that $R_0[x]$ is not π-regular. Now we give a characterization of π-regular elements of $R_0[x]$, when R is a commutative ring with $\mathrm{nil}(R)^2 = 0$.

Theorem 3.6. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$ and $(x)f \in R_0[x]$. Then the following statements are equivalent:*

(1) $(x)f \in \pi - r(R_0[x])$.
(2) $((x)f)^{(n)} \in \mathrm{vnr}(R_0[x])$ *for some* $n \geq 1$.
(3) $((x)f)^{(n)} = (x)u \circ (x)h$ *for some* $(x)u \in U(R_0[x])$ *and* $(x)h \in \mathrm{Idem}(R_0[x])$.
(4) $(x)f = (x)g + (x)w$ *for some* $(x)g \in \mathrm{vnr}(R_0[x])$ *and* $(x)w \in \mathrm{nil}(R_0[x])$.
(5) $(x)f = (x)u \circ (x)h + (x)w$ *for some* $(x)u \in U(R_0[x])$, $(x)h \in \mathrm{Idem}(R_0[x])$ *and* $(x)w \in \mathrm{nil}(R_0[x])$.
(6) $(x)f + \mathrm{nil}(R_0[x]) \in \mathrm{vnr}(R_0[x]/\mathrm{nil}(R_0[x]))$.

Proof. (1) $\Leftrightarrow$ (2) It is clear.
(2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5) It follows from Proposition 2.8.
(1) $\Rightarrow$ (5) It follows from Theorem 3.5.
(4) $\Rightarrow$ (6) It is clear.
(6) $\Rightarrow$ (1) It follows from Theorem 3.5. □

Corollary 3.7. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$. Then we have:*

(1) $\pi - r(R_0[x]) = \mathrm{vnr}(R_0[x]) + \mathrm{nil}(R_0[x])$.
(2) $\pi - r(R_0[x])/\mathrm{nil}(R_0[x]) = \mathrm{vnr}(R_0[x]/\mathrm{nil}(R_0[x]))$.
(3) $\pi - r(R_0[x]) = \mathrm{vnr}(R_0[x])$ *if and only if R is reduced.*
(4) *If $2 \in U(R)$, then every $(x)f \in \pi - r(R_0[x])$ is the sum of two units of $R_0[x]$.*

Proof. (1) This follows from the equivalence of (1) and (4) in Theorem 3.6.
(2) This follows from the equivalence of (1) and (6) in Theorem 3.6.
(3) Since by Theorem 2.1, $\mathrm{nil}(R_0[x]) \cap \mathrm{vnr}(R_0[x]) = \{0\}$, the result follows from (1).
(4) By (1), $(x)f = (x)g + (x)w$ with $(x)g \in \mathrm{vnr}(R_0[x])$ and $(x)w \in \mathrm{nil}(R_0[x])$. Then $(x)g = (x)u + (x)v$ for some $(x)u, (x)v \in U(R_0[x])$ by Theorem 2.13. Thus $(x)u' = (x)v + (x)w \in U(R_0[x])$ by Theorem 2.5. Hence $(x)f = (x)u + (x)u'$ is the sum of two units of $R_0[x]$. □

Proposition 3.8. *If R is a commutative ring with $\mathrm{nil}(R)^2 = 0$, then $\pi - r(R_0[x])$ is multiplicatively closed.*

Proof. Let $(x)f_1, (x)f_2 \in \pi - r(R_0[x])$. Thus $(x)f_1 = u_1e_1x + (x)h_1$ and $(x)f_2 = u_2e_2x + (x)h_2$ for some $u_1, u_2 \in U(R)$, $e_1, e_2 \in \mathrm{Idem}(R)$ and $(x)h_1, (x)h_2 \in \mathrm{nil}(R_0[x])$ by Corollary 3.7. Thus $(x)w_1 = u_2e_2((x)h_1)$ and $(x)w_2 = (x)f_1 \circ$

$(x)h_2$ are nilpotent elements of $R_0[x]$, since $\mathrm{nil}(R_0[x])$ is an ideal of $R_0[x]$. Hence

$$\begin{aligned}(x)f_1 \circ (x)f_2 &= (u_1e_1x + (x)h_1) \circ (u_2e_2x + (x)h_2)\\ &= (u_1e_1x + (x)h_1) \circ u_2e_2x + (u_1e_1x + (x)h_1) \circ (x)h_2\\ &= u_2e_2(u_1e_1x + (x)h_1) + (x)w_2\\ &= u_2e_2u_1e_1x + (x)w_1 + (x)w_2.\end{aligned}$$

Then by [1, Theorem 2.1], $u_2e_2u_1e_1 \in \mathrm{vnr}(R)$. Also, $(x)w_1+(x)w_2 \in \mathrm{nil}(R_0[x])$, since $\mathrm{nil}(R_0[x])$ is an ideal of $R_0[x]$. Therefore $(x)f_1 \circ (x)f_2 \in \pi - r(R_0[x])$ by Corollary 3.7. $\square$

Theorem 3.9. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$. Then $\pi - r(R_0[x]) = \mathrm{vnr}(R_0[x]) \cup \mathrm{nil}(R_0[x])$ if and only if either $\mathrm{Idem}(R) = \{0, 1\}$ or R is reduced.*

Proof. Suppose that $\pi - r(R_0[x]) = \mathrm{vnr}(R_0[x]) \cup \mathrm{nil}(R_0[x])$ and there exists $e \in \mathrm{Idem}(R) \setminus \{0, 1\}$. Thus $\mathrm{Idem}(R_0[x]) \neq \{0, x\}$ by Lemma 2.3. Let $(x)f \in \mathrm{nil}(R_0[x])$. Then $ex + (x)f \in \mathrm{vnr}(R_0[x]) + \mathrm{nil}(R_0[x]) = \pi - r(R_0[x]) = \mathrm{vnr}(R_0[x]) \cup \mathrm{nil}(R_0[x])$ by Corollary 3.7 and hypothesis. Thus $ex + (x)f \in \mathrm{vnr}(R_0[x])$, since $e \neq 0$. Hence by Theorem 2.1, $(x)f - ex \circ (x)f = (1 - e)x \circ (x)f = (1 - e)x \circ (ex + (x)f) \in \mathrm{vnr}(R_0[x])$, since idempotents of $R_0[x]$ are central. Also, $(x)f - ex \circ (x)f = (1 - e)x \circ (x)f \in \mathrm{nil}(R_0[x])$, since $\mathrm{nil}(R_0[x])$ is an ideal of $R_0[x]$. Hence by Theorem 2.1, $(x)f - ex \circ (x)f = 0$. By replacing ex with $(1-e)x$, a similar argument yields that $ex \circ (x)f = 0$, and so $(x)f = 0$. Therefore $\mathrm{nil}(R) = \{0\}$ by [3, Proposition 3.1].

Conversely, if $\mathrm{Idem}(R) = \{0, 1\}$, then $\mathrm{Idem}(R_0[x]) = \{0, x\}$ by Lemma 2.3. Hence by Theorem 2.1, $\mathrm{vnr}(R_0[x]) = U(R_0[x]) \cup \{0\}$. Thus $\pi - r(R_0[x]) = U(R_0[x]) + \mathrm{nil}(R_0[x]) = U(R_0[x])$ by Corollaries 2.6 and 3.7. Also, if $\mathrm{nil}(R) = \{0\}$, then $\mathrm{nil}(R_0[x]) = \mathrm{nil}(R)_0[x] = \{0\}$. Therefore by Corollary 3.7, $\pi - r(R_0[x]) = \mathrm{vnr}(R_0[x])$. Hence $\pi - r(R_0[x]) = \mathrm{vnr}(R_0[x]) \cup \mathrm{nil}(R_0[x])$. $\square$

Theorem 3.10. *Let R be a commutative ring with $\mathrm{nil}(R)^2 = 0$. Then*

(1) $\mathrm{cln}(R_0[x]) = (\mathrm{cln}(R))x + (\mathrm{nil}(R_0[x]))x$
$= \left\{ \sum_{i=1}^n a_ix^i \mid\ a_1 \in \mathrm{cln}(R), a_i \in \mathrm{nil}(R) \text{ for every } i \geq 2 \right\}$.

(2) $R_0[x]$ *is never a clean near-ring.*

Proof. (1) By Theorem 2.5 and Lemma 2.3, we have $\mathrm{cln}(R_0[x]) = U(R_0[x]) + \mathrm{Idem}(R_0[x]) = \left\{ \sum_{i=1}^n a_ix^i \mid a_1 = u + e \text{ for some } u \in U(R),\ e \in \mathrm{Idem}(R) \text{ and } a_i \in \mathrm{nil}(R) \text{ for every } i \geq 2 \right\} = \left\{ \sum_{i=1}^n a_ix^i \mid a_1 \in \mathrm{cln}(R),\ a_i \in \mathrm{nil}(R) \text{ for every } i \geq 2 \right\}$.

(2) It follows from (1), since $x^2 \notin \mathrm{cln}(R_0[x])$. $\square$

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