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STABILITY IN THE α-NORM FOR SOME STOCHASTIC PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this work, we study the existence, uniqueness and stability in the α-norm of solutions for some stochastic partial functional integrodifferential equations. We suppose that the linear part has an analytic resolvent operator in the sense given in Grimmer [8] and the nonlinear part satisfies a Hölder type condition with respect to the α-norm associated to the linear part. Firstly, we study the existence of the mild solutions. Secondly, we study the exponential stability in pth moment ($p > 2$). Our results are illustrated by an example. This work extends many previous results on stochastic partial functional differential equations.

1. Introduction

The purpose of this work is to study the existence, uniqueness and stability results for stochastic partial functional integrodifferential equations with finite delay of the following form

$$
\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + \displaystyle\int_0^t B(t-s)u(s)ds + F(t,u_t) \\ \qquad\qquad + G(t,u_t)dw(t) \quad \text{for } t \geq 0, \\ u_0 = \varphi \in C_{\mathcal{F}_0}\Big([-r,0], D\big((-A)^\alpha\big)\Big), \end{cases} \tag{1.1}
$$

where $r > 0$, A is the infinitesimal generator of an analytic semigroup $\big(T(t)\big)_{t\geq 0}$ on a separable Hilbert space $\mathbb{H}$, $B(t)$ is a closed linear operator with domain $D(B(t)) \supset D(A)$. For $t \geq 0$, u_t denotes, as usual, the element of C_α defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-r,0]$. The mappings $F : \mathbb{R}_+ \times C_\alpha \to \mathbb{H}$, and $G : \mathbb{R}_+ \times C_\alpha \to \mathcal{L}(\mathbb{K},\mathbb{H})$ are borel measurable. The spaces $C_{\mathcal{F}_0}([-r,0], D((-A)^\alpha))$ and C_α are defined later.

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Stochastic partial functional differential equations with finite delays are very important as stochastic models of biological, chemical, physical and economical systems. The qualitative properties (existence, stability, invariant measures, controllability and others) of solutions of these systems have been studied by many authors. It is well known that these topics have been developed mainly by using two different methods, that is, the semigroup approach (for example, Da Prato et al. [1], Liu [10], Kolmanovskii et al. [9], Wu [18] and references therein) and the variational one (for example, Pardoux [16]).

Integrodifferential equations can be used to describe a lot of natural phenomena arising from many fields such as electronics, fluid dynamics, biological models, and chemical kinetics. Most of these phenomena cannot be described through classical differential equations. That is why in recent years they have attracted more and more attention of several mathematicians, physicists, and engineers. Some topics for this kind of equations, such as existence and regularity, stability, (almost) periodicity of solutions and control problems, have been investigated by many authors, for example, we refer to [2–6, 11].

In [14], Taniguchi et al., using only a local Lipschitz condition, studied existence and p-th moment ($p > 2$) exponential stability problems in the α-norm of mild solutions of the following class of stochastic partial functional differential equations with finite delays

$$\begin{cases} dX(t) = [-AX(t) + f(t, X_t)]dt + g(t, X_t)dw(t) \ \text{ for } \ t \geq 0 \\ \quad X_0 = \phi \in L^p\Big(\Omega, C_\alpha\Big), \end{cases} \tag{1.2}$$

where ϕ is $\mathcal{F}_0$-measurable and $-A$ generates an analytic semigroup on a separable Hilbert space $\mathbb{H}$, f and g are two measurable mappings.

In [6], Govindan considered a class of stochastic partial functional differential equations, in a real separable Hilbert space, of the following form

$$\begin{cases} dx(t) = [Ax(t) + f(t, x_t)]dt + g(t, x_t)dw(t) \ \text{ for } \ t > 0 \\ \ x(t) = \varphi(t) \ \text{ for } \ t \in [-r, 0]. \end{cases} \tag{1.3}$$

The author studied the exponential stability of the quadratic moments, (that is when $p = 2$) and also studied asymptotic stability in probability of mild solutions in the α-norm, assumed general conditions of the Hölder type on the nonlinear terms, instead of the Lipschitz condition and use the method of successive approximations and a comparison principle.

Motivated by [6], we aim to study the existence, uniqueness and exponential stability of mild solutions in the α-norm of Eq. (1.1) by using the theory of resolvent operators and Picard type iteration. Recall that the resolvent operator plays an important role in solving Eq. (1.1) in the weak and strict sense, it replaces the role of the C_0-semigroup theory, for more details we refer to [7, 8].

The work is organized as follows. In Sections 2, we recall the preliminaries facts which are used throughout this work. In Section 3, we state the existence and uniqueness of a mild solution. In Section 4, we study the exponential stability in p-th moment ($p > 2$) and in Section 5, we give an example to illustrate the basic theory of this work.

2. Preliminaries

2.1. Wiener process

Throughout this work, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), $\mathbb{H}$ and $\mathbb{K}$ are two real separable Hilbert spaces; we denote by $\langle\cdot,\cdot\rangle_{\mathbb{H}}, \langle\cdot,\cdot\rangle_{\mathbb{K}}$ their inner products and by $\|\cdot\|_{\mathbb{H}}, \|\cdot\|_{\mathbb{K}}$ their vectors norms, respectively. We denote by $\mathcal{L}(\mathbb{H},\mathbb{K})$ the space of all bounded linear operator from $\mathbb{H}$ into $\mathbb{K}$, equipped with the usual operator norm $\|\cdot\|$. We use the same symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises. $C_{\mathcal{F}_0}([-r,0]; D((-A)^\alpha))$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, C_α-valued random variables. C_α will be defined later.

Let $\{w(t) : t \geq 0\}$ be a $\mathbb{K}$-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with covariance operator Q; that is,

$$\mathbb{E}\langle w(t), x\rangle_{\mathbb{K}} \langle w(s), y\rangle_{\mathbb{K}} = (t \wedge s)\langle Qx, y\rangle_{\mathbb{K}}$$

for all $x, y \in \mathbb{K}$, where Q is a positive, self-adjoint, trace class operator on $\mathbb{K}$. In particular, we denote $w(t)$ a $\mathbb{K}$-valued Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. To define the stochastic integrals with respect to the Q-Wiener process $w(t)$, we introduce the subspace $\mathbb{K}_0 = Q^{1/2}\mathbb{K}$ of $\mathbb{K}$ endowed with the inner product $\langle u, v\rangle_{\mathbb{K}_0} = \left\langle Q^{-1/2}u, Q^{-1/2}v\right\rangle_{\mathbb{K}}$ as a Hilbert space. We assume that there exist a complete orthonormal system $\{e_i\}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_i such that $Qe_i = \lambda_i e_i$, $i = 1, 2, \ldots$, and a sequence $\{\beta_i(t)\}_{i>1}$ of independent standard Brownian motions such that $w(t) = \sum_{i=1}^{+\infty}\sqrt{\lambda_i}\beta_i(t)e_i$ for $t \geq 0$ and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$ be the space of all Hilbert-Schmidt operators from $\mathbb{K}_0$ to $\mathbb{H}$. It turns out to be a separable Hilbert space equipped with the norm $\|v\|_{\mathcal{L}_2^0}^2 = tr((vQ^{1/2})(vQ^{1/2})^*)$ for any $v \in \mathcal{L}_2^0$. For any bounded operator $v \in \mathcal{L}_2^0$, its norm is reduced to $\|v\|_{\mathcal{L}_2^0}^2 = tr(vQv^*)$.

2.2. Fractional power of closed operators and partial integrodifferential equations in Banach spaces

In this section, we recall some fundamental results to establish our results.

2.2.1. *Fractional power.* Let $\mathbb{X}$ be a Banach space and $A : D(A) \subset \mathbb{X} \longrightarrow \mathbb{X}$ be the infinitesimal generator of an analytic semigroup $\left(T(t)\right)_{t\geq 0}$. $\mathbb{Y}$ is the Banach space formed from $D(-A)$ equipped with the graph norm $\|y\|_{\mathbb{Y}} = \|Ay\| + \|y\|$ for $y \in D(A)$.

Let $0 \in \rho(-A)$ (the resolvent set of -A). If $0 \notin \rho(-A)$, one can substitute the operator $-A$ by the operator $(-A+\sigma I)$ with σ large enough such that $0 \in \rho(-A+\sigma I)$. Then, without loss of generality, we can assume that $0 \in \rho(-A)$. We define the fractional power $(-A)^\alpha$ for $0<\alpha \leq 1$, as a closed linear invertible operator on its domain $D((-A)^\alpha)$ with inverse $(-A)^{-\alpha}$, by

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) dt,$$

where Γ is the gamma function.

We have the following basic properties on $(-A)^\alpha$.

Theorem 2.1 ([12]). *Let A be the infinitesimal generator of an analytic semigroup $\big(T(t)\big)_{t\geq 0}$. If $0 \in \rho(-A)$, then, for $0<\alpha<1$, the following properties hold.*

(i) $\mathbb{Y}_\alpha = D((-A)^\alpha)$ *is a Banach space with the norm* $\|x\|_\alpha = \|(-A)^\alpha x\|$ *for* $x \in D((-A)^\alpha)$;
(ii) $(-A)^{-\alpha}$ *is the closed linear operator with* $\mathrm{Im}((-A)^{-\alpha}) = D((-A)^\alpha)$ *and we have* $(-A)^\alpha = ((-A)^{-\alpha})^{-1}$;
(iii) $(-A)^{-\alpha} \in \mathcal{L}(\mathbb{X}, \mathbb{X})$;
(iv) $T(t) : \mathbb{X} \to \mathbb{Y}_\alpha$ *for every* $t>0$;
(v) $(-A)^\alpha T(t) x = T(t)(-A)^\alpha x$ *for each* $x \in D((-A)^\alpha)$ *and* $t \geq 0$;
(vi) $0<\alpha\leq\beta$ *implies* $D((-A)^\beta) \hookrightarrow D((-A)^\alpha)$;
(vii) *There exists* $M_\alpha > 1$ *such that*

$$\|(-A)^\alpha T(t) x\| \leq M_\alpha \frac{e^{-\delta t}}{t^\alpha} \|x\| \quad \text{for } x \in \mathbb{X} \text{ and } \ t>0,$$

where $\delta > 0$ *is a constant.*

We denote by $C_\alpha = C\big([-r,0]; \mathbb{Y}_\alpha\big)$ the Banach space of continuous functions $\phi : [-r,0] \to \mathbb{Y}_\alpha$ provided with the supremum norm

$$\|\phi\|_\alpha = \sup_{-r\leq\theta\leq 0} \|\phi(\theta)\|_\alpha \quad \text{for} \quad \phi \in C_\alpha.$$

2.2.2. *Resolvent operators.* Now, we collect definitions and basic results about the theory of resolvent operator, see [7, 8] for more details.

Definition 2.2 ([7]). A family of bounded linear operators $(R(t))_{t\geq 0}$ in $\mathcal{L}(\mathbb{X})$ is called resolvent operator for the following equation

$$\begin{cases} \dfrac{d}{dt} u(t) = Au(t) + \displaystyle\int_0^t B(t-s) u(t) ds \quad \text{for } t \geq 0 \\ u(0) = u_0 \in \mathbb{X}, \end{cases} \tag{2.1}$$

if

(a) $R(0) = I$ and $\|R(t)\| \leq N_1 \exp(\beta t)$ for some constants $N_1 \geq 1$ and $\beta \in \mathbb{R}$;
(b) for all $x \in \mathbb{X}$, $R(t)x$ is continuous for $t \geq 0$;

(c) $R(t) \in \mathcal{L}(\mathbb{Y})$ for $t \geq 0$. For $x \in \mathbb{Y}$, $R(\cdot)x \in C^1(\mathbb{R}_+, \mathbb{X}) \cap C(\mathbb{R}_+, \mathbb{Y})$ and for $t \geq 0$, we have

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds = R(t)Ax + \int_0^t R(t-s)B(s)xds. \tag{2.2}$$

Definition 2.3. A resolvent operator $(R(t))_{t\geq 0}$ is said to be exponentially stable if there exist positive constants M_1 and a_1 such that $\|R(t)\| \leq M_1 e^{-a_1 t}$ for $t \geq 0$.

Next, we assume the following hypotheses taken from [17].

Let $\hat{B}$ be the Laplace transform of B and $R(\lambda, A) = (\lambda I - A)^{-1}$ the resolvent operator of A.

(**H1**) The operator $A : D \subseteq \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of an analytic semigroup $\big(T(t)\big)_{t\geq 0}$ on $\mathbb{X}$ and there are constants $M_0 > 0, \sigma \in \mathbb{R}$ and $\vartheta \in (\pi/2, \pi)$ such that $\rho(A) \supseteq \Lambda_{\sigma,\vartheta} = \{\lambda \in \mathbb{C} : \lambda \neq \sigma,\ |arg(\lambda - \sigma)| < \vartheta\}$ and $\|R(\lambda, A)\| \leq M_0/|\lambda - \sigma|$ for all $\lambda \in \Lambda_{\sigma,\vartheta}$.

(**H2**) For all $t \geq 0$, $B(t) : D(B(t)) \subseteq \mathbb{X} \to \mathbb{X}$ is a closed linear operator $D(A) \subseteq D(B(t))$, and $B(\cdot)x$ is strongly measurable on $(0, +\infty)$ for each $x \in D(A)$. There exists $b(\cdot) \in L^1_{loc}(\mathbb{R}_+)$ such that $\hat{b}(\lambda)$ exists for $Re(\lambda) > 0$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\hat{B} : \Lambda_{\sigma,\pi/2} \to \mathcal{L}(D(A), \mathbb{X})$ has an analytical extension (still denoted by $\hat{B}$) to $\Lambda_{\sigma,\vartheta}$ such that $\left\|\hat{B}(\lambda)x\right\| \leq \left\|\hat{B}(\lambda)\right\| \|x\|_1$ for all $x \in D(A)$ and $\left\|\hat{B}(\lambda)\right\| = O(1/|\lambda|)$ as $\lambda \to \infty$.

(**H3**) There exist a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and a positive constant C_1 such that $A(D) \subseteq D(A)$, $\hat{B}(\lambda)(D) \subseteq D(A)$ and $\left\|A\hat{B}(\lambda)x\right\| \leq C_1\|x\|$ for every $x \in D$ and for all $\lambda \in \Lambda_{\beta_0,\vartheta}$.

In the sequel, for $r_0 > 0, \theta_0 \in (\pi/2, \vartheta)$ and $\beta_0 \in \mathbb{R}$, set

$$\Lambda_{r_0,\beta_0,\theta_0} = \{\lambda \in \mathbb{C} : |\lambda - \beta_0| > r_0, |arg(\lambda - \beta_0)| < \theta_0\},$$

and for $\beta_0 + \Gamma^i_{r_0,\theta_0}, i = 1, 2, 3$, the paths

$$\beta_0 + \Gamma^1_{r_0,\theta_0} = \{\beta_0 + te^{i\theta_0} : t \geq r_0\},$$
$$\beta_0 + \Gamma^2_{r_0,\theta_0} = \{\beta_0 + r_0 e^{i\xi} : -\theta_0 \leq \xi \leq \theta_0\},$$
$$\beta_0 + \Gamma^3_{r_0,\theta_0} = \{\beta_0 + te^{-i\theta_0} : t \geq r_0\},$$

with $\beta_0 + \Gamma_{r_0,\theta_0} = \cup_1^3 \beta_0 + \Gamma^i_{r_0,\theta_0}$ are oriented counterclockwise.

The resolvent operator plays an important role to study the existence of solutions and to give a variation of constants formula for nonlinear systems. We need to know when Eq. (2.1) has a resolvent operator. The following theorem gives a satisfactory answer to this problem.

Theorem 2.4 ([17])**.** *Suppose that the assumptions* (**H1**)-(**H3**) *hold. Then Eq.* (2.1) *admits a resolvent operator given by*

$$R(t) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\beta_0+\Gamma_{r_0,\theta_0}} e^{\lambda t}\big(\lambda I - A - \hat{B}(\lambda)\big)^{-1} d\lambda & \textit{for } t > 0, \\ I & \textit{for } t = 0. \end{cases} \tag{2.3}$$

Moreover, there exists a positive constant N_1 *such that*

$$\|(-A)^\alpha R(t)\| \leq \begin{cases} N_1 e^{(r_0+\beta_0)t} & \textit{for } t \geq 1, \\ \\ N_1 e^{(r_0+\beta_0)t} t^{-\alpha} & \textit{for } t \in (0,1), \end{cases} \tag{2.4}$$

and $R(t)$ *has as analytic extension.*

Remark 2.5. If $r_0 + \beta_0 < 0$ and $\alpha \in (0,1)$, then there exists $\varphi \in L^1(\mathbb{R}_+)$ such that

$$\|(-A)^\alpha R(t)\| \leq \varphi(t) \text{ for } t \geq 0.$$

Motivated by Grimmer [7], we adopt the following concepts of mild and strict solutions for the following non-homogeneous system.

$$\begin{cases} v^{'}(t) = Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds + q(t) \quad \text{for} \quad t \geq 0, \\ v(0) = v_0 \in \mathbb{X}, \end{cases} \tag{2.5}$$

where $q : [0, +\infty[\to \mathbb{X}$ is a continuous function.

Definition 2.6 ([7])**.** A continuous function $v : [0, +\infty) \to \mathbb{X}$ is said to be a strict solution of Eq. (2.5) if

(i) $v \in C^1([0, +\infty); \mathbb{X}) \cap C([0, +\infty); \mathbb{Y})$,
(ii) v satisfies Eq. (2.5) for $t \geq 0$.

Remark 2.7. From this definition, we deduce that $v(t) \in D(A)$, the function $B(t-s)v(s)$ is integrable for all $t \geq 0$ and $s \in [0, b]$.

Theorem 2.8 ([7])**.** *If* v *is a strict solution of Eq.* (2.5)*, then*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \quad \textit{for} \quad t \geq 0. \tag{2.6}$$

Accordingly, we make the following definition.

Definition 2.9 ([7])**.** For $v_0 \in \mathbb{X}$, a function $v : [0, +\infty) \to \mathbb{X}$ is called a mild solution of Eq. (2.5) if v satisfies the variation of constants formula (2.6).

The next theorem provides sufficient conditions for the regularity of solutions of Eq. (2.5).

Theorem 2.10 ([7])**.** *Let* $q \in C^1([0, +\infty); \mathbb{X})$ *and* v *be defined by* (2.6)*. If* $v_0 \in D(A)$*, then* v *is a strict solution of Eq.* (2.5)*.*

3. Existence and uniqueness of the mild solution of Eq. (1.1)

In this section, we consider the existence and uniqueness of a mild solution of Eq. (1.1) using Hölder type conditions.

Definition 3.1. Let $b > 0$. A process $\{u(t),\ 0 \le t \le b\}$ is called a mild solution of Eq. (1.1) if

(i) $u(t)$ is $\mathcal{F}_t$-adapted for $t \ge 0$ with $\int_0^b \|u(t)\|^p\, dt < +\infty$ a.s.
(ii) $u(t) \in \mathbb{H}$ has continuous paths on $t \in [0, b]$ a.s. and u satisfies the following system

$$\begin{cases} u(t) = R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)F(s,u_s)ds \\ \qquad\qquad + \displaystyle\int_0^t R(t-s)G(s,u_s)dw(s) \ \text{ for } \ t \in [0,b], \\ u_0 = \varphi \in C_{\mathcal{F}_0}\Big([-r,0], D\big((-A)^\alpha\big)\Big). \end{cases} \tag{3.1}$$

To guarantee the existence and uniqueness of mild solutions of Eq. (1.1), we suppose the following conditions.

(H4) There exists a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ that is integrable with respect to the first argument and is continuous monotone nondecreasing with respect to the second argument such that

$$\mathbb{E}\,\|F(t,\zeta)\|_{\mathbb{H}}^p + \mathbb{E}\,\|G(t,\zeta)\|_{\mathcal{L}_2^0}^p \le H(t, \mathbb{E}\,\|\zeta\|_\alpha^p) \ \text{ for } \ t \ge 0.$$

(H5) **(a)** There exists a function $Z : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which is locally integrable with respect to the first argument and continuous, monotone nondecreasing with respect to the second argument. Moreover, $Z(t,0) = 0$ and

$$\mathbb{E}\,\|F(t,\zeta) - F(t,\eta)\|_{\mathbb{H}}^p + \mathbb{E}\,\|G(t,\zeta) - G(t,\eta)\|_{\mathcal{L}_2^0}^p \le Z(t, \mathbb{E}\,\|\zeta - \eta\|_\alpha^p) \ \text{ for } \ t \ge 0.$$

(b) If a nonnegative, continuous function ζ satisfies $\zeta(0) = 0$ and

$$\zeta(t) \le C \int_0^t Z(s, \zeta(s))ds \ \text{ for } \ t \ge 0,$$

where $C > 0$, then $\zeta(t) = 0$ for all $t \ge 0$.

(H6) For any constant $D > 0$, the following differential equation

$$\begin{cases} \dfrac{dx}{dt} = DH(t,x) \ \text{ for } t \ge 0, \\ x(0) = x_0, \end{cases} \tag{3.2}$$

has a global solution on $\mathbb{R}_+$ for any initial value $x_0 > 0$.

One can observe that a solution of Eq. (3.2) is nondecreasing on $\mathbb{R}_+$. The following lemmas are needed in the next.

Lemma 3.2 ([1]). *For any $p \geq 2$ and for arbitrary $\mathcal{L}_2^0$-valued predictable process $\Phi(\cdot)$,*

$$\sup_{0\leq s\leq t} \mathbb{E}\left\|\int_0^s \Phi(l)dw(l)\right\|^p \leq C_p\Big(\int_0^t \mathbb{E}\left\|\Phi(l)\right\|^2 dl\Big)^{p/2} \ \ for \ \ t\geq 0,$$

where $C_p = \left(\frac{p(p-1)}{2}\right)^{p/2}$.

Lemma 3.3 ([13]). *Let $L:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ be a continuous, monotone nondecreasing function with respect to the second argument and locally integrable with respect to the first argument. Suppose that there exist two continuous functions γ_1 and γ_2 defined on $[s,\theta)$, $s\geq 0$, satisfying the following inequality*

$$\gamma_1(t)-\int_s^t L(\tau,\gamma_1(\tau))d\tau < \gamma_2(t)-\int_s^t L(\tau,\gamma_2(\tau))d\tau \ \ for \ \ t\in[s,\theta).$$

If $\gamma_1(s)<\gamma_2(s)$, then $\gamma_1(t)<\gamma_2(t)$ for all $t\in[s,\theta)$.

First of all, we introduce the following iteration procedure:

$$u^0(t)=\varphi(t) \ \text{ for } \ t\in[-r,0],$$

$$u^0(t)=R(t)\varphi(0) \ \text{ for } \ t\in[0,b]$$

and for $n\in\mathbb{N}$, $n\geq 1$,

$$u^n(t)=\varphi(t) \ \text{ for } \ t\in[-r,0],$$

$$\begin{aligned} u^n(t) &= R(t)\varphi(0)+\int_0^t R(t-s)F(s,u_s^{n-1})ds \\ &\quad+\int_0^t R(t-s)G(s,u_s^{n-1})dw(s) \ \text{ for } \ t\in[0,b]. \end{aligned} \tag{3.3}$$

with an arbitrary nonnegative initial value $\varphi\in\mathcal{B}_\alpha$. Here $\mathcal{B}_\alpha$ is the space of all $\mathcal{F}_t$-measurable stochastic process $\phi:\Omega\to C_\alpha$ endowed with the norm $\|\phi\|^p_{\mathcal{B}_\alpha}:=\mathbb{E}\|\phi\|^p_\alpha<\infty$.

Theorem 3.4. *Let $0<\alpha<\frac{p-2}{2p}$. Suppose that* **(H1)-(H6)** *hold. Then Eq. (1.1) has a unique mild solution u defined on $\mathbb{R}_+$. Moreover, for any $b>0$, we have*

$$\mathbb{E}\Big(\sup_{0\leq t\leq b}\|u^n(t)-u(t)\|^p_\alpha\Big)\to 0 \ \ as \ \ n\to\infty.$$

Proof. Let $b>1$. We claim that the solution exists on $[0,b]$.

Indeed, let D_α be the space of all continuous functions $z:[-r,b]\to\mathbb{H}$ such that

$$\|z\|_{D_\alpha}:=\sup_{0\leq t\leq b}\left(\mathbb{E}\|z_t\|^p_\alpha\right)^{1/p}<\infty.$$

Then D_α is a Banach space.

Set

$$M = \sup_{0\leq t\leq b} \|R(t)\| \quad \text{and} \quad K = N_1 \max\{e^{(r_0+\beta_0)b}, 1\}. \tag{3.4}$$

From (3.3) for $n = 1$ and for any $0 \leq s \leq b$, we have

$$\begin{aligned}
&\mathbb{E}\left\|(-A)^\alpha u^1(s)\right\|^p \\
&\leq 3^{p-1}\mathbb{E}\left\|(-A)^\alpha R(s)\varphi(0)\right\|^p + 3^{p-1}\mathbb{E}\left\|\int_0^s (-A)^\alpha R(s-l)F(l,u_l^0)dl\right\|^p \\
&\quad + 3^{p-1}\mathbb{E}\left\|\int_0^s (-A)^\alpha R(s-l)G(l,u_l^0)dw(l)\right\|^p \\
&=: 3^{p-1}(I_1^p(s) + I_2^p(s) + I_3^p(s)).
\end{aligned}$$

We have $I_1^p(t) \leq M^p\mathbb{E}\left\|\varphi\right\|_\alpha^p$.

We consider two cases : $t \in (0, 1)$ and $t \in [1, b]$.

Let $t \in (0, 1)$. By using Hölder's inequality, Lemma 3.2 and **(H4)**, we obtain that

$$I_2^p(t) \leq K^p \frac{b^{p(1-\alpha)-1}}{\left(\frac{p(1-\alpha)-1}{p-1}\right)^{p-1}} \int_0^t H(s, \mathbb{E}\left\|u_s^0\right\|_\alpha^p)ds,$$

$$I_3^p(t) \leq K^p C_p \frac{b^{\frac{p(1-2\alpha)-2}{2}}}{\left(\frac{p(1-2\alpha)-2}{p-2}\right)^{\frac{p-2}{2}}} \int_0^t H(s, \mathbb{E}\left\|u_s^0\right\|_\alpha^p)ds.$$

Hence, we have

$$\mathbb{E}\left\|u_t^1\right\|_\alpha^p \leq 3^{p-1}M^p\mathbb{E}\left\|\varphi\right\|_\alpha^p + C_0\int_0^t H(s, \mathbb{E}\left\|u_s^0\right\|_\alpha^p)ds, \tag{3.5}$$

where $C_0 = 3^{p-1}K^p\left\{\frac{b^{p(1-\alpha)-1}}{\left(\frac{p(1-\alpha)-1}{p-1}\right)^{p-1}} + C_p\frac{b^{\frac{p(1-2\alpha)-2}{2}}}{\left(\frac{p(1-2\alpha)-2}{p-2}\right)^{\frac{p-2}{2}}}\right\}$. □

Lemma 3.5. *Let x_m be the global maximum solution of Eq.* (3.2) *for* $x_0 > \max\left(3^{p-1}M^p\mathbb{E}\left\|\varphi\right\|_\alpha^p, \mathbb{E}\left\|\varphi\right\|_\alpha^p\right)$. *Then*

$$\mathbb{E}\left\|u_t^n\right\|_\alpha^p < x_m(t) \;\; \text{for} \;\; n \in \mathbb{N}, \;\; n \geq 1 \;\text{and}\; t \in [0, b]. \tag{3.6}$$

Proof of Lemma 3.5. We have

$$x_m(t) = x_0 + C_0\int_0^t H(s, x_m(s))ds. \tag{3.7}$$

By the hypothesis (**H4**), we obtain that $\int_0^t H(s, x_m(s))ds \geq 0$. Hence, $x_m(s) \geq x_0$.

Choose x_0 such that $x_0 > \max\left(3^{p-1}M^p\mathbb{E}\left\|\varphi\right\|_\alpha^p, \mathbb{E}\left\|\varphi\right\|_\alpha^p\right)$. Therefore, by (3.5) and (3.7), we obtain that

$$0 \leq C_0 \int_0^t \left[H(s, x_m(s))ds - H(s, \mathbb{E}\left\|u_s^0\right\|_\alpha^p)ds\right] < x_m(t) - \mathbb{E}\left\|u_t^1\right\|_\alpha^p,$$

which means that

$$\mathbb{E}\left\|u_t^1\right\|_\alpha^p < x_m(t) \text{ for } t \in [0, b]. \tag{3.8}$$

Proceeding as above, we show that

$$\begin{aligned} &\mathbb{E}\left\|u_t^{n+1}\right\|_\alpha^p \\ &\leq 3^{p-1}M^p\mathbb{E}\left\|\varphi\right\|_\alpha^p + C_0\int_0^t H(s, \mathbb{E}\left\|u_s^n\right\|_\alpha^p)ds \text{ for } n \in \mathbb{N}^* \text{ and } t \in [0, b]. \end{aligned}$$

Let $n \in \mathbb{N}^*$. Suppose that $\mathbb{E}\left\|u_t^k\right\|_\alpha^p < x_m(t)$ for $k \leq n$ and $t \in [0, b]$. Then

$$\begin{aligned} \mathbb{E}\left\|u_t^{k+1}\right\|_\alpha^p &\leq 3^{p-1}M^p\mathbb{E}\left\|\varphi\right\|_\alpha^p + C_0\int_0^t H(s, \mathbb{E}\left\|u_s^k\right\|_\alpha^p)ds \\ &< x_0 + C_0\int_0^t H(s, x_m(s))ds = x_m(t). \end{aligned}$$

Therefore, the inequality (3.6) holds. This ends the proof of Lemma 3.5. □

We deduce that $(u^n)_{n\geq 1}$ is uniformly bounded in D_α.

We claim that $(u^n)_{n\geq 1}$ is a Cauchy sequence in D_α. In fact, define the sequence of functions $r_n : [0, b] \to \mathbb{R}$ by

$$r_n(t) = \sup_{m\geq n} \mathbb{E}\|u_t^{m+n} - u_t^n\|_\alpha^p \text{ for } t \in [0, b] \text{ and } n \in \mathbb{N}.$$

Note that for each $n \in \mathbb{N}$, r_n is well-defined, uniformly bounded and non-decreasing (with respect to t). Since $\{r_n, n \geq 1\}$ is nonincreasing, for each $t \in [0, b]$, there exists a function $r : [0, b] \to \mathbb{R}$ such that

$$\lim_{n\to+\infty} r_n(t) = r(t) \text{ for all } t \in [0, b]. \tag{3.9}$$

Proceeding as above, we show that

$$\mathbb{E}\left\|u_t^{m+n+1} - u_t^{n+1}\right\|_\alpha^p \leq C_1\int_0^t Z(s, \mathbb{E}\left\|u_t^{m+n} - u_t^n\right\|_\alpha^p)ds,$$

where $C_1 = 2^{p-1}K^p\left\{\frac{b^{p(1-\alpha)-1}}{\left(\frac{p(1-\alpha)-1}{p-1}\right)^{p-1}} + C_p\frac{b^{\frac{p(1-2\alpha)-2}{2}}}{\left(\frac{p(1-2\alpha)-2}{p-2}\right)^{\frac{p-2}{2}}}\right\}$, which implies that

$$r(t) \leq r_{n+1}(t) \leq C_1\int_0^t Z(s, r_n(s))ds \text{ for } n \in \mathbb{N}.$$

From (3.9) and the Lebesgue dominated convergence theorem, we obtain that

$$r(t) \leq C_1 \int_0^t Z(s, r(s))ds.$$

Therefore, by **(H5)-(b)**, $\mathbb{E}\left\|u_t^{m+n} - u_t^m\right\|_\alpha^p \to 0$ as $m \to \infty$ and every n, which implies that $(u^n)_{n\geq 1}$ is a Cauchy sequence in D_α.

Let $u = \lim_{n\to\infty} u^n$. Now, we claim that u is a mild solution of Eq. (1.1). In fact, we have

$$\begin{aligned}
&\mathbb{E}\left\|u^n(t) - [R(t)\varphi(0) + \int_0^t R(t-s)F(s,u_s)ds + \int_0^t R(t-s)G(s,u_s)dw(s)]\right\|_\alpha^p \\
&\leq C_1 \int_0^t Z(s, \mathbb{E}\left\|u_s^{n-1} - u_s\right\|_\alpha^p)ds \\
&\to 0 \ \text{ as } \ n \to \infty,
\end{aligned}$$

which proving the claim.

In the case $t \in [1, b]$, by given appropriate values to the constant C_0 and C_1, the above results hold.

The uniqueness follows from assumption (**H5**). Recall that b is any positive number and hence the obtained solution u is global.

4. Almost sure exponential stability of Eq. (1.1)

In this section, we consider the almost sure exponential stability of the second moment of a trivial solution of Eq. (1.1). For this goal, we suppose that:

(H7) **(a)** The resolvent operator $(R(t))_{t\geq 0}$ given by (**H1**)-(**H3**) is exponentially stable.

(b) $-a_2 := r_0 + \beta_0 < 0$. Thus, we have the following estimation

$$\|(-A)^\alpha R(t)\| \leq \begin{cases} N_1 e^{-a_2 t} & \text{for } \ t \geq 1, \\ N_1 e^{-a_2 t} t^{-\alpha} & \text{for } \ t \in (0,1). \end{cases} \tag{4.1}$$

(H8) There exist nonnegative real numbers $Q_1, Q_2 \geq 0$ and continuous functions
$\nu_1, \nu_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathbb{E}\|F(t,\zeta)\|^p \leq Q_1 \mathbb{E}\|\zeta\|_\alpha^p + \nu_1(t) \ \text{ and } \ \mathbb{E}\|G(t,\zeta)\|^p \leq Q_2 \mathbb{E}\|\zeta\|_\alpha^p + \nu_2(t) \text{ for } t \geq 0.$$

Moreover, there exist nonnegative real numbers $P_1, P_2 \geq 0$ and $\beta > a := \min\{a_1, a_2\} > 0$ such that

$$\nu_i(t) \leq P_i e^{-\beta t} \ \text{ for } \ i = 1, 2 \ \text{ and } \ t \geq 0.$$

Now assume that $F(t, 0) = G(t, 0) = 0$ for almost every t, means that Eq. (1.1) admits a trivial solution. Let $u(t) = u(t, \varphi)$ be a solution of Eq. (1.1) where φ is any past process.

Theorem 4.1. *Assume that* **(H1)**-**(H8)** *hold. Let* $0 < \alpha < \frac{p-2}{2p}$ *and* $N = \max\{M_1, N_1\}$. *Then there exist constants* C *and* $\theta > 0$ *such that the mild solution* u *of Eq.* (1.1) *satisfies the following inequality*

$$\mathbb{E}\,\|u(t)\|^p_{C_\alpha} \le Ce^{-\theta t} \ \ \textit{for}\ \ t \in [0,b],$$

provided that

$$\lambda = 3^{p-1}N^p\Big(\lambda_1 Q_1 + \lambda_2 Q_2\Big) < a, \tag{4.2}$$

where $\theta = a - \lambda$ *and for* $t \in (0,1)$, *we have*

$$\lambda_1 = \Big(\Gamma(1-q\alpha)a^{q\alpha-1}\Big)^{\frac{p}{q}}, \ \ \lambda_2 = C_p\Big(\Gamma(1-2q'\alpha)\big((2a-\frac{2a}{p})q'\big)^{2\alpha q'-1}\Big)^{\frac{p}{2q'}}$$

and for $t \ge 1$, *we have*

$$\lambda_1 = \Big(\frac{1}{2a}\Big)^{\frac{p-2}{2}}, \ \ \lambda_2 = \frac{1}{a^{p-1}},$$

with $q = \frac{p}{p-1}$ *and* $q' = \frac{p}{p-2}$.

Proof. We have

$$\begin{aligned}\|(-A)^\alpha u(s)\| \le\ & \|(-A)^\alpha R(s)\varphi(0)\| + \left\|\int_0^s (-A)^\alpha R(s-l)F(l,u_l)dl\right\| \\ & + \left\|\int_0^s (-A)^\alpha R(s-l)G(l,u_l)dw(l)\right\|.\end{aligned}$$

Let $b > 1$. We distinguish two cases : $t \in (0,1)$ and $t \in [1,b]$.
Case 1: $t \in (0,1)$.

We have

$$\begin{aligned}\mathbb{E}\,\|u_t\|^p_\alpha \le\ & 3^{p-1}N^pe^{-pat}\mathbb{E}\,\|\varphi\|^p_\alpha + 3^{p-1}N^p\mathbb{E}\Big(\int_0^t (t-s)^{-\alpha}e^{-a(t-s)}\,\|F(s,u_s)\|\,ds\Big)^p \\ & + 3^{p-1}N^p\Big(\int_0^t (t-s)^{-2\alpha}e^{-2a(t-s)}\mathbb{E}\,\|G(s,u_s)\|^2\,ds\Big)^{\frac{p}{2}} \\ =:\ & 3^{p-1}N^p(J_1(t)+J_2(t)+J_3(t)). \end{aligned} \tag{4.3}$$

By using Hölder's inequality and (**H8**), we obtain that

$$\begin{aligned} J_2 &\le \Big(\int_0^t (t-s)^{-q\alpha}e^{-a(t-s)}ds\Big)^{\frac{p}{q}}\int_0^t e^{-a(t-s)}\mathbb{E}\,\|F(s,u_s)\|^p\,ds \\ &\le \lambda_1\int_0^t e^{-a(t-s)}\Big(Q_1\mathbb{E}\,\|u_s\|^p_\alpha + \nu_1(s)\Big)ds \\ &\le \lambda_1\int_0^t e^{-a(t-s)}\Big(Q_1\mathbb{E}\,\|u_s\|^p_\alpha + P_1e^{-\beta s}\Big)ds \end{aligned} \tag{4.4}$$

By Lemma 3.2 and proceeding as above, we deduce that

$$\begin{aligned}
J_3(t) &\leq C_p \left(\int_0^t (t-s)^{-2\alpha} e^{-2a(t-s)} \mathbb{E} \left\| G(s,u_s) \right\|^2 ds \right)^{p/2} \\
&\leq C_p \left(\int_0^t (t-s)^{-2\alpha} e^{-(2a-\frac{2a}{p})(t-s)} e^{-\frac{2a}{p}(t-s)} \mathbb{E} \left\| G(s,u_s) \right\|^2_{\mathcal{L}_2^0} ds \right)^{p/2} \\
&\leq \lambda_2 \int_0^t e^{-a(t-s)} \Big(Q_2 \mathbb{E} \left\| u_s \right\|^p_\alpha + P_2 e^{-\beta s} \Big) ds. \qquad (4.5)
\end{aligned}$$

Case 2: $t \in [1, b]$.

We have

$$\begin{aligned}
\mathbb{E} \left\| u_t \right\|^p_\alpha &\leq 3^{p-1} N^p e^{-pat} \mathbb{E} \left\| \varphi \right\|^p_\alpha + 3^{p-1} N^p \mathbb{E} \left(\int_0^t e^{-a(t-s)} \left\| F(s,u_s) \right\| ds \right)^p \\
&\quad + 3^{p-1} N^p \left(\int_0^t e^{-2a(t-s)} \mathbb{E} \left\| G(s,u_s) \right\|^2 ds \right)^{\frac{p}{2}} \\
&=: 3^{p-1} N^p (J_1^{'}(t) + J_2^{'}(t) + J_3^{'}(t)). \qquad (4.6)
\end{aligned}$$

By using Hölder's inequality and **(H4)**, we obtain that

$$\begin{aligned}
J_2^{'}(t) &\leq \left(\int_0^t e^{-a(t-s)} ds \right)^{\frac{p}{q}} \int_0^t e^{-a(t-s)} \mathbb{E} \left\| F(s,u_s) \right\|^p ds \\
&\leq \frac{1}{a^{p-1}} \int_0^t e^{-a(t-s)} \Big(Q_1 \mathbb{E} \left\| u_s \right\|^p_\alpha + \nu_1(s) \Big) ds \\
&\leq \frac{1}{a^{p-1}} \int_0^t e^{-a(t-s)} \Big(Q_1 \mathbb{E} \left\| u_s \right\|^p_\alpha + P_1 e^{-\beta s} \Big) ds. \qquad (4.7)
\end{aligned}$$

By Lemma 3.2 and proceeding as above, we deduce that

$$\begin{aligned}
J_3^{'}(t) &\leq C_p \left(\int_0^t e^{-2a(t-s)} \mathbb{E} \left\| G(s,u_s) \right\|^2 ds \right)^{p/2} \\
&\leq C_p \left(\int_0^t e^{-(2a-\frac{2a}{p})(t-s)} e^{-\frac{2a}{p}(t-s)} \mathbb{E} \left\| G(s,u_s) \right\|^2_{\mathcal{L}_2^0} ds \right)^{p/2} \\
&\leq \left(\frac{1}{2a} \right)^{\frac{p-2}{2}} \int_0^t e^{-a(t-s)} \Big(Q_2 \mathbb{E} \left\| u_s \right\|^p_\alpha + P_2 e^{-\beta s} \Big) ds. \qquad (4.8)
\end{aligned}$$

Therefore, from (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8), for $t \in [0, b]$, we obtain that

$$\begin{aligned}
\mathbb{E} \left\| u_t \right\|^p_\alpha &\leq 3^{p-1} N^p e^{-pat} \mathbb{E} \left\| \varphi \right\|^p_\alpha + \lambda_1 \int_0^t e^{-a(t-s)} \Big(Q_1 \mathbb{E} \left\| u_s \right\|^p_\alpha + P_1 e^{-\beta s} \Big) ds \\
&\quad + \lambda_2 \int_0^t e^{-a(t-s)} \Big(Q_2 \mathbb{E} \left\| u_s \right\|^p_\alpha + P_2 e^{-\beta s} \Big) ds. \qquad (4.9)
\end{aligned}$$

Thus,

$$\begin{aligned} e^{at}\mathbb{E}\left\|u_t\right\|_\alpha^p &\leq 3^{p-1}N^p\mathbb{E}\left\|\varphi\right\|_\alpha^p + 3^{p-1}N^p b(\lambda_1 P_1 + \lambda_2 P_2) \\ &\quad + 3^{p-1}N^p(\lambda_1 Q_1 + \lambda_2 Q_2)\int_0^t e^{as}\mathbb{E}\left\|u_s\right\|_\alpha^p ds \\ &\leq C + \lambda\int_0^t e^{as}\mathbb{E}\left\|u_s\right\|_\alpha^p ds, \end{aligned}$$

where $C = 3^{p-1}N^p\mathbb{E}\left\|\varphi\right\|_\alpha^p + 3^{p-1}N^p b(\lambda_1 P_1 + \lambda_2 P_2)$ and $\lambda = 3^{p-1}N^p(\lambda_1 Q_1 + \lambda_2 Q_2)$.

Letting $\theta = a - \lambda$ and invoking Gronwall's Lemma, we get that

$$\mathbb{E}\left\|u_t\right\|_\alpha^p \leq Ce^{-\theta t} \text{ for } t \in [0, b]. \qquad \square$$

5. Exponential stability in p-th moment

In this section, we study the exponential stability in p-th moment of the mild solution of Eq. (1.1).

Definition 5.1. Eq. (1.1) is said to be exponentially stable in p-th moment if there exist a pair of positive constants λ and K^* such that the mild solution satisfies the following estimation

$$\mathbb{E}\left\|u(t)\right\|_\alpha^p \leq K^*\mathbb{E}\left\|\varphi\right\|_\alpha^p e^{-\lambda t} \text{ for } t \geq 0.$$

In order to establish the exponential stability in p-th moment, firstly we consider a result that gives an estimate for the solution of Eq. (1.1). For this goal, we need the following further assumption:

(**H9**) The function H in (**H4**) satisfies

$$\delta H(t, \eta) \leq H(t, \delta\eta) \text{ for all } \delta > 1, \ \eta \in \mathbb{R}_+ \text{ and } t \geq 0.$$

Proposition 5.2. *Assume that* (**H1**)-(**H7**) *and* (**H9**) *hold. Then the mild solution u of Eq.* (1.1) *satisfies*

$$e^{at}\mathbb{E}\left\|u_t\right\|_\alpha^p < x_m(t) \text{ for } t \geq 0,$$

where $x_m(t)$ is the global solution of Eq. (3.2) *with*

$$x_0 > \max\left(3^{p-1}N^p\mathbb{E}\|\varphi\|_\alpha^p, \ \mathbb{E}\|\varphi\|_\alpha^p\right).$$

Proof. We have

$$\begin{aligned} \left\|(-A)^\alpha u(s)\right\|^p &\leq 3^{p-1}\left\|(-A)^\alpha R(s)\varphi(0)\right\|^p + 3^{p-1}\left\|\int_0^s (-A)^\alpha R(s-l)F(l, u_l)dl\right\|^p \\ &\quad + 3^{p-1}\left\|\int_0^s (-A)^\alpha R(s-l)G(l, u_l)dw(l)\right\|^p. \end{aligned}$$

By using (**H9**), Hölder's inequality, Lemma 3.2 and (**H4**), we obtain that

$$\mathbb{E}\left\|u_t\right\|_\alpha^p \leq 3^{p-1}N^p e^{-apt}\mathbb{E}\left\|\varphi\right\|_\alpha^p + 3^{p-1}N^p(\lambda_1 + \lambda_2)\int_0^t e^{-a(t-s)}H(s, \mathbb{E}\left\|u_s\right\|_\alpha^p)ds.$$

(**H9**) gives us

$$e^{at}\mathbb{E}\left\|u_t\right\|_\alpha^p \leq 3^{p-1}N^p\mathbb{E}\left\|\varphi\right\|_\alpha^p + 3^{p-1}N^p(\lambda_1+\lambda_2)\int_0^t H(s, e^{as}\mathbb{E}\left\|u_s\right\|_\alpha^p)ds.$$

Then Lemma 3.3 yields that

$$e^{at}\mathbb{E}\left\|u_t\right\|_\alpha^p < x_m(t) \text{ for } t \geq 0.$$

For the main result of this section, we need the following further assumption:

(H10) There exist positive constants P, C^*, δ and a continuous function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\kappa(t) \leq C^* e^{-\delta t}$ for all $t \geq 0$ such that the function H in **(H4)** satisfies the condition

$$H(t, \eta) \leq P\eta + \kappa(t) \quad \text{for all } \eta \in \mathbb{R}_+ \quad \text{and} \quad t \geq 0. \qquad \square$$

Moreover, we suppose that $H(t, 0) = 0$ for $t \geq 0$.

Theorem 5.3. *Assume that* (**H1**)-(**H7**), (**H9**) *and* (**H10**) *hold. Then the trivial solution of Eq.* (1.1) *is exponentially stable in pth moment provided that*

$$a > 3^{p-1}PN^p(\lambda_1+\lambda_2).$$

Proof. Let x_m be the global solution of Eq. (3.2) defined by

$$x_m(t) = x_0 + 3^{p-1}N^p(\lambda_1+\lambda_2)\int_0^t H(s, x_m(s))ds \text{ for } t \geq 0.$$

Then from (**H10**), we obtain that

$$\begin{aligned} x_m(t) &\leq x_0 + 3^{p-1}N^p(\lambda_1+\lambda_2)\int_0^t [Px_m(s)+\kappa(s)]ds \\ &\leq x_0 + \frac{C^*}{\delta}3^{p-1}N^p(\lambda_1+\lambda_2) + 3^{p-1}PN^p(\lambda_1+\lambda_2)\int_0^t x_m(s)ds. \qquad \square \end{aligned}$$

By Gronwall's Lemma, we get that

$$x_m(t) \leq x_0^* e^{\lambda_0 t} \text{ for } t \geq 0, \tag{5.1}$$

where

$$x_0^* = x_0 + \frac{C^*}{\delta}3^{p-1}N^p(\lambda_1+\lambda_2) \text{ and } \lambda_0 = 3^{p-1}PN^p(\lambda_1+\lambda_2).$$

Therefore, from (5.1) and Proposition 5.2, we have

$$\mathbb{E}\left\|u_t\right\|_\alpha^p < e^{-at}x_m(t) < e^{-at}x_0^* e^{\lambda_0 t} \text{ for } t \geq 0.$$

From (3.3), (3.5) and (**H10**), we have $\mathbb{E}\left\|\varphi\right\|_\alpha^p = 0 \Rightarrow \mathbb{E}\left\|u_t\right\|_\alpha^p = 0$. Hence,

$$\mathbb{E}\left\|u_t\right\|_\alpha^p \leq Q\mathbb{E}\left\|\varphi\right\|_\alpha^p e^{-\lambda_1 t} \text{ for } t \geq 0, \tag{5.2}$$

where $\lambda_1 = a - \lambda_0$, Q is an arbitrary positive constant if $\mathbb{E}\left\|\varphi\right\|_\alpha^p = 0$ and $Q = \frac{x_0^*}{\mathbb{E}\|\varphi\|_\alpha^p}$ if $\mathbb{E}\left\|\varphi\right\|_\alpha^p \neq 0$.

6. Application

$$(6.1)\quad \begin{cases} \dfrac{\partial}{\partial t}v(t,\xi) = \Big(\dfrac{\partial^2}{\partial \xi^2}+\mu\Big)v(t,\xi) + \displaystyle\int_0^t e^{-\gamma(t-s)}\Big(\dfrac{\partial^2}{\partial \xi^2}+\mu\Big)v(s,\xi)ds \\ \qquad\qquad + \displaystyle\int_{-r}^0 f(t,\dfrac{\partial}{\partial \xi}v(t+\theta,\xi))d\theta + g(t,\dfrac{\partial}{\partial \xi}v(t-r,\xi))dw(t) \\ \qquad\qquad \text{for } \ t\geq 0 \ \text{ and } \ \xi\in[0,\pi], \\ v(t,0)=v(t,\pi)=0 \quad \text{for } t\geq 0, \\ v(\theta,\xi)=v_0(\theta,\xi) \quad \text{for } \theta\in[-r,0] \text{ and } \xi\in[0,\pi], \end{cases}$$

where $r,\gamma>0$, $\mu<0$, x_0, f and g are continuous real values functions.

Set $\mathbb{H}=L^2([0,\pi])$ and $w(t):=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)e_n, \lambda_n>0$, where $\beta_n(t)$ are one dimensional standard Brownian motion mutually independent on a usual complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$.

Consider the operator $A: D(A)\subseteq\mathbb{H}\to\mathbb{H}$ given by $Au=u''+\mu u$ with domain $D(A)=H^2(0,\pi)\cap H_0^1(0,\pi)$. The operator A has a discrete spectrum, the eigenvalues are $-n^2+\mu$ and the corresponding normalized eigenvectors are $e_n(x)=\sqrt{(2/\pi)}\sin(nx), n\in\mathbb{N}^*$. The set $\{e_n : n\in\mathbb{N}^*\}$ is an orthonormal basis of $\mathbb{H}$. A generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ defined by:

$$T(t)u=\sum_{n=1}^{\infty}e^{-(n^2-\mu)t}\langle u,e_n\rangle e_n \ \text{ for } \ u\in\mathbb{H}.$$

For $\alpha\in(0,1)$, we define the fractional power $(-A)^{\alpha}: D\big((-A)^{\alpha}\big)\subseteq\mathbb{H}\to\mathbb{H}$ by

$$(-A)^{\alpha}u=\sum_{n=1}^{\infty}(n^2-\mu)^{\alpha}\langle u,e_n\rangle e_n$$

for each $u\in D\big((-A)^{\alpha}\big)=\Big\{x\in\mathbb{H}:\sum_{n=1}^{\infty}(n^2-\mu)^{\alpha}\langle x,e_n\rangle e_n\in\mathbb{H}\Big\}$. Next, we consider $\alpha=1/2$. Remark that $\rho(A)\supset\{\lambda\in\mathbb{C}:\mathrm{Re}(\lambda)\geq\mu\}$ and $\|R(\lambda,A)\|\leq\frac{M_1}{|\lambda|}$ for $\mathrm{Re}(\lambda)\geq\mu$. Moreover, A is a sectorial operator and there exists $M>0$ such that $\|R(\lambda,A)\|\leq\frac{M}{|\lambda-\mu|}$.

Let $B(t): D(A)\subseteq\mathbb{H}\to\mathbb{H}$ be the operator defined by $B(t)u=e^{-\gamma t}Au$ for $t\geq 0$ and $u\in D(A)$.

Proposition 6.1 ([17, Theorem 5.1])**.** *Under the above conditions, the hypothesis* (**H1**)-(**H3**) *and* (**H7**) *are satisfied* (*with* $b(t)=e^{-\gamma t}$ *and* $D=C_0^{\infty}([0,\pi])$ *the space of infinitely differentiable functions with compact supports on* $[0,\pi]$).

We suppose that:

$$(1)\qquad |f(t,\zeta)|+|g(t,\zeta)|\leq|\zeta|;$$

$$(2)\qquad |f(t,\zeta_1)-f(t,\zeta_2)|+|g(t,\zeta_1)-g(t,\zeta_2)|\leq|\zeta_1-\zeta_2|;$$

We define

$$F(t,\phi)(\xi)=\int_{-r}^{0} f\big(t,\frac{\partial}{\partial\xi}\phi(\theta)(\xi)\big)d\theta \quad \text{for} \quad \xi\in[0,\pi] \quad \text{and} \quad \phi\in\mathcal{C}_{1/2},$$

$$G(t,\phi)(\xi)=g\big(t,\frac{\partial}{\partial\xi}\phi(-r)(\xi)\big) \text{ for } \xi\in[0,\pi] \text{ and } \phi\in C_{1/2}.$$

If we put

$$u(t)(\xi)=v(t,\xi) \quad \text{for} \quad \xi\in[0,\pi] \quad \text{and} \quad t\geq 0,$$

and

$$\phi(\theta)(\xi)=v_0(\theta,\xi) \quad \text{for } \xi\in[0,\pi] \quad \text{and} \quad \theta\in[-r,0].$$

Then the system (6.1) takes the following form

$$\begin{cases} \dfrac{d}{dt}u(t)=Au(t)+\displaystyle\int_0^t B(t-s)u(s)ds+F(t,u_t) \\ \qquad\qquad +G(t,u_t)dw(t) \quad \text{for } t\geq 0, \\ \qquad u_0=\phi\in C_{1/2}=C\big([-r,0],\mathbb{Y}_{1/2}\big). \end{cases} \tag{6.2}$$

Lemma 6.2. *Let $\phi\in C_{1/2}$. Then $\|\phi\|_{1/2}=\|\nabla\phi\|_{\mathbb{H}}$.*

Proof. See [15], Example 5.1. □

By Assumption 1 and Lemma 6.2, we have

$$\begin{aligned} \|F(t,\phi_1)\|^p &= \int_0^{\pi}\left|\int_{-r}^{0} f\big(t,\frac{\partial}{\partial\xi}\phi_1(\theta)(\xi)\big)d\theta\right|^p d\xi \\ &\leq \int_0^{\pi}\Big(\int_{-r}^{0}d\theta\Big)\int_{-r}^{0}\left|f\big(t,\frac{\partial}{\partial\xi}\phi_1(\theta)(\xi)\big)\right|^p d\theta d\xi \\ &\leq \int_0^{\pi} r\int_{-r}^{0}|\frac{\partial}{\partial\xi}\phi_1(\theta)(\xi)|^p d\theta d\xi \\ &\leq r^2\,\|\phi_1\|_{1/2}^p\,. \end{aligned}$$

Likewise, we show that

$$\|G(t,\phi)\|_{\mathcal{L}_2^0}^p\leq\|\phi\|_{1/2}^p$$

and from Assumption 2, we obtain that

$$\|F(t,\phi_1)-F(t,\phi_2)\|_{\mathbb{H}}^p\leq r^2\,\|\phi_1-\phi_2\|_{1/2}^p$$

and

$$\|G(t,\phi_1)-G(t,\phi_2)\|_{\mathcal{L}_2^0}^p\leq\|\phi_1-\phi_2\|_{1/2}^p\,.$$

Set $H(t,x)=Z(t,x)=\beta x$ where $\beta=\frac{1}{1+r^2}$. Hence, all the hypotheses of Theorem 5.3 are satisfied. Then Eq. (6.2) has a unique mild solution which is exponentially stable in pth moment.

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