J. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/JKMS.j170805
pISSN: 0304-9914 / eISSN: 2234-3008

FIXED POINT THEOREMS ON GENERALIZED CONE METRIC SPACES OVER BANACH ALGEBRAS AND APPLICATIONS

Qianqian Leng and Jiandong Yin

Abstract. The aim of this paper is to introduce the concept of generalized cone metric spaces over Banach algebras as a generalization of generalized metric spaces and present several fixed point results of a class of contractive mappings in generalized cone metric spaces over Banach algebras. Moreover, in order to support our main results, one example is given at the end of this paper.

1. Introduction

The conception of generalized metric spaces was introduced in the work of Mohamed Jleli and Bessem Samet [3], who established some fixed point theorems in generalized metric spaces. The notion of generalized metric spaces was defined as follows.

Definition 1.1 (see [3]). Let X be a non-empty set. We say that $\mathcal{D}$ is a generalized metric on X if it satisfies the following conditions:
($\mathcal{D}_1$) for every $(x, y) \in X \times X$, we have $\mathcal{D}(x, y) = 0 \Rightarrow x = y$;
($\mathcal{D}_2$) for every $(x, y) \in X \times X$, we have $\mathcal{D}(x, y) = \mathcal{D}(y, x)$;
($\mathcal{D}_3$) there exists $C > 0$ such that if $(x, y) \in X \times X$ and $\{x_n\} \in C(\mathcal{D}, X, x) = \{\{x_n\} : \lim_{n\to\infty} \mathcal{D}(x_n, x) = 0\}$, then

$$\mathcal{D}(x, y) \leq C \limsup_{n\to\infty} \mathcal{D}(x_n, y).$$

Meanwhile we say the pair $(X, \mathcal{D})$ is a generalized metric space.

Clearly, the conception of generalized metric space is a generalization of that of classical metric space. Moreover, the authors of [3] proved several fixed point theorems of contractive mappings on generalized metric spaces, which also generalized some corresponding fixed point results in classical metric spaces.

Received December 21, 2017; Accepted April 24, 2018.
2010 *Mathematics Subject Classification.* 54H25, 47H10.
Key words and phrases. generalized cone metric spaces over Banach algebras, fixed point theorems, spectral radius.

Theorem 1.1 (see [3])**.** *Suppose that* $(X, \mathcal{D})$ *is a generalized metric space and the following conditions hold*:

(i) $(X, \mathcal{D})$ *is complete, i.e., every Cauchy sequence in* $(X, \mathcal{D})$ *has a limit in* X;

(ii) $f : X \to X$ *is a mapping satisfying, for some* $k \in (0, 1)$ *and every* $(x, y) \in X \times X$,

$$\mathcal{D}(f(x), f(y)) \leq k\mathcal{D}(x, y);$$

(iii) *there exists* $x_0 \in X$ *such that* $\delta(\mathcal{D}, f, x_0) = \sup\{\mathcal{D}(f^i(x_0), f^j(x_0)) : i, j \in \mathbb{N}\} < \infty$, *where* $\mathbb{N}$ *denotes the set of positive integers and similarly hereinafter. Then* $\{f^n(x_0)\}$ *converges to* $\omega \in X$, *a fixed point of* f. *Moreover, if* $\omega^{'} \in X$ *is another fixed point of* f *such that* $\mathcal{D}(\omega, \omega^{'}) < \infty$, *then* $\omega = \omega^{'}$.

Theorem 1.2 (see [3])**.** *Suppose that* $(X, \mathcal{D})$ *is a generalized metric space and the following conditions hold*:

(i) $(X, \mathcal{D})$ *is complete*;

(ii) $f : X \to X$ *is a mapping satisfying, for some* $k \in (0, 1)$ *and every* $(x, y) \in X \times X$,

$$\mathcal{D}(f(x), f(y)) \leq k \max\{\mathcal{D}(x, y), \mathcal{D}(x, f(x)), \mathcal{D}(y, f(y)), \mathcal{D}(x, f(y)), \mathcal{D}(y, f(x))\};$$

(iii) *there exists* $x_0 \in X$ *such that* $\delta(\mathcal{D}, f, x_0) = \sup\{\mathcal{D}(f^i(x_0), f^j(x_0)) : i, j \in \mathbb{N}\} < \infty$.
Then $\{f^n(x_0)\}$ *converges to* $\omega \in X$. *If* $\mathcal{D}(x_0, f(\omega)) < \infty$ *and* $\mathcal{D}(\omega, f(\omega)) < \infty$, *then* ω *is a fixed point of* f. *Moreover, if* $\omega^{'} \in X$ *is another fixed point of* f *such that* $\mathcal{D}(\omega, \omega^{'}) < \infty$, *and* $\mathcal{D}(\omega^{'}, \omega^{'}) < \infty$, *then* $\omega = \omega^{'}$.

In 2013, in order to generalize the classical Banach contractive mapping principle to a generalized form, Liu and Xu [6] introduced the concept of cone metric spaces over Banach algebras and proved Banach contractive mapping principle remains true under the setting of cone metric spaces over Banach algebras. Based on the work [6], the authors of [9] presented some fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without the assumption of normality of the involved cone, which is necessary in the proofs of the main results in [6].

Motivated by $[3, 4, 6, 11]$, by combining the concepts of generalized metric spaces over Banach algebras, we introduce a new space in this paper, which is called a generalized cone metric space over Banach algebras, and by replacing the constant k with a vector of the Banach algebra in the contractive conditions and without the assumption that the involved cone is normal, we prove some fixed point theorems of the mappings satisfying certain contractive conditions. Our results extend Theorem 1.1 and Theorem 1.2 to the case of generalized cone metric spaces over Banach algebras.

2. Preliminaries and basic concepts

Let $\mathcal{A}$ always be a real Banach algebra, that is, $\mathcal{A}$ is a real Banach space and an operation of multiplication is defined in $\mathcal{A}$ satisfying the following properties: (for all $x, y, z \in \mathcal{A}, \alpha \in \mathbb{R}$):
(i) $(xy)z = x(yz)$;
(ii) $x(y+z) = xy + xz$ and$(x+y)z = xz + yz$;
(iii) $\alpha(xy) = x(\alpha y)$;
(iv) $\|xy\| \leq \|x\|\|y\|$.

Throughout this paper, we assume that there is a unit e, a multiplicative identity, in the Banach algebra $\mathcal{A}$ such that $ex = xe = x$ for all $x \in \mathcal{A}$. $x \in \mathcal{A}$ is said to be invertible if there is an element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [6].

The following lemmas about spectral radius are needed in this paper.

Lemma 2.1 (see [9])**.** *Let $\mathcal{A}$ be a Banach algebra with a unit e and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n} < 1,$$

then $e - x$ is invertible. Actually

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Remark 2.1 (see [9]). If $r(x) < 1$, then $\|x^n\| \to 0$ $(n \to \infty)$.

Lemma 2.2 (see [9])**.** *Let $\mathcal{A}$ be a Banach algebra and x, y be vectors in $\mathcal{A}$. If x and y commute, then the following hold*:
(i) $r(xy) \leq r(x)r(y)$;
(ii) $r(x+y) \leq r(x) + r(y)$;
(iii) $|r(x) - r(y)| \leq r(x-y)$.

Lemma 2.3 (see [9])**.** *Let $\mathcal{A}$ be a Banach algebra and let k be a vector in $\mathcal{A}$. If $0 \leq r(k) < 1$, then we have*

$$r((e-k)^{-1}) \leq (1 - r(k))^{-1}.$$

Now let us recall the concepts of cone and partial ordering of the Banach algebra $\mathcal{A}$. A subset P of $\mathcal{A}$ is called a cone of $\mathcal{A}$ if
(i) P is non-empty closed and $\{\theta, e\} \subset P$;
(ii) $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
(iii)$P^2 = PP \subset P$;
(iv) $P \cap (-P) = \{\theta\}$.

Here θ stands for the null of the Banach algebra $\mathcal{A}$. For a given cone $P \subset A$, a partial ordering $\preceq$ with respect to P can be defined by $x \preceq y$ if and only if $y - x \in P$. $x < y$ means that $x \preceq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P. P is called a solid cone if $\text{int}P \neq \emptyset$.

The following lemmas that relate to the partial ordering $\preceq$ with respect to P are necessary for the proofs of our results.

Lemma 2.4 (see [8])**.** *If E is a real Banach space with a solid cone P and if $\theta \preceq \mu \ll c$ for each $\theta \ll c$, then $\mu = \theta$.*

Lemma 2.5 (see [8])**.** *Let $\mathcal{A}$ be a Banach space with a solid cone P and if $\|x_n\| \to 0$ $(n \to \infty)$, then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $x_n \ll c$.*

Now we review some theories on c-sequences.

Definition 2.1 (see [1,5])**.** Let P be a solid cone in the Banach space $\mathcal{A}$. A sequence $\{x_n\} \subset P$ is called a c-sequence if for each $\theta \ll c$ there exists $N \in \mathbb{N}$ such that $x_n \ll c$ for all $n > N$.

The properties below on c-sequences are very helpful in the proofs of the main results.

Lemma 2.6 (see [5])**.** *Let P be a solid cone in the Banach space $\mathcal{A}$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in P. If $\{x_n\}$ and $\{y_n\}$ are c-sequences and $\alpha, \beta > 0$, then $\{\alpha x_n + \beta y_n\}$ is a c-sequence.*

Lemma 2.7 (see [7])**.** *Let P be a solid cone in the Banach space $\mathcal{A}$ and let $\{x_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{x_n\}$ is a c-sequence in P. Then $\{kx_n\}$ is a c-sequence.*

Proposition 2.1. *Let P be a solid cone in the Banach algebra $\mathcal{A}$ and let x be a vector in $\mathcal{A}$. Suppose that $k \in P$ is an arbitrarily given vector and $x \ll c$ for any $\theta \ll c$, then we have $kx \ll c$ for any $\theta \ll c$.*

Proof. Fix $c \gg \theta$, then $\frac{c}{m} \gg \theta$ for all $m \in \mathbb{N}$. It is clear that $x \preceq \frac{c}{m}$ for all $m \in \mathbb{N}$, so $kx \preceq \frac{kc}{m}$. Since $\frac{kc}{m} \to \theta$ as $m \to \infty$, there exists $M \in \mathbb{N}$ such that $kx \preceq \frac{kc}{m} \ll c$ when $m > M$. This completes the proof. □

Proposition 2.2 (see [11, Proposition 3.5])**.** *Let $\mathcal{A}$ be a Banach algebra with a unit e, P be a cone in $\mathcal{A}$ and $\preceq$ be the partial ordering generated by the cone P. Let $\lambda \in P$. If the spectral radius $r(\lambda)$ of λ is less than 1, then the following assertions hold*:

(i) *Suppose that x is invertible and that $x^{-1} \succ \theta$ implies $x \succ \theta$, then for any integer $n \geq 1$, we have $\lambda^n \preceq \lambda \preceq e$.*

(ii) *For any $\mu > \theta$, we have $\mu \npreceq \lambda\mu$, i.e., $\lambda\mu - \mu \notin P$.*

(iii) *If $\lambda \succeq \theta$, then we have $(e - \lambda)^{-1} \succeq \theta$.*

3. The concept of generalized cone metric spaces over Banach algebras

In the following we always assume that P is a solid cone in the Banach algebra $\mathcal{A}$ and $\preceq$ is the partial ordering with respect to P.

Let X be a non-empty set. The conception of cone metric spaces was defined in [2,6,7] as follows.

Definition 3.1 (see [2,6,7])**.** We say that d is a cone metric on X if it satisfies the following conditions:

(d_1) for every $(x, y) \in X \times X$, $d(x, y) \geq \theta$ and $d(x, y) = \theta \Leftrightarrow x = y$;

(d_2) for every $(x, y) \in X \times X$, $d(x, y) = d(y, x)$;

(d_3) for every $(x, y, z) \in X \times X \times X$, $d(x, y) \preceq d(x, z) + d(z, y)$;

In this case, we call $(X, \mathcal{D})$ a cone metric space over the Banach algebra $\mathcal{A}$.

Now let $\mathcal{D} : X \times X \to \mathcal{A}$ be a given mapping. For every $x \in X$, write

$$C(\mathcal{D}, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} \mathcal{D}(x_n, x) = \theta\}.$$

Next we try to generalize the notion of cone metric spaces over Banach algebras and introduce the following

Definition 3.2. We say that $\mathcal{D}$ is a generalized cone metric on X if it satisfies the following conditions:

($\mathcal{D}_1$) for every $(x, y) \in X \times X$, $\mathcal{D}(x, y) \geq \theta$, and $\mathcal{D}(x, y) = \theta \Rightarrow x = y$;

($\mathcal{D}_2$) for every $(x, y) \in X \times X$, $\mathcal{D}(x, y) = \mathcal{D}(y, x)$;

($\mathcal{D}_3$) there exists $C \in P$ with $C > \theta$ satisfying that for any $\varepsilon \gg \theta$ there exists $N \in \mathbb{N}$ such that for any $n > N$, if $(x, y) \in X \times X$ and $\{x_n\} \in C(\mathcal{D}, X, x)$, then $\mathcal{D}(x, y) \preceq C(\varepsilon + \mathcal{D}(x_n, y))$.

At the same time, we call $(X, \mathcal{D})$ a generalized cone metric space over the Banach algebra $\mathcal{A}$.

To be exact, any cone metric space over Banach algebras is a generalized cone metric space over Banach algebras. So we have the following proposition.

Proposition 3.1. *Any cone metric space (X, d) over Banach algebras is a generalized cone metric space over Banach algebras.*

Proof. Let(X, d) be a cone metric space over Banach algebras, then it suffices to prove that d satisfies the property ($\mathcal{D}_3$) of Definition 3.2.

Let $\{x_n\} \in C(d, X, x)$. For any $y \in X$, by the property (d_3) of Definition 3.1, we have $d(x, y) \preceq d(x, x_n) + d(x_n, y)$ for every natural number n. Since $\lim_{n\to\infty} d(x_n, x) = \theta$ for any $\varepsilon \gg \theta$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $d(x, x_n) \preceq \varepsilon$, which implies $d(x, y) \preceq \varepsilon + d(x_n, y)$. Thus the property ($\mathcal{D}_3$) is satisfied if take $C = e$. □

In the sequel, we point out by examples that the concept of generalized cone metric spaces over Banach algebras is a proper generalization of that of cone metric spaces over Banach algebras.

Example 3.1. Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ and define a norm on $\mathcal{A}$ by $\|x\| = \|x\|_\infty + \|x^{'}\|_\infty$ for $x \in \mathcal{A}$. Define a multiplication in $\mathcal{A}$ as just the point-wise multiplication. Then $\mathcal{A}$ is a real Banach algebra with a unit $e = I$, the unit mapping in $C^1_{\mathbb{R}}[0, 1]$. Let $P = \{x \in \mathcal{A} : x \geq 0\}$, then P is a cone in $\mathcal{A}$ and is not normal.

Let $X = \{0, 1, 2\}$. Define $\mathcal{D} : X \times X \to \mathcal{A}$ by

$$(3.1)\qquad \begin{cases} \mathcal{D}(0,1)(t) = \mathcal{D}(1,0)(t) = e^t; \\ \mathcal{D}(1,2)(t) = \mathcal{D}(2,1)(t) = e^{2t}; \\ \mathcal{D}(0,2)(t) = \mathcal{D}(2,0)(t) = e^{3t}; \\ \mathcal{D}(1,1)(t) = \mathcal{D}(2,2)(t) = \theta; \\ \mathcal{D}(0,0)(t) = t \end{cases}$$

for any $t \in [0, 1]$. Then $(X, \mathcal{D})$ is a generalized cone metric space over the Banach algebra $\mathcal{A}$, but not a cone metric space over the Banach algebra $\mathcal{A}$.

Example 3.2. Let $\mathcal{A} = \mathbb{R}^2$, for each $(x_1, x_2) \in \mathcal{A}$, put $\|(x_1, x_2)\| = |x_1| + |x_2|$, then $\|\cdot\|$ is a norm in $\mathcal{A}$. A multiplication in $\mathcal{A}$ is defined by $xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ for any $(x_1, x_2), (y_1, y_2) \in \mathcal{A}$. Then $\mathcal{A}$ is a Banach algebra with unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0\}$. Then P is a cone in $\mathcal{A}$. Set $X = \{-1, 0, 1\}$. Define $\mathcal{D} : X \times X \to \mathcal{A}$ by $\mathcal{D}(x, y) = \mathcal{D}(y, x)$, for all $x, y \in X$, $\mathcal{D}(-1, -1) = \mathcal{D}(1, 1) = (0, 0)$, and $\mathcal{D}(-1, 0) = (3, 3)$, $\mathcal{D}(-1, 1) = \mathcal{D}(0, 1) = \mathcal{D}(0, 0) = (1, 1)$. Then $(X, \mathcal{D})$ is a generalized cone metric space over the Banach algebra $\mathcal{A}$, but not a cone metric space over the Banach algebra $\mathcal{A}$.

Next, we introduce the notion of convergence in a generalized cone metric space over Banach algebras.

Definition 3.3. Let $(X, \mathcal{D})$ be a generalized cone metric space over the Banach algebra $\mathcal{A}$. Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ $\mathcal{D}$-converges to x if $\{x_n\} \in C(\mathcal{D}, X, x)$.

Proposition 3.2. *Let $(X, \mathcal{D})$ be a generalized cone metric space over the Banach algebra $\mathcal{A}$. Let $\{x_n\}$ be a sequence in X and $(x, y) \in X \times X$. If $\{x_n\}$ $\mathcal{D}$-converges to x and $\mathcal{D}$-converges to y, then $x = y$.*

Proof. By the property $(\mathcal{D}_3)$ for any $\varepsilon \gg \theta$ there exist $C \in P$ with $C > \theta$ and $n_0 \in \mathbb{N}$ satisfying $\mathcal{D}(x, y) \preceq C(\varepsilon + \mathcal{D}(x_n, y))$ for any $n > n_0$. Since $\mathcal{D}(x_n, y) \to \theta (n \to \infty)$ for $\varepsilon \gg \theta$, there exists $n_1 \in \mathbb{N}$ such that $\mathcal{D}(x_n, y) \ll \varepsilon$ for any $n > n_1$. Choose $N = \max\{n_0, n_1\}$, then for any $n > N$, we have $\mathcal{D}(x, y) \ll C(\varepsilon + \varepsilon)$. According to Lemma 2.4 and Proposition 2.1 we can get $\mathcal{D}(x, y) = \theta$ which together with the property $(\mathcal{D}_1)$ implies $x = y$. □

In the following definitions, we mainly introduce the completeness of a generalized cone metric space over Banach algebras.

Definition 3.4. Let $(X, \mathcal{D})$ be a generalized cone metric space over Banach algebras. Let $\{x_n\}$ be a sequence in X.We say that $\{x_n\}$ is a $\mathcal{D}$-Cauchy sequence if $\lim\limits_{m,n\to\infty} \mathcal{D}(x_n, x_{n+m}) = \theta$.

Definition 3.5. Let $(X,\mathcal{D})$ be a generalized cone metric space over Banach algebras. $(X,\mathcal{D})$ is said to be $\mathcal{D}$-complete if every Cauchy sequence in X is convergent to some element in X.

Remark 3.1. In fact, it is easy to verify that Example 3.2 is exactly an example of $\mathcal{D}$-complete generalized cone metric spaces over Banach algebras.

4. The fixed point results of contractive mappings in generalized cone metric spaces over Banach algebras

In the section, we present our results and their detailed proofs.

Theorem 4.1. *Let $(X,\mathcal{D})$ be a generalized cone metric space over the Banach algebra $\mathcal{A}$. Suppose that P is a solid cone and the following conditions hold*:
(i) *$(X,\mathcal{D})$ is $\mathcal{D}$-complete*;
(ii) *a mapping $f : X \to X$ satisfies*

$$\mathcal{D}(f(x), f(y)) \preceq k_1\mathcal{D}(x, f(x)) + k_2\mathcal{D}(y, f(y)) + k_3\mathcal{D}(x, y) \tag{4.1}$$

for any $x, y \in X$, where $k_1, k_2, k_3 \in P$ with $0 < r(k_1) + r(k_2) + r(k_3) < 1$, $0 < r(Ck_2) < 1$, $0 < r(Ck_1) < 1$, here C is the vector satisfying Definition 3.2 and k_1, k_2, and k_3 are commute;
(iii) *there exists $x_0 \in X$ such that for any $m \in \mathbb{N}$, $\|\mathcal{D}(x_0, f^m(x_0))\| < \infty$. Then f has a unique fixed point and $\{f^n(x_0)\}$ $\mathcal{D}$-converges the fixed point of f.*

Proof. According to the formula (4.1), for all $n \in \mathbb{N}$ we have

$$\begin{aligned}&\mathcal{D}(f^{n+1}(x_0), f^n(x_0))\\ \preceq\ & k_1\mathcal{D}(f^n(x_0), f^{n+1}(x_0)) + k_2\mathcal{D}(f^{n-1}(x_0), f^n(x_0)) + k_3\mathcal{D}(f^n(x_0), f^{n-1}(x_0)).\end{aligned}$$

Hence,

$$(e - k_1)\mathcal{D}(f^{n+1}(x_0), f^n(x_0)) \preceq (k_2 + k_3)\mathcal{D}(f^n(x_0), f^{n-1}(x_0)). \tag{4.2}$$

Then multiplying both sides of (4.2) with $(e - k_1)^{-1}$ ($(e - k_1)^{-1}$ is existing by Lemma 2.1 as $0 < r(k_1) < 1$), we obtain that

$$\mathcal{D}(f^{n+1}(x_0), f^n(x_0)) \preceq (e - k_1)^{-1}(k_2 + k_3)\mathcal{D}(f^n(x_0), f^{n-1}(x_0)).$$

Thus by induction, for any positive integer m, it is easy to verify that

$$\begin{aligned}\mathcal{D}(f^n(x_0), f^{n+m}(x_0)) &\preceq (e - k_1)^{-1}(k_2 + k_3)\mathcal{D}(f^{n-1}(x_0), f^{n+m-1}(x_0))\\ &\preceq ((e - k_1)^{-1}(k_2 + k_3))^2\mathcal{D}(f^{n-2}(x_0), f^{n+m-2}(x_0))\\ &\ \ \vdots\\ &\preceq ((e - k_1)^{-1}(k_2 + k_3))^n\mathcal{D}(x_0, f^m(x_0)).\end{aligned}$$

Now we prove that $r((e - k_1)^{-1}(k_2 + k_3)) < 1$. From Lemma 2.2 and Lemma 2.3 and noting that k_1, k_2, k_3 are commute, we have

$$r((e - k_1)^{-1}(k_2 + k_3)) \leq r((e - k_1)^{-1})r((k_2 + k_3)) \leq \frac{r(k_2) + r(k_3)}{1 - r(k_1)}.$$

Since $r(k_1)+r(k_2)+r(k_3)<1$, we have $r(k_2)+r(k_3)<1-r(k_1)$ and

$$r((e-k_1)^{-1}(k_2+k_3)) \leq \frac{r(k_2)+r(k_3)}{1-r(k_1)} < 1.$$

From Lemma 2.5 and the fact that $\|((e-k_1)^{-1}(k_2+k_3))^n\mathcal{D}(x_0, f^m(x_0))\| \to 0$ as $n\to\infty$ which is yielded by Remark 2.1 since $\|((e-k_1)^{-1}(k_2+k_3))^n\| \to 0 (n\to\infty)$, and $\|\mathcal{D}(x_0, f^m(x_0))\| < \infty$, it follows that, for any $c\in\mathcal{A}$ with $\theta \ll c$, there exists $N\in\mathbb{N}$ such that for any $n, m > N$,

$$\mathcal{D}(f^n(x_0), f^{n+m}(x_0)) \preceq ((e-k_1)^{-1}(k_2+k_3))^n\mathcal{D}(x_0, f^m(x_0)) \ll c.$$

From Lemma 2.4, it follows that $\lim_{n,m\to\infty}\mathcal{D}(f^n(x_0), f^{n+m}(x_0)) = \theta$. So $\{f^n(x_0)\}$ is a $\mathcal{D}$-Cauchy sequence. The $\mathcal{D}$-completeness of $(X,\mathcal{D})$ implies that there exists $\omega\in X$ such that $\{f^n(x_0)\}$ $\mathcal{D}$-converges to ω. According to the definition of generalized cone metric spaces over Banach algebras, for any $\varepsilon \gg \theta$, there exist $C>\theta$ and $N_0\in\mathbb{N}$, such that for any $n>N_0$,

$$\mathcal{D}(\omega, f(\omega)) \preceq C(\varepsilon + \mathcal{D}(f^n(x_0), f(\omega))).$$

On the other hand, we have

$$\mathcal{D}(f^n(x_0), f(\omega)) \preceq k_1\mathcal{D}(f^{n-1}(x_0), f^n(x_0)) + k_2\mathcal{D}(\omega, f(\omega)) + k_3\mathcal{D}(f^{n-1}(x_0), \omega),$$

and

$$(e-Ck_2)\mathcal{D}(\omega, f(\omega)) \preceq Ck_1\mathcal{D}(f^{n-1}(x_0), f^n(x_0)) + Ck_3\mathcal{D}(f^{n-1}(x_0), \omega) + C\varepsilon.$$

Since $\{f^n(x_0)\}$ $\mathcal{D}$-converges to ω, for the above $\varepsilon\in\mathcal{A}$, there exists $N_1\in\mathbb{N}$ such that for any $n>N_1$, $\mathcal{D}(f^{n-1}(x_0), f^n(x_0)) \ll \varepsilon$ and $\mathcal{D}(f^{n-1}(x_0), \omega) \ll \varepsilon$. Choose $N=\max\{N_0, N_1\}$, for any $n>N$, we have

$$(e-Ck_2)\mathcal{D}(\omega, f(\omega)) \preceq Ck_1\varepsilon + Ck_3\varepsilon + C\varepsilon = (Ck_1+Ck_3+C)\varepsilon.$$

According to Proposition 2.2 and multiplying both sides of the above equation with $(e-Ck_2)^{-1}$, we have

$$\mathcal{D}(f(\omega), \omega) \preceq (e-Ck_2)^{-1}(Ck_1+Ck_3+C)\varepsilon,$$

which together with Proposition 2.1 implies that $\mathcal{D}(f(\omega), \omega) = \theta$. So we have $\omega = f(\omega)$, which yields that ω is a fixed point of f.

Suppose ω' is another fixed point of f, due to the formula (4.1), we have

$$\mathcal{D}(\omega, \omega') = \mathcal{D}(f(\omega), f(\omega')) \preceq k_1\mathcal{D}(\omega', f(\omega')) + k_2\mathcal{D}(\omega, f(\omega)) + k_3\mathcal{D}(\omega, \omega'),$$

so

$$(e-k_3)\mathcal{D}(\omega, \omega') \preceq k_1\mathcal{D}(\omega', f(\omega')) + k_2\mathcal{D}(\omega, f(\omega)).$$

Then multiplying both sides of the above formula with $(e-k_3)^{-1}$, we obtain that

$$\begin{aligned}\mathcal{D}(\omega, \omega') &\preceq (e-k_3)^{-1}k_1\mathcal{D}(\omega', f(\omega')) + (e-k_3)^{-1}k_2\mathcal{D}(\omega, f(\omega)) \\ &= (e-k_3)^{-1}k_1\mathcal{D}(\omega', \omega') + (e-k_3)^{-1}k_2\mathcal{D}(\omega, \omega).\end{aligned}$$

Now we prove $\mathcal{D}(\omega,\omega)=\theta$, and $\mathcal{D}(\omega^{'},\omega^{'})=\theta$. In fact,

$$\begin{aligned}\mathcal{D}(\omega,\omega)&=\mathcal{D}(f(\omega),f(\omega))\\&\preceq k_1\mathcal{D}(\omega,f(\omega))+k_2\mathcal{D}(\omega,f(\omega))+k_3\mathcal{D}(\omega,\omega)\\&=(k_1+k_2+k_3)\mathcal{D}(\omega,\omega),\end{aligned}$$

which implies $\mathcal{D}(\omega,\omega)=\theta$. By the same arguments, we can get $\mathcal{D}(\omega^{'},\omega^{'})=\theta$. So $\mathcal{D}(\omega,\omega^{'})=\theta$, which gives $\omega=\omega^{'}$. □

By Theorem 4.1, we can obtain the following Corollary 4.1, which is the version of Banach contraction principle in generalized cone metric spaces over Banach algebras.

Corollary 4.1. *Let $(X,\mathcal{D})$ be a generalized cone metric space over the Banach algebra $\mathcal{A}$, P be a solid cone of $\mathcal{A}$ and the following condition hold*:
(i) *$(X,\mathcal{D})$ is $\mathcal{D}$-complete*;
(ii) *a mapping $f:X\to X$ satisfies*

$$\mathcal{D}(f(x),f(y))\preceq k(\mathcal{D}(f(x),x)+\mathcal{D}(f(y),y))$$

for any $x,y\in X$, where $k\in P$ with $r(k)<\frac{1}{2}$, $0<r(Ck)<1$, where C is the vector satisfying Definition 3.2;
(iii) *there exists $x_0\in X$ such that for any $m\in\mathbb{N}$, $\|\mathcal{D}(x_0,f^m(x_0))\|<\infty$. Then f has a unique fixed point and $\{f^n(x_0)\}$ $\mathcal{D}$-converges to the fixed point of f.*

Proof. Put $k_1=k_2=k$ and $k_3=\theta$ in Theorem 4.1, then the proof is straightforward. □

Theorem 4.2. *Let $(X,\mathcal{D})$ be a generalized cone metric space over the Banach algebra $\mathcal{A}$. Suppose that P is a solid cone of $\mathcal{A}$ and the following conditions hold*:
(i) *$(X,\mathcal{D})$ is $\mathcal{D}$-complete*;
(ii) *$f:X\to X$ is a mapping, and there exists $n_0\in\mathbb{N}$ such that*

$$\mathcal{D}(f^{n_0}(x),f^{n_0}(y))\preceq k\mathcal{D}(x,y) \tag{4.3}$$

for some $k\in P$ with $r(k)<1$;
(iii) *there exists $x_0\in X$ such that for any $m\in\mathbb{N}$, $\|\mathcal{D}(x_0,f^{mn_0}(x_0))\|<\infty$. Then f has a unique fixed point in X.*

Proof. According to (4.3), for all $m,n\in\mathbb{N}$,

$$\begin{aligned}\mathcal{D}(f^{nn_0}(x_0),f^{(n+m)n_0}(x_0))&\preceq k\mathcal{D}(f^{(n-1)n_0}(x_0),f^{(n+m-1)n_0}(x_0))\\&\preceq k^2\mathcal{D}(f^{(n-2)n_0}(x_0),f^{(n+m-2)n_0}(x_0))\\&\vdots\\&\preceq k^n\mathcal{D}(x_0,f^{mn_0}(x_0)).\end{aligned}$$

From Lemma 2.5 and the fact that $\|k^n\mathcal{D}(x_0, f^m(x_0))\| \to 0$ as $\|k^n\| \to 0$ by Remark 2.1 and the given condition $\|\mathcal{D}(x_0, f^m(x_0))\| < \infty$, it follows that, for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for any $n, m > N$,

$$\mathcal{D}(f^{nn_0}(x_0), f^{(n+m)n_0}(x_0)) \preceq k^n\mathcal{D}(x_0, f^{mn_0}(x_0)) \ll c,$$

which together with Lemma 2.4 implies that

$$\lim_{n,m\to\infty} \mathcal{D}(f^{nn_0}(x_0), f^{(n+m)n_0}(x_0)) = \theta.$$

So $\{f^{nn_0}(x_0)\}$ is a $\mathcal{D}$-Cauchy sequence. The completeness of $(X, \mathcal{D})$ implies that there exists $\omega \in X$ such that $\{f^{nn_0}(x_0)\}$ $\mathcal{D}$-converges to ω.

On the other hand, according to (4.3), for all $n \in \mathbb{N}$, we have

$$\mathcal{D}(f^{(n+1)n_0}(x_0), f^{n_0}(\omega)) \preceq k\mathcal{D}(f^{nn_0}(x_0), \omega).$$

Letting $n \to \infty$ in the above inequality, we get

$$\mathcal{D}(f^{(n+1)n_0}(x_0), f(\omega)) = \theta.$$

Then $\{f^{nn_0}(x_0)\}$ $\mathcal{D}$-converges to $f^{n_0}(\omega)$. The uniqueness of the limit (see Proposition 3.2) yields that $\omega = f^{n_0}(\omega)$, that is, ω is a fixed point of f^{n_0}.

Now suppose that $\omega^{'} \in X$ is another fixed point of f^{n_0}, by (4.3) we have

$$\mathcal{D}(\omega, \omega^{'}) = \mathcal{D}(f^{n_0}(\omega), f^{n_0}(\omega^{'})) \preceq k\mathcal{D}(\omega, \omega^{'}).$$

Hence $(e - k)\mathcal{D}(\omega, \omega^{'}) \preceq \theta$ which gives that $\omega = \omega^{'}$.

Since $f^{n_0}f(\omega) = ff^{n_0}(\omega) = f(\omega)$, $f(\omega)$ is also a fixed point of f^{n_0}. The uniqueness of fixed point of f^{n_0} shows that $f(\omega) = \omega$. That means that ω is a fixed point of f. Assume that $\omega^{'}$ is another fixed point of f, then it is easy to get that $f^{n_0}(\omega^{'}) = f^{n_0-1}(\omega^{'}) = \cdots = f(\omega^{'}) = \omega^{'}$. Namely $\omega^{'}$ is also a fixed point of f^{n_0}. The uniqueness of fixed point of f^{n_0} shows again that $\omega^{'} = \omega$. So f has a unique fixed point. □

Corollary 4.2. *Let $(X, \mathcal{D})$ be a generalized cone metric space over the Banach algebra $\mathcal{A}$, P be a solid cone of $\mathcal{A}$ and the following conditions hold*:

(i) *$(X, \mathcal{D})$ is $\mathcal{D}$-complete*;

(ii) *a mapping $f : X \to X$ satisfies $\mathcal{D}(f(x), f(y)) \preceq k\mathcal{D}(x, y))$ for $k \in P$ and $r(k) < 1$*;

(iii) *there exists $x_0 \in X$, such that for any $m \in \mathbb{N}$, $\|\mathcal{D}(x_0, f^m(x_0))\| < \infty$.*

Then f has a unique fixed point and $\{f^n(x_0)\}$ $\mathcal{D}$-converges to the fixed point of f.

Proof. The proof is rightward if take $n_0 = 1$ in Theorem 4.2. □

In the following, in order to prove another fixed point result, we present firstly some notations.

Let $a_i, b_i, x, y, k \in \mathcal{A}$, write

(i) $x \preceq \bigvee_{i\geq 0}\{a_i\} \Leftrightarrow$ there exists $i \geq 0$ such that $x \preceq a_i$.

(ii) $\bigwedge_{i\geq 0}\{a_i x\} \preceq \bigvee_{j\geq 0}\{b_j y\} \Leftrightarrow$ for any $i \geq 0$ there exists $j \geq 0$ such that $a_i x \preceq b_j y$.

(iii) $\bigwedge_{i\geq 0}\{a_i\} \preceq x \Leftrightarrow$ for any $i \geq 0$, we have $a_i \preceq x$.
(iv) $k\bigvee_{i\geq 0}\{a_i\} = \bigvee_{i\geq 0}\{ka_i\}$.

Lemma 4.1. *Let $(X, \mathcal{D})$ be a generalized cone metric space over the Banach algebra $\mathcal{A}$, P be a solid cone and the following conditions hold*:
(i) *$(X, \mathcal{D})$ is $\mathcal{D}$-complete*;
(ii) *suppose that x is invertible and $x^{-1} \succ \theta$ implies $x \succ \theta$*;
(iii) *suppose the mapping $f : X \to X$ satisfies*

$$\mathcal{D}(f(x), f(y)) \preceq k\bigvee\{\mathcal{D}(x, y), \mathcal{D}(x, f(x)), \mathcal{D}(y, f(y)), \mathcal{D}(y, f(x)), \mathcal{D}(x, f(y))\} \tag{4.4}$$

for any $x, y \in X$ and an invertible $k \in P$ with $0 < r(k) < 1$.
Then for any positive integer n and $x_0 \in X$, the following assertions hold:
(i) $\bigwedge_{1\leq i,j\leq n}\{\mathcal{D}(f^i(x_0), f^j(x_0))\} \preceq k\bigvee_{0\leq i,j\leq n}\{\mathcal{D}(f^i(x_0), f^j(x_0))\}$;
(ii) $\bigwedge_{1\leq i,j\leq n}\{\mathcal{D}(f^i(x_0), f^j(x_0))\} \preceq k\bigvee_{0\leq j\leq n}\{\mathcal{D}(x_0, f^j(x_0))\}$.

Proof. According to the formula (4.4), for all i, j, $1 \leq i, j \leq n$, we have

$$\begin{aligned}
&\mathcal{D}(f^i(x_0), f^j(x_0)) = \mathcal{D}(ff^{i-1}(x_0), ff^{j-1}(x_0))\\
&\preceq k\bigvee\{\mathcal{D}(f^{i-1}(x_0), f^{j-1}(x_0)), \mathcal{D}(f^{i-1}(x_0), f^i(x_0)), \mathcal{D}(f^{i-1}(x_0), f^j(x_0)),\\
&\quad \mathcal{D}(f^{j-1}(x_0), f^j(x_0)), \mathcal{D}(f^{j-1}(x_0), f^i(x_0))\}\\
&\preceq k\bigvee_{0\leq i,j\leq n}\{\mathcal{D}(f^i(x_0), f^j(x_0))\}.
\end{aligned}$$

So (i) is true.

From (i) we know that

$$\begin{aligned}
&\bigwedge_{1\leq i,j\leq n}\{\mathcal{D}(f^i(x_0), f^j(x_0))\}\\
&\preceq k\bigvee\{\bigvee_{1\leq i,j\leq n}\{\mathcal{D}(f^i(x_0), f^j(x_0))\}, \bigvee_{0\leq j\leq n}\{\mathcal{D}(x_0, f^j(x_0))\}\}.
\end{aligned}$$

Without loss of generality, we assume that there exist $1 \leq i_0, j_0 \leq n$ such that $\mathcal{D}(f^{i_0}(x_0), f^{j_0}(x_0)) \succ \theta$. Otherwise (ii) is true.

Now we claim that

$$\bigwedge_{1\leq i,j\leq n}\{\mathcal{D}(f^i(x_0), f^j(x_0))\} \preceq k\bigvee_{0\leq j\leq n}\{\mathcal{D}(x_0, f^j(x_0))\}.$$

Otherwise

$$\bigwedge_{1\leq i,j\leq n}\{\mathcal{D}(f^i(x_0), f^j(x_0))\} \preceq k\bigvee_{1\leq i,j\leq n}\{\mathcal{D}(f^i(x_0), f^j(x_0))\}.$$

Then for any $1 \leq i, j \leq n$, there exist $1 \leq i^{(1)}, j^{(1)} \leq n$, such that

$$\mathcal{D}(f^i(x_0), f^j(x_0)) \preceq k\mathcal{D}(f^{i^{(1)}}(x_0), f^{j^{(1)}}(x_0)).$$

Also, there exist $1 \leq i^{(2)}, j^{(2)} \leq n$ such that

$$\mathcal{D}(f^{i^{(1)}}(x_0), f^{j^{(1)}}(x_0)) \preceq k\mathcal{D}(f^{i^{(2)}}(x_0), f^{j^{(2)}}(x_0)).$$

As the procedure continues, we obtain:

$$\begin{aligned}\mathcal{D}(f^{i}(x_0), f^{j}(x_0)) &\preceq k\mathcal{D}(f^{i^{(1)}}(x_0), f^{j^{(1)}}(x_0)) \\ &\preceq k^2\mathcal{D}(f^{i^{(2)}}(x_0), f^{j^{(2)}}(x_0)) \\ &\vdots \\ &\preceq k^t\mathcal{D}(f^{i^{(t)}}(x_0), f^{j^{(t)}}(x_0)) \\ &\vdots\end{aligned}$$

Clearly there exists $0 \leq s < t$ satisfying $i^{(s)} = i^{(t)}$, $j^{(s)} = j^{(t)}$, where $i = i^{(0)}$, $j = j^{(0)}$, such that

$$k^s\mathcal{D}(f^{i^{(s)}}(x_0), f^{j^{(s)}}(x_0)) \preceq k^t\mathcal{D}(f^{i^{(t)}}(x_0), f^{j^{(t)}}(x_0)) = k^t\mathcal{D}(f^{i^{(s)}}(x_0), f^{j^{(s)}}(x_0)),$$

which implies

$$(k^s - k^t)\mathcal{D}(f^{i^{(s)}}(x_0), f^{j^{(s)}}(x_0)) \preceq \theta.$$

Since $k^s - k^t$ is invertible, we obtain that $\mathcal{D}(f^{i^{(s)}}(x_0), f^{j^{(s)}}(x_0)) = \theta$ by multiplying $(k^s - k^t)^{-1}$ in both sides. This means $\mathcal{D}(f^i(x_0), f^j(x_0)) = \theta$ for any $1 \leq i, j \leq n$, which contradicts the assumption in advance that there exist $1 \leq i_0, j_0 \leq n$ such that $\mathcal{D}(f^{i_0}(x_0), f^{j_0}(x_0)) \succ \theta$. This contradiction yields that the claim is true. So (ii) is true. $\square$

Next by virtue of Lemma 4.1, we present another fixed point result in which we use an ingenious idea from [10] to deal with the possible incomparability between some elements of cone metric spaces over Banach algebras.

Theorem 4.3. *Let $(X, \mathcal{D})$ be a generalized cone metric space over the Banach algebra $\mathcal{A}$. Assume that P is a solid cone and the following conditions hold:*

(i) *$(X, \mathcal{D})$ is $\mathcal{D}$-complete;*

(ii) *suppose that $x \in X$ is invertible and that $x^{-1} \succ \theta$ implies $x \succ \theta$;*

(iii) *suppose the mapping $f : X \to X$ satisfies*

$$\begin{aligned}&\mathcal{D}(f(x), f(y)) \\ &\preceq k\bigvee\{\mathcal{D}(x, y), \mathcal{D}(x, f(x)), \mathcal{D}(y, f(y)), \mathcal{D}(y, f(x)), \mathcal{D}(x, f(y))\}\end{aligned} \tag{4.5}$$

for any $x, y \in X$ and an invertible $k \in P$ with $0 < r(Ck) < 1$, where C is the vector given in Definition 3.2;

(iv) *there exists $x_0 \in X$ such that for any $m \in \mathbb{N}$, for any $i, j \in \{0, 1, 2, \dots, m\}$ $\|\mathcal{D}(f^i(x_0), f^j(x_0))\| < \infty$.*

Then $\{f^n(x_0)\}$ $\mathcal{D}$-converges to some ω if $\mathcal{D}(\omega, f(\omega)) < \infty$ and $\mathcal{D}(x_0, f(\omega)) < \infty$, then ω is a fixed point of f. Moreover, if $\omega' \in X$ is another fixed point of f such that $\mathcal{D}(\omega, \omega') < \infty$ and $\mathcal{D}(\omega', \omega') < \infty$, then $\omega = \omega'$.

Proof. Let $x_n = f^n(x_0)$, $i, j = 1, 2, \ldots$. For any $n, m \in \mathbb{N}$, $n < m$, it follows from Lemma 4.1 and (4.5) that

$$\begin{aligned}
\mathcal{D}(x_n, x_m)) &= \mathcal{D}(f(x_{n-1}), f^{m-n+1}(x_{n-1})) \\
&\preceq k \bigvee_{0\le i,j\le m-n+1} \{\mathcal{D}(f^i(x_{n-1}), f^j(x_{n-1}))\} \\
&= k \bigvee_{n-1\le i,j\le m} \{\mathcal{D}(f^i(x_0), f^j(x_0))\} \\
&\preceq k^2 \bigvee_{n-2\le i,j\le m} \{\mathcal{D}(f^i(x_0), f^j(x_0))\} \\
&\vdots \\
&\preceq k^n \bigvee_{0\le i,j\le m} \{\mathcal{D}(f^i(x_0), f^j(x_0))\}.
\end{aligned}$$

From Lemma 2.5 and the fact that $\|k^n \bigvee_{0\le i,j\le m}\{\mathcal{D}(f^i(x_0), f^j(x_0))\}\| \to 0$ $(n \to \infty)$ which is induced by Remark 2.1 and $\|\bigvee_{0\le i,j\le m}\{\mathcal{D}(f^i(x_0), f^j(x_0))\}\| < \infty$, it follows that for any $\varepsilon \in P$ with $\theta \ll \varepsilon$, there exists $N \in \mathbb{N}$ such that for any $m > n > N$,

$$\mathcal{D}(x_n, x_m)) \preceq k^n \bigvee_{0\le i,j\le m} \{\mathcal{D}(f^i(x_0), f^j(x_0))\} \ll \varepsilon,$$

which implies

$$\lim_{n,m\to\infty} \mathcal{D}(f^n(x_0), f^m(x_0)) = \lim_{n,m\to\infty} \mathcal{D}(x_n, x_m) = \theta.$$

This gives that $\{f^n(x_0)\}$ is a $\mathcal{D}$-Cauchy sequence in X. Since $(X, \mathcal{D})$ is $\mathcal{D}$-complete, $\{f^n(x_0)\}$ $\mathcal{D}$-converges to some $\omega \in X$. According to the property $(\mathcal{D}_3)$, for the above $\varepsilon \gg \theta$, there exist a constant vector $C > \theta$ and $N_0 \in \mathbb{N}$, such that for any $n > N_0$,

$$\mathcal{D}(\omega, f(\omega)) \preceq C(\varepsilon + \mathcal{D}(f^n(x_0), f(\omega))). \tag{4.6}$$

On the other hand, it derives from (4.5) that

$$\begin{aligned}
&\mathcal{D}(f^n(x_0), f(\omega)) \\
\preceq\ & k \bigvee \{\mathcal{D}(f^{n-1}(x_0), \omega), \mathcal{D}(f^{n-1}(x_0), f^n(x_0)), \mathcal{D}(\omega, f(\omega)), \\
&\mathcal{D}(f^{n-1}(x_0), f(\omega)), \mathcal{D}(\omega, f^n(x_0))\}.
\end{aligned}$$

As $\{f^n(x_0)\}$ $\mathcal{D}$-converges to ω, for the given $\varepsilon > \theta$, there exists $N_1 \in \mathbb{N}$, such that for any $n > N_1$,

$$\mathcal{D}(f^{n-1}(x_0), \omega) \ll \varepsilon,\ \mathcal{D}(f^{n-1}(x_0), f^n(x_0)) \ll \varepsilon \ \text{ and } \ \mathcal{D}(\omega, f^n(x_0)) \ll \varepsilon. \tag{4.7}$$

Now we consider the following cases.

(1) When

$$\mathcal{D}(f^n(x_0), f(\omega)) \preceq k \bigvee \{\mathcal{D}(f^{n-1}(x_0), \omega), \mathcal{D}(f^{n-1}(x_0) f^n(x_0)), \mathcal{D}(\omega, f^n(x_0))\}.$$

Then noting (4.6), for any $n > N = \max\{N_0, N_1\}$, we have

$$\mathcal{D}(\omega, f(\omega)) \preceq C(\varepsilon + \mathcal{D}(f^n(x_0), f(\omega))) \preceq C(\varepsilon + k\varepsilon),$$

which together with Proposition 2.1 implies that $\mathcal{D}(\omega, f(\omega)) = \theta$. So we have $\omega = f(\omega)$, that is, ω is a fixed point of f.

(2) When $\mathcal{D}(f^n(x_0), f(\omega)) \preceq k\mathcal{D}(\omega, f(\omega))$, by the above analysis and noting (4.6), we obtain that $\mathcal{D}(\omega, f(\omega)) \preceq C(\varepsilon + k\mathcal{D}(\omega, f(\omega))$, which implies that $(e - Ck)\mathcal{D}(\omega, f(\omega)) \preceq C\varepsilon$. Because of $0 < r(Ck) < 1$, $\mathcal{D}(\omega, f(\omega)) = \theta$. Then $\omega = f(\omega)$, that is ω is a fixed point of f.

(3) When $\mathcal{D}(f^n(x_0), f(\omega)) \preceq k\mathcal{D}(f^{n-1}(\omega), f(\omega))$, it follows formula (4.5) that

$$\begin{aligned}
\mathcal{D}(f^n(x_0), f(\omega)) &\preceq k\mathcal{D}(f^{n-1}(\omega), f(\omega)) \\
&\preceq kk \bigvee \{\mathcal{D}(f^{n-2}(x_0), \omega), \mathcal{D}(f^{n-2}(x_0), f^{n-1}(x_0)), \mathcal{D}(\omega, f(\omega)), \\
&\quad \mathcal{D}(f^{n-2}(x_0), f(\omega)), \mathcal{D}(\omega, f^{n-1}(x_0))\} \\
&= k^2 \bigvee \{\mathcal{D}(f^{n-2}(x_0), \omega), \mathcal{D}(f^{n-2}(x_0), f^{n-1}(x_0)), \mathcal{D}(\omega, f(\omega)), \\
&\quad \mathcal{D}(f^{n-2}(x_0), f(\omega)), \mathcal{D}(\omega, f^{n-1}(x_0))\}.
\end{aligned}$$

For the sake of description, we also consider the following cases.

(i) When

$$\begin{aligned}
&\mathcal{D}(f^{n-1}(x_0), f(\omega)) \\
&\preceq k \bigvee \{\mathcal{D}(f^{n-2}(x_0), \omega), \mathcal{D}(f^{n-2}(x_0) f^{n-1}(x_0)), \mathcal{D}(\omega, f^{n-1}(x_0)), \mathcal{D}(\omega, f(\omega))\}.
\end{aligned}$$

Then we can easily get $\omega = f(\omega)$. So ω is a fixed point of f.

(ii) When $\mathcal{D}(f^{n-1}(x_0), f(\omega)) \preceq k\mathcal{D}(f^{n-2}(x_0), f(\omega))$, then

$$\mathcal{D}(f^n(x_0), f(\omega)) \preceq k\mathcal{D}(f^{n-1}(x_0), f(\omega)) \preceq k^2\mathcal{D}(f^{n-2}(x_0), f(\omega)).$$

As the procedure continues, we have $\mathcal{D}(f^n(x_0), f(\omega)) \preceq k^n\mathcal{D}(x_0, f(\omega))$. By Lemma 2.5 and the fact that $\|k^n\mathcal{D}(x_0, f(\omega))\| \to 0$ $(n \to \infty)$ as $\|k^n\| \to 0 (n \to \infty)$ and $\|\mathcal{D}(x_0, f(\omega))\| < \infty$ by Remark 2.1, we get for any $\varepsilon \in \mathcal{A}$ with $\theta \ll \varepsilon$, there exists $N_2 \in \mathbb{N}$ such that for any $n > N_2$, we have

$$\mathcal{D}(f^n(x_0), f(\omega)) \preceq k^n\mathcal{D}(x_0, f(\omega)) \ll \varepsilon.$$

Then for the above ε, there exists $C > \theta$, such that for any $n > N = \max\{N_0, N_2\}$, $\mathcal{D}(\omega, f(\omega)) \preceq C(\varepsilon + \varepsilon)$, which gives $\mathcal{D}(\omega, f(\omega)) = \theta$. That is $\omega = f(\omega)$. So ω is a fixed point of f.

Suppose ω' is another fixed point of f, then

$$\begin{aligned}
\mathcal{D}(\omega', \omega) &= k\mathcal{D}(f(\omega'), f(\omega)) \\
&\preceq k \bigvee \{\mathcal{D}(\omega, \omega'), \mathcal{D}(\omega, f(\omega)), \mathcal{D}(\omega', f(\omega')),
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}(\omega, , f(\omega^{'})), \mathcal{D}(\omega^{'}, f(\omega^{'}))\} \\
&= k \bigvee \{\mathcal{D}(\omega, \omega^{'}), \mathcal{D}(\omega, \omega), \mathcal{D}(\omega^{'}, \omega^{'})\}.
\end{aligned}$$

Now we prove $\mathcal{D}(\omega, \omega) = \mathcal{D}(\omega^{'}, \omega^{'}) = \theta$. In fact,

$$\begin{aligned}
\mathcal{D}(\omega, \omega) &= k\mathcal{D}(f(\omega), f(\omega)) \\
&\preceq k \bigvee \{\mathcal{D}(\omega, \omega), \mathcal{D}(\omega, f(\omega)), \mathcal{D}(\omega, f(\omega)), \mathcal{D}(\omega, f(\omega)), \mathcal{D}(\omega, f(\omega))\} \\
&= k\mathcal{D}(\omega, \omega).
\end{aligned}$$

So we have $\mathcal{D}(\omega, \omega) = \theta$. By using the same proof as the above, we can easily get $\mathcal{D}(\omega^{'}, \omega^{'}) = \theta$. Thus $\mathcal{D}(\omega^{'}, \omega) = \theta$, that is, $\omega^{'} = \omega$. So the fixed point of f is unique. □

We conclude the paper with an example, which can illustrate the result of Corollary 4.2.

Example 4.1. Let $\mathcal{A} = \mathbb{R}^2$, for each $(x_1, x_2) \in \mathcal{A}$, the norm of (x_1, x_2) is defined by $\|(x_1, x_2)\| = |x_1| + |x_2|$. A multiplication is defined by $xy = (x_1, x_2)(y_1, y_2) = (x_1 y_1, x_1 y_2 + x_2 y_1)$ for any pair $(x_1, x_2), (y_1, y_2) \in \mathcal{A}$. Then $\mathcal{A}$ is a Banach algebra with unit $e = (1, 0)$. Put $P = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0\}$. Then P is a cone in $\mathcal{A}$.

Let $X = [0, +\infty) \times [0, +\infty)$ and a metric $\mathcal{D}$ be defined by

$$\mathcal{D}(x, y) = \mathcal{D}((x_1, x_2), (y_1, y_2)) = (\max\{x_1, y_1\}, \max\{x_2, y_2\}).$$

Then it is easy to verify that $(X, \mathcal{D})$ is a complete generalized cone metric space over the Banach algebra $\mathcal{A}$. Now define mapping $f : X \to X$ by

$$f(x_1, x_2) = \left(\lg(x_1 + 1),\ \arctan\left(\frac{|\sin x_2|}{x_2 + 1}\right)\right).$$

Then by a simple calculation, we have

$$\mathcal{D}(f(x_1, x_2), f(y_1, y_2)) \leq \left(\frac{1}{\ln 10}, 0\right) \mathcal{D}((x_1, x_2), (y_1, y_2)),$$

where the spectral radius $r((\frac{1}{\ln 10}, 0))$ of $(\frac{1}{\ln 10}, 0)$ satisfies $0 < r((\frac{1}{\ln 10}, 0)) < 1$. By Corollary 4.2, we can get f has a unique fixed point $(0, 0)$.

Remark 4.1. Just from the definition of the mapping f, it is easy to see that $(0, 0)$ is a fixed point of f, but it is difficult to conclude that $(0, 0)$ is the unique fixed point of f. However, by Corollary 4.2, we can easily claim that $(0, 0)$ is the unique fixed point of f. Thus Example 4.1 is valid and appropriate.

Conclusion. In this paper, to generalize the concept of cone metric spaces over Banach algebras, we introduce a new notion called generalized cone metric spaces over Banach algebras and consider some existence problems of fixed points for a kind of contractive mappings in such spaces. We obtain several results which generalize the corresponding results gotten in the generalized metric spaces without the assumption that the involved cone is normal. Finally, an example is presented to illustrate the validity of our results.

Acknowledgments. The work was supported by the Foundation of Department of Education of Jiangxi Province(No. GJJ1520008).

References

[1] M. Dordević, D. Dorić, Z. Kadelburg, S. Radenović, D. Spasić, *Fixed point results under c-distance in tvs-cone metric spaces*, Fixed Point Theory Appl. **2011** (2011), 29, 9 pp.
[2] L.-G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), no. 2, 1468–1476.
[3] M. Jleli and B. Samet, *A generalized metric space and related fixed point theorems*, Fixed Point Theory Appl. **2015** (2015), 61, 14 pp.
[4] Z. Kadelburg, M. Pavlović, and S. Radenović, *Common fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces*, Comput. Math. Appl. **59** (2010), no. 9, 3148–3159.
[5] Z. Kadelburg and S. Radenović, *Common fixed point results of Das-Naik and Geraghty types in ν-generalized metric spaces*, Sarajevo J. Math. **13(25)** (2017), no. 1, 93–103.
[6] H. Liu and S. Xu, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*, Fixed Point Theory Appl. **2013** (2013), 320, 10 pp.
[7] ———, *Fixed point theorems of quasicontractions on cone metric spaces with Banach algebras*, Abstr. Appl. Anal. **2013** (2013), Art. ID 187348, 5 pp.
[8] S. Radenović and B. E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, Comput. Math. Appl. **57** (2009), no. 10, 1701–1707.
[9] W. Rudin, *Functional Analysis*, 2nd Edition, McGraw-Hill, Inc., 1991.
[10] T. Wang, J. Yin, and Q. Yan, *Fixed point theorems on cone 2-metric spaces over Banach algebras and an application*, Fixed Point Theory Appl. **2015** (2015), 204, 13 pp.
[11] S. Xu and S. Radenović, *Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality*, Fixed Point Theory Appl. **2014** (2014), 102, 12 pp.

Qianqian Leng
Department of Mathematics
Nanchang University
Nanchang 330031, P. R. China
Email address: 13517914026@163.com

Jiandong Yin
Department of Mathematics
Nanchang University
Nanchang 330031, P. R. China
Email address: yjdaxf@163.com