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ON S-COHERENCE

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Abstract. Recently, Anderson and Dumitrescu's S-finiteness has attracted the interest of several authors. In this paper, we introduce the notions of S-finitely presented modules and then of S-coherent rings which are S-versions of finitely presented modules and coherent rings, respectively. Among other results, we give an S-version of the classical Chase's characterization of coherent rings. We end the paper with a brief discussion on other S-versions of finitely presented modules and coherent rings. We prove that these last S-versions can be characterized in terms of localization.

1. Introduction

Throughout this paper all rings are commutative with identity; in particular, R denotes such a ring, and all modules are unitary. S will be a multiplicative subset of R. We use $(I : a)$, for an ideal I and an element $a \in R$, to denote the quotient ideal $\{x \in R;\ xa \in I\}$.

According to [3], an R module M is called S-finite if there exists a finitely generated submodule N of M such that $sM \subseteq N$ for some $s \in S$. Also, from [3], an R-module M is called S-Noetherian if each submodule of M is S-finite. In particular, R is said to be an S-Noetherian ring, if it is S-Noetherian as an R-module; that is, every ideal of R is S-finite. It is clear that every Noetherian ring is S-Noetherian.

The notions of S-finite modules and of S-Noetherian rings were introduced by Anderson and Dumitrescu motivated by the works done in [8] and [4]. They succeeded to generalize several well-known results on Noetherian rings including the classical Cohen's result and Hilbert basis theorem under an additional condition. Since then S-finiteness has attracted the interest of several authors (see for instance [1, 2, 10–13]). Recently, motivated by the work of Anderson and Dumitrescu, S-versions of some classical notions have been introduced (see for instance [2,10]). In this paper we are interested in S-versions of finitely presented modules and coherent rings which are called, respectively, *S-finitely*

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presented modules and *S-coherent* rings (see Definitions 2.1 and 3.1). We prove that the S-coherent rings have a characterization similar to the classical one given by Chase for coherent rings [5, Theorem 2.2]. However as the notion of S-coherent rings defined here cannot be characterized by localization we introduce another notion that does have this characterization.

The organization of the paper is as follows: In Section 2, we introduce and study an S-version of finitely presented modules. We call it an *S-finitely presented module* (see Definition 2.1). Then, we study the behavior of S-finiteness in short exact sequences (see Theorem 2.4). We end Section 2 with some change of rings results (see Proposition 2.6 and Corollary 2.7). Section 3 is devoted to the S-version of coherent rings which are called *S-coherent rings* (see Definition 3.3). Our main result represents the S-counterpart of Chase's result [5, Theorem 2.2] (see Theorem 3.8). Also an S-version of coherent modules is introduced (see Definition 3.1 and Proposition 3.2). We end the paper with a short section which presents another S-version of S-finiteness (see Definitions 4.1 and 4.4). We prove that these notions can be characterized in terms of localization (see Proposition 4.3 and Theorem 4.7). We end the paper with results which relate S-finiteness with the notion of S-saturation (see Propositions 4.9 and 4.8 and Corollary 4.10).

2. S-finitely presented modules

In this section, we introduce and investigate an S-version of the classical finitely presented modules. Another version is discussed in Section 4.

Definition 2.1. An R-module M is called S-finitely presented, if there exists an exact sequence of R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$, where K is S-finite and F is a finitely generated free R-module.

Clearly, every finitely presented module is S-finitely presented. However, the converse does not hold in general. For that, it suffices to note that when R is a non-Noetherian S-Noetherian ring, then there is an S-finite ideal I which is not finitely generated. Then, the R-module R/I is S-finitely presented but it is not finitely presented.

Also, it is evident that every S-finitely presented module is finitely generated. To give an example of a finitely generated module which is not S-finitely presented, it suffices to consider an ideal I which is not S-finite and then use Proposition 2.3 given hereinafter.

Definition 2.1 does not assume that the free module is S-finite because the notions of finitely generated free and free and S-finite free modules coincide, as seen in the following proposition.

Proposition 2.2. *Every S-finite free R-module is finitely generated.*

Proof. Let $M = \bigoplus_{i\in I} Re_i$ be an S-finite free R-module, where $(e_i)_{i\in I}$ is a basis of M and I is an index set. Then, there exist a finitely generated R-module N and an $s \in S$ such that $sM \subseteq N \subseteq M$. Then, $N = Rm_1 + \cdots + Rm_n$ for some

$m_1, \ldots, m_n \in M$ ($n > 0$ is an integer). For every $k \in \{1, \ldots, n\}$, there exists a finite subset J_k of I such that $m_k = \sum_{j \in J_k} \lambda_{kj} e_j$. Let $J = \bigcup_{k=1}^{n} J_k$. Then, the finitely generated R-module $M' = \bigoplus_{j \in J} Re_j$ contains N. We show that $M' = M$ by contradiction. There exists an $i_0 \in I \backslash J$ such that $e_{i_0} \notin M'$. But $se_{i_0} \in N \subseteq M'$ and so $se_{i_0} = \sum_{j \in J} \lambda'_j e_j$ for some $\lambda'_j \in R$. This is impossible since $(e_i)_{i \in I}$ is a basis. □

Similarly to the proof of Proposition 2.2 above, one can prove that any S-finite torsion-free module cannot be decomposed into an infinite direct sum of non-zero modules. This shows that any S-finite projective module is countably generated by Kaplansky [9, Theorem 1]. Then, naturally one would ask of the existence of an S-finite projective module which is not finitely generated. For this, consider the Boolean ring $R = \prod_{i=1}^{\infty} k_i$, where k_i is the field of two elements for every $i \in \mathbb{N}$. Consider the projective ideal $M = \bigoplus_{i=1}^{\infty} k_i$, the direct sum of principal projective ideals, and consider the element $e = (1, 0, 0, \ldots)$ (see [6, Example 2.7]). Then, $S = \{1, e\}$ is a multiplicative subset of R. Since $eM = k_1$ is a finitely generated R-module, M is the desired example of S-finite projective module which is not finitely generated.

However, determining rings over which every S-finite projective module is finitely generated could be of interest. It is worth noting that rings over which every projective module is a direct sum of finitely generated modules satisfy this condition. These rings were investigated in [14].

The next result shows that, as in the classical case [7, Lemma 2.1.1], an S-finitely presented module does not depend on one specific short exact sequence of the form given in Definition 2.1.

Proposition 2.3. *An R-module M is S-finitely presented if and only of M is finitely generated and, for every surjective homomorphism of R-modules $F \xrightarrow{f} M \longrightarrow 0$, where F is a finitely generated free R-module,* $\ker f$ *is S-finite.*

Proof. ($\Leftarrow$) Obvious.

($\Rightarrow$) Since M is S-finitely presented, there exists an exact sequence of R-modules $0 \longrightarrow K \longrightarrow F' \longrightarrow M \longrightarrow 0$, where K is S-finite and F' is finitely generated and free. Then, by Schanuel's lemma, $K \oplus F \cong \ker f \oplus F'$, then $\ker f$ is S-finite. □

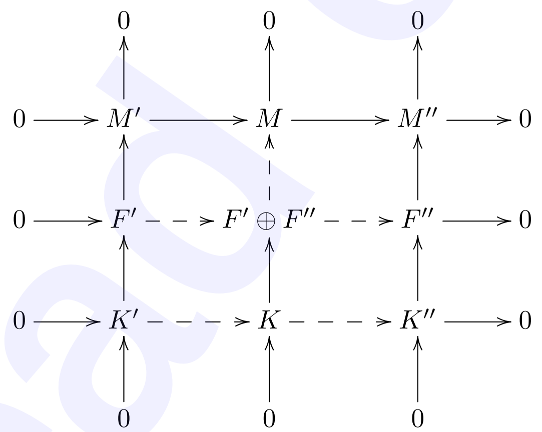
The following result represents the behavior of S-finiteness in short exact sequences. It is a generalization of [7, Theorem 2.1.2] for modules with λ-dimension at most 1. Note that one can give an S-version of the classical λ-dimension (see [7, page 32]). However, here we prefer to focus on the notion of S-finitely presented modules, and a discussion on the suitable S-version of the λ-dimension could be the subject of a further work.

Theorem 2.4. *Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence of R-modules. The following assertions hold:*

(1) *If M' and M'' are S-finite, then M is S-finite.*
In particular, every finite direct sum of S-finite modules is S-finite.
(2) *If M' and M'' are S-finitely presented, then M is S-finitely presented.*
In particular, every finite direct sum of S-finitely presented modules is S-finitely presented.
(3) *If M is S-finite, then M'' is S-finite.*
In particular, a direct summand of an S-finite module is S-finite.
(4) *If M' is S-finite and M is S-finitely presented, then M'' is S-finitely presented.*
(5) *If M'' is S-finitely presented and M is S-finite, then M' is S-finite.*

Proof. (1) Since M'' is S-finite, there exist a finitely generated submodule N'' of M'' and an $s \in S$ such that $sM'' \subseteq N''$. Let $N'' = \sum_{i=1}^{n} Re_i$ for some $e_i \in M''$ and $n \in \mathbb{N}$. Since g is surjective, there exists an $m_i \in M$ such that $g(m_i) = e_i$ for every $i \in \{1, \dots, n\}$. Let $x \in M$, so $sx \in N = g^{-1}(N'')$. Then $g(sx) \in g(N) = N''$, and so $g(sx) = \sum_{i=1}^{n} \alpha_i e_i = \sum_{i=1}^{n} \alpha_i g(m_i) = g(\sum_{i=1}^{n} \alpha_i m_i)$. Then, $g(sx - \sum_{i=1}^{n} \alpha_i m_i) = 0$. Thus, $(sx - \sum_{i=1}^{n} \alpha_i m_i) \in \ker g = \mathrm{Im} f$ which is S-finite. So there exist a finitely generated submodule N' of $\mathrm{Im} f$ and an $s' \in S$ such that $s'\mathrm{Im} f \subseteq N'$. Then, $s'sx \in N' + \sum_{i=1}^{n} Rm_i$ and so $s'sM$ is a submodule of $N' + \sum_{i=1}^{n} Rm_i$ which is a finitely generated submodule of M. Therefore, M is S-finite.

(2) Since M' and M'' are S-finitely presented, there exist two short exact sequences: $0 \longrightarrow K' \longrightarrow F' \longrightarrow M' \longrightarrow 0$ and $0 \longrightarrow K'' \longrightarrow F'' \longrightarrow M'' \longrightarrow 0$, with K' and K'' are S-finite R-modules and F' and F'' are finitely generated free R-modules. Then, by the Horseshoe Lemma, we get the following diagram:

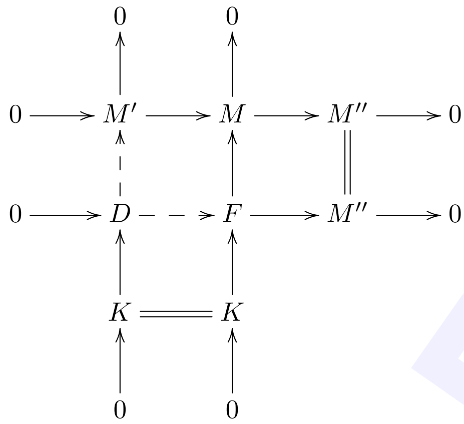


By the first assertion, K is S-finite. Therefore, M is S-finitely presented.

(3) Obvious.

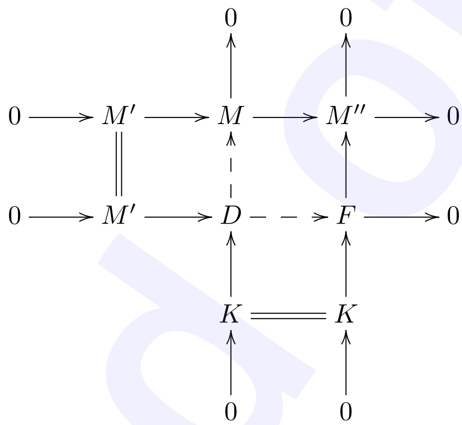
(4) Since M is S-finitely presented, there exists a short exact sequence of R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$, where K is S-finite and F is a finitely

generated free R-module. Consider the following pullback diagram:



By (1), D is S-finite. Therefore, M'' is S-finitely presented.

(5) Since M'' is S-finitely presented, there exists a short exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M'' \longrightarrow 0$ where K is S-finite and F is a finitely generated free R-module. Consider the following pullback diagram:



Since F is free, $D \cong M' \oplus F$, and so D is S-finite (since M' and F are S-finite). Therefore, M' is S-finite. □

As a simple consequence, we get the following result which extends [7, Corollary 2.1.3].

Corollary 2.5. *Let N_1 and N_2 be two S-finitely presented submodules of an R-module. Then, $N_1 + N_2$ is S-finitely presented if only if $N_1 \cap N_2$ is S-finite.*

Proof. Use the short exact sequence of R-modules $0 \longrightarrow N_1 \cap N_2 \longrightarrow N_1 \oplus N_2 \longrightarrow N_1 + N_2 \longrightarrow 0$. □

We end this section with the following change of rings results.

The following result extends [7, Theorem 2.1.7].

Proposition 2.6. *Let A and B be rings, let $\phi : A \longrightarrow B$ be a ring homomorphism making B a finitely generated A-module and let V be a multiplicative subset of A such that $0 \notin \phi(V)$. Every B-module which is V-finitely presented as an A-module it is $\phi(V)$-finitely presented as a B-module.*

Proof. Let M be a B-module which is V-finitely presented as an A-module. Then M is a finitely generated A-module, so M is a finitely generated B-module. Thus there is an exact sequence of B-modules $0 \longrightarrow K \longrightarrow B^n \longrightarrow M \longrightarrow 0$, where $n > 0$ is an integer. This sequence is also an exact sequence of A-modules. Since M is an V-finitely presented A-module and B^n is a finitely generated A-module (since B is a finitely generated A-module), K is a V-finite A-module (cf. Part 5 of Theorem 2.4), and so K is a $\phi(V)$-finite B-module. Therefore, M is a $\phi(V)$-finitely presented B-module. □

The following result extends [7, Theorem 2.1.8(2)].

Proposition 2.7. *Let I be an ideal of R and let M be an R/I-module. Assume that $I \cap S = \emptyset$ so that $T := \{s + I \in R/I;\ s \in S\}$ is a multiplicative subset of R/I. Then,*

(1) *M is an S-finite R-module if and only if M is a T-finite R/I-module.*
(2) *If M is an S-finitely presented R-module, then M is a T-finitely presented R/I-module. The converse holds when I is an S-finite ideal of R.*

Proof. (1) Easy.

(2) Use the canonical ring surjection $R \longrightarrow R/I$ and Proposition 2.6.

Conversely, if M is a T-finitely presented R/I-module. Then, there is an exact sequence of R/I-modules, and then of R-modules

$$0 \longrightarrow K \longrightarrow (R/I)^n \longrightarrow M \longrightarrow 0,$$

where $n > 0$ is an integer and K is a T-finite R/I-module. By the first assertion, K is also an S-finite R-module. And since I is an S-finite ideal of R, $(R/I)^n$ is an S-finitely presented R-module. Therefore, by Theorem 2.4(4), M is an S-finitely presented R-module. □

3. S-coherent rings

Before giving the definition of S-coherent rings, we give, following the classical case, the definition of S-coherent modules.

Definition 3.1. An R-module M is said to be S-coherent, if it is finitely generated and every finitely generated submodule of M is S-finitely presented.

Clearly, every coherent module is S-coherent.

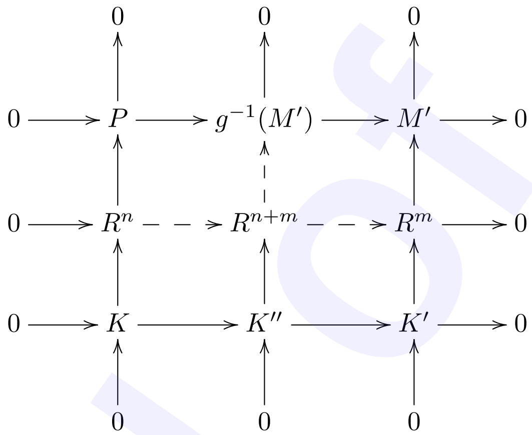
The reason why we consider finitely generated submodules rather than S-finite submodules is explained in Remark 3.4(4).

The following result studies the behavior of S-coherence of modules in short exact sequences. It generalizes [7, Theorem 2.2.1].

Proposition 3.2. *Let* $0 \longrightarrow P \xrightarrow{f} N \xrightarrow{g} M \longrightarrow 0$ *be an exact sequence of* R*-modules. The following assertions hold*:

1. *If* P *is finitely generated and* N *is* S*-coherent, then* M *is* S*-coherent.*
2. *If* M *is coherent and* P *is* S*-coherent, then is* N S*-coherent.*
3. *If* N *is* S*-coherent and* P *is finitely generated, then* P *is* S*-coherent.*

Proof. (1) It is clear that M is finitely generated. Let M' be a finitely generated submodule of M. There exist two short exact sequences of R-modules $0 \longrightarrow K \longrightarrow R^n \longrightarrow P \longrightarrow 0$ and $0 \longrightarrow K' \longrightarrow R^m \longrightarrow M' \longrightarrow 0$, where n and m are two positive integers. Then, by the Horseshoe Lemma, we get the following diagram:



Since $g^{-1}(M')$ is a finitely generated submodule of the S-coherent module N, $g^{-1}(M')$ is S-finitely presented. Then, using Theorem 2.4(5), K'' is S-finite, and so K' is S-finite. Therefore, M' is S-finitely presented.

(2) Clearly N is finitely generated. Let N' be a finitely generated submodule of N. Consider the exact sequence $0 \longrightarrow \mathrm{Ker}(g_{/N'}) \xrightarrow{f} N' \xrightarrow{g} g(N') \longrightarrow 0$. Then, $g(N')$ is a finitely generated submodule of the coherent module M. Then, $g(N')$ is finitely presented. Then, $\mathrm{Ker}(g_{/N'})$ is finitely generated, and since P is S-coherent, $\mathrm{Ker}(g_{/N'})$ is S-finitely presented. Therefore, by (2) of Theorem 2.4, N' is S-finitely presented.

(3) Evident since a submodule of P can be seen as a submodule of N. □

The following questions raise naturally: Let $0 \longrightarrow P \xrightarrow{f} N \xrightarrow{g} M \longrightarrow 0$ be an exact sequence of R-modules. When are the following assertions true?

(1) If P is S-finitely generated and N is S-coherent, then M is S-coherent.

(2) M and P are S-coherent, then N is S-coherent.
(3) Every finite direct sum of S-coherent modules is S-coherent.

Now we set the definition of an S-coherent ring.

Definition 3.3. A ring R is called S-coherent, if it is S-coherent as an R-module; that is, if every finitely generated ideal of R is S-finitely presented.

Remark 3.4.

(1) Note that every S-Noetherian ring is S-coherent. Indeed, this follows from the fact that when R is S-Noetherian, every finitely generated free R-module is S-Noetherian (see the discussion before [3, Lemma 3]). Next, in Example 3.6, we give an example of an S-coherent ring which is not S-Noetherian.
(2) Clearly, every coherent ring is S-coherent. The converse is not true in general. As an example of an S-coherent ring which is not coherent, we consider the trivial extension $A = \mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$ and the multiplicative set $V = \{(2,0)^n; n \in \mathbb{N}\}$. Since $(0 : (2,0)) = 0 \ltimes M$ is not finitely generated, A is not coherent. Now, for every ideal I of A, $(2,0)I$ is finitely generated; in fact, $(2,0)I = 2J \ltimes 0$, where $J = \{a \in \mathbb{Z}; \exists b \in (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}, (a,b) \in I\}$. Since J is an ideal of Z, $J = a\mathbb{Z}$ for some element $a \in \mathbb{Z}$. Then, $(2,0)I = 2J \ltimes 0 = (2a,0)A$. This shows that A is V-Noetherian and so V-coherent.
(3) It is easy to show that, if M is an S-finitely presented R-module, then M_S is a finitely presented R_S-module. Thus, if R is an S-coherent ring, R_S is a coherent ring. However, it seems not evident to give a condition so that the converse holds, as done for S-Noetherian rings (see [3, Proposition 2(f)]). In Section 4, we give another S-version of coherent rings which can be characterized in terms of localization.
(4) One would propose for an S-version of coherent rings, the following condition "S-C: every S-finite ideal of R is S-finitely presented". However, if R satisfies the condition S-C, then in particular, every S-finite ideal of R is finitely generated. So, every S-finite ideal of R is finitely presented; in particular, R is coherent. This means that the notion of rings with the condition S-C cannot be considered as an S-version of the classical coherence. Nevertheless, these rings could be of particular interest as a new class of rings between the class of coherent rings and the class of Noetherian rings.

    To give an example of a coherent ring which does not satisfy the condition S-C, one could consider the ring $B = \prod_{i=1}^{\infty} k_i$, where k_i is the field of two elements for every $i \in \mathbb{N}$, and the multiplicative subset $V = \{1, e\}$ of B, where $e = (1, 0, 0, \dots) \in B$. Indeed, the ideal $B = \bigoplus_{i=1}^{\infty} k_i$ is V-finite but not finitely generated.

    Also, note that the following condition "S-c: every S-finite ideal of R is finitely generated" could be of interest. Indeed, clearly one can show the following equivalences:

(a) A ring R satisfies the condition S-C if and only if R is coherent and satisfies the condition S-c.
(b) A ring R is coherent if and only if R is S-coherent and satisfies the condition S-c.
(c) A ring R is Noetherian if and only if R is S-Noetherian and satisfies the condition S-c.

To give an example of an S-coherent ring which is not S-Noetherian, we use the following result.

Proposition 3.5. *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings R_i ($n \in \mathbb{N}$) and $S = \prod_{i=1}^{n} S_i$ be a cartesian product of multiplicative sets S_i of R_i. Then, R is S-coherent if and only if R_i is S_i-coherent for every $i \in \{1, \ldots, n\}$.*

Proof. The result is proved using standard arguments. □

Example 3.6. Consider the ring A given in Remark 3.4(2). Let B be a coherent ring which has a multiplicative set W such that B_W is not Noetherian. Then, $A \times B$ is $V \times W$-coherent (by Proposition 3.5), but it is not $V \times W$-Noetherian (by [3, Proposition 2(f)]).

Now, we give our main result. It is the S-counterpart of the classical Chase's result [5, Theorem 2.2]. As Theorem 3.8 mimics the proof of [7, Theorem 2.3.2], we use the following lemma.

Lemma 3.7 ([7, Lemma 2.3.1])**.** *Let R be a ring, let $I = (u_1, u_2, \ldots, u_r)$ be a finitely generated ideal of R ($r \in \mathbb{N}$) and let $a \in R$. Set $J = I + Ra$. Let F be a free module on generators $x_1, x_2, \ldots, x_{r+1}$ and let $0 \longrightarrow K \longrightarrow F \xrightarrow{f} J \longrightarrow 0$, be an exact sequence with $f(x_i) = u_i$ ($1 \leq i \leq r$) and $f(x_{r+1}) = a$. Then there exists an exact sequence $0 \longrightarrow K \cap F' \longrightarrow K \xrightarrow{g} (I : a) \longrightarrow 0$, where $F' = \bigoplus_{i=1}^{r} Rx_i$.*

Theorem 3.8. *The following assertions are equivalent:*
(1) *R is S-coherent.*
(2) *$(I : a)$ is an S-finite ideal of R for every finitely generated ideal I of R and $a \in R$.*
(3) *$(0 : a)$ is an S-finite ideal of R for every $a \in R$ and the intersection of two finitely generated ideals of R is an S-finite ideal of R.*

Proof. The proof is similar to that of [5, Theorem 2.2] (see also [7, Theorem 2.3.2]). However, for the sake of completeness we give its proof here.

(1)⇒(2) Let I be a finitely generated ideal of R. Then, I is S-finitely presented. Consider $J = I + Ra$, where $a \in R$. Then, J is finitely generated, and so it is S-finitely presented. Thus, there exists an exact sequence $0 \longrightarrow K \longrightarrow R^{n+1} \longrightarrow J \longrightarrow 0$, where K is S-finite. By Lemma 3.7, there exists a surjective homomorphism $g : K \longrightarrow (I : a)$ which shows that $(I : a)$ is S-finite.

(2)⇒(1) This is proved by induction on n, the number of generators of a finitely generated ideal I of R. For $n = 1$, use assertion (2) and the exact sequence $0 \longrightarrow (0:I) \longrightarrow R \longrightarrow I \longrightarrow 0$. For $n > 1$, use assertion (2) and Lemma 3.7.

(1)⇒(3) Since R is S-coherent, Proposition 2.3 applied on the exact sequence $0 \longrightarrow (0:a) \longrightarrow R \longrightarrow aR \longrightarrow 0$ shows that the ideal $(0:a)$ is S-finite. Now, let I and J be two finitely generated ideals of R. Then, $I+J$ is finitely generated and so S-finitely presented. Then, applying Theorem 2.4(5) on the short the exact sequence $0 \longrightarrow I \cap J \longrightarrow I \oplus J \longrightarrow I+J \longrightarrow 0$, we get that $I \cap J$ is S-finite.

(3)⇒(1) This is proved by induction on the number of generators of a finitely generated ideal I of R, using the two short exact sequences used in (1) ⇒ (3). □

It is worth noting that, in Chase's paper [5], coherent rings were characterized using the notion of flat modules. Then, naturally one can ask of an S-version of flatness that characterizes S-coherent rings similarly to the classical case. We leave it as an interesting open question.

Also, one could ask, as done in the classical case, when does the condition "R is S-coherent" implies (and then equivalent to) the condition "every finitely presented R-module is S-coherent". It is clear that this hold true if R satisfies the condition "R^n is an S-coherent R-module for every positive integer n". However, in general, the equivalent deserves investigating.

We end this section with some change of rings results.

The following results extends [7, Theorem 2.4.1].

Proposition 3.9. *Let I be an S-finite ideal of R. Assume that $I \cap S = \emptyset$ so that $T := \{s + I \in R/I;\ s \in S\}$ is a multiplicative subset of R/I. Then, an R/I-module M is T-coherent if and only if it is an R-module S-coherent. In particular, the following assertions hold:*

1. *If R is an S-coherent ring, then R/I is a T-coherent ring.*
2. *If R/I is a T-coherent ring and I is an S-coherent R-module, then R is an S-coherent ring.*

Proof. Use Proposition 2.7. □

Next result generalizes [7, Theorem 2.4.2]. It studies the transfer of S-coherence under localizations.

Lemma 3.10. *Let $f : A \to B$ be a ring homomorphism such that B is a flat A-module, and let V be a multiplicative set of A. If an A-module M is V-finite (resp., a V-finitely presented), then $M \otimes_A B$ is an $f(V)$-finite (resp., $f(V)$-finitely presented) B-module.*

Proof. Follows using the fact that flatness preserves injectivity. □

Proposition 3.11. *If R is S-coherent, then R_T is an S_T-coherent ring for every multiplicative set T of R.*

Proof. Let J be a finitely generated ideal of R_T. Then, there is a finitely generated ideal I of R such that $J = I_T$. Since R is S-coherent, I is S-finitely presented. Then, using Lemma 3.10, the ideal $J = I \otimes_R R_T$ of R_T is S_T-finitely presented, as desired. □

4. Another S-version of finiteness

In this short section, we present another S-version of S-finiteness and we prove that this notion can be characterized in terms of localization.

The following definition gives another S-version of finitely presented modules.

Definition 4.1. An R module M is called c-S-finitely presented, if there exists a finitely presented submodule N of M such that $sM \subseteq N \subseteq M$ for some $s \in S$.

Remark 4.2.

(1) Clearly, every finitely presented module is c-S-finitely presented. However, the converse does not hold in general. For that it suffices to consider a coherent ring which has an S-finite module which is not finitely generated. An example of a such ring is given in Remark 3.4 (4).

(2) The inclusions in Definition 4.1 complicate the study of the behavior of c-S-finitely presented modules in short exact sequences as done in Theorem 2.4. This is why we think that c-S-finitely presented modules will be mostly used by commutative rings theorists rather than researchers interested in notions of homological algebra. This is the reason behind the use of the letter "c" in "c-S-finitely presented".

(3) It seems that there is not any relation between the two notions of c-S-finitely presented and S-finitely presented modules. Nevertheless, we can deduce that in a c-S-coherent ring (defined below), every S-finitely presented ideal is c-S-finitely presented.

It is well-known that if, for an R-module M, M_S is a finitely presented R_S-module, then there is a finitely presented R-module N such that $M_S = N_S$. This result doesn't generalize to S-finitely presented modules because the module N which satisfies $M_S = N_S$ is not necessarily a submodule of M. For c-S-finitely presented modules we give the following result.

Proposition 4.3.

(1) *If an R-module M is c-S-finitely presented, then M_S is a finitely presented R_S-module.*

(2) *A finitely generated R-module M is c-S-finitely presented if and only if there is a finitely presented submodule N of M such that $M_S = N_S$.*

Proof. (1) Obvious.

(2) ($\Rightarrow$) Clear.

(⇐) Since M is finitely generated and $M_S = N_S$, there is an $s \in S$ such that $sM \subseteq N$, as desired. □

Now we define the other S-version of the classical coherence of rings.

Definition 4.4. A ring R is called c-S-coherent, if every S-finite ideal of R is S-finitely presented.

Clearly, every coherent ring is c-S-coherent. The converse is not true in general. The ring given in Remark 3.4(2) can be used as an example of a c-S-coherent ring which is not coherent.

Also, it is evident that every S-Noetherian ring is c-S-coherent. As done in Example 3.6, we use the following result to give an example of a c-S-coherent ring which is not S-Noetherian.

Proposition 4.5. *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings R_i ($n \in \mathbb{N}$) and $S = \prod_{i=1}^{n} S_i$ be a cartesian product of multiplicative sets S_i of R_i. Then, R is c-S-coherent if and only if R_i is c-S_i-coherent for every $i \in \{1, \dots, n\}$.*

Proof. The result is proved using standard arguments. □

Example 4.6. Consider the ring A given in Remark 3.4(2) (it is c-V-coherent but not coherent). Let B be a coherent ring which has a multiplicative set W such that B_W is not Noetherian. Then, $A \times B$ is c-$V \times W$-coherent (by Proposition 4.5), but it is not $V \times W$-Noetherian (by [3, Proposition 2(f)]).

The following result gives a characterization of c-S-coherent rings in terms of localization.

Theorem 4.7. *The following assertions are equivalent:*

(1) *R is c-S-coherent.*
(2) *Every finitely generated ideal of R is c-S-finitely presented.*
(3) *For every finitely generated ideal I of R, there is a finitely presented ideal $J \subseteq I$ such that $I_S = J_S$. In particular, R_S is a coherent ring.*

Proof. (1)⇒(2)⇒(3) Straightforward.

(3)⇒(1) Let I be an S-finite ideal of R. Then, there exist an $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$. By assertion (3), there is a finitely presented ideal $K \subseteq J$ such that $K_S = J_S$. Then, there is a $t \in S$ such that $tJ \subseteq K$. Therefore, $tsI \subseteq K \subseteq I$, as desired. □

We end the paper with a result which relates c-S-coherent rings with the notion of S-saturation.

In [3], the notion of S-saturation is used to characterize S-Noetherian rings. Assume that R is an integral domain. Let $Sat_S(I)$ denotes the S-saturation of an ideal I of R; that is, $Sat_S(I) := IR_S \cap R$. In [3, Proposition 2(b)], it is proved that if $Sat_S(I)$ is S-finite, then I is S-finite and $Sat_S(I) = (I : s)$ for some $s \in S$. This fact was used to prove that a ring R is S-Noetherian if and only if R_S is Noetherian and, for every finitely generated ideal of R, $Sat_S(I) = (I : s)$

for some $s \in S$ (see [3, Proposition 2(f)]). The following result shows that the implication of [3, Proposition 2(b)] is in fact an equivalence in more general context.

Consider $N \subseteq M$ an inclusion of R-modules. Let $f : M \to M_S$ be the canonical R-module homomorphism. Denote by $f(N)R_S$ the R_S-submodule of M_S generated by $f(N)$. We set $Sat_{S,M}(N) := f^{-1}(f(N)R_S)$ and $(N :_M s) := \{m \in M;\ sm \in N\}$.

Proposition 4.8. *Let N be an R-submodule of an R-module M. $Sat_{S,M}(N)$ is S-finite if and only if N is S-finite and $Sat_{S,M}(N) = (N :_M s)$ for some $s \in S$.*

Proof. ($\Rightarrow$) Set $K = Sat_{S,M}(N)$. Since K is S-finite, there exist an $s \in S$ and a finitely generated R-module J such that $sK \subseteq J \subseteq K$. Thus, $sN \subseteq sK \subseteq J$. We can write $J = Rx_1 + Rx_2 + \cdots + Rx_n$ for some $x_1, x_2, \ldots, x_n \in J$. For each x_i, there exists a $t_i \in S$ such that $t_i x_i \in N$. We set $t = \prod_{i=1}^{n} t_i$. Then, $tsN \subseteq tsK \subseteq tJ \subseteq N$. Then, N is S-finite. On the other hand, since $sK \subseteq tJ \subseteq N \subseteq K$, $K \subseteq (N :_M s)$. Conversely, let $x \in (N :_M s)$. Then, $sx \in N$, so $x \in K$, as desired.

($\Leftarrow$) Since N is S-finite, there exist a $t \in S$ and a finitely generated R-module J such that $tN \subseteq J \subseteq N$. On the other hand, since $K = (N : s)$ for some $s \in S$, $sK \subseteq N$. Consequently, $tsK \subseteq tN \subseteq J \subseteq N \subseteq K$. Therefore, K is S-finite. □

The following result is proved similarly to the proof of Proposition 4.8. However, to guarantee the preservation of finitely presented modules when multiplying by elements of S, we assume that S does not contain any zero-divisor of R.

Proposition 4.9. *Assume that every element of S is regular. Let N be an R-submodule of an R-module M. Then $Sat_{S,M}(N)$ is c-S-finitely presented if and only if N is c-S-finitely presented and $Sat_{S,M}(N) = (N :_M s)$ for some $s \in S$.*

Corollary 4.10. *Assume that every element of S is regular. The following assertions are equivalent:*

(1) *For every finitely generated ideal I of R, $Sat_S(I)$ is c-S-finitely presented.*
(2) *R is c-S-coherent and, for every finitely generated ideal I of R, $Sat_S(I)$ $= (I : s)$ for some $s \in S$.*

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