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SINGULARITY ESTIMATES FOR ELLIPTIC SYSTEMS OF m-LAPLACIANS

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Abstract. This paper is concerned about several quasilinear elliptic systems with m-Laplacians. According to the Liouville theorems of those systems on $\mathbb{R}^n$, we obtain the singularity estimates of the positive C^1-weak solutions on bounded or unbounded domain (but it is not $\mathbb{R}^n$) and their decay rates on the exterior domain when $|x| \to \infty$. The doubling lemma which is developed by Polacik-Quittner-Souplet plays a key role in this paper. In addition, the corresponding results of several special examples are presented.

1. Introduction

In 2002, Serrin and Zou studied the existence results on the following elliptic equation of m-Laplacian (cf. [22])

(1) $$-div(|\nabla u|^{m-2}\nabla u) = u^p, \quad u > 0 \text{ in } \mathbb{R}^n,$$

where $n > 1$, $p > 1$, and $m \in (1, n)$. They pointed out that the Sobolev exponent $m^* - 1$ is critical. Namely, (1) exists C^1-solution if and only if $p \geq m^* - 1$. Here $m^* = \frac{nm}{n-m}$. In addition, if u is a positive solution of (1) on $B_R(0)$ with $1 < p < \frac{nm}{n-m} - 1$, the following singularity estimate holds (cf. Theorem IV in [22])

(2) $$u(x) \leq C[dist(x, \partial B_R(0))]^{\frac{-m}{p-m+1}}.$$

When $m = 2$, (1) is reduced to the Lane-Emden equation. Such an equation represents many scientific phenomena in astrophysics and mathematical physics. In addition, it comes into play in the study of the conformal geometry and the classical inequalities. The corresponding singularity estimate rate was obtained by Dancer (cf. [5]).

Clearly, those singularity estimate rates can be viewed as boundary blowing-up results, which imply the Liouville theorems if we let $R \to \infty$ in (2).

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On the contrary, Polacik, Quittner and Souplet [18] proved that Liouville theorems also imply boundary blowing-up estimates (2). The following doubling lemma plays a key role.

Lemma 1.1. *Let (X, d) be a complete metric space and let $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Set $\Gamma = \Sigma \backslash D$. Finally let $M : D \to (0, \infty)$ be bounded on compact subsets of D and fix a real $k > 0$. If $y \in D$ is such that*

$$M(y)dist(y, \Gamma) > 2k, \tag{3}$$

then there exists $x \in D$ such that

$$M(x)dist(x, \Gamma) > 2k, \quad M(x) \geq M(y), \tag{4}$$

and

$$M(z) \leq 2M(x) \quad \text{for all } z \in D \cap \overline{B}_X(x, kM^{-1}(x)).$$

Remark 1.1. (a) If $\Gamma = \emptyset$, then $dist(x, \Gamma) := +\infty$.

(b) Take $X = \mathbb{R}^n$, Ω an open subset of $\mathbb{R}^n$, and put $D = \Omega$, $\Sigma = \overline{D}$, hence $\Gamma = \partial\Omega$. Then we have $\overline{B}_X(x, kM^{-1}(x)) \subset D$. Indeed, since D is open, (3) implies that $dist(x, D^c) = dist(x, \Gamma) > 2kM^{-1}(x)$.

This lemma was originally employed by Hu [8] to estimate blow-up rates of nonglobal solutions of parabolic problems. Indeed, it comes into play not only in establishing equivalence between the Liouville theorems and the rates of the singularity estimate, but also in the study of the Lane-Emden conjecture. Such an open problem is that the Lane-Emden system

$$\begin{cases} -\triangle u = v^q, \ u, v > 0 \text{ in } \mathbb{R}^n, \\ -\triangle v = u^p, \ p, q > 0, \end{cases} \tag{5}$$

has no classical solution as long as the subcritical condition $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}$ holds. It was solved for the solutions with radial structure by Mitidieri (cf. [16]). In 1996, Serrin and Zou [21] pointed out that Lane-Emden system (5) has no solution with polynomial growth when $n = 3$. Polacik, Quittner and Souplet [18] employed the doubling lemma to prove that nonexistence of bounded solutions implies estimates of boundary blowing-up rate

$$\begin{aligned} u(x) &\leq C[dist(x, \partial B_R(0))]^{-\frac{n(q+1)}{pq-1}}, \\ v(x) &\leq C[dist(x, \partial B_R(0))]^{-\frac{n(p+1)}{pq-1}}. \end{aligned} \tag{6}$$

Combining with the result of Serrin-Zou, and letting $R \to \infty$, one can see that the Lane-Emden conjecture is solved in case of $n = 3$. More results about the Lane-Emden conjecture can be found in [6], [23] and the references therein.

Motivated by the results above, we consider the system with m-Laplacian

$$\begin{cases} -\triangle_m u(x) = u^{p_1} v^{q_1}, \\ -\triangle_m v(x) = u^{p_2} v^{q_2}, \end{cases} \tag{7}$$

where $p_1, p_2, q_1, q_2 > 0$, $m > 1$, and $\triangle_m u := div(|\nabla u|^{m-2} \nabla u)$.

Remark 1.2. If it is not stated specially, the solution of (7) in this paper is the positive C^1-weak solution. The definition is analogous to that in [22]. Namely, we here do not consider the trivial and semi-trivial solutions of (7).

Write

$$\alpha = \frac{m(q_2 - q_1 - m + 1)}{(m-1-p_1)(m-1-q_2) - q_1 p_2},$$

$$\beta = \frac{m(p_1 - p_2 - m + 1)}{(m-1-p_1)(m-1-q_2) - q_1 p_2},$$

and we always assume in this paper that p_1, p_2, q_1, q_2 satisfy $\max\{\alpha, \beta\} > 0$.

The main result of this paper is the following theorem.

Theorem 1.2. *Assume that* (7) *does not admit any bounded entire positive solution in* $\mathbb{R}^n$*. Let* $\Omega \neq \mathbb{R}^n$ *be a domain of* $\mathbb{R}^n$*. There exists* $C = C(m, n, p_1, p_2, q_1, q_2) > 0$ (*independent of* Ω *and* (u, v)) *such that any solution* (u, v) *of* (7) *in* Ω *satisfies*

$$u(x) \leq C dist^{-\alpha}(x, \partial\Omega), \quad v(x) \leq C dist^{-\beta}(x, \partial\Omega), \quad x \in \Omega. \tag{8}$$

If Ω *is an exterior domain, i.e.,* $\Omega \supset \{x \in \mathbb{R}^n, \ |x| > R\}$ *for some* $R > 0$*, then it follows that*

$$u(x) \leq C|x|^{-\alpha}, \quad v(x) \leq C|x|^{-\beta}, \quad |x| \geq 2R. \tag{9}$$

Recall the Liouville theorem in [4]. Let $q_1 p_2 > (p_1 - m + 1)(q_2 - m + 1)$. If one of the following

$$\min\{p_1, q_2\} > m - 1, \quad \max\{p_1 + q_1, p_2 + q_2\} \leq \frac{n(m-1)}{n-m}, \tag{10}$$

$$\max\{p_1, q_2\} \leq m - 1, \quad \max\{\alpha, \beta\} > \frac{n-m}{m-1}, \tag{11}$$

$$\max\{p_1, q_2\} < m - 1, \quad \max\{\alpha, \beta\} \geq \frac{n-m}{m-1}, \tag{12}$$

holds, then (7) has no entire solution on $\mathbb{R}^n$.

By the nonexistence results, we have the corollary.

Corollary 1.3. *Let* $\Omega \neq \mathbb{R}^n$ *be a domain of* $\mathbb{R}^n$*, and assume* $q_1 p_2 > (p_1 - m + 1)(q_2 - m + 1)$*. If one of* (10)-(12) *holds, then there exists* $C = C(m, n, p_1, p_2, q_1, q_2) > 0$ (*independent of* Ω *and* (u, v)) *such that any solution* (u, v) *of* (7) *in* Ω *satisfies* (8)*. If* Ω *is an exterior domain, then* (9) *is true.*

A special example of (7) is the following system

$$\begin{cases} -\triangle_m u = v^q, \ u, v > 0 \text{ in } \mathbb{R}^n, \\ -\triangle_m v = u^p, \ p, q > 0, \ m > 1. \end{cases} \tag{13}$$

If $m=2$, (13) becomes (5). In the following, we consider quasilinear systems with more general right hand sides form than (13):

$$\begin{cases} -\triangle_m u(x) = f(v), \\ -\triangle_m v(x) = g(u), \end{cases} \tag{14}$$

where the functions $f:[0,\infty)\to R$ and $g:[0,\infty)\to R$ are continuous.

Theorem 1.4. *Let $p,q>0$ and $pq>(m-1)^2$. Assume that* (13) *does not admit any bounded entire solution in $\mathbb{R}^n$, and assume that*

$$\lim_{v\to\infty} v^{-q} f(v) = l_1 \in (0,\infty), \tag{15}$$

$$\lim_{u\to\infty} u^{-p} g(u) = l_2 \in (0,\infty). \tag{16}$$

Let Ω be an arbitrary domain of $\mathbb{R}^n$. Then there exists positive constant $C=C(m,n,f,g)>0$ (independent of Ω and (u,v)) such that for any solution (u,v) of (14) *in Ω, there holds*

$$u(x) \le C(1 + dist^{-\frac{m(m+q-1)}{pq-(m-1)^2}}(x,\partial\Omega)), \quad x\in\Omega, \tag{17}$$

$$v(x) \le C(1 + dist^{-\frac{m(m+p-1)}{pq-(m-1)^2}}(x,\partial\Omega)), \quad x\in\Omega. \tag{18}$$

2. Proofs of theorems

Proof of Theorem 1.2. Assume that the Theorem fails. Then, there exist sequences Ω_k, (u_k,v_k), $y_k\in\Omega_k$, such that (u_k,v_k) solves (7) on Ω_k. It is easy to see that

$$M_k := u_k^{1/\alpha} + v_k^{1/\beta}, \ k=1,2,\dots \tag{19}$$

satisfies

$$M_k(y_k) > 2k\, dist^{-1}(y_k,\partial\Omega_k). \tag{20}$$

By Lemma 1.1 and Remark 1.1(b), there exists $x_k\in\Omega_k$ such that

$$M_k(x_k) > 2k\, dist^{-1}(y_k,\partial\Omega_k) \tag{21}$$

and

$$M_k(z) \le 2M_k(x_k), \quad |z-x_k| \le kM^{-1}(x_k). \tag{22}$$

Now we rescale (u_k,v_k) by setting

$$\begin{cases} \lambda_k = M_k^{-1}(x_k), \\ \widetilde{u}_k(y) := \lambda_k^{\alpha} u_k(x_k+\lambda_k y), \quad \widetilde{v}_k(y) := \lambda_k^{\beta} v_k(x_k+\lambda_k y), \quad |y|\le k. \end{cases} \tag{23}$$

Since $(\alpha+1)(m-1)+1 = \alpha p_1 + \beta q_1$ and $(\beta+1)(m-1)+1 = \alpha p_2 + \beta q_2$, we see that $(\widetilde{u}_k,\widetilde{v}_k)$ still solves

$$\begin{cases} -\triangle_m \widetilde{u}_k(y) = \widetilde{u}_k^{p_1}(y)\widetilde{v}_k^{q_1}(y), \\ -\triangle_m \widetilde{v}_k(y) = \widetilde{u}_k^{p_2}(y)\widetilde{v}_k^{q_2}(y), \end{cases} \tag{24}$$

for $|y| \le k$. In addition,

$$[\widetilde{u}_k^{1/\alpha} + \widetilde{v}_k^{1/\beta}](0) = 1 \tag{25}$$

and

$$[\widetilde{u}_k^{1/\alpha} + \widetilde{v}_k^{1/\beta}](y) \le 2, \quad |y| \le k. \tag{26}$$

According to the estimate in [25], there exists $\gamma \in (0,1)$ such that $\widetilde{u}_k, \widetilde{v}_k$ are bounded in $C^{1+\gamma}_{loc}(\mathbb{R}^n)$. Thus, we can find some subsequence of $(\widetilde{u}_k, \widetilde{v}_k)$ denoted by itself converging in $C^1_{loc}(\mathbb{R}^n)$ to a pair of solutions $(\widetilde{u}, \widetilde{v})$ of (7) on $\mathbb{R}^n$. Moreover $[\widetilde{u}^{1/\alpha} + \widetilde{v}^{1/\beta}](0) = 1$ by (25), hence $(\widetilde{u}, \widetilde{v})$ is positive, and moreover, $\widetilde{u}, \widetilde{v}$ are bounded due to (26). This contradicts the assumption of Theorem 1.2. □

Proof of Theorem 1.4. Assume that the Theorem fails. Keeping the same notations as in the proof of Theorem 1.2, we have sequences Ω_k, (u_k, v_k), $y_k \in \Omega_k$, such that (u_k, v_k) solves (14) on Ω_k. In addition,

$$M_k(y_k) > 2k(1 + dist^{-1}(y_k, \partial\Omega_k)) > 2k\ dist^{-1}(y_k, \partial\Omega_k). \tag{27}$$

Then, formulas (19)–(26) are unchanged except that (24) is replaced by

$$\begin{cases} -\triangle_m u_k(y) = f_k(v_k(y)) := \lambda_k^{(\alpha+1)(m-1)+1} f(\lambda_k^{-\alpha} v_k(y)), \\ -\triangle_m v_k(y) = g_k(u_k(y)) := \lambda_k^{(\beta+1)(m-1)+1} g(\lambda_k^{-\beta} u_k(y)). \end{cases} \tag{28}$$

Here $|y| \le k$, $\alpha = \frac{m(m+q-1)}{pq-(m-1)^2}$ and $\beta = \frac{m(m+p-1)}{pq-(m-1)^2}$. In view of $M_k(x_k) \ge M_k(y_k) > 2k$, we also have

$$\lambda_k \to 0, \quad k \to \infty. \tag{29}$$

Clearly, for $s \ge 0$,

$$\begin{cases} -C_1 \le f(s) \le C_1(1 + s^q), \\ -C_2 \le g(s) \le C_2(1 + s^p). \end{cases} \tag{30}$$

Therefore, by using (16) and (17), and noticing the continuity of f, g, we know that for $|y| \le k$, $k = 1, 2, \ldots$,

$$\begin{cases} -C_1 \lambda_k^{(\alpha+1)(m-1)+1} \le f_k(v_k(y)) \le C_1', \\ -C_2 \lambda_k^{(\beta+1)(m-1)+1} \le g_k(u_k(y)) \le C_2'. \end{cases} \tag{31}$$

According to the estimate in [25], there exists $\gamma \in (0,1)$, such that u_k, v_k are bounded in $C^{1+\gamma}_{loc}(R^n)$. Then there exists some subsequence of (u_k, v_k) denoted by itself converging in $C^1_{loc}(R^n)$ to a pair of positive functions (u, v) which satisfies

$$\begin{cases} -\triangle_m u \ge 0, \\ -\triangle_m v \ge 0, \end{cases} \tag{32}$$

on $\mathbb{R}^n$. Moreover $[u^{1/\alpha} + v^{1/\beta}](0) = 1$ by (25). Therefore, (u, v) is nontrivial, hence $u(y), v(y) > 0$, $y \in \mathbb{R}^n$ by the strong maximum principle for the single inequality of m-Laplacian (see Lemma 2.1 in [22]). Using assumption (15) and (16) again, we deduce that for each $y \in \mathbb{R}^n$,

$$\begin{cases} f_k(v_k(y)) \to l_1 v^q(y), \\ g_k(u_k(y)) \to l_2 u^p(y), \end{cases} \tag{33}$$

as $k \to \infty$. Consequently, (u, v) is a solution of

$$\begin{cases} -\triangle_m u = l_1 v^q, \\ -\triangle_m v = l_2 u^p, \end{cases} \tag{34}$$

with $y \in \mathbb{R}^n$. Clearly, there exist $C_1, C_2 > 0$ such that $(U, V) := (C_1 u, C_2 v)$ is a pair of entire solution of (13). Furthermore, U and V are bounded due to (26). This contradicts the assumption of Theorem 1.4. □

3. Several special cases

According to Theorem 1.2, to obtain the singularity estimates, we should be concerned with the conditions for the nonexistence of (7). Clearly, the critical conditions of Sobolev type and Serrin type play the important role. In this section, we mainly consider several special cases.

Case I: $m = 2$.

Now, (7) becomes

$$\begin{cases} -\triangle u(x) = u^{p_1} v^{q_1}, \\ -\triangle v(x) = u^{p_2} v^{q_2}. \end{cases} \tag{35}$$

This system appears in the study of the two coupled Schrödinger equations (cf. [13], [14]). The reference [20] shows that the nonexistence of positive solutions on the bounded domain. The reference [15] shows the radial symmetry of entire solutions on $\mathbb{R}^n$.

Recall a Liouville theorem related to the Serrin exponent in [1]. Write

$$\alpha = \frac{2(q_2 - q_1 - 1)}{(1 - p_1)(1 - q_2) - q_1 p_2},$$
$$\beta = \frac{2(p_1 - p_2 - 1)}{(1 - p_1)(1 - q_2) - q_1 p_2}.$$

If (35) has positive entire solutions on $\mathbb{R}^n$, then

$$\max\{p_1 + q_1, p_2 + q_2\} > n/(n-2), \quad \text{if } p_1, q_2 > 1; \tag{36}$$

$$\max(\alpha, \beta) \le n - 2, (\alpha, \beta) \ne (n-2, n-2), \quad \text{if } p_1, q_2 \le 1; \tag{37}$$

$$p_1 + q_1 \ge n/(n-2), \alpha \le n - 2, \quad \text{if } q_2 \le 1 < p_1; \tag{38}$$

$$p_2 + q_2 \ge n/(n-2), \beta \le n - 2, \quad \text{if } p_1 \le 1 < q_2, \tag{39}$$

and one inequality is strict in (38) and (39).

Moreover, we assume $p_1 + q_1 = p_2 + q_2 > 1$. Quittner and Souplet [19] proved the nonexistence of the entire solutions on $\mathbb{R}^n$ when $p_1 + q_1$ is not larger than the Serrin exponent $\frac{n}{n-2}$. The Sobolev exponent $2^* - 1$ may be not critical for the existence of positive entire solutions of (35) on exterior domain. In fact, [2] shows that there exists a pair of solutions of

$$\begin{cases} -\triangle u(x) = u^p v^{q+1}, \\ -\triangle v(x) = u^{p+1} v^q, \end{cases} \tag{40}$$

in the case of subcritical, which one converges to a positive constant and the other decays to zero when $|x| \to \infty$. Recently, Li and Lei [12] proved that (40) has no positive classical solution when $p + q + 1 < 2^* - 1$. Meanwhile, the singularity estimate of the positive solution (u, v) of (40) on the domain $\Omega(\neq \mathbb{R}^n)$

$$u(x), v(x) \leq C dist^{-\frac{2}{p+q}}(x, \partial\Omega), \quad \forall\ x \in \Omega,$$

was obtained by the doubling lemma.

In the critical case $p + q = 2^* - 1$, Li and Ma [11] classified the positive integrable solutions of

$$\begin{cases} -\triangle u(x) = u^p v^q, \\ -\triangle v(x) = u^q v^p. \end{cases} \tag{41}$$

In addition, an analogous result for (40) was obtained in [12].

Come back to (35). Besides Corollary 1.3 with $m = 2$, we have another corollary based on Theorem 1.2 and the Liouville type results in [1] and [19] mentioned above.

Corollary 3.1. *Let $\Omega \neq \mathbb{R}^n$ be a domain of $\mathbb{R}^n$. If either $1 < p_1 + q_1 = p_2 + q_2 \leq \frac{n}{n-2}$ and $\max\{\alpha, \beta\} > 0$, or one of the items in (36)-(39) fails, then there exists $C = C(m, n, p_1, p_2, q_1, q_2) > 0$ (independent of Ω and (u, v)) such that any solution (u, v) of (35) in Ω satisfies*

$$u(x) \leq C dist^{-\alpha}(x, \partial\Omega),\ v(x) \leq C dist^{-\beta}(x, \partial\Omega), \quad x \in \Omega.$$

If Ω is an exterior domain, then it follows that

$$u(x) \leq C|x|^{-\alpha},\ v(x) \leq C|x|^{-\beta}, \quad |x| \geq 2R.$$

Case II: $m \neq 2$.

Subcase II.1: $p_1 = q_2 := p, q_1 = p_2 := q$.

Now, (7) becomes

$$\begin{cases} -\triangle_m u(x) = u^p v^q, \\ -\triangle_m v(x) = u^q v^p. \end{cases} \tag{42}$$

Here $p, q > 0$ and $p + q > m - 1$. In particular, when $m = 2$, (42) is reduced to (41).

When $p + q = m^* - 1$ and $m \in (1, 2]$, the positive entire L^{m^*}-solutions of (42) on $\mathbb{R}^n$ converge to zero with the fast decay rate $\frac{n-m}{m-1}$ (cf. [9]). Namely, when $|x| \to \infty$,

$$u(x), v(x) \sim |x|^{\frac{m-n}{m-1}}. \tag{43}$$

Here, (43) means that when $|x| \to \infty$, there exists $C > 1$ such that $C^{-1}|x|^{\frac{m-n}{m-1}} \leq u(x),\ v(x) \leq C|x|^{\frac{m-n}{m-1}}$.

Except for the Liouville theorem in [4], we have another nonexistence result on the radial solution. When $p + q > m - 1$ and $p \leq \min\{\frac{n(m-1)}{n-m}, q\}$ (here $\frac{n(m-1)}{n-m}$ is the Serrin exponent), (42) has no positive entire radial solution (u, v) on $\mathbb{R}^n$ (cf. Corollary 1.3 in [26]). By Theorem 1.2, for all positive radial solutions (u, v) on the bounded domain and the exterior domain, (8) and (9) still hold (see the following corollary).

Corollary 3.2. *Let $\Omega = B_R(0)$ and $p + q > m - 1$ and $p \leq \min\{\frac{n(m-1)}{n-m}, q\}$. There exists $C = C(m, n, p, q) > 0$ such that any positive radial solution (u, v) of (42) in $B_R(0)$ satisfies*

$$u(x), v(x) \leq C dist^{-\frac{m}{p+q+1-m}}(x, \partial B_R(0)), \quad x \in B_R(0).$$

If $\Omega = B_R^c(0)$ is an exterior domain, then it follows that

$$u(x), v(x) \leq C|x|^{-\frac{m}{p+q+1-m}}, \quad |x| \geq 2R.$$

Remark 3.1. Clearly, $p \leq \min\{\frac{n(m-1)}{n-m}, q\}$ is different from (10)-(12). On the other hand, not all positive solutions are radial except for the ground states (cf. [3]). Therefore, Corollaries 1.3 and 3.2 do not cover each other.

Remark 3.2. There is a gap between $\frac{(n-1)m}{n-m} - 1$ and $m^* - 1$ where the nonexistence is not understood. Different from the results of the single equation (1), $p + q < m^* - 1$ may be no more the critical condition for the nonexistence for (42) even if $m = 2$. It is predictable that the Serrin exponent $\frac{(n-1)m}{n-m} - 1$ is critical. However, there is only the related conclusion for radial solutions (cf. [26]) to our knowledge.

Subcase II.2: $p_1 = q_2 = 0$.

Since $p_1 = q_2 = 0$, it implies from (7) that

$$\begin{cases} -\triangle_m u(x) = v^q, \\ -\triangle_m v(x) = u^p. \end{cases} \tag{44}$$

Here $p, q > 0$ and $pq > (m - 1)^2$.

Different from (1) and (5), the conditions related to the nonexistence of positive entire solutions of (44) are not understood completely. According to Theorem 5.1 (2) in [10], the Sobolev type condition is degenerate (cf. (5.4) in [10])

$$m = 2 \quad \text{or} \quad p = q = m^* - 1.$$

It was obtained by the scaling invariant of system (44) and the norms $\|u\|_{p+1}$ and $\|v\|_{q+1}$.

However, Corollary 1.8(2) in [10] shows that the Serrin type condition is critical for the nonexistence of the positive entire integrable solution on $\mathbb{R}^n$ and the positive entire $C^1(\mathbb{R}^n)$-solution satisfying $\inf_{\mathbb{R}^n} u = \inf_{\mathbb{R}^n} v = 0$. Namely, the Liouville theorem holds as long as one of the following two items is true

$$0 < pq \leq (m-1)^2; \tag{45}$$

$$pq > (m-1)^2, \quad \max\{\alpha, \beta\} \geq \frac{n-m}{m-1}. \tag{46}$$

Here,

$$\alpha = \frac{m(m+q-1)}{pq-(m-1)^2}, \quad \beta = \frac{m(m+p-1)}{pq-(m-1)^2}.$$

On the other hand, (5.3) in [10] provides another critical condition

$$\frac{1}{p+m-1} + \frac{1}{q+m-1} = \frac{n-m}{n(m-1)}. \tag{47}$$

It was obtained by the scaling invariant of system (44) and other norms $\|u\|_{p+m-1}$ and $\|v\|_{q+m-1}$ (cf. [10]).

Under this condition and the assumption $m \in (1,2]$, fast decay rates when $|x| \to \infty$ of the entire solution on $\mathbb{R}^n$ are showed in [24]. Namely, if $1 < p \leq q$, then as $|x| \to \infty$,

$$u(x) \sim |x|^{\frac{m-n}{m-1}};$$

and

$$\begin{aligned} v(x) &\sim |x|^{\frac{m-n}{m-1}}, && \text{if} \quad p\tfrac{n-m}{m-1} > n; \\ v(x) &\sim |x|^{\frac{m-n}{m-1}} (\ln|x|)^{\frac{1}{m-1}}, && \text{if} \quad p\tfrac{n-m}{m-1} = n; \\ v(x) &\sim |x|^{-\frac{n}{m-1}\frac{p+m-1}{q+m-1}}, && \text{if} \quad p\tfrac{n-m}{m-1} < n. \end{aligned}$$

If $1 < q \leq p$, then the results above still hold by exchanging u and v, as well as p and q.

When $m = 2$, these results were obtained in [7].

When $p = q$, (47) implies

$$p = \frac{n+m}{n-m}(m-1).$$

Clearly, it is not the Sobolev exponent $m^* - 1$ except for $m = 2$. But it is also an important exponent in the study of the separation property of radial solutions (cf. [17]) and the existence of stable solutions of quasilinear elliptic equations (cf. [27]).

Corollary 3.3. *Let $\Omega \neq \mathbb{R}^n$ be a domain of $\mathbb{R}^n$. If either* (45) *or* (46) *holds, then there exists $C = C(m,n,p,q) > 0$ (independent of Ω and (u,v)) such that any solution (u,v) of* (44) *in Ω satisfies*

$$u(x) \leq C dist^{-\alpha}(x, \partial\Omega), \quad v(x) \leq C dist^{-\beta}(x, \partial\Omega), \quad x \in \Omega.$$

If Ω is an exterior domain, then it follows that

$$u(x) \leq C|x|^{-\alpha}, \quad v(x) \leq C|x|^{-\beta}, \quad |x| \geq 2R.$$

Remark 3.3. Clearly, (44) is a special case of (14). Subcase II.2 is also an example for Theorem 1.4.

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