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GORENSTEIN WEAK INJECTIVE MODULES WITH RESPECT TO A SEMIDUALIZING BIMODULE

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Abstract. In this paper, we introduce the notion of C-Gorenstein weak injective modules with respect to a semidualizing bimodule ${}_SC_R$, where R and S are arbitrary associative rings. We show that an iteration of the procedure used to define G_C-weak injective modules yields exactly the G_C-weak injective modules, and then give the Foxby equivalence in this setting analogous to that of C-Gorenstein injective modules over commutative Noetherian rings. Finally, some applications are given, including weak co-Auslander-Buchweitz context, model structure and dual pair induced by G_C-weak injective modules.

1. Introduction

Auslander and Bridger introduced [1] the G-dimension for finitely generated modules over Noetherian rings. In [3], Enochs and Jenda introduced Gorenstein projective modules for arbitrary modules over a general ring, which is a generalization of finitely generated modules of G-dimension 0. As a dual of Gorenstein projective modules, Gorenstein injective modules were also introduced in [3]. Furthermore, Enochs, Jenda and Torrecillas in [5] introduced the notion of Gorenstein flat modules. It is well known that Gorenstein projective, injective and flat modules share many nice properties analogous to projective, injective and flat modules, respectively, and the homological properties of some generalized versions of these modules have been studied by many authors (e.g. [9, 18, 23–25]). In particular, Holm and Jørgensen in [13] introduced the notions of C-Gorenstein projective, C-Gorenstein injective and C-Gorenstein flat

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modules (G_C-projective, G_C-injective and G_C-flat respectively for short) with respect to a semidualizing module C over a commutative Noetherian ring R.

In [21], Sather-Wagstaff et al. further studied the properties of the category of G_C-flat modules, where C is a semidualizing module over a commutative Noetherian ring R. They showed that the category of all G_C-flat modules is part of a weak AB-context in the terminology of Hashimoto. Also in [21], they proved the stability of the category of G_C-flat modules. Let S and R be rings and ${}_SC_R$ a semidualizing bimodule. Inspired by [21], Hu and Zhang in [17] introduced and studied G_C-FP-injective left R-modules. It was proven that the G_C-FP-injective left R-modules have nice properties when S is a left coherent ring and ${}_SC_R$ a faithfully semidualizing bimodule, and the category of G_C-FP-injective left R-modules is part of a weak AB-context, which is dual of weak AB-context in the terminology of Hashimoto.

More recently, Bravo, Gillespie and Hovey in [2] described how Gorenstein homological algebra should work for general rings, and they introduced the notions of FP_∞-injective (or absolutely clean) and level modules. Independently, in [6, 7], the FP_∞-injective and level modules were also called weak injective and weak flat modules respectively. Along the same lines, it seems natural to investigate certain generalization of the G_C-(FP-)injective modules in a general setting. The purpose of this paper is to study the homological theory of G_C-weak injective modules with respect to a semidualizing bimodule ${}_SC_R$, where R and S are arbitrary associative rings, and to show that many parts of the homological theory on the G_C-(FP-)injective modules can be generalized directly to the similar theory on G_C-weak injective modules. The paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

In Section 3, we introduce and study G_C-weak injective modules with respect to a semidualizing bimodule ${}_SC_R$, where R and S are arbitrary associative rings. It is proven that the class of G_C-weak injective modules is closed under extensions, cokernels of monomorphisms, direct summands and direct products. We also show that an iteration of the procedure used to define G_C-weak injective modules yields exactly the G_C-weak injective modules.

Theorem 1. *The following are equivalent for a left R-module M:*

(1) *M is G_C-weak injective;*

(2) *There is a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact exact sequence*

$$\mathbb{G} = \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow \cdots$$

with each $G_i \in \mathcal{G}_C\mathcal{WI}(R)$ such that $M \cong \mathrm{Ker}(G_{-1} \to G_{-2})$;

(3) *There is a $\mathrm{Hom}_R(\mathcal{G}_C\mathcal{WI}(R), -)$-exact exact sequence*

$$\mathbb{G} = \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow \cdots$$

with each $G_i \in \mathcal{G}_C\mathcal{WI}(R)$ such that $M \cong \mathrm{Ker}(G_{-1} \to G_{-2})$.

In Section 4, we mainly discuss the Foxby equivalence in this setting. Especially, we prove that the subcategory of G_C-weak injective left R-modules in the Auslander class $\mathcal{A}_C(R)$ and that of G-weak injective left S-modules in the Bass class $\mathcal{B}_C(S)$ are equivalent under Foxby equivalence.

Theorem 2. *There are equivalences of categories*

$$\begin{tikzcd}
\mathcal{H}_C(\mathcal{W}(R)) \arrow[r, shift left=1.5, "C\otimes_R -"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & \mathcal{H}(\mathcal{W}(S)) \arrow[l, shift left=1.5, "\mathrm{Hom}_S(C,-)"] \arrow[d, hook] \\
\mathcal{WI}_C(R) \arrow[r, shift left=1.5, "C\otimes_R -"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & \mathcal{WI}(S) \arrow[l, shift left=1.5, "\mathrm{Hom}_S(C,-)"] \arrow[d, hook] \\
\mathcal{G}_C\mathcal{WI}(R)\cap\mathcal{A}_C(R) \arrow[r, shift left=1.5, "C\otimes_R -"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & \mathcal{GWI}(S)\cap\mathcal{B}_C(S) \arrow[l, shift left=1.5, "\mathrm{Hom}_S(C,-)"] \arrow[d, hook] \\
\mathcal{A}_C(R) \arrow[r, shift left=1.5, "C\otimes_R -"] \arrow[r, phantom, "\sim"] & \mathcal{B}_C(S). \arrow[l, shift left=1.5, "\mathrm{Hom}_S(C,-)"]
\end{tikzcd}$$

In Section 5, we give some applications of G_C-weak injective modules. After introducing the notion of co-Auslander-Buchweitz context dual to that of Auslander-Buchweitz context in [11, p. 34], we prove that every module in $\mathcal{G}_C\widehat{\mathcal{WI}(R)}$ admits a special $\mathcal{H}_C\widehat{(\mathcal{W}(R)})$-precover and a special $\mathcal{G}_C\mathcal{WI}(R)$-preenvelope, and that the triple

$$\Big(\mathcal{G}_C\mathcal{WI}(R), \mathcal{H}_C\widehat{(\mathcal{W}(R)}), \mathcal{H}_C(\mathcal{W}(R))\Big)$$

satisfies the weak co-Auslander-Buchweitz context. In addition, we give a new model structure in the category of left R-modules and a dual pair induced by G_C-weak injective modules.

2. Preliminaries

In this section, we give some terminology and some preliminary results needed in the sequel. For more details the reader can consult [4, 7, 10, 13, 15].

2.1. Throughout this paper, R and S are fixed associative rings with identity elements, and all modules are unitary. We use $\mathrm{Mod}\,R$ or $\mathrm{Mod}\,S$ to stand for the class of left R- or S-modules. Right R- or S-modules are identified with left modules over the opposite rings R^{op} or S^{op}. The notation ${}_SM_R$ is used to indicate that M is an (S,R)-bimodule, and the structures are compatible in the sense that $s(mr) = (sm)r$ for all $s \in S, r \in R, m \in M$. For an R-module M, $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

2.2. Let $\mathcal{X}$ be a subcategory of $\mathrm{Mod}\,R$. Denote by

$${}^{\perp}\mathcal{X} = \{M \mid \mathrm{Ext}^i_R(M, X) = 0 \text{ for all } X \in \mathcal{X} \text{ and all } i \geq 1\}.$$

In particular, we denote by ${}^{\perp_1}\mathcal{X} = \{M \mid \mathrm{Ext}^1_R(M,X) = 0 \text{ for all } X \in \mathcal{X}\}$. The notations $\mathcal{X}^{\perp}$ and $\mathcal{X}^{\perp_1}$ can be defined dually. Also, if $\mathcal{X}$ and $\mathcal{Y}$ are subcategories of $\mathrm{Mod}\,R$, we write $\mathcal{X} \perp \mathcal{Y}$ if $\mathrm{Ext}^i_R(X,Y) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and all $i \geq 1$.

Let M be a left R-module. An $\mathcal{X}$-coresolution of M is an exact sequence

$$\mathbb{X} = \ 0 \to M \to X^0 \to X^1 \to X^2 \to \cdots$$

in $\mathrm{Mod}\,R$ with each $X^i \in \mathcal{X}$. The $\mathcal{X}$-injective dimension of M is defined as

$$\mathcal{X}\text{-}\mathrm{id}(M) = \inf\{\sup\{n \geq 0 \mid X^n \neq 0\} \mid \mathbb{X} \text{ is an } \mathcal{X}\text{-coresolution of } M\}.$$

Dually, the $\mathcal{X}$-resolution and $\mathcal{X}$-projective dimension of M are defined.

Let $\mathcal{H} \subseteq \mathcal{X}$ be subcategories of $\mathrm{Mod}\,R$. We say that $\mathcal{H}$ is a *generator* for $\mathcal{X}$ provided that for each $X \in \mathcal{X}$ there is an exact sequence $0 \to \Omega \to H \to X \to 0$ with $H \in \mathcal{H}$ and $\Omega \in \mathcal{X}$; and moreover, $\mathcal{H}$ is a *projective generator* for $\mathcal{X}$ provided that $\mathcal{H}$ is a generator for $\mathcal{X}$ and $\mathcal{H} \perp \mathcal{X}$.

2.3. An (S,R)-bimodule $C = {}_SC_R$ is *semidualizing* if

(a1) ${}_SC$ admits a degreewise finite S-projective resolution.
(a2) C_R admits a degreewise finite R^{op}-projective resolution.
(b1) The homothety map ${}_SS_S \xrightarrow{{}_S\gamma} \mathrm{Hom}_{R^{\mathrm{op}}}(C,C)$ is an isomorphism.
(b2) The homothety map ${}_RR_R \xrightarrow{\gamma_R} \mathrm{Hom}_S(C,C)$ is an isomorphism.
(c1) $\mathrm{Ext}^i_S(C,C) = 0$ for all $i \geq 1$.
(c2) $\mathrm{Ext}^i_{R^{\mathrm{op}}}(C,C) = 0$ for all $i \geq 1$.

A semidualizing bimodule ${}_SC_R$ is *faithfully semidualizing* if it satisfies the following conditions for all modules ${}_SN$ and M_R:

(1) If $\mathrm{Hom}_S(C,N) = 0$, then $N = 0$.
(2) If $\mathrm{Hom}_{R^{\mathrm{op}}}(C,M) = 0$, then $M = 0$.

We always assume that ${}_SC_R$ is a faithfully semidualizing bimodule in this sequel.

2.4. The *Auslander class* $\mathcal{A}_C(R)$ with respect to C consists of all modules M in $\mathrm{Mod}\,R$ satisfying:

(A1) $\mathrm{Tor}^R_i(C,M) = 0$ for all $i \geq 1$.
(A2) $\mathrm{Ext}^i_S(C, C \otimes_R M) = 0$ for all $i \geq 1$.
(A3) The natural evaluation homomorphism $\mu_{_M} : M \longrightarrow \mathrm{Hom}_S(C, C \otimes_R M)$ is an isomorphism (of R-modules).

The *Bass class* $\mathcal{B}_C(S)$ with respect to C consists of all modules $N \in \mathrm{Mod}\,S$ satisfying:

(B1) $\mathrm{Ext}^i_S(C,N) = 0$ for all $i \geq 1$.
(B2) $\mathrm{Tor}^R_i(C, \mathrm{Hom}_S(C,N)) = 0$ for all $i \geq 1$.
(B3) The natural evaluation homomorphism $\nu_{_N} : C \otimes_R \mathrm{Hom}_S(C,N) \longrightarrow N$ is an isomorphism (of S-modules).

There are equivalences of categories ([15, Proposition 4.1]):

$$\begin{tikzcd}\mathcal{A}_C(R) \arrow[r, shift left, "C\otimes_R-"] \arrow[r, phantom, "\sim"] & \mathcal{B}_C(S). \arrow[l, shift left, "\mathrm{Hom}_S(C,-)"]\end{tikzcd}$$

2.5. Recall that a left R-module F is called *super finitely presented* [7] if it admits a degreewise finite R-projective resolution, and a left R-module M (resp. right R-module N) is called *weak injective* (resp. *weak flat*) if $\mathrm{Ext}_R^1(F,M)$ $= 0$ (resp. $\mathrm{Tor}_1^R(N,F)=0$) for all super finitely presented left R-modules F. Moreover, we will say a left R-module M is *weak projective* if $\mathrm{Ext}_R^1(M,Q)=0$ for all weak injective left R-modules Q. Denote by $\mathcal{WI}(R)$ and $\mathcal{WP}(R)$ the full subcategories of $\mathrm{Mod}\,R$ consisting of weak injective modules and weak projective modules respectively. One can easily verify that $\mathcal{WP}(R) = {}^{\perp}\mathcal{WI}(R)$ by definition and basic homological methods.

From [4, Definition 7.1.2], it follows that $(\mathcal{WP}(R),\mathcal{WI}(R))$ is a cotorsion theory which is cogenerated by the representative set of all super finitely presented left R-modules. So, by [4, Theorem 7.4.1 and Definition 7.1.5], every left R-module M has a special $\mathcal{WI}(R)$-preenvelope, that is, there is an exact sequence $0\to M\to W\to L\to 0$ with W weak injective and L weak projective. Meanwhile, every left R-module M has a special $\mathcal{WP}(R)$-precover, that is, there is an exact sequence $0\to K\to Q\to M\to 0$ with Q weak projective and K weak injective.

2.6. A module in $\mathrm{Mod}\,R$ is called C-*weak injective* [8] if it has the form $\mathrm{Hom}_S(C,I)$ for some $I\in\mathcal{WI}(S)$. We denote all C-weak injective modules in $\mathrm{Mod}\,R$ by $\mathcal{WI}_C(R)$. It has been shown in [8, Proposition 3.1] that there are equivalences of categories

$$\begin{tikzcd}\mathcal{WI}_C(R) \arrow[r, shift left, "C\otimes_R-"] \arrow[r, phantom, "\sim"] & \mathcal{WI}(S). \arrow[l, shift left, "\mathrm{Hom}_S(C,-)"]\end{tikzcd}$$

We denote the kernel of $(\mathcal{WP}(R),\mathcal{WI}(R))$ by $\mathcal{H}(\mathcal{W}(R)) := \mathcal{WI}(R)\cap\mathcal{WP}(R)$. Then this class of modules is not trivial. Indeed, we can take a nonzero weak injective R-module M, then there is an exact sequence $0\to K\to Q\to M\to 0$ with Q weak projective and K weak injective. Since $\mathcal{WI}(R)$ is closed under extensions by [6, Proposition 2.6(1)], we have Q is also weak injective. Therefore Q belongs to $\mathcal{H}(\mathcal{W}(R))$. We call the elements of $\mathcal{H}(\mathcal{W}(R))$ *weak injective-projective* R-modules. Also we will say a module in $\mathrm{Mod}\,R$ is C-*weak injective-projective* if it has the form $\mathrm{Hom}_S(C,I)$ for some $I\in\mathcal{H}(\mathcal{W}(S))$.

In what follows, we use $\mathcal{H}_C(\mathcal{W}(R))$ to denote the subcategory of C-weak injective-projective left R-modules. Moreover, let $\mathcal{WP}_C(R) = {}^{\perp}\mathcal{WI}_C(R)$, and the modules in $\mathcal{WP}_C(R)$ are called C-*weak projective.*

2.7. Let R be a commutative ring, and C a semidualizing module.

1. A module $M\in\mathrm{Mod}\,R$ is called C-*Gorenstein injective* (G_C-*injective* for short) [13] if

(1) $\mathrm{Ext}^i_R(\mathrm{Hom}_R(C,I),M)=0$ for all injective R-modules I and all $i\geq 1$;
(2) There exist injective R-modules $I_0, I_1, \dots$ together with an exact sequence

$$\cdots \to \mathrm{Hom}_R(C,I_1)\to \mathrm{Hom}_R(C,I_0)\to M\to 0$$

such that it stays exact after applying the functor $\mathrm{Hom}_R(\mathrm{Hom}_R(C,J), -)$ for each injective R-module J.

2. A module $M\in \mathrm{Mod}\,R$ is called *C-Gorenstein flat* (*G_C-flat* for short) [13] if

(1) $\mathrm{Tor}^R_i(\mathrm{Hom}_R(C,I),M)=0$ for all injective R-modules I and all $i\geq 1$;
(2) There exist flat R-modules $F^0, F^1, \dots$ together with an exact sequence

$$0\to M\to C\otimes_R F^0\to C\otimes_R F^1\to \cdots$$

such that it stays exact after applying the functor $\mathrm{Hom}_R(C,I)\otimes_R -$ for each injective R-module I.

2.8. Let ${}_SC_R$ be a semidualizing bimodule. A *complete $\mathcal{FI}_C\mathcal{I}$-resolution* is a complex $\mathbf{Y}$ of left R-modules satisfying the following:

(1) $\mathbf{Y}$ is exact and $\mathrm{Hom}_R(\mathcal{FI}^{\mathrm{fp}}_C(R),-)$-exact, and
(2) Y_i is C-FP-injective when $i\geq 0$ and Y_i is injective when $i<0$,

where $\mathcal{FI}^{\mathrm{fp}}_C(R)$ is the subcategory of left R-modules $\mathrm{Hom}_S(C,E)$ with $E\in {}^{\perp_1}\mathcal{FI}(S)\cap\mathcal{FI}(S)$.

A module $M\in\mathrm{Mod}\,R$ is called *G_C-FP-injective* [17] if there exists a complete $\mathcal{FI}_C\mathcal{I}$-resolution $\mathbf{Y}$ such that $M\cong \mathrm{Ker}(I^0\to I^1)$. In this case, $\mathbf{Y}$ is called a *complete $\mathcal{FI}_C\mathcal{I}$-resolution* of M. Denote by $\mathcal{G}_C\mathcal{FI}(R)$ the class of G_C-FP-injective left R-modules.

3. G_C-weak injective modules

In this section, we give a treatment of G_C-weak injective modules. It is shown that the G_C-weak injective modules share many nice properties of G_C-(FP-)injective modules in [13, 17].

We start with the following:

Lemma 3.1. *The following are equivalent for a left R-module M:*

(1) $M\in\mathcal{WP}_C(R)$.
(2) $C\otimes_R M\in\mathcal{WP}(S)$ *and* $\mathrm{Tor}^R_i(C,M)=0$ *for any* $i\geq 1$.

In particular, if $M\in\mathcal{A}_C(R)$, then $M\in\mathcal{WP}_C(R)\Longleftrightarrow C\otimes_R M\in\mathcal{WP}(S)$.

Proof. (1) ⇒ (2). Let $M\in\mathcal{WP}_C(R)$. Firstly, suppose E is a faithfully injective left S-module, then $\mathrm{Hom}_S(C,E)\in\mathcal{WI}_C(R)$. So $\mathrm{Ext}^i_R(M,\mathrm{Hom}_S(C,E))=0$ for any $i\geq 1$ by definition. Moreover, from the isomorphism ([4, Theorem 3.2.1]):

$$\mathrm{Ext}^i_R(M,\mathrm{Hom}_S(C,E))\cong \mathrm{Hom}_S(\mathrm{Tor}^R_i(C,M),E)\quad \text{for any } i\geq 1,$$

it follows that $\mathrm{Tor}_i^R(C, M) = 0$ for any $i \geq 1$.

Let $N \in \mathcal{WI}(S)$. Then $N \in \mathcal{B}_C(S)$ by [8, Theorem 2.2], and so $N \cong C \otimes_R \mathrm{Hom}_S(C, N)$ by definition, and $\mathrm{Hom}_S(C, N) \in \mathcal{A}_C(R)$ by [15, Proposition 4.1]. Hence, by [15, Theorem 6.4(a)], we have

$$\begin{aligned}\mathrm{Ext}_S^1(C \otimes_R M, N) &\cong \mathrm{Ext}_S^1(C \otimes_R M, C \otimes_R \mathrm{Hom}_S(C, N))\\ &\cong \mathrm{Ext}_R^1(M, \mathrm{Hom}_S(C, N)).\end{aligned}$$

Note that $\mathrm{Hom}_S(C, N) \in \mathcal{WI}_C(R)$, we have $\mathrm{Ext}_S^1(C \otimes_R M, N) = 0$, and so $C \otimes_R M \in \mathcal{WP}(S)$.

(2) ⇒ (1). Assume that $C \otimes_R M \in \mathcal{WP}(S)$ and $\mathrm{Tor}_i^R(C, M) = 0$ for any $i \geq 1$. For any C-weak injective left R-module N, we have $C \otimes_R N$ is weak injective over S by [8, Proposition 3.1]. It follows from the fact $\mathcal{WP}(S) = {}^{\perp}\mathcal{WI}(S)$ that $\mathrm{Ext}_R^i(M, N) \cong \mathrm{Ext}_S^i(C \otimes_R M, C \otimes_R N) = 0$ for any $i \geq 1$. Therefore, M is C-weak projective, as desired. ☐

Proposition 3.2. *The following are equivalent for a left R-module M:*

(1) $M \in \mathcal{H}_C(\mathcal{W}(R))$;
(2) $M \in \mathcal{WI}_C(R) \cap \mathcal{WP}_C(R)$;
(3) $C \otimes_R M \in \mathcal{H}(\mathcal{W}(S))$.

Proof. (1) ⇒ (3). Let $M \in \mathcal{H}_C(\mathcal{W}(R))$. Then there exists a module $W \in \mathcal{H}(\mathcal{W}(S))$ such that $M = \mathrm{Hom}_S(C, W)$. Since W is weak injective, we have that $W \in \mathcal{B}_C(S)$ by [8, Theorem 2.2]. In particular, $W \cong C \otimes_R \mathrm{Hom}_S(C, W)$. Consequently, $C \otimes_R M \cong C \otimes_R \mathrm{Hom}_S(C, W) \cong W$. Thus $C \otimes_R M \in \mathcal{H}(\mathcal{W}(S))$.

(3) ⇒ (2). Assume that $C \otimes_R M \in \mathcal{H}(\mathcal{W}(S))$. Then $C \otimes_R M \in \mathcal{B}_C(S)$ by [8, Theorem 2.2]. So $M \in \mathcal{A}_C(R)$ by [8, Lemma 2.9], and thus $\mathrm{Tor}_i^R(C, M) = 0$ for any $i \geq 1$. Since $C \otimes_R M \in \mathcal{WP}(S)$, we have $M \in \mathcal{WP}_C(R)$ by Lemma 3.1. Moreover, from the isomorphism $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$, we get that $M \in \mathcal{WI}_C(R)$. So (2) holds.

(2) ⇒ (1). Since M is C-weak injective, there exists a weak injective left S-module W such that $M = \mathrm{Hom}_S(C, W)$. Let N be any weak injective left S-module. Note that $W, N \in \mathcal{B}_C(S)$ by [8, Theorem 2.2]. Then, by [15, Theorem 6.4(b)], we have

$$\mathrm{Ext}_S^1(W, N) \cong \mathrm{Ext}_R^1(\mathrm{Hom}_S(C, W), \mathrm{Hom}_S(C, N)) \cong \mathrm{Ext}_R^1(M, \mathrm{Hom}_S(C, N)).$$

Since M is C-weak projective, $\mathrm{Ext}_R^1(M, \mathrm{Hom}_S(C, N)) = 0$. It follows that $\mathrm{Ext}_S^1(W, N) = 0$, and hence W is weak projective.

Consequently, $M \in \mathcal{H}_C(\mathcal{W}(S))$, and so (1) holds. ☐

Remark 3.3. From the equivalence between (1) and (2) in Proposition 3.2, we can see that the class of C-weak injective-projective left R-modules is just the intersection of the class of C-weak injective left R-modules and that of C-weak projective left R-modules. In particular, we have $\mathcal{H}_C(\mathcal{W}(R)) \perp \mathcal{WI}_C(R)$ and $\mathcal{WP}_C(R) \perp \mathcal{H}_C(\mathcal{W}(R))$.

Definition 3.4. The *C-weak injective-projective dimension* of a module $M \in \operatorname{Mod} R$ is defined that $\mathcal{H}_C(\mathcal{W}(R))\text{-}\operatorname{id}(M) \leq n$ if and only if there is an exact sequence

$$0 \to M \to \operatorname{Hom}_S(C, W^0) \to \operatorname{Hom}_S(C, W^1) \to \cdots \to \operatorname{Hom}_S(C, W^n) \to 0$$

in $\operatorname{Mod} R$ where each W^i is both weak injective and weak projective.

If no such n exists, set $\mathcal{H}_C(\mathcal{W}(R))\text{-}\operatorname{id}(M) = \infty$.

Next we give some characterization of the modules of finite C-weak injective-projective dimension.

Lemma 3.5. *The following are equivalent for a left R-module M:*

1. $\mathcal{H}_C(\mathcal{W}(R))\text{-}\operatorname{id}(M) \leq n$.
2. $\mathcal{H}(\mathcal{W}(S))\text{-}\operatorname{id}(C \otimes_R M) \leq n$.
3. *$M \cong \operatorname{Hom}_S(C, N)$ for some left S-module N with $\mathcal{H}(\mathcal{W}(S))\text{-}\operatorname{id}(N) \leq n$.*
4. *M is C-weak projective and $\mathcal{WI}_C(R)\text{-}\operatorname{id}(M) \leq n$.*
5. *M is C-weak projective and $\mathcal{WI}(S)\text{-}\operatorname{id}(C \otimes_R M) \leq n$.*

Proof. (1) ⇒ (2). Assume that $\mathcal{H}_C(\mathcal{W}(R))\text{-}\operatorname{id}(M) \leq n$, then there is an exact sequence

$$0 \to M \to \operatorname{Hom}_S(C, W^0) \to \operatorname{Hom}_S(C, W^1) \to \cdots \to \operatorname{Hom}_S(C, W^n) \to 0$$

with each W^i weak injective and weak projective. Since each W^i is weak injective, we have $W^i \in \mathcal{B}_C(S)$ by [8, Theorem 2.2]. Then $W^i \cong C \otimes_R \operatorname{Hom}_S(C, W^i)$ by definition, and hence $\operatorname{Hom}_S(C, W^i) \in \mathcal{A}_C(R)$. It follows from [15, Corollary 6.3] that $M \in \mathcal{A}_C(R)$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C \otimes_R M \arrow[r] \arrow[d, equal] & C \otimes_R \operatorname{Hom}_S(C, W^0) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & C \otimes_R \operatorname{Hom}_S(C, W^n) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & C \otimes_R M \arrow[r] & W^0 \arrow[r] & \cdots \arrow[r] & W^n \arrow[r] & 0.
\end{tikzcd}$$

Since M and $\operatorname{Hom}_S(C, W^i)$ are in $\mathcal{A}_C(R)$, it is easy to verify that the upper row is exact, and so is the lower row. Hence $\mathcal{H}(\mathcal{W}(S))\text{-}\operatorname{id}(C \otimes_R M) \leq n$.

(2) ⇒ (3). Assume $\mathcal{H}(\mathcal{W}(S))\text{-}\operatorname{id}(C \otimes_R M) \leq n$, that is, there is an exact sequence

$$0 \longrightarrow C \otimes_R M \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots \longrightarrow W^n \longrightarrow 0$$

with each $W^i \in \mathcal{H}(\mathcal{W}(S))$. In particular, each W^i is weak injective, and hence $W^i \in \mathcal{B}_C(S)$. So $C \otimes_R M \in \mathcal{B}_C(S)$ by [15, Corollary 6.3], and $M \in \mathcal{A}_C(R)$. It follows that $M \cong \operatorname{Hom}_S(C, C \otimes_R M)$. Therefore, (3) holds by setting $N = C \otimes_R M$.

(3) ⇒ (4). By the hypothesis, we have an exact sequence

$$0 \longrightarrow N \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots \longrightarrow W^n \longrightarrow 0$$

in $\operatorname{Mod} S$ with each $W^i \in \mathcal{H}(\mathcal{W}(S))$. In particular, each W^i is weak projective. By dimension shifting, we get that N is weak projective. Moreover, since each W^i is in $\mathcal{B}_C(S)$, then so is N by [15, Corollary 6.3]. From this, we can easily get that the sequence

$$0 \to M \to \operatorname{Hom}_S(C, W^0) \to \operatorname{Hom}_S(C, W^1) \to \cdots \to \operatorname{Hom}_S(C, W^n) \to 0$$

is exact, and $M \cong \operatorname{Hom}_S(C, N) \in \mathcal{A}_C(R)$ and each $\operatorname{Hom}_S(C, W^i) \in \mathcal{WI}_C(R)$. Thus, $\mathcal{WI}_C(R)\text{-}\operatorname{id}(M) \leq n$. Furthermore, since $C \otimes_R M \cong C \otimes_R \operatorname{Hom}_S(C, N) \cong N$ is weak projective, and $\operatorname{Tor}_i^R(C, M) = 0$ for any $i \geq 1$, we have that M is C-weak projective by Lemma 3.1.

(4) ⇒ (5). It follows from the fact $\mathcal{WI}_C(R)\text{-id}(M) = \mathcal{WI}(R)\text{-id}(C \otimes_R M)$ ([8, Proposition 3.5]).

(5) ⇒ (1). Let M be C-weak projective. Then $C \otimes_R M$ is weak projective by Lemma 3.1. Note that $(\mathcal{WP}(S), \mathcal{WI}(S))$ is a complete cotorsion pair, so there is an exact sequence $0 \to C \otimes_R M \to W^0 \to V^1 \to 0$ in $\operatorname{Mod} S$ with W^0 weak injective and V^1 weak projective. Since V^1 is weak projective, we have an exact sequence $0 \to V^1 \to W^1 \to V^2 \to 0$ in $\operatorname{Mod} S$ with W^1 weak injective and V^2 weak projective. Continuing this process, one can easily get the following exact sequence

$$0 \longrightarrow C \otimes_R M \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots \longrightarrow W^n \longrightarrow \cdots$$

such that each W^i is weak injective. Moreover, since $C \otimes_R M$ and V^1 are weak projective, it follows that W^0 is weak projective. Similarly, we can get that each W^i is weak projective. By the hypothesis $\mathcal{WI}(S)\text{-}\operatorname{id}(C \otimes_R M) \leq n$, so $V^n = \operatorname{Ker}(W^n \to W^{n+1})$ is weak injective, and hence $V^n \in \mathcal{B}_C(S)$. This implies $C \otimes_R M \in \mathcal{B}_C(S)$, and thus $M \in \mathcal{A}_C(R)$. By definition, $M \cong \operatorname{Hom}_S(C, C \otimes_R M)$. Now applying the functor $\operatorname{Hom}_S(C, -)$ to the above exact sequence, we can get the following exact sequence

$$0 \to M \to \operatorname{Hom}_S(C, W^0) \to \operatorname{Hom}_S(C, W^1) \to \cdots \to \operatorname{Hom}_S(C, V^n) \to 0.$$

From the above argument, we have that W^i and V^n are weak injective and weak projective. Thus (1) follows. □

Now we give our main definition of this paper as follows.

Definition 3.6. Let ${}_SC_R$ be a semidualizing bimodule. A *complete $\mathcal{WI}_C\mathcal{I}$-resolution* is an exact complex

$$\mathbb{Y} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

of left R-modules satisfying:

(1) $\mathbb{Y}$ is $\operatorname{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact;

(2) Each W_i is C-weak injective and each I^i is injective for any $i \geq 0$.

A module $M \in \operatorname{Mod} R$ is called *C-Gorenstein weak injective* (or *G_C-weak injective* for short) if there exists a complete $\mathcal{WI}_C\mathcal{I}$-resolution $\mathbb{Y}$ such that

$M \cong \mathrm{Ker}(I^0 \to I^1)$. In this case, $\mathbb{Y}$ is called a *complete* $\mathcal{WI}_C\mathcal{I}$*-resolution* of M. We denote by $\mathcal{G}_C\mathcal{WI}(R)$ the class of G_C-weak injective left R-modules.

Remark 3.7.

1. (1) It is clear that every C-weak injective module is G_C-weak injective by definition and the fact that $\mathcal{H}_C(\mathcal{W}(R)) \perp \mathcal{WI}_C(R)$. In particular, if ${}_SC_R = {}_RR_R$, then we call G_C-weak injective modules G-weak injective modules, and in this case, every weak injective left R-module is G-weak injective.
2. (2) One can easily check that a module $M \in \mathrm{Mod}\, R$ is G_C-weak injective if and only if $M \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$ and there is a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact exact sequence $\cdots \to W_1 \to W_0 \to M \to 0$ with each $W_i \in \mathcal{WI}_C(R)$.

Proposition 3.8. *Given an exact sequence* $0 \to L \to M \to N \to 0$. *We have*

1. (1) *If* $L \in \mathcal{G}_C\mathcal{WI}(R)$ *and* $M \in \mathcal{WI}_C(R)$, *then* $N \in \mathcal{G}_C\mathcal{WI}(R)$.
2. (2) *If* $L \in \mathcal{WI}_C(R)$ *and* $N \in \mathcal{G}_C\mathcal{WI}(R)$, *then* $M \in \mathcal{G}_C\mathcal{WI}(R)$.

Proof. (1) Since $L \in \mathcal{G}_C\mathcal{WI}(R)$, the sequence $0 \to L \to M \to N \to 0$ is $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact. Moreover, there is a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact exact sequence $\cdots \to W_1 \to W_0 \to L \to 0$ with $W_i \in \mathcal{WI}_C(R)$. So, by assembling the two sequences, we get an exact sequence $\cdots \to W_1 \to W_0 \to M \to N \to 0$. Thus one gets a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact exact sequence $\cdots \to W_1 \to W_0 \to M \to N \to 0$. Finally, from the fact $\mathcal{H}_C(\mathcal{W}(R)) \perp \mathcal{WI}_C(R)$ and the exact sequence

$$\mathrm{Ext}^i_R(W, M) \to \mathrm{Ext}^i_R(W, N) \to \mathrm{Ext}^{i+1}_R(W, L) \to \mathrm{Ext}^{i+1}_R(W, M),$$

it follows that $\mathrm{Ext}^i_R(W, N) \cong \mathrm{Ext}^{i+1}_R((W, L) = 0$ for any $i \geq 1$ and any $W \in \mathcal{H}_C(\mathcal{W}(R))$. Thus N is G_C-weak injective.

(2) Let $N \in \mathcal{G}_C\mathcal{WI}(R)$. Then by definition there is an exact sequence $0 \to N_1 \to W \to N \to 0$ in $\mathrm{Mod}\, R$ with $W \in \mathcal{WI}_C(R)$ and $N_1 \in \mathcal{G}_C\mathcal{WI}(R)$. Consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & N_1 \arrow[r, equal] \arrow[d] & N_1 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $L, W \in \mathcal{WI}_C(R)$, we have $Q \in \mathcal{WI}_C(R)$. Moreover, since $N_1 \in \mathcal{G}_C\mathcal{WI}(R)$, we have that M is G_C-weak injective by (1) and the middle column of the above diagram. □

The following result gives some characterization of the G_C-weak injective modules by replacing C-weak injective part in the sequence $\mathbb{Y}$ with C-weak injective-projective modules.

Proposition 3.9. *The following are equivalent for a left R-module M:*

1. *M is G_C-weak injective;*
2. *There is a* $\operatorname{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$*-exact exact sequence*

$$\mathbb{Y} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

with $W_i \in \mathcal{H}_C(\mathcal{W}(R))$ and I^i injective for any $i \geq 0$ such that $M \cong Ker(I^0 \to I^1)$;

3. *$M \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$ and there is a* $\operatorname{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$*-exact exact sequence*

$$\cdots \to W_1 \to W_0 \to M \to 0$$

with $W_i \in \mathcal{H}_C(\mathcal{W}(R))$.

Proof. (3) ⇒ (2) ⇒ (1) are trivial.

(1) ⇒ (3). Assume that M is a G_C-weak injective left R-module. Then by definition there is an exact sequence $0 \to M_1 \to W \to M \to 0$ with $W \in \mathcal{WI}_C(R)$ and $M_1 \in \mathcal{G}_C\mathcal{WI}(R)$. So there exists a weak injective left S-module E_0 such that $W = \operatorname{Hom}_S(C, E_0)$. Note that the cotorsion pair $(\mathcal{WP}(S), \mathcal{WI}(S))$ is complete, so there exists an exact sequence

$$(\diamond) : 0 \to E_1 \to W_0 \to E_0 \to 0$$

such that $W_0 \to E_0$ is a weak projective precover and E_1 is weak injective. Moreover, since the class of weak injective modules is closed under extensions, so W_0 is weak injective. This implies that $W_0 \in \mathcal{H}(\mathcal{W}(S))$ and hence $\operatorname{Hom}_S(C, W_0) \in \mathcal{H}_C(\mathcal{W}(R))$. By applying the functor $\operatorname{Hom}_S(C, -)$ to the sequence $(\diamond)$, we can get that the sequence

$$0 \to \operatorname{Hom}_S(C, E_1) \to \operatorname{Hom}_S(C, W_0) \to W \to 0$$

is exact. Now consider the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathrm{Hom}_S(C,E_1) \arrow[r, equal] \arrow[d] & \mathrm{Hom}_S(C,E_1) \arrow[d] & & \\
0 \arrow[r] & Q \arrow[r] \arrow[d] & \mathrm{Hom}_S(C,W_0) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\mathrm{Hom}_S(C,E_1) \in \mathcal{WI}_C(R)$ and $M_1 \in \mathcal{G}_C\mathcal{WI}(R)$, then $Q \in \mathcal{G}_C\mathcal{WI}(R)$ by Proposition 3.8(1). Thus $Q \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$, and hence the middle row in the above diagram is $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)),-)$-exact. By repeating the above argument to Q, one easily gets an exact sequence

$$0 \to T \to \mathrm{Hom}_S(C,W_1) \to Q \to 0$$

with $W_1 \in \mathcal{H}(\mathcal{W}(S))$. Continuing this process, we may obtain an exact sequence

$$\cdots \to \mathrm{Hom}_S(C,W_2) \to \mathrm{Hom}_S(C,W_1) \to \mathrm{Hom}_S(C,W_0) \to M \to 0$$

with each $W_i \in \mathcal{H}(\mathcal{W}(S))$, as desired. □

Proposition 3.10.

1. *The class $\mathcal{G}_C\mathcal{WI}(R)$ is closed under extensions, cokernels of monomorphisms, direct summands and direct products.*
2. *Given an exact sequence $0 \to L \to M \to N \to 0$ with $M, N \in \mathcal{G}_C\mathcal{WI}(R)$. Then $L \in \mathcal{G}_C\mathcal{WI}(R)$ if and only if $\mathrm{Ext}^1_R(W,L) = 0$ for any $W \in \mathcal{H}_C(\mathcal{W}(R))$.*

Proof. (1) Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathrm{Mod}\,R$. If $L, N \in \mathcal{G}_C\mathcal{WI}(R)$, then by Proposition 3.9, there are $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)),-)$-exact exact sequences:

$$\cdots \longrightarrow W_1' \xrightarrow{d_1'} W_0' \xrightarrow{d_0'} L \longrightarrow 0$$

$$\cdots \longrightarrow W_1'' \xrightarrow{d_1''} W_0'' \xrightarrow{d_0''} N \longrightarrow 0\ ,$$

such that all W_i' and W_i'' are in $\mathcal{H}_C(\mathcal{W}(R))$, and all kernels of d_i' and d_i'' are in $\mathcal{G}_C\mathcal{WI}(R)$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & W_0' \arrow[r, "\binom{1}{0}"] \arrow[d, "d_0'"] & W_0' \oplus W_0'' \arrow[r, "(0\ 1)"] & W_0'' \arrow[r] \arrow[d, "d_0''"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

Since L is G_C-weak injective, $\mathrm{Ext}_R^1(W_0'', L) = 0$, and thus we have the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(W_0'', L) \longrightarrow \mathrm{Hom}_R(W_0'', M) \longrightarrow \mathrm{Hom}_R(W_0'', N) \longrightarrow 0\ .$$

So there exists $\alpha : W_0'' \to M$ such that $d_0'' = g\alpha$. For any $(e_0', e_0'') \in W_0' \oplus W_0''$, we define $d_0 : W_0' \oplus W_0'' \to M$ by $d_0(e_0', e_0'') = fd_0'(e_0') + \alpha(e_0'')$. Then it is easy to verify that d_0 makes the above diagram commute. By Snake Lemma, we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Ker}d_0' \arrow[r] \arrow[d] & \mathrm{Ker}d_0 \arrow[r] \arrow[d] & \mathrm{Ker}d_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_0' \arrow[r, "\binom{1}{0}"] \arrow[d, "d_0'"] & W_0' \oplus W_0'' \arrow[r, "(0\ 1)"] \arrow[d, "d_0"] & W_0'' \arrow[r] \arrow[d, "d_0''"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since $\mathrm{Ker}d_0'$ and $\mathrm{Ker}d_0''$ are G_C-weak injective, $\mathrm{Ker}d_0', \mathrm{Ker}d_0'' \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$, and so $\mathrm{Ker}d_0 \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$. In particular, the exact sequence $0 \to \mathrm{Ker}d_0 \to W_0' \oplus W_0'' \to M \to 0$ is $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact. Repeating this process, we may get a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact exact sequence

$$\cdots \longrightarrow W_1' \oplus W_1'' \longrightarrow W_0' \oplus W_0'' \longrightarrow M \longrightarrow 0\ ,$$

where each $W_i' \oplus W_i''$ is in $\mathcal{H}_C(\mathcal{W}(R))$. Moreover, since $L, N \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$, we have $M \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$. Therefore, M is in $\mathcal{G}_C\mathcal{WI}(R)$ by Proposition 3.9.

Now assume $L, M \in \mathcal{G}_C\mathcal{WI}(R)$. Then there is a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact exact sequence $0 \to M_1 \to W \to M \to 0$ with $W \in \mathcal{WI}_C(R)$ and

$M_1 \in \mathcal{G}_C\mathcal{WI}(R)$. Consider the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M_1 \arrow[r, equal] \arrow[d] & M_1 \arrow[d] & & \\
0 \arrow[r] & Q \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

From the previous argument and the second column in the above diagram, we have that Q is G_C-weak injective. Moreover, by Proposition 3.8(2) and the middle row in the above diagram, we have that N is in $\mathcal{G}_C\mathcal{WI}(R)$.

The closure of direct products follows directly from the definition.

Finally, since the class $\mathcal{G}_C\mathcal{WI}(R)$ is closed under cokernels of monomorphisms and direct products, then it is closed under direct summands from the proof of [12, Proposition 1.4].

(2) "If " part is trivial.

"Only if " part. Since $N \in \mathcal{G}_C\mathcal{WI}(R)$, there is an exact sequence $0 \to N_1 \to W \to N \to 0$ with $W \in \mathcal{H}_C(\mathcal{W}(R))$ and $N_1 \in \mathcal{G}_C\mathcal{WI}(R)$ by definition. Consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & N_1 \arrow[r, equal] \arrow[d] & N_1 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

By the middle column in the above diagram and (1), we have $Q \in \mathcal{G}_C\mathcal{WI}(R)$. Moreover, by assumption, $\mathrm{Ext}_R^1(W, L) = 0$, that is, the middle row in the above diagram is split. So L is in $\mathcal{G}_C\mathcal{WI}(R)$ by (1). $\square$

Now we give some equivalent descriptions of G_C-weak injective modules, which shows that an iteration of the procedure used to describe the class of G_C-weak injective modules yields exactly the class of G_C-weak injective modules.

Theorem 3.11. *The following are equivalent for a left R-module M:*

1. M *is G_C-weak injective*;
2. *There is a* $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)),-)$*-exact exact sequence*
$$\mathbb{G} = \ \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow \cdots$$
with each $G_i \in \mathcal{G}_C\mathcal{WI}(R)$ *such that* $M \cong \mathrm{Ker}(G_{-1} \to G_{-2})$;
3. *There is a* $\mathrm{Hom}_R(\mathcal{G}_C\mathcal{WI}(R),-)$*-exact exact sequence*
$$\mathbb{G} = \ \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow \cdots$$
with each $G_i \in \mathcal{G}_C\mathcal{WI}(R)$ *such that* $M \cong \mathrm{Ker}(G_{-1} \to G_{-2})$.

Proof. (1)⇒ (3) holds by setting $\mathbb{G} = \ \cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}} M \longrightarrow 0 \longrightarrow \cdots$.

(3) ⇒ (2) is trivial.

(2) ⇒ (1). Given a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)),-)$-exact exact sequence

$$\mathbb{G} = \ \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow \cdots$$

with each $G_i \in \mathcal{G}_C\mathcal{WI}(R)$ and $M \cong \mathrm{Ker}(G_{-1} \to G_{-2})$. Since $\mathcal{H}_C(\mathcal{W}(R)) \perp G_i$, then $\mathcal{H}_C(\mathcal{W}(R)) \perp M$ by [21, Lemma 2.9]. Set $M_1 = \mathrm{Ker}(G_0 \to G_{-1})$. Then we have a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)),-)$-exact exact sequence $0 \to M_1 \to G_0 \to M \to 0$ such that $M_1 \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$. Moreover, since $G_0 \in \mathcal{G}_C\mathcal{WI}(R)$, there is a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)),-)$-exact exact sequence $0 \to G_0' \to W_0 \to G_0 \to 0$ with $W_0 \in \mathcal{WI}_C(R)$ and $G_0' \in \mathcal{G}_C\mathcal{WI}(R)$. Consider the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G_0' \arrow[r, equal] \arrow[d] & G_0' \arrow[d] & & \\
0 \arrow[r] & Q \arrow[r] \arrow[d] & W_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $G_0', M_1 \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$, we have $Q \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$. Thus the sequence $0 \to Q \to W_0 \to M \to 0$ is $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)),-)$-exact. Let $M_2 = \mathrm{Ker}(G_1 \to G_0)$. Then we get a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)),-)$-exact exact sequence $0 \to M_2 \to G_1 \to M_1 \to 0$ with $M_2 \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$. Now consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G'_0 \arrow[r, equal] \arrow[d] & G'_0 \arrow[d] & \\
0 \arrow[r] & M_2 \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_2 \arrow[r] & G_1 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Now that $G'_0, G_1 \in \mathcal{G}_C\mathcal{WI}(R)$, we have $G \in \mathcal{G}_C\mathcal{WI}(R)$. Moreover, since $M_2 \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$, the sequence $0 \to M_2 \to G \to Q \to 0$ is $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact. Hence we have a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact exact sequence $\cdots \to G_3 \to G_2 \to G \to Q \to 0$. We repeat the argument by replacing M with Q to get $W_1 \in \mathcal{WI}_C(R)$ and a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact exact sequence $0 \to T \to W_1 \to Q \to 0$. Continuing this process, we may obtain a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact exact sequence

$$\cdots \to W_2 \to W_1 \to W_0 \to M \to 0,$$

where each W_i is C-weak injective. This shows that M is G_C-weak injective, which completes the proof. ☐

4. Foxby equivalence of the modules with finite G_C-weak injective dimension

In this section, we investigate Foxby equivalence relative to the modules of finite G_C-weak injective dimension. Some known results in [19] are generalized.

Proposition 4.1. *There are equivalences of categories*

$$\mathcal{H}_C(\mathcal{W}(R)) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\underset{\sim}{\rightleftarrows}}} \mathcal{H}(\mathcal{W}(S)).$$

Proof. We have that the functor $\mathrm{Hom}_S(C, -)$ maps $\mathcal{H}(\mathcal{W}(S))$ to $\mathcal{H}_C(\mathcal{W}(R))$ by definition, and the functor $C \otimes_R -$ maps $\mathcal{H}_C(\mathcal{W}(R))$ to $\mathcal{H}(\mathcal{W}(S))$ by Proposition 3.2. Since $\mathcal{H}_C(\mathcal{W}(R)) \subseteq \mathcal{A}_C(R)$ and $\mathcal{H}(\mathcal{W}(S)) \subseteq \mathcal{B}_C(S)$, if $M \in \mathcal{H}_C(\mathcal{W}(R))$ and $N \in \mathcal{H}(\mathcal{W}(S))$, then we have two natural isomorphisms: $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$ and $C \otimes_R \mathrm{Hom}_S(C, N) \cong N$, which completes the proof. ☐

Proposition 4.2. *If $M \in \mathcal{A}_C(R)$, then M is G_C-weak injective if and only if $C \otimes_R M$ is G-weak injective.*

Proof. Let M be G_C-weak injective. Then there exists a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact exact sequence

$$\cdots \to \mathrm{Hom}_S(C, W_1) \to \mathrm{Hom}_S(C, W_0) \to M \to 0$$

in $\mathrm{Mod}\, R$ with $W_i \in \mathcal{H}(\mathcal{W}(S))$. Since, in this sequence, each kernel and M are in $\mathcal{A}_C(R)$, by applying the functor $C \otimes_R -$ to it, we will get an exact sequence $\cdots \to C \otimes_R \mathrm{Hom}_S(C, W_1) \to C \otimes_R \mathrm{Hom}_S(C, W_0) \to C \otimes_R M \to 0$. Moreover, since each W_i is in $\mathcal{B}_C(S)$, there is a natural isomorphism $C \otimes_R \mathrm{Hom}_S(C, W_i) \cong W_i$, and hence there is an exact sequence $\cdots \to W_1 \to W_0 \to C \otimes_R M \to 0$. Now let $W \in \mathcal{H}(\mathcal{W}(S))$, and consider the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & {}_S(W, W_1) \arrow[r] \arrow[d] & {}_S(W, W_0) \arrow[r] \arrow[d] & {}_S(W, C \otimes_R M) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & {}_R({}_S(C, W), {}_S(C, W_1)) \arrow[r] & {}_R({}_S(C, W), {}_S(C, W_0)) \arrow[r] & {}_R({}_S(C, W), M) \arrow[r] & 0,
\end{tikzcd}$$

where the symbols ${}_R(-, -)$ and ${}_S(-, -)$ stand for $\mathrm{Hom}_R(-, -)$ and $\mathrm{Hom}_S(-, -)$. By the hypothesis and the isomorphism

$$\begin{aligned}
\mathrm{Hom}_S(W, W_i) &\cong \mathrm{Hom}_S(C \otimes_R \mathrm{Hom}_S(C, W), W_i) \\
&\cong \mathrm{Hom}_R(\mathrm{Hom}_S(C, W), \mathrm{Hom}_S(C, W_i)),
\end{aligned}$$

we have that the top row is exact, that is, the sequence $\cdots \to W_1 \to W_0 \to C \otimes_R M \to 0$ is $\mathrm{Hom}_R(\mathcal{H}(\mathcal{W}(S)), -)$-exact. In addition, since

$$\begin{aligned}
\mathrm{Ext}^i_S(W, C \otimes_R M) &\cong \mathrm{Ext}^i_S(C \otimes_R \mathrm{Hom}_S(C, W), C \otimes_R M) \\
&\cong \mathrm{Ext}^i_S(\mathrm{Hom}_S(C, W), \mathrm{Hom}_S(C, C \otimes_R M)) \qquad (1) \\
&\cong \mathrm{Ext}^i_R(\mathrm{Hom}_S(C, W), M)
\end{aligned}$$

for any $i \geq 1$, it follows that $C \otimes_R M \in \mathcal{H}(\mathcal{W}(S))^{\perp}$. This shows that $C \otimes_R M \in \mathcal{GWI}(S)$.

Conversely, if $C \otimes_R M$ is G-weak injective, then there is a $\mathrm{Hom}_R(\mathcal{H}(\mathcal{W}(S)), -)$-exact exact sequence $\cdots \to W_1 \to W_0 \to C \otimes_R M \to 0$ with $W_i \in \mathcal{H}(\mathcal{W}(S))$. Since $M \in \mathcal{A}_C(R)$, $C \otimes_R M \in \mathcal{B}_C(S)$. Moreover, since each W_i is in $\mathcal{B}_C(S)$, we can get that all kernels in the above sequence are in $\mathcal{B}_C(S)$. By applying the functor $\mathrm{Hom}_S(C, -)$, we obtain the following exact sequence

$$\cdots \to \mathrm{Hom}_S(C, W_1) \to \mathrm{Hom}_S(C, W_0) \to \mathrm{Hom}_S(C, C \otimes_R M) \cong M \to 0.$$

Now we will prove that this sequence is also $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)),-)$-exact. Indeed, for any $W \in \mathcal{H}(\mathcal{W}(S))$, consider the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & {}_S(W, W_1) \arrow[r] \arrow[d, "\mu_1"'] & {}_S(W, W_0) \arrow[r] \arrow[d, "\mu_0"'] & {}_S(W, C \otimes_R M) \arrow[r] \arrow[d, "\mu"'] & 0 \\
\cdots \arrow[r] & {}_R(C \otimes_R {}_S(C, W), W_1) \arrow[r] \arrow[d, "\nu_1"'] & {}_R(C \otimes_R {}_S(C, W), W_0) \arrow[r] \arrow[d, "\nu_0"'] & {}_R(C \otimes_R {}_S(C, W), C \otimes_R M) \arrow[r] \arrow[d, "\nu"'] & 0 \\
\cdots \arrow[r] & {}_R({}_S(C, W), {}_S(C, W_1)) \arrow[r] & {}_R({}_S(C, W), {}_S(C, W_0)) \arrow[r] & {}_R({}_S(C, W), M) \arrow[r] & 0
\end{tikzcd}$$

where the morphisms μ, μ_0 and μ_1 are isomorphisms since $W \in \mathcal{H}(\mathcal{W}(S)) \subseteq \mathcal{B}_C(S)$, and the morphisms ν, ν_0 and ν_1 are also isomorphisms by [20, Theorem 2.76]. Since the top row is exact by hypothesis, this implies that the middle row is exact, and hence the bottom row is also exact. Finally, it follows from the isomorphism: $\mathrm{Ext}^i_R(\mathrm{Hom}_S(C, W), M) \cong \mathrm{Ext}^i_R(W, C \otimes_R M)$ given in (1) that M is a G_C-weak injective R-module. ☐

Proposition 4.3. *There are equivalences of categories*

$$\mathcal{G}_C\mathcal{WI}(R) \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}}} \mathcal{GWI}(S) \cap \mathcal{B}_C(S).$$

Proof. By Proposition 4.2, we first have that the image of the functor $C \otimes_R -$ under $\mathcal{G}_C\mathcal{WI}(R) \cap \mathcal{A}_C(R)$ is in $\mathcal{GWI}(S)$. Also $M \in \mathcal{A}_C(R)$ implies $C \otimes_R M \in \mathcal{B}_C(S)$. Therefore, if $M \in \mathcal{G}_C\mathcal{WI}(R) \cap \mathcal{A}_C(R)$, then $C \otimes_R M \in \mathcal{GWI}(S) \cap \mathcal{B}_C(S)$.

We next show that the image of the functor $\mathrm{Hom}_S(C, -)$ under $\mathcal{GWI}(S) \cap \mathcal{B}_C(S)$ is in $\mathcal{G}_C\mathcal{WI}(R) \cap \mathcal{A}_C(R)$. In fact, let $M \in \mathcal{GWI}(S) \cap \mathcal{B}_C(S)$. Then $M \in \mathcal{H}(\mathcal{W}(S))^{\perp}$ and there is a $\mathrm{Hom}_S(\mathcal{H}(\mathcal{W}(S)), -)$-exact exact sequence $\cdots \to W_1 \to W_0 \to M \to 0$ with $W_i \in \mathcal{H}(\mathcal{W}(S))$. By applying the functor $\mathrm{Hom}_S(C, -)$ to it, since M and all kernels in this sequence are in $\mathcal{B}_C(S)$, we get an exact sequence

$$\cdots \to \mathrm{Hom}_S(C, W_1) \to \mathrm{Hom}_S(C, W_0) \to \mathrm{Hom}_S(C, M) \to 0$$

and clearly $\mathrm{Hom}_S(C, W_i) \in \mathcal{H}_C(\mathcal{W}(R))$ for any $i \geq 0$. Now let $\widetilde{W} \in \mathcal{H}_C(\mathcal{W}(R))$. Then there exists $W \in \mathcal{H}(\mathcal{W}(S))$ such that $\widetilde{W} = \mathrm{Hom}_S(C, W)$. By applying the functor $\mathrm{Hom}_R(\widetilde{W}, -)$, we can get the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & {}_R(\widetilde{W}, {}_S(C, W_1)) \arrow[r] \arrow[d, equal] & {}_R(\widetilde{W}, {}_S(C, W_0)) \arrow[r] \arrow[d, equal] & {}_R(\widetilde{W}, {}_S(C, M)) \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & {}_R({}_S(C, W), {}_S(C, W_1)) \arrow[r] \arrow[d] & {}_R({}_S(C, W), {}_S(C, W_0)) \arrow[r] \arrow[d] & {}_R({}_S(C, W), {}_S(C, M)) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & {}_S(W, W_1) \arrow[r] & {}_S(W, W_0) \arrow[r] & {}_S(W, M) \arrow[r] & 0
\end{tikzcd}$$

Since the bottom row is exact, it follows that the top one is also exact. Moreover, from the isomorphism

$$\operatorname{Ext}^i_R(\widetilde{W}, \operatorname{Hom}_S(C,M)) = \operatorname{Ext}^i_R(\operatorname{Hom}_S(C,W), \operatorname{Hom}_S(C,M)) \cong \operatorname{Ext}^i_R(W,M),$$

we have $\operatorname{Hom}_S(C,M) \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$. Therefore, $\operatorname{Hom}_S(C,M) \in \mathcal{G}_C\mathcal{WI}(R) \cap \mathcal{A}_C(R)$.

Finally, if $M \in \mathcal{G}_C\mathcal{WI}(R) \cap \mathcal{A}_C(R)$ and $N \in \mathcal{GWI}(S) \cap \mathcal{B}_C(S)$, then by definition we have two natural isomorphisms: $M \cong \operatorname{Hom}_S(C, C \otimes_R M)$ and $C \otimes_R \operatorname{Hom}_S(C,N) \cong N$, which complete the proof. ☐

Following Propositions 4.1, 4.3 and the classical Foxby equivalence, we have:

Theorem 4.4. (Foxby Equivalence) *There are equivalences of categories*

$$\begin{tikzcd}
\mathcal{H}_C(\mathcal{W}(R)) \arrow[r, shift left, "C\otimes_R -"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & \mathcal{H}(\mathcal{W}(S)) \arrow[l, shift left, "\operatorname{Hom}_S(C,-)"] \arrow[d, hook] \\
\mathcal{WI}_C(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & \mathcal{WI}(S) \arrow[l, shift left, "\operatorname{Hom}_S(C,-)"] \arrow[d, hook] \\
\mathcal{G}_C\mathcal{WI}(R) \cap \mathcal{A}_C(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & \mathcal{GWI}(S) \cap \mathcal{B}_C(S) \arrow[l, shift left, "\operatorname{Hom}_S(C,-)"] \arrow[d, hook] \\
\mathcal{A}_C(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[r, phantom, "\sim"] & \mathcal{B}_C(S). \arrow[l, shift left, "\operatorname{Hom}_S(C,-)"]
\end{tikzcd}$$

Let n be a positive integer. For convenience, we set some notations as follows:

$$\begin{aligned}
\mathcal{H}(\mathcal{W}(S))_{\leq n} &= \text{the subcategory of those } M \text{ with } \mathcal{H}(\mathcal{W}(S))\text{-}\operatorname{id}(M) \leq n;\\
\mathcal{H}_C(\mathcal{W}(R))_{\leq n} &= \text{the subcategory of those } M \text{ with } \mathcal{H}_C(\mathcal{W}(R))\text{-}\operatorname{id}(M) \leq n;\\
\mathcal{GWI}(S)_{\leq n} &= \text{the subcategory of those } M \text{ with } \mathcal{GWI}(S)\text{-}\operatorname{id}(M) \leq n;\\
\mathcal{G}_C\mathcal{WI}(R)_{\leq n} &= \text{the subcategory of those } M \text{ with } \mathcal{G}_C\mathcal{WI}(R)\text{-}\operatorname{id}(M) \leq n.
\end{aligned}$$

where $\mathcal{G}_C\mathcal{WI}(R)\text{-}\operatorname{id}(M) \leq n$ means that there is an exact sequence

$$0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0$$

in $\operatorname{Mod} R$ where each G^i is G_C-weak injective.

Proposition 4.5. *There are equivalences of categories*

$$\begin{tikzcd}
\mathcal{H}_C(\mathcal{W}(R))_{\leq n} \arrow[r, shift left, "C\otimes_R -"] \arrow[r, phantom, "\sim"] & \mathcal{H}(\mathcal{W}(S))_{\leq n}. \arrow[l, shift left, "\operatorname{Hom}_S(C,-)"]
\end{tikzcd}$$

Proof. First, if $M \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n}$, then $C \otimes_R M \in \mathcal{H}(\mathcal{W}(S))_{\leq n}$ by Proposition 3.5. Now let $M \in \mathcal{H}(\mathcal{W}(S))_{\leq n}$. By definition there is an exact sequence $0 \to M \to W_0 \to W_1 \to \cdots \to W_n \to 0$ with $W_i \in \mathcal{H}(\mathcal{W}(S))$. Since M and all cokernels are in $\mathcal{B}_C(S)$, we obtain that the following sequence

$$0 \to \mathrm{Hom}_S(C, M) \to \mathrm{Hom}_S(C, W_0) \to \cdots \to \mathrm{Hom}_S(C, W_n) \to 0$$

is exact. Moreover, $\mathrm{Hom}_S(C, W_0) \in \mathcal{H}_C(\mathcal{W}(R))$ shows that $\mathrm{Hom}_S(C, M) \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n}$, as desired. □

Proposition 4.6. *There are equivalences of categories*

$$\begin{tikzcd}
\mathcal{G}_C\mathcal{WI}(R)_{\leq n} \cap \mathcal{A}_C(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[r, phantom, "\sim"] & \mathcal{GWI}(S)_{\leq n} \cap \mathcal{B}_C(S). \arrow[l, shift left, "\mathrm{Hom}_S(C,-)"]
\end{tikzcd}$$

Proof. Let $M \in \mathcal{G}_C\mathcal{WI}(R)_{\leq n} \cap \mathcal{A}_C(R)$. We first claim that there is an exact sequence $(\rhd):\ 0 \to M \to G^0 \to W^1 \to \cdots \to W^n \to 0$ with $G^0 \in \mathcal{G}_C\mathcal{WI}(R)$ and $W^i \in \mathcal{H}_C(\mathcal{W}(R))$. Indeed, since $M \in \mathcal{G}_C\mathcal{WI}(R)_{\leq n}$, there is an exact sequence

$$0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0$$

in $\mathrm{Mod}\, R$ where each G^i is G_C-weak injective. Since G^n is G_C-weak injective, there is an exact sequence $0 \to N \to W^n \to G^n \to 0$ such that $W^n \in \mathcal{H}_C(\mathcal{W}(R))$ and $N \in \mathcal{G}_C\mathcal{WI}(R)$. Let $V^i = \mathrm{Ker}(G^i \to G^{i+1})$ for $1 \leq i \leq n-1$ and consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & V^{n-1} \arrow[r, equal] \arrow[d] & V^{n-1} \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & H^{n-1} \arrow[r] \arrow[d] & G^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & W^n \arrow[r] \arrow[d] & G^n \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since N and G^{n-1} are G_C-weak injective, so is H^{n-1}. By assembling the middle column in the above diagram and the exact sequence $\cdots \to G^{n-3} \to G^{n-2} \to V^{n-1} \to 0$, we get the following exact sequence

$$\cdots \to G^{n-3} \to G^{n-2} \to H^{n-1} \to W^n \to 0.$$

Now that H^{n-1} is G_C-weak injective, so there is an exact sequence $0 \to N' \to W^{n-1} \to H^{n-1} \to 0$ such that $W^{n-1} \in \mathcal{H}_C(\mathcal{W}(R))$ and $N' \in \mathcal{G}_C\mathcal{WI}(R)$. Continuing this process, we may obtain the exact sequence $(\rhd)$ as claimed.

Since M and W^i are in $\mathcal{A}_C(R)$ in the sequence ($\triangleright$), it is easy to verify that G^0 and all kernels are in $\mathcal{A}_C(R)$. Thus we obtain the following exact sequence

$$0 \to C \otimes_R M \to C \otimes_R G^0 \to C \otimes_R W^1 \to \cdots \to C \otimes_R W^n \to 0.$$

By Propositions 4.1 and 4.3, we have $C \otimes_R G^0 \in \mathcal{GWI}(S)$ and $C \otimes_R W^i \in \mathcal{H}(\mathcal{W}(S)) \subseteq \mathcal{GWI}(S)$, which induces that $C \otimes_R M \in \mathcal{GWI}(S)_{\leq n} \cap \mathcal{B}_C(S)$.

Now let $M \in \mathcal{GWI}(S)_{\leq n} \cap \mathcal{B}_C(S)$. Then, as a special case for $C = R$, there is an exact sequence $0 \to M \to G^0 \to W^1 \to \cdots \to W^n \to 0$ with $G^0 \in \mathcal{GWI}(R)$ and $W^i \in \mathcal{H}(\mathcal{W}(S))$. Since M and W^i are in $\mathcal{B}_C(S)$, it is easy to verify that G^0 and all cokernels are in $\mathcal{B}_C(S)$. Thus we obtain the following exact sequence

$$0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(C, G^0) \to \mathrm{Hom}(C, W^1) \to \cdots \to \mathrm{Hom}(C, W^n) \to 0.$$

By Propositions 4.1 and 4.3, $\mathrm{Hom}(C, G^0) \in \mathcal{G}_C\mathcal{WI}(R)$ and $\mathrm{Hom}(C, W^i) \in \mathcal{H}_C(\mathcal{W}(R)) \subseteq \mathcal{G}_C\mathcal{WI}(R)$, which induces $\mathrm{Hom}(C, M) \in \mathcal{G}_C\mathcal{WI}(R)_{\leq n} \cap \mathcal{A}_C(R)$. □

By Propositions 4.5 and 4.6, one can easily obtain the following result, which is the counterpart of [19, Theorem 4.6] in the present context.

Theorem 4.7. *There are equivalences of categories*

$$\begin{tikzcd}
\mathcal{H}_C(\mathcal{W}(R))_{\leq n} \arrow[r, shift left=2, "C\otimes_R-"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & \mathcal{H}(\mathcal{W}(S))_{\leq n} \arrow[l, shift left=2, "\mathrm{Hom}_S(C,-)"] \arrow[d, hook] \\
\mathcal{WI}_C(R)_{\leq n} \arrow[r, shift left=2, "C\otimes_R-"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & \mathcal{WI}(S)_{\leq n} \arrow[l, shift left=2, "\mathrm{Hom}_S(C,-)"] \arrow[d, hook] \\
\mathcal{G}_C\mathcal{WI}(R)_{\leq n} \cap \mathcal{A}_C(R) \arrow[r, shift left=2, "C\otimes_R-"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & \mathcal{GWI}(S)_{\leq n} \cap \mathcal{B}_C(S) \arrow[l, shift left=2, "\mathrm{Hom}_S(C,-)"] \arrow[d, hook] \\
\mathcal{A}_C(R) \arrow[r, shift left=2, "C\otimes_R-"] \arrow[r, phantom, "\sim"] & \mathcal{B}_C(S). \arrow[l, shift left=2, "\mathrm{Hom}_S(C,-)"]
\end{tikzcd}$$

We end this section with some description of $\mathcal{GWI}(S) \cap \mathcal{B}_C(S)$, which is of independent interest.

Definition 4.8. Let ${}_SC_R$ be a semidualizing bimodule. A *complete* $\mathcal{H}$*om-*$\mathcal{WII}$*-resolution* is an exact complex

$$\mathbb{Y} = \ \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

of left S-modules satisfying:

(1) $\mathbb{Y}$ is both $\mathrm{Hom}_S(\mathcal{H}(\mathcal{W}(S)), -)$-exact and $\mathrm{Hom}_S(C, -)$-exact;
(2) Each W_i is weak injective and each I^i is injective for any $i \geq 0$.

We denote by $\mathcal{GWI}^C(S)$ the modules M which arise in the following way:

There exists a complete $\mathcal{H}$om-$\mathcal{WII}$-resolution $\mathbb{Y}$ as above, such that $M \cong \mathrm{Ker}(I^0 \to I^1)$.

We note that, $\mathcal{WI}(S) \subseteq \mathcal{GWI}^C(S) \subseteq \mathcal{GWI}(S)$; and if $C = R$, then $\mathcal{GWI}^C(S) = \mathcal{GWI}(S)$.

Proposition 4.9. $\mathcal{GWI}(S) \cap \mathcal{B}_C(S) = \mathcal{GWI}^C(S)$.

Proof. Let $M \in \mathcal{GWI}(S) \cap \mathcal{B}_C(S)$. Then there exists a $\mathrm{Hom}_S(\mathcal{H}(\mathcal{W}(S)), -)$-exact exact complex

$$\mathbb{Y} = \quad \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

in $\mathrm{Mod}\, S$ with each W_i weak injective and each I^i injective such that $M \cong \mathrm{Ker}(I^0 \to I^1)$. Note that $M \in \mathcal{B}_C(S)$, and I^i and W_i belong to $\mathcal{B}_C(S)$ by [8, Theorem 2.2], one can easily verify that the complex $\mathbb{Y}$ is $\mathrm{Hom}_S(C, -)$-exact. Thus $M \in \mathcal{GWI}^C(S)$.

Conversely, if $M \in \mathcal{GWI}^C(S)$, then by definition there exists a exact complex

$$\mathbb{Y} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

in $\mathrm{Mod}\, S$ with each W_i weak injective and each I^i injective such that $M \cong \mathrm{Ker}(I^0 \to I^1)$ and $\mathbb{Y}$ is both $\mathrm{Hom}_S(\mathcal{H}(\mathcal{W}(S)), -)$-exact and $\mathrm{Hom}_S(C, -)$-exact. Clearly, $\mathrm{Ext}^i_S(C, M) = 0$ for any $i \geq 1$. By definition, the sequence $\mathrm{Hom}_S(C, \mathbb{Y})$:

$$\cdots \to \mathrm{Hom}_S(C, W_1) \to \mathrm{Hom}_S(C, W_0) \to \mathrm{Hom}_S(C, I^0) \to \mathrm{Hom}_S(C, I^1) \to \cdots$$

is exact and $\mathrm{Hom}_S(C, M) = \mathrm{Ker}(\mathrm{Hom}_S(C, I^0) \to \mathrm{Hom}_S(C, I^1))$. Moreover, since all W_i and I^i belong to $\mathcal{B}_C(S)$ by [8, Theorem 2.2], the natural evaluation homomorphisms $\nu_{W_i} : C \otimes_R \mathrm{Hom}_S(C, W_i) \longrightarrow W_i$ and $\nu_{I^i} : C \otimes_R \mathrm{Hom}_S(C, I^i) \longrightarrow I^i$ are isomorphisms. Now consider the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C \otimes_R {}_S(C, W_1) \arrow[r] \arrow[d, "\nu_{W_1}", "\cong"'] & C \otimes_R {}_S(C, W_0) \arrow[r] \arrow[d, "\nu_{W_0}", "\cong"'] & C \otimes_R {}_S(C, I^0) \arrow[r] \arrow[d, "\nu_{I^0}", "\cong"'] & C \otimes_R {}_S(C, I^1) \arrow[r] \arrow[d, "\nu_{I^1}", "\cong"'] & \cdots \\
\cdots \arrow[r] & W_1 \arrow[r] & W_0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

Then the sequence $C \otimes_R \mathrm{Hom}_S(C, \mathbb{Y})$ is exact and that

$$\begin{aligned} C \otimes_R \mathrm{Hom}_S(C, M) &\cong \mathrm{Ker}(C \otimes_R \mathrm{Hom}_S(C, I^0) \to C \otimes_R \mathrm{Hom}_S(C, I^1)) \\ &\cong \mathrm{Ker}(I^0 \to I^1) \cong M. \end{aligned}$$

Moreover, since all $\mathrm{Hom}_S(C, I^i)$ and $\mathrm{Hom}_S(C, W^i)$ belong to $\mathcal{A}_C(R)$, we have

$$\mathrm{Tor}^R_i(C, \mathrm{Hom}_S(C, I^i)) = 0 = \mathrm{Tor}^R_i(C, \mathrm{Hom}_S(C, W^i)) \text{ for any } i \geq 1.$$

By [21, Lemma 2.9(3)], we get that $\mathrm{Tor}^R_i(C, \mathrm{Hom}_S(C, M)) = 0$ for any $i \geq 1$. Thus $M \in \mathcal{B}_C(S)$, and so $M \in \mathcal{GWI}(S) \cap \mathcal{B}_C(S)$, which shows that $\mathcal{GWI}(S) \cap \mathcal{B}_C(S) = \mathcal{GWI}^C(S)$. $\square$

By Propositions 4.3 and 4.9, we immediately get the following result.

Corollary 4.10. *There are equivalences of categories*

$$\begin{tikzcd}\mathcal{G}_C\mathcal{WI}(R)\cap\mathcal{A}_C(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[r, phantom, "\sim"] & \mathcal{GWI}^C(S). \arrow[l, shift left, "\mathrm{Hom}_S(C,-)"]\end{tikzcd}$$

5. Applications

In this section, we give some applications of G_C-weak injective modules. We show that every module in $\mathcal{G}_C\widehat{\mathcal{WI}(R)}$ admits a special $\mathcal{H}_C(\widehat{\mathcal{W}(R)})$-precover and a special $\mathcal{G}_C\mathcal{WI}(R)$-preenvelope, and that the triple

$$\Big(\mathcal{G}_C\mathcal{WI}(R), \mathcal{H}_C(\widehat{\mathcal{W}(R)}), \mathcal{H}_C(\mathcal{W}(R))\Big)$$

satisfies weak co-Auslander-Buchweitz context. In addition, we give a new model structure in $\mathrm{Mod}\,R$ and a dual pair induced by G_C-weak injective modules.

5.1. Weak co-Auslander-Buchweitz context and approximations

In [11, p. 34], Hashimoto introduced the terminology of weak Auslander-Buchweitz context (or weak AB-context for short). It is a triple $(\mathcal{X},\mathcal{Y},\omega)$ of full subcategories of an abelian category $\mathcal{A}$ which satisfies the following conditions:

(AB1) $\mathcal{X}$ is closed under extensions, epikernels and direct summands in $\mathcal{A}$;
(AB2) $\mathcal{Y}$ is closed under monocokernels, extensions and direct summands in $\mathcal{A}$, and one has $\mathcal{Y}\subseteq\widetilde{\mathcal{X}}$ where $\widetilde{\mathcal{X}}$ is the subcategory of left R-modules with finite $\mathcal{X}$-projective dimension;
(AB3) $\omega=\mathcal{X}\cap\mathcal{Y}$ and ω is an injective cogenerator of $\mathcal{X}$.

If, moreover $\widetilde{\mathcal{X}}=\mathcal{A}$, then $(\mathcal{X},\mathcal{Y},\omega)$ is called Auslander-Buchweitz context (or AB-context for short).

As a duality, we give the following definition.

Definition 5.1. A triple $(\mathcal{X},\mathcal{Y},\omega)$ of full subcategories of an abelian category $\mathcal{A}$ is called *weak co-Auslander-Buchweitz context* (*weak co-AB-context* for short) if this triple satisfies the following conditions:

(coAB1) $\mathcal{X}$ is closed under extensions, monocokernels and direct summands in $\mathcal{A}$;
(coAB2) $\mathcal{Y}$ is closed under epikernels, extensions and direct summands in $\mathcal{A}$, and one has $\mathcal{Y}\subseteq\widehat{\mathcal{X}}$ where $\widehat{\mathcal{X}}$ is the subcategory of left R-modules with finite $\mathcal{X}$-injective dimension;
(coAB3) $\omega=\mathcal{X}\cap\mathcal{Y}$ and ω is a projective generator of $\mathcal{X}$.

If, moreover $\widehat{\mathcal{X}}=\mathcal{A}$, then $(\mathcal{X},\mathcal{Y},\omega)$ is called *co-Auslander-Buchweitz context* (co-AB-context for short).

Lemma 5.2. *The subcategory* $\widehat{\mathcal{WI}_C(R)}$ *of left* R*-modules with finite* $\mathcal{WI}_C(R)$*-injective dimension is closed under kernels of epimorphisms, extensions and summands.*

Proof. Easy. ☐

Lemma 5.3. *The subcategory* $\mathcal{H}_C\widehat{(\mathcal{W}(R))}$ *is closed under kernels of epimorphisms, extensions and summands. Moreover,* $\mathcal{H}_C\widehat{(\mathcal{W}(R))} \subseteq \mathcal{G}_C\widehat{\mathcal{WI}(R)}$*.*

Proof. Because $\mathcal{H}_C\widehat{(\mathcal{W}(R))} = \widehat{\mathcal{WI}_C(R)} \cap \mathcal{WP}_C(R)$ by Lemma 3.5, and $\mathcal{WP}_C(R)$ is closed under kernels of epimorphisms, extensions and summands by definition. Thus the assertion follows from Lemma 5.2. ☐

Lemma 5.4. $\mathcal{H}_C(\mathcal{W}(R)) = \mathcal{G}_C\mathcal{WI}(R) \bigcap \mathcal{H}_C\widehat{(\mathcal{W}(R))}$ *and* $\mathcal{H}_C(\mathcal{W}(R))$ *is a projective generator of* $\mathcal{G}_C\mathcal{WI}(R)$*.*

Proof. The containment $\mathcal{H}_C(\mathcal{W}(R)) \subseteq \mathcal{G}_C\mathcal{WI}(R) \bigcap \mathcal{H}_C\widehat{(\mathcal{W}(R))}$ is trivial.

Now let $M \in \mathcal{G}_C\mathcal{WI}(R) \bigcap \mathcal{H}_C\widehat{(\mathcal{W}(R))}$. Then there is an exact sequence

$$0 \to M \to Q \to N \to 0$$

with $Q \in \mathcal{H}_C(\mathcal{W}(R))$ and $N \in \mathcal{H}_C\widehat{(\mathcal{W}(R))}$. It is easy to verify that $\mathrm{Ext}_R^1(N, M) = 0$ since $M \in \mathcal{G}_C\mathcal{WI}(R)$. So $M \in \mathcal{H}_C(\mathcal{W}(R))$ and hence the equality holds.

Moreover, $\mathcal{H}_C(\mathcal{W}(R))$ is a projective generator of $\mathcal{G}_C\mathcal{WI}(R)$ by Proposition 3.9. ☐

Theorem 5.5. *The triple* $\Big(\mathcal{G}_C\mathcal{WI}(R), \mathcal{H}_C\widehat{(\mathcal{W}(R))}, \mathcal{H}_C(\mathcal{W}(R))\Big)$ *satisfies the weak co-AB-context.*

Proof. The assertion follows immediately from Proposition 3.10 and Lemmas 5.3, 5.4. ☐

By Theorem 5.5 and a dual result of [11, Theorem 1.12.10], we get immediately the following result.

Corollary 5.6.

(1) $\mathcal{H}_C(\mathcal{W}(R))$ *is a unique additive projective generator for* $\mathcal{G}_C\mathcal{WI}(R)$ *in the sense that if* $\mathcal{P}$ *is a projective generator for* $\mathcal{G}_C\mathcal{WI}(R)$*, then* $\mathrm{add}(\mathcal{P}) = \mathcal{H}_C(\mathcal{W}(R))$*.*

(2) *Let* $M \in \mathcal{G}_C\widehat{\mathcal{WI}(R)}$*. Then*
  - (i) *there exists an exact sequence* $0 \to L \to W \to M \to 0$ *with* $W \in \mathcal{H}_C\widehat{(\mathcal{W}(R))}$ *and* $L \in \mathcal{G}_C\mathcal{WI}(R)$*;*
  - (ii) *there exists an exact sequence* $0 \to M \to N \to W \to 0$ *with* $W \in \mathcal{H}_C\widehat{(\mathcal{W}(R))}$ *and* $N \in \mathcal{G}_C\mathcal{WI}(R)$*.*

(3) *Let* $M \in \mathcal{G}_C\widehat{\mathcal{WI}(R)}$*. Then the following are equivalent:*
  - (i) $M \in \mathcal{G}_C\mathcal{WI}(R)$*;*
  - (ii) $M \in \mathcal{H}_C\widehat{(\mathcal{W}(R))}^{\perp}$*;*

   - (iii) $M \in \mathcal{H}_C\widehat{(\mathcal{W}(R))}^{\perp_1}$;
   - (iv) $M \in \mathcal{H}_C(\mathcal{W}(R))^{\perp}$.
- (4) *Let* $M \in \mathcal{G}_C\widehat{\mathcal{WI}(R)}$. *Then the following are equivalent*:
   - (i) $M \in \mathcal{H}_C\widehat{(\mathcal{W}(R))}$;
   - (ii) $M \in {}^{\perp}\mathcal{G}_C\mathcal{WI}(R)$;
   - (iii) $M \in {}^{\perp_1}\mathcal{G}_C\mathcal{WI}(R)$;
   - (iv) $\inf\{n \mid \mathrm{Ext}_R^{n+1}(M,G) = 0$ *for any* $G \in \mathcal{G}_C\mathcal{WI}(R)\} < \infty$ *and* $M \in {}^{\perp}\mathcal{H}_C\widehat{(\mathcal{W}(R))}$.
- (5) *Let* $M \in \mathcal{G}_C\widehat{\mathcal{WI}(R)}$. *Then we have the equalities*

$$\begin{aligned}\mathcal{G}_C\mathcal{WI}(R)\text{-}\mathrm{id}(M) &= \inf\{n \mid \mathrm{Ext}_R^{n+1}(W,M) = 0 \textit{ for any } W \in \mathcal{H}_C(\mathcal{W}(R))\}\\ &= \inf\{n \mid \mathrm{Ext}_R^{n+1}(W,M) = 0 \textit{ for any } W \in \mathcal{H}_C\widehat{(\mathcal{W}(R))}\}.\end{aligned}$$

- (6) *Let* $M \in \mathcal{H}_C\widehat{(\mathcal{W}(R))}$. *Then* $\mathcal{G}_C\mathcal{WI}(R)\text{-}\mathrm{id}(M) = \mathcal{H}_C(\mathcal{W}(R))\text{-}\mathrm{id}(M)$.
- (7) *Given an exact sequence* $0 \to L \to M \to N \to 0$. *If any two of* L, M *and* N *belong to* $\mathcal{G}_C\widehat{\mathcal{WI}(R)}$, *then the third also belongs to* $\mathcal{G}_C\widehat{\mathcal{WI}(R)}$.

Remark 5.7.

- (1) From the conditions (2)(i) and (3) of Corollary 5.6, we see that every modules in $\mathcal{G}_C\widehat{\mathcal{WI}(R)}$ admits a special $\mathcal{H}_C\widehat{(\mathcal{W}(R))}$-precover in the sense that: Given a class $\mathcal{F}$ of modules and a module M, a special $\mathcal{F}$-precover of M is an epimorphism $F \to M$ such that $F \in \mathcal{F}$ and its kernel is in $\mathcal{F}^{\perp_1}$ (see [10, Definition 5.12]).
- (2) From the conditions (2)(i) and (4) of Corollary 5.6, we see that every modules in $\mathcal{G}_C\widehat{\mathcal{WI}(R)}$ admits a special $\mathcal{G}_C\mathcal{WI}(R)$-preenvelope in the sense that: Given a class $\mathcal{F}$ of modules and a module M, a special $\mathcal{F}$-preenvelope of M is a monomorphism $M \to F$ such that $F \in \mathcal{F}$ and its cokernel is in ${}^{\perp_1}\mathcal{F}$ (see [10, Definition 5.12]).

In fact, we can get a more refined version of Corollary 5.6(2) as follows.

Theorem 5.8. *Let* M *be a left* R*-module and* n *a nonnegative integer. The following are equivalent*:

- (1) $M \in \mathcal{G}_C\mathcal{WI}(R)_{\leq n}$;
- (2) *There exists an exact sequence* $0 \to L \to W \to M \to 0$ *with* $W \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n}$ *and* $L \in \mathcal{G}_C\mathcal{WI}(R)$;
- (3) *There exists an exact sequence* $0 \to M \to N \to W \to 0$ *with* $W \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n-1}$ *and* $N \in \mathcal{G}_C\mathcal{WI}(R)$ *(*$n \geq 1$*).*

Proof. (1) $\Rightarrow$ (2). We use induction on n. The case $n = 0$ holds by Proposition 3.9. Assume that it is true for the case $n-1$ ($n \geq 1$). Let $M \in \mathcal{G}_C\mathcal{WI}(R)_{\leq n}$. Then there exists an exact sequence

$$0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0$$

such that each $G^i \in \mathcal{G}_C\mathcal{WI}(R)$. Set $\mho := \mathrm{Im}(G^0 \to G^1)$. We get two exact sequences

$$0 \to M \to G^0 \to \mho \to 0 \text{ and } 0 \to \mho \to G^1 \to \cdots \to G^n \to 0,$$

that is, $\mho \in \mathcal{G}_C\mathcal{WI}(R)_{\leq n-1}$. By induction hypothesis, there exists an exact sequence $0 \to L' \to W' \to \mho \to 0$ with $W' \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n-1}$ and $L' \in \mathcal{G}_C\mathcal{WI}(R)$.

Consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & L' \arrow[r, equal] \arrow[d] & L' \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & G^0 \arrow[r] \arrow[d] & \mho \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

By the middle column and Proposition 3.10, we know that $T \in \mathcal{G}_C\mathcal{WI}(R)$. Thus there is an exact sequence $0 \to L \to W_0 \to T \to 0$ with $W_0 \in \mathcal{H}_C(\mathcal{W}(R))$ and $L \in \mathcal{G}_C\mathcal{WI}(R)$ by Proposition 3.9. Consider the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\
0 \arrow[r] & W \arrow[r] \arrow[d] & W_0 \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & W' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By the middle row and the fact $W' \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n-1}$, we have

$$W \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n}.$$

Thus the second column shows that the result holds for the case n.

(2) ⇒ (1). Assume that there exists an exact sequence $0 \to L \to W \to M \to 0$ with $W \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n}$ and $L \in \mathcal{G}_C\mathcal{WI}(R)$. Since $W \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n}$, there is an exact sequence $0 \to W \to W^0 \to W^1 \to \cdots \to W^n \to 0$ with each $W^i \in \mathcal{H}_C(\mathcal{W}(R))$. Set $V := \mathrm{Im}(W^0 \to W^1)$. Then we have an exact sequence

$0 \to W \to W^0 \to V \to 0$ with $W^0 \in \mathcal{H}_C(\mathcal{W}(R))$ and $V \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n-1}$. Consider the following pushout diagram:

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & W^0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
& & V \arrow[r, equal] \arrow[d] & V \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

By Proposition 3.2, $W^0 \in \mathcal{WI}_C(R)$, and hence by the middle row and Proposition 3.8, we get $Q \in \mathcal{G}_C\mathcal{WI}(R)$. Thus $M \in \mathcal{G}_C\mathcal{WI}(R)_{\leq n}$ by considering the exact sequence $0 \to M \to Q \to V \to 0$.

(1) ⇒ (3) As a similar argument to the proof of (1) ⇒ (2), we have an exact sequence $0 \to M \to G^0 \to \mho \to 0$ with $G^0 \in \mathcal{G}_C\mathcal{WI}(R)$ and $\mho \in \mathcal{G}_C\mathcal{WI}(R)_{\leq n-1}$. Applying (1) ⇒ (2) to $\mho$, we can get an exact sequence $0 \to L \to W \to \mho \to 0$ with $W \in \mathcal{H}_C(\mathcal{W}(R))_{\leq n-1}$ and $L \in \mathcal{G}_C\mathcal{WI}(R)$. Consider the following pullback diagram:

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & G^0 \arrow[r] \arrow[d] & \mho \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd}$$

By the middle column and Proposition 3.10, we know that $T \in \mathcal{G}_C\mathcal{WI}(R)$. The middle row of this diagram is just our desired sequence.

(3) ⇒ (1) is trivial. □

5.2. Model structure

Proposition 5.9. *If* $\widehat{\mathcal{GWI}(R)} = \operatorname{Mod} R$*, that is, the triple*

$$\Big(\mathcal{GWI}(R), \mathcal{H}(\widehat{\mathcal{W}(R)}), \mathcal{H}(\mathcal{W}(R))\Big)$$

satisfies the co-AB-context, then $(\mathcal{WP}(R)\cap\widehat{\mathcal{WI}(R)},\mathcal{GWI}(R))$ *and* $(\mathcal{WP}(R),$ $\mathcal{GWI}(R)\cap\widehat{\mathcal{WI}(R)})$ *are complete cotorsion pairs.*

Proof. By Lemma 3.5, $\mathcal{WP}(R)\cap\widehat{\mathcal{WI}(R)}=\mathcal{H}(\widehat{\mathcal{W}(R)})$. Moreover, by hypothesis $\widehat{\mathcal{GWI}(R)}=\operatorname{Mod}R$, and by Corollary 5.6, every left R-module has a special $\mathcal{H}(\widehat{\mathcal{W}(R)})$-precover and has a special $\mathcal{GWI}(R)$-preenvelope, and $\mathcal{GWI}(R)=\mathcal{H}(\widehat{\mathcal{W}(R)})^{\perp_1}$, $\mathcal{H}(\widehat{\mathcal{W}(R)})={}^{\perp_1}\mathcal{GWI}(R)$. Hence $(\mathcal{WP}(R)\cap\widehat{\mathcal{WI}(R)},\mathcal{GWI}(R))$ is a complete cotorsion pair.

Since every weak injective left R-module is G-weak injective by definition, we have $\mathcal{WI}(R)\subseteq\mathcal{GWI}(R)\cap\widehat{\mathcal{WI}(R)}$. Next we will show that $\mathcal{GWI}(R)\cap\widehat{\mathcal{WI}(R)}\subseteq\mathcal{WI}(R)$ by induction, and thus $\mathcal{GWI}(R)\cap\widehat{\mathcal{WI}(R)}=\mathcal{WI}(R)$. Now let $M\in\mathcal{GWI}(R)\cap\widehat{\mathcal{WI}(R)}$. The case $\mathcal{WI}(R)$-id$(M)=0$ is trivial. Suppose that $\mathcal{WI}(R)$-id$(M)\leq n<\infty$. Then there is an exact sequence $0\to M\to W\to N\to 0$ with W weak injective and $\mathcal{WI}(R)$-id$(N)\leq n-1$. It is easy to see that N is G-weak injective by Proposition 3.8. Thus N is weak injective by induction. Moreover, since $(\mathcal{WP}(R),\mathcal{WI}(R))$ is a complete cotorsion pair, there is an exact sequence $0\to L\to Q\to N\to 0$ with L weak injective, and Q weak injective-projective. Consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & W \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since L and W are injective, T is weak injective. Furthermore, since M is G-weak injective and Q is weak injective, we have $\operatorname{Ext}^1_R(Q,M)=0$. Therefore the middle row in the above diagram is split. Thus M is weak injective. Consequently, $(\mathcal{WP}(R),\mathcal{GWI}(R)\cap\widehat{\mathcal{WI}(R)})=(\mathcal{WP}(R),\mathcal{WI}(R))$ is, of course, a complete cotorsion pair. ☐

Combining with a relation between model structures and cotorsion pairs given by Hovey in [16, Theorem 2.2], we get directly the following result.

Theorem 5.10. *If* $\widehat{\mathcal{GWI}(R)}=\operatorname{Mod}R$*, then there is a model structure in which the cofibrant objects are the weak projective left R-modules, the fibrant objects are the G_C-weak injective left R-modules and the trivial objects are the left R-module with finite weak injective dimension.*

5.3. A dual pair induced by $\mathcal{G}_C\mathcal{WI}$

In what follows, we assume that R is commutative, and C is a semidualizing R-module.

In [14], Holm and Jøgensen introduced the notion of a duality pair and demonstrated how the left half of such a pair is covering and preenveloping.

Let R be a ring. A *duality pair* over R is a pair $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are two classes of R-modules, subject to the following conditions:

1. For an R-module M, one has $M \in \mathcal{X}$ if and only if $M^+ \in \mathcal{Y}$.
2. $\mathcal{Y}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{X}, \mathcal{Y})$ is called *(co)product-closed* if the class $\mathcal{X}$ is closed under (co)products in the category of all R-modules.

A duality pair $(\mathcal{X}, \mathcal{Y})$ is called *perfect* if it is coproduct-closed, the class $\mathcal{X}$ is closed under extensions and R belongs to $\mathcal{X}$.

Lemma 5.11 ([14, Theorem 3.1])**.** *Let $(\mathcal{X}, \mathcal{Y})$ be a duality pair. Then $\mathcal{X}$ is closed under pure submodules, pure quotients and pure extensions. Furthermore, the following hold*:

1. *If $(\mathcal{X}, \mathcal{Y})$ is product-closed, then $\mathcal{X}$ is preenveloping.*
2. *If $(\mathcal{X}, \mathcal{Y})$ is coproduct-closed, then $\mathcal{X}$ is covering.*
3. *If $(\mathcal{X}, \mathcal{Y})$ is perfect, then $(\mathcal{X}, \mathcal{X}^{\perp})$ is a perfect cotorsion pair.*

Next we will construct a suitable dual pair induced by $\mathcal{G}_C\mathcal{WI}$. Before that, we recall that the subcategory of C-weak flat R-modules is denoted by $\mathcal{WF}_C(R) = \{C \otimes_R F \mid F \text{ is a weak flat } R\text{-module}\}$ in [8, Definition 2.1]. Also we give the following definition.

Definition 5.12. Let C be a semidualizing R-module. A complete $\mathcal{FWF}_C$-resolution is an exact complex

$$\mathbb{X} = \ \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

of R-modules satisfying the following:

1. $\mathbb{X}$ is $\mathcal{H}_C(\mathcal{W}(R)) \otimes_R -$ exact;
2. Each W^i is C-weak flat and each F_i is flat for any $i \geq 0$.

An R-module M is called *G_C-weak flat* if there exists a complete $\mathcal{FWF}_C$-resolution $\mathbb{X}$ such that $M \cong \operatorname{Ker}(W^0 \to W^1)$, in which $\mathbb{X}$ is called a complete $\mathcal{FWF}_C$-resolution of M.

We denote by $\mathcal{G}_C\mathcal{WF}(R)$ the subcategory consisting of G_C-weak flat R-modules.

Proposition 5.13. *Let C be a faithfully semidualizing bimodule.*

1. *M is a G_C-weak flat R-module if and only if $\operatorname{Tor}_i^R(M, \mathcal{H}_C(\mathcal{W}(R))) = 0$ for any $i \geq 1$ and there exists a $\mathcal{H}_C(\mathcal{W}(R)) \otimes_R -$ exact exact sequence $0 \to M \to W^0 \to W^1 \to \cdots$ with each $W^i \in \mathcal{WF}_C(R)$.*
2. *If M is a C-weak flat R-module, then $M \in \mathcal{G}_C\mathcal{WF}(R)$.*

Proof. Easy. ☐

Proposition 5.14. *The following statements are equivalent for an R-module M:*

(1) *M is G_C-weak flat R-module;*
(2) *M^+ is G_C-weak injective R-module.*

Proof. (1) ⇒ (2). Let M be a G_C-weak flat R-module. Then, by Proposition 5.13 and [20, Theorem 11.54], we have

$$\mathrm{Ext}_R^i(\mathcal{H}_C(\mathcal{W}(R)), M^+) \cong \mathrm{Tor}_i^R(\mathcal{H}_C(\mathcal{W}(R)), M)^+ = 0$$

for any $i \geq 1$. Next it suffices to show that there exists a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), -)$-exact $\mathcal{WI}_C(R)$-resolution of M^+ by Remark 3.7. Since M is G_C-weak flat, there exists an exact sequence

$$\mathbb{F} = 0 \longrightarrow M \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots$$

such that the complex $\mathcal{H}_C(\mathcal{W}(R)) \otimes_R \mathbb{F}$ is exact, where each F^i is weak flat. This implies that the complex

$$\mathbb{F}^+ = \cdots \longrightarrow (C \otimes_R F^1)^+ \longrightarrow (C \otimes_R F^0)^+ \longrightarrow M^+ \longrightarrow 0$$

is exact, and thus the complex

$$\mathbb{F}^+ = \cdots \longrightarrow \mathrm{Hom}_R(C, (F^1)^+) \longrightarrow \mathrm{Hom}_R(C, (F^0)^+) \longrightarrow M^+ \longrightarrow 0$$

is exact with each $(F^i)^+$ weak injective. Since $\mathcal{H}_C(\mathcal{W}(R)) \otimes_R \mathbb{F}$ is exact, it follows that the complex $(\mathcal{H}_C(\mathcal{W}(R)) \otimes_R \mathbb{F})^+$ is exact. Moreover, we have

$$\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), (C \otimes_R F^i)^+) \cong (\mathcal{H}_C(\mathcal{W}(R)) \otimes_R C \otimes_R F^i)^+,$$

This implies that the complex $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{W}(R)), \mathbb{F}^+)$ is exact. Therefore M^+ is G_C-weak injective.

(2) ⇒ (1). Let M^+ be a G_C-weak injective R-module. To prove that M is a G_C-weak flat R-module, we will construct a $\mathcal{H}_C(\mathcal{W}(R)) \otimes_R -$ exact $\mathcal{WF}_C(R)$-coresolution of M. Now if we can construct a short exact sequence

$$(\dagger) \qquad 0 \longrightarrow M \longrightarrow F^0 \longrightarrow L^1 \longrightarrow 0,$$

where F^0 is C-weak flat and $(L^1)^+$ is G_C-weak injective, then the $\mathcal{WF}_C(R)$-coresolution of M can be constructed recursively. That is, we can conclude an exact sequence

$$\mathbb{F} = \quad 0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots,$$

where each F^n is C-weak flat, and $(L^n)^+$ is G_C-weak injective for each $L^n = \mathrm{Ker}(F^n \to F^{n+1})$. Let $I \in \mathcal{H}_C(\mathcal{W}(R))$. Then

$$(\mathrm{Tor}_i^R(L^n, I))^+ \cong \mathrm{Ext}_R^i(I, (L^n)^+) = 0$$

for any $i \geq 1$. It follows that $\mathrm{Tor}_i^R(L^n, I) = 0$ for any $i \geq 1$. Therefore $\mathbb{F}$ is $\mathcal{H}_C(\mathcal{W}(R)) \otimes_R -$ exact and $\mathrm{Tor}_i^R(M, \mathcal{H}_C(\mathcal{W}(R))) = 0$ for any $i \geq 1$.

Next it suffices to construct the short exact sequence (†). By assumption, the module M^+ is G_C-weak injective. So we have a short exact sequence

$0 \longrightarrow Z \longrightarrow E \longrightarrow M^+ \longrightarrow 0$, where E is C-weak injective by definition. It follows that the sequence $0 \longrightarrow M^{++} \longrightarrow E^+ \longrightarrow Z^+ \longrightarrow 0$ is exact. Note that M, being a pure submodule of M^{++} (see [22, Exercise 41, p. 48]), embeds in a C-weak flat R-module (for E^+ is C-weak flat by [8, Proposition 2.6]). Since every module admits a C-weak flat preenvelope by [8, Theorem 2.12(1)], we assume that $\varphi : M \longrightarrow F^0$ is a C-weak flat preenvelope of M, then φ is injective. Set $L^1 = \mathrm{Coker}\varphi$. Then we have an exact sequence

$$(\ddagger) \qquad 0 \longrightarrow M \longrightarrow F^0 \longrightarrow L^1 \longrightarrow 0.$$

Next we will prove that $(L^1)^+$ is G_C-weak injective. From (‡) we get a short exact sequence

$$0 \longrightarrow (L^1)^+ \longrightarrow (F^0)^+ \xrightarrow{\varphi^+} M^+ \longrightarrow 0,$$

where $(F^0)^+$ is C-weak injective by [8, Proposition 2.6]. Since M^+ is G_C-weak injective by assumption, it is enough to show that $\mathrm{Ext}^1_R(J, (L^1)^+) = 0$ for all $J \in \mathcal{H}_C(\mathcal{W}(R))$ by Proposition 3.10(2). But $\mathrm{Ext}^1_R(J, (L^1)^+)$ vanishes if and only if the map

$$\mathrm{Hom}_R(J, \varphi^+)) : \mathrm{Hom}_R(J, (F^0)^+) \longrightarrow \mathrm{Hom}_R(J, M^+)$$

is surjective (where $\mathrm{Ext}^1_R(J, (F^0)^+) = 0$ since $(F^0)^+$ is C-weak injective), so we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(J, (F^0)^+) \arrow[r, "\mathrm{Hom}_R(J,\varphi^+)"] \arrow[d, "\cong"] & \mathrm{Hom}_R(J, M^+) \arrow[d, "\cong"] \\
\mathrm{Hom}_R(F^0, J^+) \arrow[r, "\mathrm{Hom}_R(\varphi, J^+)"] & \mathrm{Hom}_R(M, J^+).
\end{tikzcd}$$

The module J^+ is C-weak flat, and φ is a C-weak flat preenvelope of M, so the map $\mathrm{Hom}_R(\varphi, J^+)$ is surjective, and hence so is $\mathrm{Hom}_R(J, \varphi^+)$. This completes the proof. □

Now we are able to give the main result in this subsection.

Theorem 5.15. *The pair $(\mathcal{G}_C\mathcal{WF}(R), \mathcal{G}_C\mathcal{WI}(R))$ is a coproduct-closed duality pair. In particular, $\mathcal{G}_C\mathcal{WF}(R)$ is covering.*

Proof. By Propositions 5.14 and 3.10, we have $(\mathcal{G}_C\mathcal{WF}(R), \mathcal{G}_C\mathcal{WI}(R))$ is a duality pair. It is easy to check that the class $\mathcal{G}_C\mathcal{WF}(R)$ is closed under coproducts, so $(\mathcal{G}_C\mathcal{WF}(R), \mathcal{G}_C\mathcal{WI}(R))$ is coproduct-closed. Thus the assertion holds by Lemma 5.11. □

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