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INFINITELY MANY SMALL ENERGY SOLUTIONS FOR EQUATIONS INVOLVING THE FRACTIONAL LAPLACIAN IN $\mathbb{R}^N$

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Abstract. We are concerned with elliptic equations in $\mathbb{R}^N$, driven by a non-local integro-differential operator, which involves the fractional Laplacian. The main aim of this paper is to prove the existence of small solutions for our problem with negative energy in the sense that the sequence of solutions converges to 0 in the L^∞-norm by employing the regularity type result on the L^∞-boundedness of solutions and the modified functional method.

1. Introduction

In this paper we consider the existence of infinitely many small weak solutions for non-local integro-differential equations in $\mathbb{R}^N$, whose is given by

$$(P_\lambda) \qquad -\mathcal{L}_K u + u = \lambda f(x,u) \quad \text{in} \quad \mathbb{R}^N,$$

where λ is a real parameter, $0 < s < 1$, $2s < N$, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. Here $\mathcal{L}_K$ is the non-local integro-differential operator defined as

$$\mathcal{L}_K u(x) = 2\int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)dy \quad \text{for all } x \in \mathbb{R}^N,$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a kernel function satisfying properties that

(K1) $mK \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^2, 1\}$;
(K2) there exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(N+2s)}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;
(K3) $K(x) = K(-x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

The integro-differential operator $\mathcal{L}_K$ is a generalization of the fractional Laplacian, since, taking $K(x) = |x|^{n+2s)}$, we get $\mathcal{L}_K = -(-\Delta)^s$.

In the last years a great attention has been drawn to the study of fractional and non-local problems of elliptic type, both for the pure mathematical research and in view of concrete real-world applications. The interest in such

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operators has consistently increased in view of the mathematical theory to concrete some phenomena such as, among the others, social sciences, fractional quantum mechanics, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, continuum mechanics, phase transition phenomena, image process, game theory and Lévy processes; see [5, 7, 8, 15, 20, 22, 23] and the references therein. Especially, in terms of fractional quantum mechanics, the nonlinear fractional Schrödinger equation was originally discovered by Laskin in [20, 21] as an extension of the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In these directions, many researchers have been extensively studied the fractional Laplacian type problems in various way; see [4, 6–8, 25, 26, 28, 35] and the references therein. Especially, R. Servadei and E. Valdinoci in [29] gave applications to Dirichlet problems involving non-local integro-differential operators of fractional Laplacian type as observing mountain pass theorem [2] in the non-local framework; see also [28] and [3, 16, 27] for the whole spaces. The study for superlinear non-local fractional problems with infinitely many solutions via the fountain theorems in [34, 36] can be found in [9, 13, 14, 28, 31, 33].

Motivated by large interest in the current literature, exploiting variational methods, we investigate the existence of nontrivial weak solutions for elliptic equations in $\mathbb{R}^N$, driven in a non-local integro-differential operator, which involves the fractional Laplacian. More precisely, the main aim is to prove the existence of small solutions for the problem (P_λ) with negative energy in the sense that the sequence of solutions converges to 0 in the L^∞-norm, relies only on local behavior and assumptions on $f(x,t)$ only for sufficiently small t are required. To do this, we utilize the global variational formulations and the modified functional method which is initially observed by Z.-Q. Wang [32]. In all the papers quoted in [9, 13, 14, 28, 31, 33], the global property of $f(x,t)$ for t large enough was used in an crucial way to derive the existence of infinitely many solutions with negative energy. In contrast to other papers which yield large solutions in the sense that they form an unbounded sequence, as modifying and extending the functional $f(x,t)$ to an appropriate $\tilde{f}(x,t)$, Wang obtained the existence of infinitely many solutions for nonlinear boundary value problems which is rather a local phenomenon and this phenomenon is forced by the sublinear term. Utilizing the argument in [32], Z. Guo [17] showed that elliptic equations with indefinite concave nonlinearities have infinitely many solutions whose L^∞-norms converge to zero. In this direction, many authors considered the results for the nonlinear equations on a bounded domain in $\mathbb{R}^N$; see [10, 19, 24, 30]. As we know, such a result for non-local integro-differential equations on the whole space $\mathbb{R}^N$ has not been much studied, and we are only aware of the interesting paper [14] in this direction. We point out that the strategy for obtaining this multiplicity is to give a regularity type result on the L^∞-boundedness of solutions for the problem (P_λ) by applying the Nash-Moser bootstrap iteration method based on the work of P. Drábek, A. Kufner, and F. Nicolosi in [12]. Also it is worth mentioning that the conditions on f which will be given are imposed near zero and there are no conditions assumed on $f(x,t)$ at infinity.

This paper is structured as follows. In Section 2, we recall briefly some basic results for the fractional Sobolev spaces. And under certain conditions on f, we establish existence result of infinitely many small weak solutions for problem (P_λ) by employing the regularity type result on the L^∞-boundedness of solutions and the modified functional method.

2. Preliminaries and main results

First we briefly recall the definitions and some elementary properties of the fractional Sobolev spaces. We refer the reader to [1,11,15] for further references.

Let $s \in (0,1)$. We define the fractional Sobolev space $H^s(\mathbb{R}^N)$ as follows:

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy < +\infty \right\},$$

endowed with the Gagliardo norm

$$\|u\|_{H^s(\mathbb{R}^N)} := \left(\|u\|^2_{L^p(\mathbb{R}^N)} + |u|^2_{H^s(\mathbb{R}^N)} \right)^{\frac{1}{2}},$$

where

$$|u|^2_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy.$$

Then $H^s(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$\langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\, dxdy + \int_{\mathbb{R}^N} u(x)\varphi(x)\, dx.$$

Also, the space $C_0^\infty(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$, that is $H_0^s(\mathbb{R}^N) = H^s(\mathbb{R}^N)$ (see e.g. [1,11]).

Lemma 2.1 ([11])**.** *Let $0 < s < 1$ with $2s < N$. Then there exists a positive constant $C = C(N,s)$ such that for all $u \in H^s(\mathbb{R}^N)$,*

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)} \le C\, |u|_{H^s(\mathbb{R}^N)},$$

where $2_s^* = \frac{2N}{N-2s}$ is the fractional critical exponent. Moreover, the space $H^s(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [2, 2_s^*]$ and compactly embedded into $L^q_{loc}(\mathbb{R}^N)$ for any $q \in [2, 2_s^*)$.

By the condition (K1), the function

$$(x,y) \mapsto (u(x)-u(y))K(x-y)^{\frac{1}{2}} \in L^2(\mathbb{R}^{2N})$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. Let us denote by $H_K^s(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{H_K^s(\mathbb{R}^N)} := \left(\|u\|_{L^2(\mathbb{R}^N)}^2 + |u|_{H_K^s(\mathbb{R}^N)}^2\right)^{\frac{1}{2}},$$

where

$$|u|_{H_K^s(\mathbb{R}^N)}^2 := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 K(x-y)\,dxdy.$$

Then $H_K^s(\mathbb{R}^N)$ is also a Hilbert space with the inner product

$$\begin{aligned}\langle u,\varphi\rangle_{H_K^s(\mathbb{R}^N)} := &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dxdy \\ &+ \int_{\mathbb{R}^N} u(x)\varphi(x)\,dx.\end{aligned}$$

Lemma 2.2 ([29])**.** *Let $0 < s < 1$ with $2s < N$ and $K : \mathbb{R}^N \setminus \{0\} \to (0,\infty)$ is a kernel function satisfying the conditions* (K1)–(K3)*. Then if $\varphi \in H_K^s(\mathbb{R}^N)$, then $\varphi \in H^s(\mathbb{R}^N)$. Moreover*

$$\|\varphi\|_{H^s(\mathbb{R}^N)} \le \max\{1, \theta^{-\frac{1}{2}}\}\|\varphi\|_{H_K^s(\mathbb{R}^N)}.$$

Combining with Lemmas 2.1 and 2.2, we can obtain the following assertion immediately.

Lemma 2.3 ([29])**.** *Let $0 < s < 1$ with $2s < N$ and $K : \mathbb{R}^N \setminus \{0\} \to (0,\infty)$ satisfy the conditions* (K1)–(K3)*. Then there exists a positive constant $C_0 = C_0(N,s)$ such that for any $\varphi \in H_K^s(\mathbb{R}^N)$ and $2 \le q \le 2_s^*$*

$$\begin{aligned}\|\varphi\|_{L^q(\mathbb{R}^N)}^2 &\le C_0\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}dxdy \\ &\le \frac{C_0}{\theta}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi(x)-\varphi(y)|^2K(x-y)\,dxdy.\end{aligned}$$

Definition 2.4. Let $0 < s < 1$ with $2s < N$. We say that $u \in H_K^s(\mathbb{R}^N)$ is a weak solution of the problem (P_λ) if

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dxdy + \int_{\mathbb{R}^N}u(x)\varphi(x)\,dx \\ &= \lambda\int_{\mathbb{R}^N} f(x,u)\varphi\,dx\end{aligned} \tag{2.1}$$

for all $\varphi \in H_K^s(\mathbb{R}^N)$.

Let us define a functional $\Phi_s : H_K^s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Phi_s(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-u(y)|^2K(x-y)\,dxdy + \frac{1}{2}\int_{\mathbb{R}^N}|u(x)|^2\,dx.$$

It is a simple exercise to check that the functional Φ_s is well defined on $H^s_K(\mathbb{R}^N)$, $\Phi_s \in C^1(H^s_K(\mathbb{R}^N),\mathbb{R})$ and its Fréchet derivative is given by for any $\varphi \in H^s_K(\mathbb{R}^N)$,

$$\langle \Phi_s'(u),\varphi\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy + \int_{\mathbb{R}^N} u(x)\varphi(x)dx.$$

Lemma 2.5 ([3])**.** *Let $0 < s < 1$ with $2s < N$. Then the functional Φ_s is convex and weakly lower semicontinuous on $H^s_K(\mathbb{R}^N)$. Furthermore, the operator $\Phi_s' : H^s_K(\mathbb{R}^N) \to (H^s_K(\mathbb{R}^N))^*$ is a linear bounded homeomorphism onto $(H^s_K(\mathbb{R}^N))^*$.*

Denoting $F(x,t) = \int_0^t f(x,s)\,ds$ and we assume that for $2 < q < 2^*_s$ and $x \in \mathbb{R}^N$,

(F1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\cdot,t)$ is measurable for all $t \in \mathbb{R}$ and $f(x,\cdot)$ is continuous for almost all $x \in \mathbb{R}^N$.

(F2) There exist nonnegative functions $\rho \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $a \in L^{\frac{2^*_s}{2^*_s-q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that
$$|f(x,t)| \le \rho(x) + a(x)\,|t|^{q-1}$$
for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(F3) There exists a constant $s_0 > 0$ such that $2F(x,t) - f(x,t)t > 0$ for all $x \in \mathbb{R}^N$ and for $0 < |t| < s_0$.

(F4) $\lim_{|t|\to 0} \frac{f(x,t)}{t} = +\infty$ uniformly for all $x \in \mathbb{R}^N$.

Under the assumptions (F1) and (F2), we define the functional $\Psi : H^s_K(\mathbb{R}^N) \to \mathbb{R}$ by
$$\Psi(u) = \int_{\mathbb{R}^N} F(x,u)\,dx.$$
Then it follows from the same arguments as those of Proposition 1.12 in [34] that $\Psi \in C^1(H^s_K(\mathbb{R}^N),\mathbb{R})$ and its Fréchet derivative is
$$\langle \Psi'(u),\varphi\rangle = \int_{\mathbb{R}^N} f(x,u)\varphi\,dx$$
for any $u,\varphi \in H^s_K(\mathbb{R}^N)$. Next we define a functional $I_\lambda : H^s_K(\mathbb{R}^N) \to \mathbb{R}$ by
$$I_\lambda(u) = \Phi_s(u) - \lambda\Psi(u).$$
Then we know that the functional $I_\lambda \in C^1(H^s_K(\mathbb{R}^N),\mathbb{R})$ and its Fréchet derivative is
$$\begin{aligned}\langle I_\lambda'(u),\varphi\rangle = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dxdy\\ &+\int_{\mathbb{R}^N} u(x)\varphi(x)\,dx - \lambda\int_{\mathbb{R}^N} f(x,u)\varphi(x)\,dx\end{aligned}$$
for any $u,\varphi \in H^s_K(\mathbb{R}^N)$.

We first give the L^∞-boundedness of the weak solution which is based on the Nash-Moser bootstrap iteration technique in [12, Theorem 4.1].

Proposition 2.6. *Let $0<s<1$ with $2s<N$. Assume that* (F1)–(F2) *hold. If u is a weak solution of the problem (P_λ), then $u \in L^r(\mathbb{R}^N)$ for all $r \in [2_s^*, \infty]$.*

Proof. Let us assume first that u is nonnegative. For a positive constant M, we define

$$\varphi_M(x) = \min\{u(x), M\}$$

and use $\varphi = \varphi_M^{2\tau+1}$ $(\tau \geq 0)$ as a test function in (2.1). Then it is obvious that $\varphi \in H_K^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))(\varphi_M^{2\tau+1}(x)-\varphi_M^{2\tau+1}(y))K(x-y)\,dxdy \\
&\quad + \int_{\mathbb{R}^N} u(x)\varphi_M^{2\tau+1}(x)\,dx = \lambda\int_{\mathbb{R}^N} f(x,u)\varphi_M^{2\tau+1}(x)\,dx.
\end{aligned} \tag{2.2}$$

With the aid of Lemma 2.3, a straightforward calculation shows that the left-hand side of (2.2) becomes

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))(\varphi_M^{2\tau+1}(x)-\varphi_M^{2\tau+1}(y))K(x-y)\,dxdy \\
&\quad + \int_{\mathbb{R}^N} u(x)\varphi_M^{2\tau+1}(x)\,dx \\
&\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|\left|\varphi_M^{2\tau+1}(x)-\varphi_M^{2\tau+1}(y)\right|K(x-y)\,dxdy \\
&\quad + \int_{\mathbb{R}^N} \varphi_M^{2(\tau+1)}(x)\,dx \\
&\geq C_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi_M^{\tau+1}(x)-\varphi_M^{\tau+1}(y)|^2K(x-y)\,dxdy + \int_{\mathbb{R}^N}\varphi_M^{2(\tau+1)}(x)\,dx \\
&\geq \min\{C_1,1\}\|\varphi_M^{\tau+1}\|^2_{H_K^s(\mathbb{R}^N)} \geq \min\{C_1,1\}C_2\left(\int_{\mathbb{R}^N}|\varphi_M|^{(\tau+1)2_s^*}(x)\,dx\right)^{\frac{2}{2_s^*}}
\end{aligned} \tag{2.3}$$

for some positive constants C_1, C_2. Taking into account the assumption (F2) and the Hölder inequality, we observe that the right-hand side of (2.2) can be formally bounded from above as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(x,u)\varphi_M^{2\tau+1}(x)\,dx \\
&\leq \int_{\mathbb{R}^N} |f(x,u)||u|^{2\tau+1}\,dx \\
&\leq \|\rho\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}|u|^{2(\tau+1)}\,dx + \|\rho\|_{L^2(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}|u|^{2(\tau+1)}\,dx\right)^{\frac{1}{2}} \\
&\quad + \left(\int_{\mathbb{R}^N} a(x)^{\gamma_1}\,dx\right)^{\frac{1}{\gamma_1}}\left(\int_{\mathbb{R}^N}|u|^{2(\tau+1)\gamma_1'}|u|^{(q-2)\gamma_1'}\,dx\right)^{\frac{1}{\gamma_1'}}
\end{aligned} \tag{2.4}$$

$$\leq \|\rho\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^{2(\tau+1)}\,dx + \|\rho\|_{L^2(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} |u|^{2(\tau+1)}\,dx\right)^{\frac{1}{2}}$$
$$+ \left(\int_{\mathbb{R}^N} a(x)^{\gamma_1}\,dx\right)^{\frac{1}{\gamma_1}} \left(\int_{\mathbb{R}^N} |u|^{(\tau+1)\alpha}\,dx\right)^{\frac{2}{\alpha}} \left(\int_{\mathbb{R}^N} |u|^{(q-2)\gamma_1'\frac{\alpha}{\alpha-2\gamma_1'}}\,dx\right)^{\frac{\alpha-2\gamma_1'}{\alpha\gamma_1'}},$$

where $\gamma_1' > \frac{2_s^*}{2_s^*-q}$ and $\alpha = \frac{2_s^* 2\gamma_1'}{2_s^*-(q-2)\gamma_1'}$. Obviously

$$\alpha \leq 2_s^*, \quad 1 < \frac{\alpha}{2\gamma_1'}, \quad \text{and} \quad \frac{(q-2)\gamma_1'\alpha}{\alpha-2\gamma_1'} = 2_s^*,$$

and thus the relation (2.4) implies

$$\lambda \int_{\mathbb{R}^N} f(x,u)\varphi_M^{2\tau+1}(x)\,dx \tag{2.5}$$
$$\leq \lambda\|\rho\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^{2(\tau+1)}\,dx + \|\rho\|_{L^2(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} |u|^{2(\tau+1)}\,dx\right)^{\frac{1}{2}}$$
$$+ \lambda \left(\int_{\mathbb{R}^N} a(x)^{\gamma_1}\,dx\right)^{\frac{1}{\gamma_1}} \left(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\right)^{\frac{\alpha-2\gamma_1'}{\alpha\gamma_1'}} \left(\int_{\mathbb{R}^N} |u|^{(\tau+1)\alpha}\,dx\right)^{\frac{2}{\alpha}}.$$

Now it follows from relations (2.2), (2.3), (2.5) and the Sobolev inequality that there exist positive constants C_3, C_4 and C_5 (independent of M and $\tau > 0$) such that

$$\left(\int_{\mathbb{R}^N} |\varphi_M|^{(\tau+1)2_s^*}\,dx\right)^{\frac{2}{2_s^*}}$$
$$\leq C_3 \int_{\mathbb{R}^N} |u|^{2(\tau+1)}\,dx + C_4 \left(\int_{\mathbb{R}^N} |u|^{2(\tau+1)}\,dx\right)^{\frac{1}{2}} + C_5 \left(\int_{\mathbb{R}^N} |u|^{(\tau+1)\alpha}\,dx\right)^{\frac{2}{\alpha}},$$

in other words,

$$\|\varphi_M\|_{L^{(\tau+1)2_s^*}(\mathbb{R}^N)} \tag{2.6}$$
$$\leq C_3\|u\|^{2(\tau+1)}_{L^{2(\tau+1)}(\mathbb{R}^N)} + C_4\|u\|^{\tau+1}_{L^{2(\tau+1)}(\mathbb{R}^N)} + C_5^{\frac{1}{2(\tau+1)}}\|u\|_{L^{(\tau+1)\alpha}(\mathbb{R}^N)}.$$

To apply the bootstrap argument which plays an important role in L^∞-estimates, we first assume that $\|u\|_{L^{2(\tau+1)}(\mathbb{R}^N)} > 1$. From the relation (2.6), we have

$$\|\varphi_M\|^{2(\tau+1)}_{L^{(\tau+1)2_s^*}(\mathbb{R}^N)} \tag{2.7}$$
$$\leq C_3\|u\|^{2(\tau+1)}_{L^{2(\tau+1)}(\mathbb{R}^N)} + C_4\|u\|^{\tau+1}_{L^{2(\tau+1)}(\mathbb{R}^N)} + C_5\|u\|^{2(\tau+1)}_{L^{(\tau+1)\alpha}(\mathbb{R}^N)}$$
$$\leq (C_3+C_4)\|u\|^{2(\tau+1)}_{L^{2(\tau+1)}(\mathbb{R}^N)} + C_5\|u\|^{2(\tau+1)}_{L^{(\tau+1)\alpha}(\mathbb{R}^N)},$$

which implies

$$\|\varphi_M\|_{L^{(\tau+1)2_s^*}(\mathbb{R}^N)} \leq C_6^{\frac{1}{2(\tau+1)}}\|u\|_{L^{(\tau+1)\kappa}(\mathbb{R}^N)} \tag{2.8}$$

for some positive constant C_6 and for any positive constant M, where κ is either 2 or α. The expression (2.8) is a starting point for Nash-Moser bootstrap iterations which plays an important role in obtaining L^∞-estimates. Since $u \in H^s_K(\mathbb{R}^N)$ and so $u \in L^{2^*_s}(\mathbb{R}^N)$, we can choose $\tau := \tau_1$ in (2.8) such that $(\tau_1+1)\kappa = 2^*_s$, *i.e.*, $\tau_1 = \frac{2^*_s}{\kappa} - 1$. Then we have

$$\|\varphi_M\|_{L^{(\tau_1+1)2^*_s}(\mathbb{R}^N)} \leq C_6^{\frac{1}{2(\tau_1+1)}} \|u\|_{L^{(\tau_1+1)\kappa}(\mathbb{R}^N)} \tag{2.9}$$

for any positive constant M. Since $u(x) = \lim_{M\to\infty} \varphi_M(x)$ for almost every $x \in \mathbb{R}^N$, the Fatou lemma and relation (2.9) imply

$$\|u\|_{L^{(\tau_1+1)2^*_s}(\mathbb{R}^N)} \leq C_6^{\frac{1}{2(\tau_1+1)}} \|u\|_{L^{(\tau_1+1)\kappa}(\mathbb{R}^N)}. \tag{2.10}$$

Thus, we can choose $\tau = \tau_2$ in (2.8) such that $(\tau_2+1)\kappa = (\tau_1+1)2^*_s = \frac{(2^*_s)^2}{\kappa}$ and repeating the same argument we get

$$\|u\|_{L^{(\tau_2+1)2^*_s}(\mathbb{R}^N)} \leq C_6^{\frac{1}{2(\tau_2+1)}} \|u\|_{L^{(\tau_1+1)2^*_s}(\mathbb{R}^N)}.$$

By induction we obtain

$$\|u\|_{L^{(\tau_n+1)2^*_s}(\mathbb{R}^N)} \leq C_6^{\frac{1}{2(\tau_n+1)}} \|u\|_{L^{(\tau_{n-1}+1)2^*_s}(\mathbb{R}^N)} \tag{2.11}$$

for any $n \in \mathbb{N}$, where $\tau_n + 1 = \left(\frac{2^*_s}{\kappa}\right)^n$. It follows from (2.10) and (2.11) that

$$\begin{aligned} \|u\|_{L^{(\tau_n+1)2^*_s}(\mathbb{R}^N)} &\leq C_6^{\frac{1}{2}\sum_{j=1}^n \frac{1}{\tau_j+1}} \|u\|_{L^{(\tau_1+1)\kappa}(\mathbb{R}^N)} \\ &\leq C_6^{\frac{1}{2}\sum_{j=1}^n \frac{1}{\tau_j+1}} \|u\|_{L^{2^*_s}(\mathbb{R}^N)}. \end{aligned} \tag{2.12}$$

However $\sum_{j=1}^n \frac{1}{\tau_j+1} = \sum_{j=1}^n \left(\frac{\kappa}{2^*_s}\right)^j$ and $\frac{\kappa}{2^*_s} < 1$. Hence it follows from (2.12) that there exists a constant $C_7 > 0$ such that

$$\|u\|_{L^{r_n}(\mathbb{R}^N)} \leq C_7 \|u\|_{L^{2^*_s}(\mathbb{R}^N)} \tag{2.13}$$

for $r_n = (\tau_n+1)2^*_s \to \infty$ when $n \to \infty$. Let us assume that $\|u\|_\infty > C_7\|u\|_{L^{2^*_s}(\mathbb{R}^N)}$. Then there exist $\eta > 0$ and a set Ω_0 of positive measure in $\mathbb{R}^N$ such that $u(x) \geq C_7\|u\|_{L^{2^*_s}(\mathbb{R}^N)} + \eta$ for $x \in \Omega_0$. It follows that

$$\begin{aligned} \liminf_{r_n\to\infty} \left(\int_{\mathbb{R}^N} |u(x)|^{r_n}\, dx\right)^{\frac{1}{r_n}} &\geq \liminf_{r_n\to\infty} \left(\int_{\Omega_0} |u(x)|^{r_n}\, dx\right)^{\frac{1}{r_n}} \\ &\geq \liminf_{r_n\to\infty} \left(C_7\|u\|_{L^{2^*_s}(\mathbb{R}^N)} + \eta\right) (\mathrm{meas}\Omega_0)^{\frac{1}{r_n}} \\ &= C_7\|u\|_{L^{2^*_s}(\mathbb{R}^N)} + \eta, \end{aligned}$$

which contradicts (2.13). Therefore taking into account Lemmas 2.1 and 2.3 we have

$$\|u\|_\infty \leq C_7\|u\|_{L^{2^*_s}(\mathbb{R}^N)} \leq C_8$$

for some constant $C_8 > 0$.

On the other hand we assume that $\|u\|_{L^{2(\tau+1)}(\mathbb{R}^N)} \leq 1$. From the relation (2), we have

$$\|\varphi_M\|^{2(\tau+1)}_{L^{(\tau+1)2^*_s}(\mathbb{R}^N)} \leq C_9\|u\|^{2(\tau+1)}_{L^{(k+1)\alpha}(\mathbb{R}^N)},$$

which implies

$$\|\varphi_M\|_{L^{(\tau+1)2^*_s}(\mathbb{R}^N)} \leq C_9^{\frac{1}{(2\tau+1)}}\|u\|_{L^{(\tau+1)\alpha}(\mathbb{R}^N)}$$

for some positive constant C_9. Repeating again the iterations as in the above arguments, we derive $\|u\|_{L^\infty(\mathbb{R}^N)} \leq C_{10}$ for some positive constant C_{10}.

If u changes sign, we set $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \max\{-u(x), 0\}$. Then it is clear that $u^+ \in H^s_K(\mathbb{R}^N)$ and $u^- \in H^s_K(\mathbb{R}^N)$. Define for each $M > 0$, $\varphi_M(x) = \min\{u^+(x), M\}$. Taking again $\varphi = \varphi_M^{2\tau+1}$ as a test function in $H^s_K(\mathbb{R}^N)$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(\varphi_M^{2\tau+1}(x)-\varphi_M^{2\tau+1}(y))K(x-y)\,dxdy\\&+\int_{\mathbb{R}^N}u(x)\varphi_M^{2\tau+1}(x)\,dx = \lambda\int_{\mathbb{R}^N}f(x,u)\varphi_M^{2\tau+1}(x)\,dx.\end{aligned}$$

which implies

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u^+(x)-u^+(y))(\varphi_M^{2\tau+1}(x)-\varphi_M^{2\tau+1}(y))K(x-y)\,dxdy\\&+\int_{\mathbb{R}^N}u^+(x)\varphi_M^{2\tau+1}(x)\,dx = \lambda\int_{\mathbb{R}^N}f(x,u^+)\varphi_M^{2\tau+1}(x)\,dx.\end{aligned}$$

Proceeding the same way as above we obtain $u^+ \in L^\infty(\mathbb{R}^N)$. Likewise, we get $u^- \in L^\infty(\mathbb{R}^N)$ after taking $\varphi = -\varphi_M^{2\tau+1}$ as a test function, where $\varphi_M(x) = \min\{u^-(x), M\}$. Therefore $u = u^+ - u^-$ is in $L^\infty(\mathbb{R}^N)$. This completes the proof. □

The following lemma is quoted from [18].

Lemma 2.7 ([18]). *Let $I \in C^1(X, \mathbb{R})$ where X is a Banach space. Assume that I satisfies the (PS)-condition, is even and bounded from below, and $I(0) = 0$. If for any $n \in \mathbb{N}$, there exists an n-dimensional subspace X_n and $\rho_n > 0$ such that*

$$\sup_{X_n \cap S_{\rho_n}} I < 0,$$

where $S_\rho := \{u \in X : \|u\|_X = \rho\}$, then I has a sequence of critical values $c_n < 0$ satisfying $c_n \to 0$ as $n \to \infty$.

We are ready to prove the existence of small solutions for the problem (P_λ) with negative energy in the sense that the sequence of solutions converges to zero in the L^∞-norm. To do this, the main tool is the modified functional method which is given in [10, 32].

Theorem 2.8. *Let $0 < s < 1$ with $2s < N$. Assume that* (F1)–(F4) *hold and $f(x,t)$ is odd in t for t small. Then there exists a positive constant λ^* such that for every $\lambda \in [0, \lambda^*)$, the problem (P_λ) has a sequence of weak solutions $\{u_n\}$ satisfying $\|u_n\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $n \to \infty$.*

Proof. Let us define a cut-off function $\varrho \in C^1(\mathbb{R}, \mathbb{R})$ satisfying $\varrho(t) = 1$ for $|t| \le t_0$, $\varrho(t) = 0$ for $|t| \ge 2t_0$, $|\varrho'(t)| \le 2/t_0$, and $\varrho'(t)t \le 0$ for a constant $t_0 \in (0, 1/2)$ with $t_0 < s_0/2$. From the analogous arguments as in [32, Lemma 2.3], we can modify and extend the given function $f(x,t)$ to $\tilde{f} \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying all properties that $\tilde{f}(x,t)$ is odd in t and satisfy

$$\tilde{\mathcal{F}}(x,t) := 2\tilde{F}(x,t) - \tilde{f}(x,t)t \ge 0,$$

$$\tilde{\mathcal{F}}(x,t) = 0 \quad \text{if and only if} \quad t = 0 \quad \text{or} \quad |t| \ge 2t_0,$$

where

$$\tilde{F}(x,t) = \varrho(t)F(x,t) + (1 - \varrho(t))\xi|t|^2 \quad \text{and} \quad \tilde{f}(x,t) = \frac{\partial}{\partial t}\tilde{F}(x,t)$$

for some $\xi > 0$. Then it is easy to show that $\tilde{I}_\lambda(0) = 0$, $\tilde{I}_\lambda \in C^1(H^s_K(\mathbb{R}^N), \mathbb{R})$, and $\tilde{I}_\lambda$ is even on $H^s_K(\mathbb{R}^N)$, where

$$\tilde{I}_\lambda(u) := \Phi_s(u) - \lambda \int_{\mathbb{R}^N} \tilde{F}(x,u)\,dx.$$

First of all, we will show that $\tilde{I}_\lambda$ is coercive on $H^s_K(\mathbb{R}^N)$. Let $u \in H^s_K(\mathbb{R}^N)$ and $\|u\|_{H^s_K(\mathbb{R}^N)} > 1$. Set

$$\begin{aligned}
\Omega_1 &:= \left\{x \in \mathbb{R}^N : |u(x)| \le t_0\right\}, \\
\Omega_2 &:= \left\{x \in \mathbb{R}^N : t_0 \le |u(x)| \le 2t_0\right\}, \text{ and} \\
\Omega_3 &:= \left\{x \in \mathbb{R}^N : 2t_0 \le |u(x)|\right\}.
\end{aligned}$$

Then it follows from (F2) that for $x \in \Omega_1 \cup \Omega_2$, there exists a positive constant C_{11} such that $|F(x,u)| \le \rho(x)|u| + C_{11}|u|^2$. One has

$$\begin{aligned}
\tilde{I}_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)\,dxdy + \frac{1}{2}\int_{\mathbb{R}^N} |u(x)|^2\,dx \\
&\quad - \lambda \int_{\mathbb{R}^N} \tilde{F}(x,u)\,dx \\
&\ge \frac{1}{2}\|u\|^2_{H^s_K(\mathbb{R}^N)} - \lambda \int_{\Omega_1} F(x,u)\,dx \\
&\quad - \lambda \int_{\Omega_2} \left\{\varrho(u)F(x,u) + (1 - \varrho(u))\xi|u|^2\right\}\,dx - \lambda \int_{\Omega_3} \xi|u|^2\,dx \\
&\ge \frac{1}{2}\|u\|^2_{H^s_K(\mathbb{R}^N)} - \lambda \int_{\Omega_1} F(x,u)\,dx - \lambda \int_{\Omega_2} F(x,u)\,dx \\
&\quad - \lambda \int_{\Omega_2} \xi|u|^2\,dx - \lambda \int_{\Omega_3} \xi|u|^2\,dx
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2}\|u\|^2_{H^s_K(\mathbb{R}^N)} - \lambda \int_{\Omega_1\cup\Omega_2} \rho(x)|u|\,dx - \lambda \int_{\Omega_1\cup\Omega_2} C_6|u|^2\,dx \\
&\quad - \lambda \int_{\Omega_2\cup\Omega_3} \xi|u|^2\,dx \\
&\geq \frac{1}{2}\|u\|^2_{H^s_K(\mathbb{R}^N)} - \lambda\|\rho\|_{L^2(\mathbb{R}^N)}\|u\|_{L^2(\mathbb{R}^N)} - \lambda\,(C_6+\xi)\int_{\mathbb{R}^N}|u|^2\,dx \\
&\geq \frac{1}{2}\|u\|^2_{H^s_K(\mathbb{R}^N)} - \lambda\left(\|\rho\|_{L^2(\mathbb{R}^N)} + C_6 + \xi\right)\|u\|^2_{H^s_K(\mathbb{R}^N)}.
\end{aligned}$$

If we set

$$\lambda^* := \frac{1}{2(\|\rho\|_{L^2(\mathbb{R}^N)} + C_6 + \xi)},$$

then we deduce that the functional $\tilde{I}_\lambda$ is coercive for every $\lambda \in [0, \lambda^*)$, that is, $\tilde{I}_\lambda(u) \to \infty$ as $\|u\|_{H^s_K(\mathbb{R}^N)} \to \infty$, as required.

Next we claim that the functional $\tilde{\Psi}' : H^s_K(\mathbb{R}^N) \to (H^s_K(\mathbb{R}^N))^*$, defined by

$$\left\langle \tilde{\Psi}'(u), \varphi \right\rangle = \int_{\mathbb{R}^N} \tilde{f}(x,u)\varphi\,dx \quad \text{for any} \quad \varphi \in H^s_K(\mathbb{R}^N),$$

is compact in $H^s_K(\mathbb{R}^N)$. Let us assume that $u_n \rightharpoonup u$ in $H^s_K(\mathbb{R}^N)$ as $n \to \infty$. Since meas(Ω_2) and meas(Ω_3) are finite, we get that $\Omega_2 = \widetilde{\Omega}_2 \cup N_2$ and $\Omega_3 = \widetilde{\Omega}_3 \cup N_3$, where the sets $\widetilde{\Omega}_2$ and $\widetilde{\Omega}_3$ are bounded, and the sets N_2 and N_3 are measure zero. Let us denote $B^N_R(0)$ by the open N-dimensional ball centered at the origin with radius R whose contains in the bounded sets $\widetilde{\Omega}_2$ and $\widetilde{\Omega}_3$ for sufficiently large $R \in \mathbb{R}$, where $B^N_R(0) := \{x \in \mathbb{R}^N : |x| \leq R\}$. Then it follows upon the definition of $\tilde{f}$ that $\tilde{f}(x,u) = f(x,u)$ on $\mathbb{R}^N \setminus (B^N_R(0) \cup N_2 \cup N_3)$. Thus we deduce that for any $\varphi \in H^s_K(\mathbb{R}^N)$

$$\begin{aligned}
(2.14)\qquad &\sup_{\|\varphi\|_{H^s_K(\mathbb{R}^N)}\leq 1} \left|\langle \tilde{\Psi}'(u_n) - \tilde{\Psi}'(u), \varphi\rangle\right| \\
&= \sup_{\|\varphi\|_{H^s_K(\mathbb{R}^N)}\leq 1} \left|\int_{\mathbb{R}^N} (\tilde{f}(x,u_n) - \tilde{f}(x,u))\varphi\,dx\right| \\
&\leq \sup_{\|\varphi\|_{H^s_K(\mathbb{R}^N)}\leq 1} \left|\int_{B^N_R(0)} (\tilde{f}(x,u_n) - \tilde{f}(x,u))\varphi\,dx\right| \\
&\quad + \sup_{\|\varphi\|_{H^s_K(\mathbb{R}^N)}\leq 1} \left|\int_{\mathbb{R}^N\setminus(B^N_R(0)\cup N_2\cup N_3)} (f(x,u_n) - f(x,u))\varphi\,dx\right|.
\end{aligned}$$

Due to Lemma 2.1, the compact embedding

$$H^s(\mathbb{R}^N) \hookrightarrow L^2(B^N_R(0)) \quad \text{implies} \quad u_n \to u \quad \text{in} \quad L^2(B^N_R(0)) \quad \text{as} \quad n \to \infty.$$

This together with the continuity of the Nemytskij operator with $\tilde{f}$ and acting from $L^2(B^N_R(0))$ into $L^{q'}(B^N_R(0))$ yields that it is easy to see that the first term in the right side of the inequality (2.14) tends to 0 as $n \to \infty$. For the second term in (2.14), we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N\setminus(B_R^N(0)\cup N_2\cup N_3)} (f(x,u_n)-f(x,u))\varphi dx\Big| \\
&\le \int_{\mathbb{R}^N\setminus(B_R^N(0)\cup N_2\cup N_3)} (a(x)\,|u_n(x)|^{q-1})\,|\varphi|\;dx \\
&\quad + \int_{\mathbb{R}^N\setminus(B_R^N(0)\cup N_2\cup N_3)} (a(x)\,|u(x)|^{q-1})\,|\varphi|\;dx \\
&\le \|a\|_{L^{\frac{2_s^*}{2_s^*-q}}(\mathbb{R}^N\setminus(B_R^N(0)\cup N_2\cup N_3))}\big(\|u_n\|^{q-1}_{L^{2_s^*}(\mathbb{R}^N)}+\|u\|^{q-1}_{L^{2_s^*}(\mathbb{R}^N)}\big)\|\varphi\|_{L^{2_s^*}(\mathbb{R}^N)}.
\end{aligned}$$

From the assumption (F2), for above $\varepsilon > 0$, there exists $N(R) \in \mathbb{N}$ such that

$$\|a\|_{L^{\frac{2_s^*}{2_s^*-q}}(\mathbb{R}^N\setminus(B_R^N(0)\cup N_2\cup N_3))} < \varepsilon$$

for $R > N(R)$. As the sequence $\{u_n\}$ is bounded in $H_K^s(\mathbb{R}^N)$, according to Lemma 2.1, one has $\{u_n\}$ is bounded in $L^{2_s^*}(\mathbb{R}^N)$. Thus, it is immediate that

$$\left|\int_{\mathbb{R}^N\setminus(B_R^N(0)\cup N_2\cup N_3)} (f(x,u_n)-f(x,u))\varphi\,dx\right| \le C_12\varepsilon \tag{2.15}$$

for a positive constant C_12. Due to (2.15), we can deduce that

$$\int_{\mathbb{R}^N} (f(x,u_n)-f(x,u))\varphi\,dx \to 0 \quad \text{as} \quad n\to\infty.$$

This implies that $\tilde{\Psi}'$ is compact in $H_K^s(\mathbb{R}^N)$, as claimed.

Since the derivative of Φ_s has a continuous inverse by Lemma 2.5 and the derivative of $\tilde{\Psi}_\lambda$ is compact, it follows from the coercivity of $\tilde{I}_\lambda$ that the functional $\tilde{I}_\lambda$ satisfies the (PS)-condition. The weak lower semicontinuity and the coercivity of $\tilde{I}_\lambda$ ensure that $\tilde{I}_\lambda$ is bounded from below. In order to apply Lemma 2.7, we only need to find for any $n \in \mathbb{N}$, a subspace X_n and $\rho_n > 0$ such that $\sup_{X_n\cap S_{\rho_n}} \tilde{I}_\lambda < 0$. For any $n \in \mathbb{N}$ we find n independent smooth functions ϕ_i for $i = 1,\dots,n$, and define $X_n := \operatorname{span}\{\phi_1,\dots,\phi_n\}$. When $\|u\|_{H_K^s(\mathbb{R}^N)} < 1$ we have that

$$\begin{aligned}
\tilde{I}_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2K(x-y)\,dxdy \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} |u(x)|^2\;dx - \lambda\int_{\mathbb{R}^N}\tilde{F}(x,u)\,dx \\
&\le \frac{1}{2}\|u\|^2_{H_K^s(\mathbb{R}^N)} - \lambda C_{13}\int_{\mathbb{R}^N} F(x,u)\,dx
\end{aligned}$$

for a positive constant C_{13}. It follows from the assumption (F4) that for a sufficiently large $C_{14} > 0$, there exists $\delta_0 > 0$ such that $|t| < \delta_0$ implies

$$\int_{\mathbb{R}^N} F(x,t)\,dx \geq \frac{C_{14}}{2} \int_{\mathbb{R}^N} |t|^2\,dx.$$

By this relation and the fact that all norms on X_n are equivalent, choosing a suitable constant C_{13} and sufficiently small $\rho_n > 0$, we can obtain

$$\sup_{X_n \cap S_{\rho_n}} \tilde{I}_\lambda < 0.$$

By Lemma 2.7, we get a sequence $c_n < 0$ for $\tilde{I}_\lambda$ satisfying $c_n \to 0$ when n goes to ∞. Then for any (PS)-sequence $\{u_n\}$ in $H^s_K(\mathbb{R}^N)$ for $\tilde{I}(u)$ satisfying $\tilde{I}_\lambda(u_n) = c_n$ and $\tilde{I}'_\lambda(u_n) = 0$ has a convergent subsequence. The above modified functional methods imply that 0 is the only critical point with 0 energy and the subsequence of $\{u_n\}$ has to converge to 0 as $n \to \infty$. Invoking an indirect argument, we deduce that the sequence $\{u_n\}$ has to converge to 0 as $n \to \infty$. On the other hand, we get $u_n \in L^r(\mathbb{R}^N)$ for all $2^*_s \leq r \leq \infty$ due to Proposition 2.6. Since $\|u_n\|_{L^\infty(\mathbb{R}^N)} \to 0$, we deduce $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq t_0$ for large n. Thus the sequence $\{u_n\}$ are weak solutions of the problem (P_λ). The proof is complete. □

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