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STRUCTURE OF UNIT-IFP RINGS

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Abstract. In this article we first investigate a sort of unit-IFP ring by which Antoine provides very useful information to ring theory in relation with the structure of coefficients of zero-dividing polynomials. Here we are concerned with the whole shape of units and nilpotent elements in such rings. Next we study the properties of unit-IFP rings through group actions of units on nonzero nilpotent elements. We prove that if R is a unit-IFP ring such that there are finite number of orbits under the left (resp., right) action of units on nonzero nilpotent elements, then R satisfies the descending chain condition for nil left (resp., right) ideals of R and the upper nilradical of R is nilpotent.

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. The group of units in R is written by $U(R)$. The Jacobson radical, Wedderburn radical (i.e., sum of all nilpotent ideals), the upper nilradical (i.e., the sum of all nil ideals), the lower nilradical (i.e., the intersection of all prime ideals), and the set of all nilpotent elements in R are denoted by $J(R)$, $N_1(R)$, $N^*(R)$, $N_*(R)$, and $N(R)$, respectively. It is well-known that $N_1(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). The polynomial ring with an indeterminate x over R is denoted by $R[x]$.

In Section 1, first our study concerns a sort of unit-IFP ring in [2, Theorem 4.7] by which Antoine could provide very excellent and fruitful information to the study of the structure of zero-divisors in polynomial rings. Let K be a field, $n \geq 2$, and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^n and set $R = A/I$. Kim et al. showed that R is a unit-IFP ring for the case of $n = 2$ in [7, Example 1.1]. We show that such kind *of rings are* unit-IFP and prime in a more general situation. In the procedure we examine the structure of units and nilpotent elements in them. So the investigation of the structure of this

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kind of ring may give widely applicable information to the research of various topics related to zero-divisors.

In Section 2, we study the properties of unit-IFP rings through group actions of units on nilpotent elements. We prove the following. Let R be a unit-IFP ring and suppose that there are finite number of orbits under the left (resp., right) action of the group of units on the nonzero nilpotent elements in R. Then R satisfies the descending chain condition for nil left (resp., right) ideals of R, and the upper nilradical of R is nilpotent.

1. Units and nilpotent elements in Antoine's unit-IFP rings

In this section we investigate a ring property of an example constructed by Antoine in [2, Example 4.8]. This extends the argument of [7, Example 1.1] to the general case, extending the class of unit-IFP rings more deeply. So the results obtained here may be useful to the study of the structure of units and nilpotent elements.

Due to Bell [3], a ring R is said to be *IFP* if $ab = 0$ for $a, b \in R$ implies $aRb = 0$. A ring is usually called *reduced* if it has no nonzero nilpotent elements. A ring is usually called *Abelian* if every idempotent is central. It is easily checked that commutative rings and reduced rings are contained in the class of IFP rings. IFP rings are shown easily to be Abelian. It is also easily shown that if R is an IFP ring, then RaR is nilpotent for all $a \in N(R)$, entailing $N_1(R) = N_*(R) = N^*(R) = N(R)$.

Following Kim et al. [7], a ring R is said to be *unit-IFP* if $ab = 0$ for $a, b \in R$ implies $aU(R)b = 0$. IFP rings are clearly unit-IFP, and the converse need not hold by [7, Example 1.1]. Kim et al. provide various results for units and nilpotent elements which are useful to the research of related topics. For example, they show that Köthe's conjecture (i.e., the upper nilradical contains every nil left ideal) holds for unit-IFP rings in [7, Theorem 1.3(1)]. The facts in the following are necessary through our study in this article.

Lemma 1.1. (1) *A ring R is unit-IFP if and only if*

$$a_1U(R)a_2U(R)a_3 \cdots a_{n-1}U(R)a_n = 0$$

whenever $a_1a_2 \cdots a_n = 0$ for $a_1, a_2, \ldots, a_n \in R$.

(2) *If R is a unit-IFP ring, then $N_1(R) = N_*(R) = N^*(R)$.*

(3) *Unit-IFP rings are Abelian.*

(4) *If R is a unit-IFP ring, then $N(R)$ forms a subring of R.*

(5) *If R is a unit-IFP ring, then so is $R/N^*(R)$.*

Proof. (1) is shown by applying the definition inductively. (2), (3), and (4) are proved by [7, Theorem 1.3(2)], [7, Lemma 1.2(2)], and [7, Lemma 1.2(5)], respectively.

(5) Let R be a unit-IFP ring. Then $us \in N(R)$ for all $u \in U(R)$ and $s \in N(R)$ by (1). This implies $u+s \in U(R)$ (indeed, $(u+s)^{-1} = (1+u^{-1}s)^{-1}u^{-1}$).

Write $\bar{r} = r + N^*(R)$ for $r \in R$. Since $N^*(R)$ is nil, it is easily checked that $U(R/N^*(R)) = \{u + N^*(R) \mid u \in U(R)\}$.

Suppose $\bar{a}\bar{b} = 0$ for $a, b \in R$. Then $ab \in N^*(R)$ and $RabR \subseteq N^*(R)$ follows. Let $r, s \in R$ be arbitrary. Then since R is unit-IFP, $raubs \in N(R)$ for all $u \in U(R)$ by (1). But this implies $Raub$ is a nil left ideal of R. Since R is unit-IFP, Köthe's conjecture holds for R by [7, Theorem 1.3(1)]. So $Raub$ is contained in $N^*(R)$, entailing $aub \in N^*(R)$. This implies $\bar{a}\bar{u}\bar{b} = 0$, and $\bar{a}U(R/N^*(R))\bar{b} = 0$ follows. Therefore $R/N^*(R)$ is also a unit-IFP ring. □

Recall that a ring R is said to be *directly finite* (or *Dedekind finite*) if $ab = 1$ for $a, b \in R$ implies $ba = 1$. It is easily checked that Abelian rings are directly finite, and hence unit-IFP rings are directly finite by Lemma 1.1(3). We use this fact freely throughout.

In what follows, we analyze the structure of a kind of unit-IFP ring which is provided by Antoine in [2, Example 4.8]. An element u of R is called *right regular* if $ur = 0$ for $r \in R$ implies $r = 0$. The *left regular* can be defined similarly. An element is *regular* if it is both left and right regular. In the following we deal with a general situation of [7, Example 1.1], which extend the class of unit-IFP rings widely. For the purpose, we refer to the construction of Antoine in [2, Theorem 4.7]. Let K be a field and R, S be algebras over K. $R *_K S$ denotes the ring coproduct of R and S.

Theorem 1.2. *Let K be a field, $n \geq 2$, and $A = K\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^n and set $R = A/I$, where each element of A is identified with its image in R for simplicity. Then R is a unit-IFP ring such that*

$$U(R) = \{k + g + b^p f b^q \mid k \in K\backslash\{0\},\ g \in bK[b],\ f \in R,\ \text{and}\ p, q \geq 1\ \text{with}\ p + q \geq n\}.$$

Moreover R is a prime ring.

Proof. We follow the construction in [2, Example 4.8]. The ring R is isomorphic to $K[a] *_K \frac{K[b]}{b^n K[b]}$ that is the ring coproduct of $K[a]$ and $K[b]/b^n K[b]$ over K.

By [4, Corollary 2.16(i)], $U(R)$ is generated by the units of $R_1 = K[a]$ and $R_2 = K[b]/b^n K[b]$, together with elements of the form $1 - \gamma\delta\epsilon$, where $\delta \in R$ and $\gamma, \epsilon \in R_i$ for some i, such that $\epsilon\gamma = 0$. Clearly $U(R_1) = U(K[a]) = K\backslash\{0\}$. So, if both γ and ϵ are nonzero, then they are contained in R_2, and hence we have that $\epsilon = b^p f(b)$ and $\gamma = b^q g(b)$ with $p + q \geq n$. But $\gamma\delta\epsilon = b^q g(b)\delta b^p f(b) = b^q[g(b)\delta f(b)]b^p$ with $g(b)\delta f(b) \in R$. Therefore

$$U(R) = \{k + g + b^p f b^q \mid k \in K\backslash\{0\},\ g \in bK[b],\ f \in R,\ \text{and}\ p, q \geq 1\ \text{with}\ p + q \geq n\}.$$

Next suppose that $\alpha\beta = 0$ for $\alpha, \beta \in R\backslash\{0\}$. Then, by [4, Corollary 2.16(ii)], there exist $u \in U(R)$ and sets C, D in some R_i with $CD = 0$ such that $\alpha \in RCu$

and $\beta \in u^{-1}DR$. $CD = 0$ implies $C, D \subseteq R_2$. Furthermore, $CD = 0$ implies that

$$C \subseteq b^l R_2 \text{ and } D \subseteq b^m R_2 \text{ for some } l, m \geq 1 \text{ with } l + m \geq n.$$

Then α is a sum of nonzero elements of the form $r_1 b^l g_1(b) u$ and β is a sum of nonzero elements of the form $u^{-1} b^m g_2(b) r_2$, where $g_1(b), g_2(b) \in K[b]$ and $r_1, r_2 \in R$. Consequently $\alpha = r b^l u$ and $\beta = u^{-1} b^m s$ with $r, s \in R$.

We will show $\alpha v \beta = [rb^l u]v[u^{-1}b^m s] = 0$ for all $v \in U(R)$. Then $\alpha U(R)\beta = 0$ and hence R is unit-IFP. Here $uvu^{-1} \in U(R)$, say $w = uvu^{-1}$. By the argument above, $w = k_1 + bg(b) + b^p f b^q$ with $k_1 \in K\backslash\{0\}$, $g(b) \in K[b]$, and $f \in R$, where $p + q \geq n$. Then we obtain

$$\begin{aligned}[rb^l u]v[u^{-1}b^m s] &= [rb^l]w[b^m s] \\ &= [rb^l][k_1 + bg(b) + b^p f b^q][b^m s] \\ &= [rb^l][k_1 + bg(b)][b^m s] + [rb^l][b^p f b^q][b^m s] \\ &= [rb^l][b^p f b^q][b^m s],\end{aligned}$$

noting $[rb^l][k_1 + bg(b)][b^m s] = 0$ because $l + m \geq n$. But $l + m \geq n$ and $p + q \geq n$, so $l + p \geq n$ or $q + m \geq n$. Thus $[rb^l][b^p f b^q][b^m s] = 0$, proving that R is a unit-IFP ring. But R is not IFP since $b^n = 0$ and $b^{n-1}ab \neq 0$.

Next we prove that R is prime. Suppose that $\alpha R \beta = 0$ for $\alpha, \beta \in R$. Assume on the contrary that α and β are both nonzero. Note $\alpha\beta = 0$. Then, by the argument above, α is a sum of nonzero elements of the form $r_1 b^l g_1(b) u$ and β is a sum of nonzero elements of the form $u^{-1} b^m g_2(b) r_2$, where $l + m \geq n$, $u \in U(R)$, $g_1(b), g_2(b) \in K[b]$ and $r_1, r_2 \in R$. Furthermore, $\alpha a \beta$ is a sum of nonzero elements of the form $[r_1 b^l g_1(b) u]a[u^{-1} b^m g_2(b) r_2]$.

Next we use the method in [6, Example 14]. Observe that nonzero monomials in R can be embedded into the set of natural numbers through the corresponding "$a \to 1, b \to 2$", σ say. Then they are totally ordered via σ (for example, $a < b < aa < ab < ba < aba < aab < aaaa$ because $1 < 2 < 11 < 12 < 21 < 121 < 112 < 1111$). Note that 2^n does not occur because $b^n = 0$. We identify f with $\sigma(f)$ for a nonzero monomial f in R.

We can write

$$\alpha = c_1 d_1 + \cdots + c_{v_1} d_{v_1} \quad \text{and} \quad a\beta = c'_1 d'_1 + \cdots + c'_{w_1} d'_{w_1}$$

with $d_1 < \cdots < d_{v_1}$, $d'_1 < \cdots < d'_{w_1}$, where d_i's, d'_j's, are nonzero monomials of coefficient 1, and $c_i, c'_j \in K\backslash\{0\}$. Then every $(c_i d_i)(c'_j d'_j) = (c_i c'_j)(d_i d'_j)$ is of the form $[r_1 b^l g_1(b) u]a[u^{-1} b^m g_2(b) r_2]$, hence nonzero. Furthermore $d_{v_1} d'_{w_1}$ is the largest in the set $\{d_i d'_j\}$; hence this must be zero because $\alpha a \beta = 0$, contrary to $d_{v_1} d'_{w_1} \neq 0$. So $\alpha = 0$ or $\beta = 0$. Therefore R is prime. □

In Theorem 1.2, we can also write $U(R) = \{k + g + b^p f b^q \mid k \in K\backslash\{0\},\ g \in bK[b],\ f \in R, \text{ and } p, q \geq 1 \text{ with } p + q \geq n\}$, where a occurs in every nonzero term of f. The results in [7, Example 1.1] are obtained form Theorem 1.2 as the case of $n = 2$.

In the following we study the structure of zero divisors in the unit-IFP ring R in Theorem 1.2 further.

Remark. Let R be the unit-IFP ring in Theorem 1.2. If $\alpha\beta = 0$ for $0 \neq \alpha, \beta \in R$, then $\alpha = rb^l u$ and $\beta = u^{-1}b^m s$ with $l + m \geq n$, $r, s \in R$ and $u \in U(R)$. Say $u = k + g + b^p f b^q$ with $k \in K\backslash\{0\}$, $g \in bK[b]$, $p + q \geq n$, and $f \in R$. Note that $l + p \geq n$ or $q + m \geq n$. We can write $u = g_1 + b^p f b^q$ and $u^{-1} = g_2 + b^p f' b^q$ with $g_i \in K[b]$ and $f' \in R$.

Suppose $l + p \geq n$. Then

$$\alpha = rb^l u = rb^l(g_1 + b^p f b^q) = r(g_1 b^l + b^{l+p} f b^q) = r g_1 b^l$$

and

$$\beta = (g_2 + b^p f' b^q) b^m s = (b^m g_2 + b^p f' b^{q+m}) s = b^{m_1} s_1$$

with $m_1 = \min\{m, p\}$ and $s_1 \in R$.

But since $l + m, l + p \geq n$, we have $l + m_1 \geq n$.

Suppose $q + m \geq n$. Then

$$\alpha = rb^l u = rb^l(g_1 + b^p f b^q) = r(g_1 b^l + b^{l+p} f b^q) = r_1 b^{l_1}$$

with $l_1 = \min\{l, q\}$ and $r_1 \in R$ and

$$\beta = (g_2 + b^p f' b^q) b^m s = (b^m g_2 + b^p f' b^{q+m}) s = b^m g_2 s.$$

But since $l + m, q + m \geq n$, we have $l_1 + m \geq n$.

Therefore $\alpha = r' b^{l'}$ and $\beta = b^{m'} s'$ for some $r', s' \in R$ and $l', m' \geq 1$ with $l' + m' \geq n$.

In the following we observe the shape of nilpotent elements in the ring in Theorem 1.2.

Theorem 1.3. *Let K be a field, $n \geq 2$, and $A = K\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^n and set $R = A/I$, where each element of A is identified with its image in R for simplicity. Then*

$$N(R) = \{g + b^p f b^q \mid g \in bK[b],\ f \in R,\ \text{and } p, q \geq 1 \text{ with } p + q \geq n\} \text{ and}$$
$$N(R)^n = 0.$$

Proof. Let $c \in N(R)$ and $u \in U(R)$. Since R is unit-IFP by Theorem 1.2, $uc \in N(R)$ by Lemma 1.1(1). Then we also have $u^{-1}c \in N(R)$ (say $(u^{-1}c)^k = 0$), and this implies $u + c \in U(R)$, noting that $(u + c)^{-1} = (1 - u^{-1}c + \cdots + (-1)^{k-1}(u^{-1}c)^{k-1})u^{-1}$. Next, by Theorem 1.2, $U(R) = \{k + g + b^p f b^q \mid k \in K\backslash\{0\},\ g \in bK[b],\ f \in R, \text{ and } p, q \geq 1 \text{ with } p + q \geq n\}$. So $u + c \in U(R)$ implies that c is of the form $g + b^p f b^q$.

Consider the expansion of $(g + b^p f b^q)^n$ and let e be a term of it. Note that $g^n = 0$ because $b^n = 0$.

If $b^p f b^q$ does not occur in e, then $e = g^n = 0$.

If two or more b^pfb^q's occur in e, then e is of the form $r_1(b^pfb^q)g^h(b^pfb^q)r_2$ with $h \geq 0$ and $r_i \in R$; hence e is zero because every term of $b^qg^hb^p$ contains $b^{p+q} = 0$.

Assume that exactly one b^pfb^q occurs in e, i.e., $e = g^h(b^pfb^q)g^k$ with $h, k \geq 0$ and $h + k = n - 1$. Then every term of e is of the form $r_1'b^{h+p}f'b^{q+k}r_2'$ with $h \geq 0$ and $f', r_i' \in R$. But $h+p+q+k \geq 2n-1$ and so $h+p \geq n$ or $q+k \geq n$; hence e is zero.

This implies $(g + b^pfb^q)^n = 0$, and therefore we can say that every nilpotent element in R is exactly of the form $g + b^pfb^q$. This concludes

$$N(R) = \{g + b^pfb^q \mid g \in bK[b],\ f \in R, \text{ and } p, q \geq 1 \text{ with } p + q \geq n\}.$$

Consider next $N(R)^n$ and let F be an element of the form

$$(g_1 + b^{p_1}f_1b^{q_1})(g_2 + b^{p_2}f_2b^{q_2}) \cdots (g_n + b^{p_n}f_nb^{q_n})$$

with $g_i + b^{p_i}f_ib^{q_i} \in N(R)$. Say that $g_i + b^{p_i}f_ib^{q_i} \neq 0$ for all i. Note that $b^n = 0$ implies $g_1 \cdots g_n = 0$. Let E be a product of n-times of g_i's and $b^{p_i}f_ib^{q_i}$'s in the expansion of F.

If any $b^{p_i}f_ib^{q_i}$ does not occur in E, then $E = g_1 \cdots g_n = 0$.

Assume that exactly one $b^{p_i}f_ib^{q_i}$ occurs in E, i.e.,

$$E = g_{i_1} \cdots g_{i_s}(b^{p_i}f_ib^{q_i})g_{i_{s+1}} \cdots g_{i_{n-1}}.$$

Then every term of E is of the form $r_1b^{s+p_i}f'b^{q_i+(n-1-s)}r_2$ with $f', r_1, r_2 \in R$. But $s + p_i + q_i + (n - 1 - s) = p_i + q_i + n - 1 \geq 2n - 1$, and so $s + p_i \geq n$ or $q_i + (n - 1 - s) \geq n$; hence E is zero.

Assume that two or more $b^{p_i}f_ib^{q_i}$'s occur in E. Then E is of the form

$$\begin{aligned} &g_{i_1} \cdots g_{i_{s_1}}(b^{p_{j_1}}f_{j_1}b^{q_{j_1}})g_{i_{s_1+1}} \cdots g_{i_{s_2}}(b^{p_{j_2}}f_{j_2}b^{q_{j_2}})g_{i_{s_2+1}} \cdots \\ &g_{i_{s_3}} \cdots g_{i_{s_{(t-1)}+1}} \cdots g_{i_{s_t}}(b^{p_{j_t}}f_{j_t}b^{q_{j_t}})g_{i_{s_t+1}} \cdots g_{i_{n-t}} \end{aligned}$$

with $2 \leq t \leq n$. So every term of E contains

$$b^{s_1+p_{j_1}}f_1'b^{q_{j_1}+(s_2-s_1)+p_{j_2}}f_2'b^{q_{j_2}+(s_3-s_2)+p_{j_3}}f_3' \cdots b^{q_{j_{t-1}}+(s_t-s_{t-1})+p_{j_t}}f_t'b^{q_{j_t}+n-t-s_t}$$

where $f_j' \in R$. Here assume that $m_l = q_{j_l} + (s_{l+1} - s_l) + p_{j_{l+1}} \leq n - 1$ for all $l = 1, 2, \ldots, t - 1$. But

$$\begin{aligned} &(s_1 + p_{j_1}) + (m_1 + \cdots + m_{t-1}) + (q_{j_t} + n - t - s_t) \geq tn + (n - t) \text{ and} \\ &\sum_{l=1}^{t-1} m_l \leq (t - 1)n - (t - 1). \end{aligned}$$

This yields

$$(s_1 + p_{j_1}) + (q_{j_t} + n - t - s_t) \geq [tn + n - t] - [(t - 1)n - (t - 1)] = 2n - 1,$$

entailing $s_1+p_{j_1} \geq n$ or $q_{j_t}+n+1-t-s_t \geq n$. So $b^{s_1+p_{j_1}} = 0$ or $b^{q_{j_t}+n-t-s_t} = 0$; hence $E = 0$.

Therefore $F = 0$ and this implies $N(R)^n = 0$. □

Consider the unit $r = k + g + b^p f b^q$ in Theorem 1.2. Then the inverse of r is

$$r^{-1} = k^{-1}(1 - [k^{-1}(g + b^p f b^q)] + \cdots + (-1)^{n-1}[k^{-1}(g + b^p f b^q)]^{n-1})$$

by the argument in the proof of Theorem 1.3, noting $[k^{-1}(g + b^p f b^q)]^n = 0$.

Next consider the structure of $0 \neq r = g + b^p f b^q \in N(R)$. Then $r^k = 0$ and $r^{k-1} \neq 0$ for some $k \geq 2$. In what follows, we use [4, Corollary 2.16(ii)] and the argument in the proof of Theorem 1.2 without referring.

Consider $r(r^{k-1}) = 0$. Then there exist $u \in U(R)$ and sets C, D in R_2 with $CD = 0$ such that $r \in RCu$ and $r^{k-1} \in u^{-1}DR$. Here $C, D \subseteq R_2$ and

$$C \subseteq b^l R_2 \text{ and } D \subseteq b^m R_2 \text{ for some } l, m \geq 1 \text{ with } l + m \geq n.$$

Consider $(r^{k-1})r = 0$. Then there exist $u' \in U(R)$ and sets C', D' in R_2 with $C'D' = 0$ such that $r^{k-1} \in RC'u'$ and $r \in u'^{-1}D'R$. Here $C', D' \subseteq R_2$ and

$$C' \subseteq b^s R_2 \text{ and } D' \subseteq b^t R_2 \text{ for some } s, t \geq 1 \text{ with } s + t \geq n.$$

Summarizing, r is a sum of nonzero elements of the form $u'^{-1}b^t r_1 b^l u$. Furthermore, by Remark above, $r = b^{t'} r' b^{l'}$ with $t', l' \geq 1$ and $r' \in R$.

Next we apply Theorem 1.2 to extend the class of unit-IFP rings. The direct product of unit-IFP rings is also unit-IFP by [7, Proposition 1.10(1)]. So we can obtain the following by Theorem 1.2.

Corollary 1.4. *Let K be a field, $n \geq 2$, and $A = K\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over K. Let I_n be the ideal of A generated by b^n and set $R_n = A/I_n$. Then the direct product $\prod_{n=2}^{\infty} R_n$ of R_n's is a unit-IFP ring.*

Note that the direct product in Corollary 1.4 is a unit-IFP ring that is semiprime. In the following we consider a direct (or injective) limit of rings, each of which is of the same structure as in Theorem 1.2, such that it is also prime.

Example 1.5. Let K be a field, $n \geq 2$, and $A = K\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over K. Let I_i be the ideal of A generated by b^{2^i} and set $R_i = A/I_i$, where $i = 1, 2, \ldots$. Identify each element of A with its image in R_i for simplicity. Consider $R_i = A/I_i$ for all $i \geq 1$, and define a map

$$\sigma : R_i \to R_{i+1} \text{ by } b \mapsto b^2.$$

Then R_i can be considered as a subring of R_{i+1} via σ.

Let R be the direct limit of $\{R_i, \sigma_{st}\}$, where $\sigma_{st} = \sigma^{t-s}$ for $s \leq t$. Note $R = \cup_{i=1}^{\infty} R_i$. We first show that R is unit-IFP. Recall that every R_i is unit-IFP by Theorem 1.2. Let $\alpha\beta = 0$ for $\alpha, \beta \in R$, and take any $\gamma \in U(R)$. Then there exists $j \geq 1$ such that $\alpha, \beta, \gamma \in R_j$. Since γ is also a unit in R_j, we have $\alpha\gamma\beta = 0$ because R_j is unit-IFP. This implies that R is unit-IFP.

Next we show that R is prime. Suppose that $\alpha R \beta = 0$ for $\alpha, \beta \in R$. Assume on the contrary that both α and β are nonzero. There exists $j \geq 1$ such that

$\alpha, \beta \in R_j$. Then we also have $\alpha\beta = 0$ in R_j. But, by the argument in the proof of Theorem 1.2, $\alpha a \beta \neq 0$, entailing $\alpha R_j \beta \neq 0$. This implies $\alpha R \beta \neq 0$, contrary to $\alpha R \beta = 0$. Thus $\alpha = 0$ or $\beta = 0$, and so R is prime.

By Theorem 1.3, we have

$$N(R_i) = \{g + b^p f b^q \mid g \in bK[b],\ f \in R_i,\ \text{and } p, q \geq 1 \text{ with } p + q \geq 2^i\}$$

and $N(R) = \cup_{i=1}^{\infty} N(R_i)$.

2. Group actions in unit-IFP rings

In this section we study the structure of unit-IFP rings in relation with group actions of units on nonzero nilpotent elements. Let R be a unit-IFP ring. Then, by Lemma 1.1(1), both ua and au are contained in $N(R)$ for all $a \in N(R)$ and $u \in U(R)$. This enables us to consider group actions of $U(R)$ on $N(R)'$, i.e., maps from $U(R) \times N(R)'$ to $N(R)'$ given by $(u, a) \mapsto ua$ (the left action) and $(u, a) \mapsto au^{-1}$ (the right action), where $N(R)' = N(R) \backslash \{0\}$. Under these actions, we use the following usual expression. The *orbit* of $a \in N(R)'$ is $o_l(a) = \{ua \mid u \in U(R)\} = U(R)a$ (resp., $o_r(a) = \{au^{-1} \mid u \in U(R)\} = aU(R)$) under the left (resp., right) action of $U(R)$ on $N(R)'$. We write $o(a)$ when $o_l(a) = o_r(a)$. We use these definitions and notations freely throughout this section.

Han and Park [5] investigate the structure of rings in which the number of orbits is finite under the left (resp., right) actions of units on nonzero nonunits. Indeed they prove that such kind of ring is left (resp., right) Artinian (hence Jacobson radicals are nilpotent) in [5, Lemma 3.1]. We prove here that the upper nilradicals are nilpotent when the number of orbits is finite under the left (right) action of units on nonzero nilpotent elements.

Theorem 2.1. *Let R be a unit-IFP ring with $N(R) \neq 0$. Suppose that there are finite number of orbits under the left* (*resp., right*) *action of $U(R)$ on $N(R)'$. Then R satisfies the descending chain condition for nil left* (*resp., right*) *ideals of R. Moreover if n is the number of orbits, then $N^*(R)^{n+1} = 0$.*

Proof. Let n be the number of orbits under the left action of $U(R)$ on $N(R)'$. We apply the proof of [5, Lemma 3.1] in the first half part of this proof.

Assume on the contrary that there exists a non-stationary descending chain of nil left ideals of R,

$$I_1 \supsetneq I_2 \supsetneq \cdots I_i \supsetneq I_{i+1} \supsetneq \cdots \ (i = 1, 2, \ldots)$$

say. Take $a_i \in I_i \backslash I_{i+1}$ for all $i = 1, 2, \ldots$, and consider the orbits

$$o_l(a_1), o_l(a_2), \ldots, o_l(a_i), \ldots$$

under the left action of $U(R)$ on $N(R)'$. Then

$$a_i \in o_l(a_i) = U(R)a_i \subseteq Ra_i \subseteq I_i$$

for all i, and so these orbits $o_l(a_i)$ are all distinct because $a_i \in I_i$ and $a_i \notin I_{i+1}$. This implies that there are infinitely many orbits $o_l(a_i)$ $(i = 1, 2, \ldots)$, a contradiction. Therefore R satisfies the descending chain condition for nil left ideals of R.

By this result, for any strictly descending chain $J_1 \supsetneq J_2 \supsetneq \cdots$ of nil left ideals of R, we obtain $J_{n+1} = J_{n+2} = \cdots$ because n is the number of orbits under the left action. The proof for the nil right ideals is analogous.

Next, we claim that $N^*(R)$ is nilpotent. Consider the descending chain of nil ideals

$$N^*(R) \supseteq N^*(R)^2 \supseteq \cdots \supseteq N^*(R)^i \supseteq N^*(R)^{i+1} \supseteq \cdots$$

in R for $i = 1, 2, \ldots$. Then $N^*(R)^{n+1} = N^*(R)^{n+2} = \cdots$ by the preceding result.

Assume $N^*(R)^{n+1} \neq 0$. Here we apply the well-known method related to the nilpotency of Jacobson radical. Consider the set S of all nil left ideals J such that $N^*(R)^{n+1}J \neq 0$. Then S is nonempty because $N^*(R)^{n+1}N^*(R) = N^*(R)^{n+1} \neq 0$, i.e., $N^*(R) \in S$. Since R satisfies the descending chain condition for nil left ideals of R, S has a minimal element J_0. Then $N^*(R)^{n+1}J_0 \neq 0$. Take $x \in J_0$ such that $N^*(R)^{n+1}x \neq 0$. This yields

$$N^*(R)^{n+1}N^*(R)^{n+1}x = N^*(R)^{2n+2}x = N^*(R)^{n+1}x \neq 0,$$

entailing $N^*(R)^{n+1}x \in S$. But $N^*(R)^{n+1}x = J_0$ by the minimality of J_0 because $N^*(R)^{n+1}x \subseteq J_0$. Then $yx = x$, i.e., $(1-y)x = 0$, for some $y \in N^*(R)^{n+1}$. Since $1 - y \in U(R)$, we have $0 = (1-y)^{-1}(1-y)x = x$. This contradicts $x \neq 0$. Thus we must have $N^*(R)^{n+1} = 0$. The proof for the right case is similar. □

The converse of Theorem 2.1 need not hold by the following.

Example 2.2. Let R be the unit-IFP ring in Theorem 1.2. Then, by Theorems 1.2 and 1.3, $U(R) = \{k + g + b^p f b^q \mid k \in K\backslash\{0\},\ g \in bK[b],\ f \in R, \text{ and } p, q \geq 1 \text{ with } p + q \geq n\}$ and $N(R) = \{g + b^p f b^q \mid g \in bK[b],\ f \in R, \text{ and } p, q \geq 1 \text{ with } p + q \geq n\}$.

Clearly $N(R) \neq 0$. But $N^*(R) = 0$ by Theorem 1.2, whence R has no nonzero nil left (right) ideals because Köthe's conjecture holds for R by [7, Theorem 1.3(1)]. We claim that there are infinitely many orbits under the left (right) action of $U(R)$ on $N(R)'$.

Let $c_i = ba^ib^{n-1}$ for $i = 1, 2, \ldots$. Then $0 \neq c_i \in N(R)$. Suppose that $c_i = (k + g_1 + b^{p_1}f_1b^{q_1})(g + b^pfb^q)$ for some $g + b^pfb^q \in N(R)$ and $k + g_1 + b^{p_1}f_1b^{q_1} \in U(R)$, i.e., $c_i \in o_l(g + b^pfb^q)$. Then we first get $kg = 0$ and $g_1g = 0$, from the equality

$$c_i = kg + g_1g + (b^pkfb^q + b^{p_1}f_1b^{q_1}g + g_1b^pfb^q + b^{p_1}f_1b^{q_1}b^pfb^q).$$

So $k = 0$ or $g = 0$. Assume $g \neq 0$. Then $k = 0$ and so $c_i = b^{p_1}f_1b^{q_1}g + g_1b^pfb^q + b^{p_1}f_1b^{q_1}b^pfb^q$. This induces a contradiction because every term of the right hand side of this equality contains $n+1$ or more number of b's. Thus $g = 0$ and this yields $c_i = b^pkfb^q + g_1b^pfb^q + b^{p_1}f_1b^{q_1}b^pfb^q$. Here $g_1b^pfb^q + b^{p_1}f_1b^{q_1}b^pfb^q =$

$(g_1 + b^{p_1} f_1 b^{q_1}) b^p f b^q$ must be zero because every term contains $n+1$ or more number of b's, entailing $c_i = b^p k f b^q$. This implies $p = 1$, $q = n-1$, and $kf = a^i$. Thus

$$ba^ib^{n-1} = (k + g_1 + b^{p_1} f_1 b^{q_1})(bab^{n-1}) \in o_l(ba^ib^{n-1}),$$

where $(g_1 + b^{p_1} f_1 b^{q_1})(ba^ib^{n-1}) = 0$. From this result, we can say that $o_l(ba^ib^{n-1})$ and $o_l(ba^jb^{n-1})$ are distinct orbits when $i \neq j$. Therefore there are infinitely many orbits under the left action of $U(R)$ on $N(R)'$. The argument for the right case is analogous.

The condition, that there are finite number of orbits under the left (resp., right) action of $U(R)$ on $N(R)'$, in Theorem 2.1 is not superfluous as the following shows. Let $R_i = \mathbb{Z}_p^{i+1}$ for all $i \geq 1$, where p is a prime. Then the subring S of the direct product R of R_i's which is generated by the direct sum of R_i's and 1_R is commutative (hence unit-IFP). But there are infinitely many orbits under the action of $U(S)$ on $N(S)'$, and $N^*(S)$ is not nilpotent. We see an application of Theorem 2.1 in the following.

Corollary 2.3. (1) *Let R be a unit-IFP ring with $N(R) \neq 0$. Suppose that there are finite number of orbits under the left (right) action of $U(R)$ on $N(R)'$. Then both $J(R[x])$ and $J(\bar{R}[x])$ are nilpotent, where $\bar{R} = R/N^*(R)$.*

(2) *Let R be a unit-IFP ring in which $N(R) \neq 0$. Suppose that every nonunit is nilpotent, and there are finite number of orbits under the left (resp., right) action of $U(R)$ on $N(R)'$. Then R is left (resp., right) Artinian. Especially, $R/J(R)$ is a finite direct product of division rings.*

Proof. (1) Suppose that there are finite number of orbits under the left action of $U(R)$ on $N(R)'$. Then $N^*(R)$ is nilpotent by Theorem 2.1. So $J(R[x])$ is nilpotent because $J(R[x]) \subseteq N^*(R)[x]$ by [1, Theorem 1]. Since R is unit-IFP, $\bar{R}$ is also unit-IFP by Lemma 1.1(5). Note that there are finite number of orbits under the left regular action of $U(\bar{R})$ on $N(\bar{R})'$. So $J(\bar{R}[x])$ is also nilpotent by the same argument as above. The proof for the right case is similar.

(2) Suppose that $N(R)'$ contains all nonzero nonunits in R, and there are finite number of orbits under the left (resp., right) action of $U(R)$ on $N(R)'$. Then R is left (resp., right) Artinian by Theorem 2.1. Since R is unit-IFP, R is Abelian by Lemma 1.1(3). This yields that $R/J(R)$ is a finite direct product of division rings by help of Wedderburn-Artin theorem. □

Under the hypothesis of Theorem 2.1, Jacobson radicals need not be nilpotent. Let $R = A[[x]]$ be the power series ring over a division ring A. Then R is a domain and hence the number of orbits is zero. But $J(R) = xA[[x]]$ is not nilpotent. For a given ring R with a group action of $U(R)$ on $N(R)'$, note that there are finite number of orbits if and only if $N(R)'$ is a finite union of orbits.

Theorem 2.4. *Let R be a unit-IFP ring with $N(R) \neq 0$, and n be the number of orbits under the left (resp., right) action of $U(R)$ on $N(R)'$, where $n < \infty$.*

If $N^*(R)^m \neq 0$ for some $m \leq n$, then there are at most $n-m$ orbits under the left (resp., right) action of $U(R/N^*(R))$ on $N(R/N^*(R))'$.

Proof. Since R is unit-IFP, $R/N^*(R)$ is also unit-IFP by Lemma 1.1(5). So the left action of $U(R/N^*(R))$ on $N(R/N^*(R))'$ can be considered. Let n be the number of orbits under the left action of $U(R)$ on $N(R)'$. Then $N^*(R)^{n+1} = 0$ by Theorem 2.1. Suppose $N^*(R)^m \neq 0$ for some $m \geq 1$. Then $m < n+1$. We apply the proof of [5, Lemma 3.5] in the remainder of the proof. We can take $a_i \in N^*(R)^i \backslash N^*(R)^{i+1}$ for each $i = 1, 2, \ldots, m$. Then the orbits $o_l(a_1), o_l(a_2), \ldots, o_l(a_m)$ are distinct. Since n is the number of orbits, there exist $b_1, \ldots, b_{n-m} \in N(R)'$ such that

$$N(R)' = o_l(a_1) \cup o_l(a_2) \cup \ldots \cup o_l(a_m) \cup o_l(b_1) \cup \cdots \cup o_l(b_{n-m}).$$

But $o_l(a_1), \ldots, o_l(a_m) \subseteq N^*(R)$, and so we can obtain

$$N(R/N^*(R))' = o_l(\bar{b}_1) \cup \cdots \cup o_l(\bar{b}_{n-m})$$

under the left action of $U(R/N^*(R))$ on $N(R/N^*(R))'$. But these orbits $o_l(\bar{b}_1), \ldots, o_l(\bar{b}_{n-m})$ may be not distinct. Thus we can say that there are at most $(n-m)$ orbits under the left action of $U(R/N^*(R))$ on $N(R/N^*(R))'$. The proof for the right case is analogous. □

Following Marks [9], a ring R is called *NI* if $N^*(R) = N(R)$. It is obtained by definition that a ring R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. IFP rings are easily shown to be NI, but this direction is irreversible by [8, Example 1.3]. Note that the classes of unit-IFP rings and NI rings do not imply each other as can be seen by the existence of the NI ring $T_n(A)$ that is non-Abelian (hence not unit-IFP by Lemma 1.1(3)); and the unit-IFP ring R in Theorem 1.2 which is not NI (indeed $b \in N(R) \backslash N^*(R)$), where A is a reduced ring and $n \geq 2$. We see a condition under which unit-IFP rings may be NI.

Corollary 2.5. *Let R be a unit-IFP ring with $N^*(R) \neq 0$, and n be the number of orbits under the left (right) action of $U(R)$ on $N(R)'$, where $n < \infty$. Suppose that $N^*(R)^{n-1} \neq 0$. Then R is either NI or satisfies that*

$$aRb \nsubseteq N^*(R) \text{ for all } a, b \in N(R) \backslash N^*(R).$$

Proof. Note first that R is NI if and only if $R/N^*(R)$ is a reduced ring if and only if $N(R/N^*(R))'$ is empty. Suppose $N^*(R)^{n-1} \neq 0$. Then $N(R/N^*(R))'$ is empty or $U(R/N^*(R))$ is transitive on $N(R/N^*(R))'$ by Theorem 2.4. Write $\bar{R} = R/N^*(R)$ and $\bar{r} = r + N^*(R)$. If $N(\bar{R})'$ is empty, then $R/N^*(R)$ is reduced (hence R is NI). Let $U(\bar{R})$ is transitive on $N(\bar{R})'$. Then R is not NI obviously. Note that $U(R/N^*(R)) = \{u + N^*(R) \mid u \in U(R)\}$ because $N^*(R)$ is nil. So, for all $x, y \in N(R) \backslash N^*(R)$ there exists $u \in U(R)$ such that $\bar{y} = \bar{u}\bar{x}$.

Assume on the contrary that $aRb \subseteq N^*(R)$ for some $a, b \in N(R) \backslash N^*(R)$. By the preceding argument, $\bar{b} = \bar{u}\bar{a}$ for some $u \in U(R)$. Then $aRb \subseteq N^*(R)$

implies

$$0 = \bar{a}\bar{R}\bar{b} = \bar{a}\bar{R}(\bar{u}\bar{a}) = \bar{a}(\bar{R}\bar{u})\bar{a} = \bar{a}\bar{R}\bar{a}.$$

But $\bar{R}$ is semiprime, and $\bar{a} = 0$ follows. This is contrary to $\bar{a} \neq 0$. Thus $aRb \nsubseteq N^*(R)$ for all $a, b \in N(R)\backslash N^*(R)$. □

Recall that the unit-IFP ring R in Theorem 1.2 is not NI and has infinitely many orbits under the left (right) actions of $U(R)$ on $N(R)'$. But we can construct an NI subring of R which has finite number of orbits as we see in the following.

Example 2.6. Let R be the unit-IFP ring in Theorem 1.2. Let K be a finite field. Consider a subring

$$S = K + bK[b] + \{b^p a b^q \mid p, q \geq 1 \text{ and } p + q \geq n\}$$

of R. Then

$$N(S) = bK[b] + \{b^p a b^q \mid p, q \geq 1 \text{ and } p + q \geq n\} = N^*(S),$$

and S is unit-IFP by [7, Lemma 1.2(4)]. Note that $N^*(R)^n = 0$ by Theorem 1.3 and $N^*(R)^{n-1} \neq 0$. Since $S/N^*(S) \cong K$, S is an NI ring. The orbits under the left action of $U(S)$ on $N(S)'$ are

$$o_l(g) \text{ and } o_l(g + b^p a b^q) \text{ with } g \in bK[b].$$

So the number of orbits is finite. The computation for the right case is similar.

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