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WHEN NILPOTENTS ARE CONTAINED IN JACOBSON RADICALS

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Abstract. We focus our attention on a ring property that nilpotents are contained in the Jacobson radical. This property is satisfied by NI and left (right) quasi-duo rings. A ring is said to be *NJ* if it satisfies such property. We prove the following: (i) Köthe's conjecture holds if and only if the polynomial ring over an NI ring is NJ; (ii) If R is an NJ ring, then R is exchange if and only if it is clean; and (iii) A ring R is NJ if and only if so is every (one-sided) corner ring of R.

In this article we consider a ring property that is satisfied by NI and left (right) quasi-duo rings, in relation with nilpotents and Jacobson radicals. In fact we study the structures of various situations when nilpotents are contained in the Jacobson radical. Throughout every ring is an associative ring with identity unless otherwise stated. Let R be a ring. The lower nilradical (or the prime radical), the upper nilradical, the Jacobson radical, and the set of all nilpotent elements of R are denoted by $N_*(R)$, $N^*(R)$, $J(R)$, and $N(R)$, respectively. A nilpotent element is also called *nilpotent* for simplicity. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and use E_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. Denote the polynomial ring and the power series ring over a ring R by $R[x]$ and $R[[x]]$, respectively. A subring S of R is said to be *unital* ([23]) if $1_S = 1_R$, where 1_S and 1_R are identity elements of S and R, respectively. The set of all idempotents (resp., the group of all units) in R is written by $I(R)$ (resp., $U(R)$). $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Following Lam [22] and Leroy et al. [26], a subring S of a ring R is called a *right* (resp., *left*) *corner ring* of R if there exists an additive subgroup C of R such that $R = S \oplus C$, $CS \subseteq C$ (resp., $SC \subseteq C$). The subgroup C is called a *complement* of S. A (resp., unital) subring S is called a (resp., *unital*) *corner ring* if it has a complement C such that $SC \cup CS \subseteq C$. A corner ring S of a ring R is called *Peirce corner* if there is $e \in I(R)$ such that $S = eRe$.

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1. Rings in which Jacobson radicals contain all nilpotents

A ring is usually called *reduced* if it contains no nonzero nilpotent elements. Following Marks [27], a ring R is called *NI* if $N(R) = N^*(R)$. It is straightforward that a ring R is NI if and only if $N(R)$ forms an ideal of R if and only if $R/N^*(R)$ is reduced. Hong and Kwak [14, Corollary 13] proved that a ring is NI if and only if every minimal strongly prime ideal of R is completely prime. For any ring R, the upper nilradical of R is contained in Jacobson radical of R.

Following Yu [33] a ring is called *left* (resp., *right*) *quasi-duo* if every maximal left (resp., right) ideal is two-sided. We immediately obtain that every factor ring of a left quasi-duo ring is again left quasi-duo. We obtain from the definition that a ring is left (resp., right) quasi-duo if and only if every left (resp., right) primitive factor ring is a division ring. Moreover if a ring R is left or right quasi-duo, then $N(R) \subseteq J(R)$ and $R/J(R)$ is reduced by [33, Lemma 2.3 and Corollary 2.4].

Every ring property in the preceding argument is closely related to the Köthe conjecture (i.e., the upper nil radical contains all nil left ideals) by the arguments in [29, 30]. As a generalization of them we now consider the following definition.

Definition 1.1. A ring R (possibly without identity) is said to be *NJ* if $N(R) \subseteq J(R)$.

Chen et al. [5] also used *J-reduced* for the concept of NJ. But in this article we shall use NJ to argue about the relations between NI rings and NJ rings. We immediately observe that a ring R is NJ if and only if for all $r \in R$ and $a \in N(R)$, $1 - ra$ is left invertible in R. NI rings are obviously NJ, and left (right) quasi-duo rings are NJ as noted above. We see that each converse of the preceding facts is not true in general and the concepts of NI and right quasi-duo are independent of each other by the following example.

Example 1.2. (1) There is an NJ ring but not NI. Let F be a simple domain, $A = \mathrm{Mat}_2(F)$, and $B = D_2(F)$. Note $N(B) = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$. Set $R = B + A[[x]]x$, where $A[[x]]$ denotes the formal power series ring with an indeterminate x over a ring A. Then

$$N(R) \subsetneq J(R) = N(B) + A[[x]]x \text{ and } R/J(R) \cong F.$$

This means that R is NJ. On the other hand, consider $f(x) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) x$ and $g(x) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) x$ in $N(R)$. Then $f(x) + g(x) \notin N(R)$ because $(f(x) + g(x))^k = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^k x^k \neq 0$ for all $k \geq 1$. This implies R being not NI.

(2) There exists an NI (hence NJ) ring that is neither left nor right quasi-duo. Let F be a field of characteristic zero and R be the first Weyl algebra over F. Then R is a simple domain that is not a division ring, and so R is neither left nor right quasi-duo. But domains are clearly NI.

(3) There exists a right quasi-duo ring that is not NI. To see that, we refer to the argument in [25, Example 1.1] and [18, Example 2]. Let V be an infinite

dimensional left vector space over a field F with a basis $\{v_1, v_2, \ldots\}$. For the endomorphism ring $A = \mathrm{End}_F(V)$, define $A_1 = \{f \in A \mid \mathrm{rank}(f) < \infty$, and $f(v_i) = a_1v_1 + \cdots + a_iv_i$ for any i with $a_i \in F\}$. Let R be the F-subalgebra of A generated by A_1 and 1_A. Then R is right quasi-duo by the argument in [18, Example 2], and moreover $R[[x]]$ is right quasi-duo by [15, Proposition 4]. However $R[[x]]$ is not NI by the argument in [25, Example 1.1].

Recall that a ring R is said to be *directly finite* (or *Dedekind-finite* [23]) if $ab = 1$ for $a, b \in R$ implies $ba = 1$. NI rings are directly finite by [16, Proposition 2.7(1)].

Lemma 1.3. (1) *Every NJ ring is directly finite.*

(2) *If a ring R is NJ and $J(R) = 0$, then R is reduced.*

(3) *Let R be a ring. $R/J(R)$ is NJ if and only if $R/J(R)$ is reduced if and only if $R/J(R)$ is NI.*

(4) *Let R be a ring. If $R/J(R)$ is NJ, then R is NJ.*

(5) *A ring R is NJ if and only if so is any ideal of R as a ring (possibly without identity).*

(6) [5, Lemma 3.1] *A ring R is NJ if and only if so is eRe for all $e \in I(R)$.*

(7) *Let $\sigma : R \to T$ be a ring epimorphism such that* $\ker(\sigma)$ *is nil. If R is an NJ ring, then T is NJ. Especially R/I is NJ for any nil ideal I of an NJ ring R.*

Proof. (1) Let R be an NJ ring and assume on the contrary that there exist $a, b \in R$ such that $ab = 1$ but $ba \neq 1$. Then $ba \in I(R)$. Let $c = bab(1 - ba) = b(1 - ba)$. Then $c \in N(R)$. Letting $c = 0$, we have $b = bba$ and $1 = ab = abba = ba$ follows, a contradiction. Thus $c \neq 0$. Since R is NJ, $c \in J(R)$ and hence $1 - ac \in U(R)$. This yields

$$1 - ac = 1 - ab(1 - ba) = 1 - 1 + ba = ba \in U(R),$$

a contradiction. Thus R is directly finite.

(2) is obtained from the definition. (3) and (4) are immediate consequences of (2).

(5) is obtained from the fact that the Jacobson radical property is hereditary.

(7) Note that $\sigma(J(R)) = J(T)$ by [2, Corollary 15.8] and $\sigma(N(R)) = N(T)$. Since R is NJ, $\ker(\sigma) \subseteq N(R) \subseteq J(R)$ and

$$N(T) = \sigma(N(R)) \subseteq \sigma(J(R)) = J(T)$$

follows. Thus T is NJ. Letting $T = R/I$ and $\sigma : R \to T$ with $\sigma(r) = r + I$, R/I is NJ by the preceding result. This result is also obtained by [5, Lemma 2.12]. □

If a ring R is NI, then $N(R)$ forms a subring of R without identity. Hence it is natural to ask whether $N(R)$ of an NJ ring R forms a subring of R. However the following example shows that the answer is negative and that rings whose nilpotents form a subring need not be NJ.

Example 1.4. (1) Let R be the ring in Example 1.2(1). Then R is NJ, but $N(R)$ is not closed under addition as can be seen by the nilpotents $f(x) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) x$ and $g(x) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) x$.

(2) We refer to [3, Example 4.8]. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^2 and set $R = A/I$. Identify a and b with their images in R for simplicity. Then $N(R)$ forms a subring of R by [3, Corollary 3.3 and Example 4.8]. Note $b \in N(R)$. Assume $b \in J(R)$. Then there exists $u \in U(R)$ such that $1 = u(1-ab)$. Note that $u \notin K$ and $u = k+c$ for some $0 \neq k \in K$ and $0 \neq c \in B$, where $B = \{f \in A \mid \text{ the constant term of } f \text{ is zero}\}$. This yields

$$1 = u(1-ab) = (k+c)(1-ab) = k + (c - kab + cab) \text{ and } 1 - k = c - kab + cab.$$

But $1 - k = c - kab + cab \neq 0$ because $cab \neq 0$ and the degrees of $c - kab$ and cab are different. This result induces a contradiction because $0 \neq 1 - k \in K$ and $0 \neq c - kab + cab \in B$. Therefore R is not NJ.

(3) The converse of Lemma 1.3(4) need not hold. We apply the ring in [12, Example 3]. Let R_0 be the localization of $\mathbb{Z}$ at the prime ideal $q\mathbb{Z}$, where q is an odd prime. Set R be the quaternions over R_0. Then R is clearly a domain (hence NJ) and $J(R) = qR$. But $R/J(R)$ is isomorphic to $Mat_2(\mathbb{Z}_q)$ by the argument in [10, Exercise 2A]. But $Mat_2(\mathbb{Z}_q)$ is semiprimitive but not reduced. Thus $R/J(R)$ is not NJ. Moreover this shows that the class of NJ rings are not closed under homomorphic images (see also Example 1.4(2)).

A ring is called *Abelian* if every idempotent is central. Abelian rings are also directly finite. But NJ and Abelian properties are independent of each other. The ring R in Example 1.4(2) is Abelian by the arguments in [3], but R is not NJ; and $U_n(R)$, over a reduced ring R for $n \geq 2$, is not Abelian clearly but it is NI by [16, Proposition 4.1(1)].

Proposition 1.5. *Let $\{R_\lambda \mid \lambda \in \Lambda\}$ be a class of NJ rings. Then we have the following*:

(1) *The direct product $\prod_{\lambda\in\Lambda} R_\lambda$ of R_λ's is NJ.*

(2) *If Λ is a finite set, then the subdirect product of R_λ's is NJ.*

(3) *There exists an NJ ring which is neither NI nor right quasi-duo.*

Proof. (1) is shown by the fact $N\left(\prod_{\lambda\in\Lambda} R_\lambda\right) \subseteq \prod_{\lambda\in\Lambda} N(R_\lambda) \subseteq \prod_{\lambda\in\Lambda} J(R_\lambda)$ $= J\left(\prod_{\lambda\in\Lambda} R_\lambda\right)$, using the hypothesis that every R_λ is NJ.

(2) It suffices to show that the subdirect product R of two NJ rings R_1 and R_2 is also NJ. By the property of subdirect products, there are two ideals A_1 and A_2 of R such that $A_1 \cap A_2 = 0$ and $R_i \cong R/A_i$ for any $i = 1, 2$.

Let $r \in R$ and $x \in N(R)$. We use the assumption that R_1 and R_2 are NJ freely. Since $x + A_1 \in J(R_1)$ and $x + A_2 \in J(R_2)$, we have that for all $r \in R$, $b_1(1 - rx) = 1 + a_1$ and $b_2(1 - rx) = 1 + a_2$ for some b_1, $b_2 \in R$, $a_1 \in A_1$, and

$a_2 \in A_2$. This implies

$$(b_1 + b_2 - b_1(1 - rx)b_2)(1 - rx) = 1$$

because $a_1a_2 \in A_1 \cap A_2 = 0$. Therefore, $1 - rx$ is left invertible, entailing that R is NJ.

(3) Let R_1 be an NI ring but not right quasi-duo and R_2 be a right quasi-duo ring but not NI as in Example 1.2(2,3). Set $R = R_1 \oplus R_2$. Then R is an NJ ring by (1). However R is neither NI nor right quasi-duo by [16, Proposition 2.4(2)] and [23, Corollary 3.6(1)]. □

Following [9], a ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$. Following Feller [8], a ring R is called *left* (resp., *right*) *duo* if every left (resp., right) ideal is two-sided. It is easily checked that right or left duo rings are Abelian, and right duo rings are clearly right quasi-duo. But the converse need not hold because there exist right quasi-duo rings but not Abelian.

Proposition 1.6. *Let R be a regular ring. Then the following conditions are equivalent*:

(1) *R is NJ*;
(2) *R is NI*;
(3) *R is right* (*left*) *quasi-duo*;
(4) *R is right* (*left*) *duo*;
(5) *R is reduced*;
(6) *$N(R)$ is a subring of R* (*without identity*).

Proof. (2) ⇒ (1), (4) ⇒ (3), (3) ⇒ (1), (5) ⇒ (2), and (2) ⇒ (6) are obvious. (4) ⇔ (5) and (6) ⇒ (2) are obtained by [9, Theorem 3.2] and [17, Theorem 13], respectively.

(2) ⇒ (5): is obtained by [17, Theorem 13].

(1) ⇒ (5): It directly follows from [9, Corollary 1.2]. □

The equivalence of the conditions (1) and (3) in Proposition 1.6 is also proved by [5]. In fact, [5, Corollary 4.2] gives that an exchange rings is NJ if and only if it is right (left) quasi-duo, and note that regular rings are exchange. A ring R is usually called *π-regular* if for each $a \in R$ there exist $n \geq 1$ and $b \in R$ such that $a^n = a^nba^n$. Regular rings are clearly π-regular. However the converse need not hold as can be seen by $R = U_n(D)$ $(n \geq 2)$ over a division ring D. Moreover R is both NI and quasi-duo, but not reduced. This implies that π-regular NJ rings need not be reduced.

A ring R is usually called *right* (*resp., left*) *weakly π-regular* if for each a in R there exists $n \geq 1$ such that $a^n \in a^nRa^nR$ (resp., $a^n \in Ra^nRa^n$). The Jacobson radical of a left or right weakly π-regular ring is nil by [11, Proposition 3.3]. A ring is called *weakly π-regular* if it is both left and right weakly π-regular. A π-regular ring is clearly weakly π-regular.

Proposition 1.7. *Let R be a right weakly π-regular ring. Then the following conditions are equivalent*:

(1) *R is NJ*;

(2) *R is NI*;

(3) *$R/J(R)$ is reduced.*

Proof. This is routine, since $J(R)$ is nil for a right weakly π-regular ring R as noted above. □

Note that the ring in Example 1.2(2) is weakly π-regular (since it is simple). So we can conclude that weakly π-regular NJ rings need not be right quasi-duo. But for π-regular rings, we have an affirmative result by the following theorem.

Warfield [32] called a ring R *exchange* if $_RR$ has the exchange property, and proved that a module $_RM$ has the exchange property if and only if the endomorphism ring $\mathrm{End}_R(M)$ is an exchange ring. Nicholson [28] called a ring R *clean* if every element of R is a sum of a unit and an idempotent; and proved that clean rings are exchange and the converse holds when R is Abelian. Yu [33, Theorem 4.2] showed that one-sided quasi-duo exchange ring is clean by help of [33, Proposition 4.1].

Now we extend [33, Theorem 4.2] to the case of NJ rings, noting that right or left quasi-duo rings are NJ.

Theorem 1.8. *Let R be an NJ ring.*

(1) *If R is π-regular, then R is both NI and quasi-duo.*

(2) *R is exchange if and only if it is clean.*

Proof. (1) Let R be π-regular. Then R is NI by Proposition 1.7, and $J(R)$ is nil by [11, Proposition 3.3]. Consequently we have $N(R) = N^*(R) = J(R)$ and hence $R/J(R)$ is both π-regular and reduced.

Then every prime factor ring of $R/J(R)$ is a division ring by [24, Lemma 4], entailing that every right primitive factor ring of $R/J(R)$ is a division ring. Since the set of all right primitive factor rings of R and one of $R/J(R)$ coincide, we get that every right primitive factor ring of R is a division ring. Thus R is right quasi-duo by [15, Proposition 1]. The proof for the left quasi-duo is similar.

(2) It suffices to show the necessity. Let R be exchange. Then $R/J(R)$ is exchange and idempotents are lifted modulo $J(R)$ by [28, Proposition 1.5]. First we show that $\bar{R} = R/J(R)$ is Abelian. Let $\bar{f} = f + J(R) \in I(\bar{R})$. Then there exists $e \in I(R)$ such that $\bar{f} = \bar{e}$. Since R is NJ, $er(1-e), (1-e)re \in N(R) \subseteq J(R)$ for all $r \in R$. This yields $\bar{e}\bar{r} = \bar{e}\bar{r}\bar{e} = \bar{r}\bar{e}$. So $\bar{R}$ is Abelian.

Then $\bar{R}$ is clean by [28, Proposition 1.8(2)]. Thus, for any $x \in R$, there exist $\bar{g} \in I(\bar{R})$ and $\bar{u} \in U(\bar{R})$ such that $\bar{x} = \bar{g} + \bar{u}$. Here $\bar{g} = \bar{e}$ for some $e \in I(R)$, and note $u \in U(R)$. Then $x = e + u + s$ for some $s \in J(R)$. But since $u + s$ is also a unit in R, R is clean. □

By Theorem 1.8(2), we can also obtain the following result.

Corollary 1.9. *If R is an NI ring, then R is exchange if and only if it is clean.*

2. More properties of NJ rings

In this section, we observe the structure of polynomial rings in relation with Köthe's conjecture and the NJ property. Next the structure of subrings of NJ rings is also investigated.

For a ring R with a ring endomorphism θ of R, the skew polynomial ring with an indeterminate x over R, denoted by $R[x;\theta]$, is the ring of polynomials in $R[x]$, only subject to $xa = \theta(a)x$ for all $a \in R$. Recall that an element $a \in R$ is called *θ-nilpotent* if for every $m \geq 1$ there exists $n \geq 1$ such that $a\theta^m(a)\cdots\theta^{mn}(a) = 0$, and a subset S of R is called *θ-nil* if every element of S is θ-nilpotent. Following the literature, θ is said to be *of locally finite order* if for any $r \in R$, there exists $n \geq 1$ such that $\theta^n(r) = r$.

Lemma 2.1. *Let θ be an automorphism of a ring R and suppose that θ is of locally finite order.*

(1) *If $R[x;\theta]$ is NJ, then R is NI and $J(R[x;\theta]) = N(R)[x;\theta]$.*

(2) *$R[x;\theta]$ is NJ such as $R/N(R)$ is a commutative ring if and only if $R[x;\theta]$ is right quasi-duo.*

Proof. (1) By [4, Theorem 3.1], $J(R[x;\theta]) = I \cap J(R) + I[x;\theta]x$ for some θ-nil ideal I of R. If θ is of locally finite order, then I is nil by definition of θ-nilpotent, entailing $I \subseteq J(R)$. Thus $J(R[x;\theta]) = I[x;\theta]$ follows.

Here if $R[x;\theta]$ is NJ, then $N(R) \subseteq I[x;\theta]$ and $N(R) \subseteq I$. This implies $I = N(R) = N^*(R)$ (hence R is NI) and $J(R[x;\theta]) = N(R)[x;\theta]$.

(2) is proved by (1) and [26, Theorem 4.1]. □

Lemma 2.1(1) can be applied to polynomial rings.

Proposition 2.2. *For a ring R, the following conditions are equivalent*:

(1) *$R[x]$ is NJ*;

(2) *R is NI and $J(R[x]) = N(R)[x]$*;

(3) *$R[x]/J(R[x])$ is reduced.*

Proof. It suffices to prove (2) $\Rightarrow$ (3) by help of Lemma 1.3(3, 4) and Lemma 2.1(1). Suppose that R is NI and $J(R[x]) = N[x]$, where $N = N(R) = N^*(R)$. Then $\frac{R[x]}{J(R[x])} = \frac{R[x]}{N[x]} \cong (R/N)[x]$ is a reduced ring. □

Following Krempa [21], a ring R is said to be *θ-rigid* if $a\theta(a) = 0$ implies $a = 0$ for each $a \in R$, where θ is an endomorphism of R. By [13, Proposition 5], a ring R is θ-rigid if and only if $R[x;\theta]$ is reduced. The following example shows that Proposition 2.2 cannot be extended to the case of arbitrary skew polynomial rings.

Example 2.3. Let $R = \mathbb{Z} \oplus \mathbb{Z}$ and define $\theta : R \to R$ by $\theta(a,b) = (b,a)$. Then θ is an automorphism of order 2, but R is not θ-rigid as can be seen by $(1,0)\theta(1,0) = 0$. Since R is reduced, $J(R[x;\theta]) = 0$ by [4, Theorem 3.1] and $R[x;\theta] \cong R[x;\theta]/J(R[x;\theta])$ follows. Since $0 \neq (1,0)x \in N(R[x;\theta])$ and $J(R[x;\theta]) = 0$, $R[x;\theta]$ is not NJ.

Let A be an algebra over a commutative ring K. Recall that the *Dorroh extension* of A by K is the ring $K \times A$ with operations $(n_1, a_1) + (n_2, a_2) = (n_1 + n_2, a_1 + a_2)$ and $(n_1, a_1)(n_2, a_2) = (n_1 n_2, n_1 a_2 + a_1 n_2 + a_1 a_2)$, where $a_i \in A$ and $n_i \in K$.

It is shown in the following that the converse of partial part of Lemma 2.1 is equivalent to Köthe's conjecture being true. Recall that a ring R without identity is said to be *Jacobson radical* if $J(R) = R$.

Proposition 2.4. *The following are equivalent for a ring R:*
(1) *Köthe's conjecture holds;*
(2) *$R_0[x]$ is a Jacobson radical ring for any nil ring R_0;*
(3) *If R is NI, then $R[x]$ is NJ.*

Proof. (1) $\Leftrightarrow$ (2) is proved by [20, Theorem 2].

(2) $\Rightarrow$ (3): Suppose that R is NI and consider $N^*(R)[x]$. Then $N^*(R)[x]$ is Jacobson radical (i.e., $J(N^*(R)[x]) = N^*(R)[x]$) by the condition (2); hence by a theorem of Amitsur [1, Theorem 1], $J(R[x]) = N^*(R)[x]$. This implies $R[x]/J(R[x]) = R[x]/N^*(R)[x] \cong (R/N^*(R))[x]$ is reduced, and therefore $R[x]$ is NJ by Lemma 1.3(4).

(3) $\Rightarrow$ (2): Let R_0 be a nil ring and D be the Dorroh extension of R_0 by $\mathbb{Z}$. Then D is an NI ring with $N(D) = N^*(D) = R_0$. Hence $D[x]$ is NJ by the condition (3), and Proposition 2.2 gives $J(D[x]) = N(D)[x] = R_0[x]$. This yields $J(R_0[x]) = J(D[x]) \cap R_0[x] = R_0[x]$. Thus $R_0[x]$ is a Jacobson radical ring. $\square$

By Proposition 2.4, if Köthe's conjecture has a negative answer, then there exists a nil ring R_0 such that $R_0[x]$ is not Jacobson radical. In fact, Chen proved that if Köthe's conjecture has a negative answer, then there exists a nil algebra S over some countable field such that $J(S[x]) = 0$ ([6, Lemma 3.6]).

The skew power series ring $R[[x;\theta]]$ over R is defined similarly, i.e., it is the ring of power series in $R[[x]]$, only subject to $xa = \theta(a)x$ for all $a \in R$. Note that every maximal left ideal of $R[[x;\theta]]$ forms $A + R[[x;\theta]]x$, where A is a maximal left ideal of R. This entails $J(R[[x;\theta]]) = J(R) + R[[x;\theta]]$. By using this fact, Chen et al. [5] obtained the following:

Proposition 2.5 ([5, Corollary 3.8])**.** *Let θ be an endomorphism of a ring R. Then R is NJ if and only if $R[[x;\theta]]$ is NJ.*

Next we investigate the NJ property of various sorts of subrings of given NJ rings.

Proposition 2.6. *Let S be a unital subring of an NJ ring R.*
(1) *If $S \cap J(R) \subseteq J(S)$, then S is NJ.*
(2) *If S satisfies one of the following conditions, then S is NJ:*

(i) *Every element of S which is invertible in R is already invertible in S;*
(ii) *S is right Artinian.*

Proof. (1) Note that $N(S) = S \cap N(R) \subseteq S \cap J(R)$ for any subring S of an NJ ring R. Then the result follows when $S \cap J(R) \subseteq J(S)$.

(2) is shown by (1) and [31, Proposition 2.5.17]. □

By help of Proposition 2.6, we can conclude that the class of NJ rings is not closed under subrings.

Example 2.7. Let R be the ring R in Example 1.2(1). Then R is NJ but not NI. Thus, $S = R[[x]]$ is also NJ by Proposition 2.5. However the subring $R[x]$ is not NJ by Proposition 2.2 because R is not NI.

Lam [22] showed that every corner ring of a ring R is a unital corner of some Peirce corner of R and is also a Peirce corner of some unital corner of R. It is well-known that NI ring are closed under subrings and every left corner ring of right quasi-duo ring is also right quasi-duo by [16, Proposition 2.4 (2)] and [26, Theorem 1.2]. Now we give a similar results for NJ rings.

Theorem 2.8. *A ring R is NJ if and only if so is every (one-sided) corner ring of R.*

Proof. If we choose $e = 1 \in R$, then $R = eRe$ is a corner ring of itself. Thus, we only prove the necessary condition of this theorem.

Let R be an NJ ring. By Lemma 1.3(6), every Peirce corner ring of R is NJ. Now consider the case of right unital corner ring of R.

Let S be a right unital corner ring of R. Let $a \in N(S)$ and $s \in S$. Since R is NJ, there is an element $1-r \in R$ such that $(1-r)(1-sa) = 1$. By the definition $r = a'+c$ where $a' \in S$ and $c \in C$. This means that $(1-a')(1-sa)+c(1-sa) = 1+0$. Thus $1-(1-a')(1-sa) = c(1-sa) \in S \cap C = 0$, hence $(1-a')(1-sa) = 1$. Therefore S is NJ. The case of left unital corner is similar.

It is well-known that every right (resp., left) corner ring of a ring R is a right (resp., left) unital corner ring of a Peirce corner of R. Thus every (one-sided) corner ring of R is NJ. □

Note that a ring R is a (left) corner of $R[x;\theta]$ and of $U_n(R)$ by choosing their complements as $R[x;\theta]x$ and the set of all matrices in $U_n(R)$ whose $(1,1)$-entries are zero. Thus we obtain the following by Theorem 2.8.

Corollary 2.9. *Let R be a ring and θ be an endomorphism of R.*
(1) *If $R[x;\theta]$ is NJ, then so is R.*
(2) *R is NJ if and only if $U_n(R)$ is NJ for any $n \geq 2$.*

For given a skew polynomial ring $R[x;\theta]$, a subring S of R is called a *θ-subring* if $\theta(S) \subseteq S$. Note that S is a θ-subring if and only if $S[x;\theta]$ is a subring of $R[x;\theta]$.

Theorem 2.10. *Let θ be an endomorphism of an NI ring R. If $R[x;\theta]$ is NJ, then so is $S[x;\theta]$ for any θ-subring S of R.*

Proof. Let $f(x) \in N(S[x;\theta])$ $(\subseteq N(R[x;\theta]))$ and $s(x) \in S[x;\theta](\subseteq R[x;\theta])$. Then $1 - s(x)f(x)$ is left invertible in $R[x;\theta]$ by the hypothesis, that is,

$$g(x)(1 - s(x)f(x)) = 1 \text{ for some } g(x) \in R[x;\theta]. \tag{1}$$

Put $g(x) = b_0 + b_1x + \cdots + b_mx^m$ and $s(x)f(x) = a_0 + a_1x + \cdots + a_nx^n$, where $a_i \in S$ and $b_j \in R$. Since R is NI, so is S by [16, Proposition 2.4(2)]. Thus we have $a_0 \in N(S)$. If $a_0 \neq 0$, with $a_0^l = 0 \neq a_0^{l-1}$ for some $l \geq 1$, then the constant coefficient of the polynomial in (1) is given by $b_0(1 - a_0) = 1$.

Multiplying this equality by 1_S on the right side, we get $b_01_S(1_S - a_0) = 1_S$, where 1_S is the identity element of S. This means that $b_01_S = 1_S + a_0 + \cdots + a_0^{l-1} \in S$. Now suppose that $b_01_S, b_11_S, \ldots, b_{k-1}1_S \in S$, where $1 \leq k \leq m$. Then we have

$$b_01_Sa_k + b_11_S\theta(a_{k-1}) + \cdots + b_{k-1 1_S}\theta^{k-1}(a_1) - b_k1_S\theta^k(1_S - a_0) = 0$$

from the coefficient of k-th term in (1). Thus

$$b_k1_S = (b_01_Sa_k + b_11_S\theta(a_{k-1}) + \cdots + b_{k-1}1_S\theta^{k-1}(a_1))\theta^k(b_01_S) \in S.$$

Inductively we obtain $g(x)1_S \in S[x;\theta]$, and $S[x;\theta]$ is NJ. The case of $a_0 = 0$ is obtained by the analogous way. □

When θ is of locally finite order we obtain the same result that Theorem 2.10 states.

Proposition 2.11. *Let R be a ring and θ be an automorphism of R of locally finite order. If $R[x;\theta]$ is NJ, then so is $S[x;\theta]$ for any θ-subring S of R.*

Proof. Since θ is of locally finite order, R is NI by Lemma 2.1(1). Thus $S[x;\theta]$ for any θ-subring S of R is NJ by Theorem 2.10. □

Recall that a ring R is said to be of *bounded index of nilpotency* if there exists a positive integer n such that $x^n = 0$ for all $x \in N(R)$. Klein (see [19, Theorem 9]) proved that if a ring R is of bounded index of nilpotency, then $R[x]$ and $R[[x]]$ are also bounded index of nilpotency. We use this fact in the following.

Proposition 2.12. *Let R be a ring of bounded index of nilpotency. Then the following conditions are equivalent*:

(1) *R is NI*;
(2) *$R[x]$ is NJ*;
(3) *$R[x]$ is NI*;
(4) *$R[[x]]$ is NI.*

Proof. (1) $\Leftrightarrow$ (3) and (4) $\Rightarrow$ (3) are shown by [16, Proposition 4.3 and Proposition 2.4(2)]. (2) $\Rightarrow$ (1) comes from Proposition 2.2 and (3) $\Rightarrow$ (2) follows from definition. It suffices to show (1) $\Rightarrow$ (4). Let R be an NI ring. By a similar method to [16, Proposition 4.4], we have $N^*(R)[[x]] = N^*(R[[x]])$, since $N^*(R)$ is of bounded index of nilpotency. Thus $R/N^*(R)$ and $R[[x]]/N^*(R[[x]]) \cong (R/N^*(R))[[x]]$ are reduced, implying that $R[[x]]$ is NI. □

Lastly we summarize the relations between NJ rings and related ring properties in the following diagram for a given ring R:

$$\begin{tikzcd}
(1.1) : N(R) = J(R) \arrow[r, dotted, no head] \arrow[d] \arrow[dr] & (1.2) : R \text{ is one-sided quasi-duo} \arrow[d] \\
(2.1) : R \text{ is NI} \arrow[r, dotted, no head] \arrow[d] & (2.2) : R/J(R) \text{ is NJ} \arrow[d] \\
(3.1) : a \in N(R) \Rightarrow Ra \text{ is nil} \arrow[r] & (3.2) : R \text{ is NJ}
\end{tikzcd}$$

Trivially we have (1.1) $\Rightarrow$ (2.1) $\Rightarrow$ (3.1), (1.2) $\Rightarrow$ (2.2) $\Rightarrow$ (3.2), (1.1) $\Rightarrow$ (2.2), and (3.1) $\Rightarrow$ (3.2). In the following we consider some examples for the reverse implications in the diagram.

Example 2.13. (1) [7, Example 10] Let $W = \{\frac{y}{x} \mid y$ is an even integer and x is an odd integer$\}$ be a subalgebra of the field $\mathbb{Q}$ of rational numbers and R the Dorroh extension of W by $\mathbb{Q}$. Then it satisfies all condition in the diagram except (1.1).

(2) Let R be the subring of real Hamilton quaternions having integer coefficients. Then it is a semiprimitive domain. Thus it satisfies all condition in the diagram except (1.2).

(3) Let $R = D[[x]]$ over a division ring D. Then it satisfies (2.1) but $J(R) = Rx \neq 0$ is not nil. Thus R does not satisfy (1.1).

(4) Let $S = Mat_2(F)$ over a field F. Put $R = F \oplus S[[x]]x$ then $J(R) = S[[x]]x$. This means $R/J(R)$ is reduced. However R is not NI. So (2.2) does not imply (2.1).

We elaborate the preceding diagram as follows.

Proposition 2.14. *Let R be a ring with nil $J(R)$. Then the following conditions are equivalent*:

(1) $N(R) = J(R)$;
(2) *R is NI*;
(3) *$a \in N(R)$ implies Ra is nil*;
(4) *$R/J(R)$ is NJ*;
(5) *R is NJ.*

Proof. We use the hypothesis of $J(R)$ being nil (equivalently, $N^*(R) = J(R)$) freely. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (4), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (5), and (5) $\Rightarrow$ (1) are obvious. (4) $\Rightarrow$ (5) is done by Lemma 1.3(4). $\square$

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