

## **$q$ -ADDITION THEOREMS FOR THE $q$ -APPELL POLYNOMIALS AND THE ASSOCIATED CLASSES OF $q$ -POLYNOMIALS EXPANSIONS**

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ABSTRACT. Several addition formulas for a general class of  $q$ -Appell sequences are proved. The  $q$ -addition formulas, which are derived, involved not only the generalized  $q$ -Bernoulli, the generalized  $q$ -Euler and the generalized  $q$ -Genocchi polynomials, but also the  $q$ -Stirling numbers of the second kind and several general families of hypergeometric polynomials. Some  $q$ -umbral calculus generalizations of the addition formulas are also investigated.

### **1. Introduction**

Throughout this paper, we adopt the following notations:

$$\mathbb{N} = \{1, 2, 3, \dots\}, \quad \mathbb{N}_0 = \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}.$$

The classical  $q$ -Bernoulli polynomials  $B_n(x; q)$ , the classical  $q$ -Euler polynomials  $E_n(x; q)$  and the classical  $q$ -Genocchi polynomials  $G_n(x; q)$  together with their generalizations  $B_n^{(\alpha)}(x; q)$ ,  $E_n^{(\alpha)}(x; q)$  and  $G_n^{(\alpha)}(x; q)$  of (real or complex) order  $\alpha$ , are usually defined by means of the following generating functions (see for details, [?], and the references therein):

$$(1.1) \quad \left( \frac{t}{e_q(t) - 1} \right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x) \frac{t^n}{[n]_q!}, \quad (|t| < 2\pi; \quad 1^\alpha := 1),$$

$$(1.2) \quad \left( \frac{2}{e_q(t) + 1} \right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)}(x) \frac{t^n}{[n]_q!}, \quad (|t| < \pi; \quad 1^\alpha := 1),$$

and

$$(1.3) \quad \left( \frac{2t}{e_q(t) + 1} \right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x) \frac{t^n}{[n]_q!}, \quad (|t| < \pi; \quad 1^\alpha := 1),$$

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so that, obviously, the  $q$ -Bernoulli polynomials  $B_{n,q}(x)$ , the  $q$ -Euler polynomials  $E_{n,q}(x)$  and the  $q$ -Genocchi polynomials  $G_{n,q}(x)$  are given respectively, by

$$B_{n,q}(x) := B_{n,q}^{(1)}(x), \quad E_{n,q}(x) := E_{n,q}^{(1)}(x), \quad \text{and} \quad G_{n,q}(x) := G_{n,q}^{(1)}(x), \quad (n \in \mathbb{N}_0).$$

For the  $q$ -Bernoulli numbers  $B_{n,q}$ , the  $q$ -Euler numbers  $E_{n,q}$  and the  $q$ -Genocchi numbers  $G_{n,q}$  of order  $n$ , we have

$$B_{n,q} = B_{n,q}(0), \quad E_{n,q} = E_{n,q}(0), \quad \text{and} \quad G_{n,q} = G_{n,q}(0), \quad (n \in \mathbb{N}_0),$$

respectively.

The Roger Szégo polynomials  $H_n(x; q)$  (see [?, Eq. (1)]) and the Al-Salam Carlitz polynomials  $U_n^{(a)}(x; q)$  (see [?, p. 534]) are defined by the generating functions

$$(1.4) \quad e_q(t)e_q(xt) = \sum_{n=0}^{\infty} H_n(x; q) \frac{t^n}{[n]_q!},$$

and

$$(1.5) \quad \frac{e_q(xt)}{e_q(t)e_q(at)} = \sum_{n=0}^{\infty} U_n^{(a)}(x; q) \frac{t^n}{[n]_q!},$$

respectively.

In these definitions,  $[n]_q$  is the so-called  $q$ -number defined by [?, ?]

$$[n]_q = \frac{1 - q^n}{1 - q},$$

$e_q$  is the  $q$ -exponential function defined by [?, ?]

$$(1.6) \quad e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!},$$

where  $[n]_q!$  denotes the so-called  $q$ -factorial

$$[n]_q! = \prod_{k=1}^n [k]_q, \quad \text{and} \quad [0]_q! = 1.$$

There is another  $q$ -exponential function  $E_q(x)$  defined by [?, ?]

$$(1.7) \quad E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_q!} x^n.$$

Both  $e_q(x)$  and  $E_q(x)$  satisfy the fundamental relation  $e_q(x)E_q(-x) = 1$ .

The  $q$ -analogue of the binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \quad 0 \leq k \leq n,$$

here,  $(q; q)_n$  is the so-called  $q$ -Pochhammer symbol defined by

$$(a; q)_n = \prod_{k=1}^n (1 - aq^k) \text{ for } n \geq 1, \quad (a; q)_0 = 1.$$

Various interesting and potentially useful properties and relations involving the Bernoulli, Euler, Genocchi, Roger-Szégő and Al-Salam Carlitz polynomials have been investigated in the literature.

In [?], the authors gave several addition formulas for a general class of Appell polynomials. In this work, we extend these results to a general class of  $q$ -Appell sequences.

**Definition 1.1** (see [?]). A polynomial sequence  $\{P_n(x)\}_{n \in \mathbb{N}_0}$  is said to be a  $q$ -Appell sequence if

$$(1.8) \quad D_q P_0(x) = 0 \quad \text{and} \quad D_q P_n(x) = [n]_q P_{n-1}(x), \quad (n \in \mathbb{N}),$$

or equivalently, if

$$(1.9) \quad A(t)e_q(xt) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{[n]_q!},$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_q!},$$

is a formal power series with  $a_0 \neq 0$ .

From this definition, it is clear that the  $q$ -Bernoulli polynomials  $B_{n,q}(x)$ , the  $q$ -Euler polynomials  $E_{n,q}(x)$ , the  $q$ -Genocchi polynomials  $G_{n,q}(x)$ , the Roger-Szégő polynomials  $H_n(x; q)$  and the Al-Salam Carlitz polynomials  $U_n^{(a)}(x; q)$  are  $q$ -Appell sequences. Other definitions and notations for  $q$ -Appell sequences can be found in the literature (see for example [?]).

**Definition 1.2.** Let  $a$  and  $b$  two real or complex numbers. Then, the Ward  $q$ -addition of  $a$  and  $b$  is given by

$$(1.10) \quad (a \oplus_q b)^n := \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q a^k b^{n-k}, \quad n = 0, 1, 3, \dots$$

The following  $q$ -Stirling numbers will be also needed.

**Definition 1.3** (see [?, p. 173]). The  $q$ -Stirling numbers of the first kind  $s_q(n, k)$  and the  $q$ -Stirling numbers of the second kind  $S_q(n, k)$  are defined by

$$(1.11) \quad (x)_{n,q} := \sum_{k=0}^n s_q(n, k)x^k,$$

and

$$(1.12) \quad x^n = \sum_{k=0}^n S_q(n, k)(x)_{k,q},$$

where the polynomial  $(x)_{k,q}$  is defined by

$$(x)_{k,q} = \prod_{m=0}^{k-1} (x - [m]_q).$$

## 2. Some $q$ -addition theorems

Let  $\{f_n^{(\alpha)}(x)\}$  ( $\alpha \in \mathbb{C}$ ) be the *one-parameter  $q$ -Appell* sequence generated by

$$(2.1) \quad (f(t))^\alpha e_q(xt) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) \frac{t^n}{[n]_q!}, \quad (f(0) \neq 0; \quad 1^\alpha = 1).$$

It is not difficult to see that comparing to (??), we have  $f_n^{(1)}(x) = f_n(x)$ , ( $n \in \mathbb{N}_0$ ). Also, replacing  $\alpha$  by 0 in (??) and use the series expansion (??), we obtain

$$f_n^{(0)}(x) = x^n, \quad n \in \mathbb{N}_0.$$

Now we state the following important lemma.

**Lemma 2.1.** *For the one-parameter  $\{f_n^{(\alpha)}(x)\}$  generated by (??), the following  $q$ -addition formula holds:*

$$(2.2) \quad f_n^{(\alpha+\beta)}(x \oplus_q y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q f_k^{(\alpha)}(x) f_{n-k}^{(\beta)}(y).$$

*Proof.* Using the generating function (??), we have

$$\begin{aligned} \sum_{n=0}^{\infty} f_n^{(\alpha+\beta)}(x \oplus_q y) \frac{t^n}{[n]_q!} &= (f(t))^{\alpha+\beta} e_q((x \oplus y)t) \\ &= (f(t))^\alpha e_q(xt) \times (f(t))^\beta e_q(yt) \\ &= \left( \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} f_n^{(\beta)}(y) \frac{t^n}{[n]_q!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q f_k^{(\alpha)}(x) f_{n-k}^{(\beta)}(y) \right) \frac{t^n}{[n]_q!}. \end{aligned}$$

This proves the lemma. □

As a direct consequence of this lemma the following proposition holds.

**Proposition 2.2.** For the one-parameter  $\{f_n^{(\alpha)}(x)\}$  generated by (??), the following  $q$ -addition equation applies

$$(2.3) \quad f_n^{(\alpha)}(x \oplus_q y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q f_{n-k}^{(\alpha)}(y) x^k = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q f_k^{(\alpha)}(y) x^{n-k}.$$

Next, we need the following inversion formulas for the Roger-Szégő and the Al-Salam Carlitz polynomials.

**Proposition 2.3** (see [?, ?]). The following inversion formulas hold for the Roger-Szégő polynomials  $H_k(x; q)$  and the Al-Salam Carlitz polynomials  $U_n^{(a)}(x; q)$ :

$$(2.4) \quad x^n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} H_k(x; q),$$

$$(2.5) \quad (x \ominus 1)_q^n = \sum_{k=0}^n a^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q U_k^{(a)}(x; q),$$

$$(2.6) \quad x^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \left( \sum_{i=0}^{n-k} \begin{bmatrix} n-k \\ i \end{bmatrix}_q a^i \right) U_k^{(a)}(x; q).$$

*Proof.* From the generating function (??), we have

$$\begin{aligned} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} &= E_q(-t) e_q(t) e_q(xt) \\ &= \left( \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{[n]_q!} t^n \right) \left( \sum_{n=0}^{\infty} H_n(x; q) \frac{t^n}{[n]_q!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} H_k(x; q) \right) \frac{t^n}{[n]_q!}. \end{aligned}$$

This prove the first equation. For the second one, we first remark that [?, (5.19)]

$$(x \ominus y)_q^n = \sum_{k=0}^n (-y)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k.$$

Next, taking into account that  $e_q(x)E_q(-x) = 1$  and multiplying the generating function (??) by  $e_q(at)$ , the left-hand side gives

$$\begin{aligned} \frac{e_q(xt)}{e_q(t)} &= e_q(xt) E_q(-t) \\ &= \left( \sum_{k=0}^{\infty} \frac{x^k t^k}{[k]_q!} \right) \left( \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{[k]_q!} (-t)^k \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \right) \frac{t^n}{[n]_q!} \end{aligned}$$

$$= \sum_{n=0}^{\infty} (x \ominus 1)_q^n \frac{t^n}{[n]_q!},$$

the right-hand side gives

$$\begin{aligned} e_q(at) \sum_{n=0}^{\infty} U_n^{(a)}(x; q) \frac{t^n}{[n]_q!} &= \left( \sum_{n=0}^{\infty} a^n \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} U_n^{(a)}(x; q) \frac{t^n}{[n]_q!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q U_k^{(a)}(x; q) \right) \frac{t^n}{[n]_q!}. \end{aligned}$$

Hence we have

$$\sum_{n=0}^{\infty} (x \ominus 1)_q^n \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q U_k^{(a)}(x; q) \right) \frac{t^n}{[n]_q!}.$$

So (??) is proved. Note that this result is proved in [?] using the Verma's  $q$ -extension [?] of Filds and Wimp inversion formula [?].  $\square$

**Lemma 2.4** ( $q$ -Analogue of [?, p. 5707, Lemma 2]). *The following relation between the  $q$ -Genocchi polynomials and the  $q$ -Euler polynomials holds true:*

$$(2.7) \quad E_{n,q}^{(\ell)}(x) = \frac{[n]_q!}{[n+\ell]_q!} G_{n+\ell,q}^{(\ell)}(x), \quad n, \ell \in \mathbb{N}_0, \quad 0 \leq \ell \leq n.$$

*Proof.* Let  $\ell$  such that  $0 \leq \ell \leq n$ . Then, from the generating functions (??) and (??), we have

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,q}^{(\ell)}(x) \frac{t^n}{[n]_q!} &= t^\ell \left( \frac{2}{e_q(t) + 1} \right)^\ell e_q(xt) \\ &= \sum_{n=0}^{\infty} E_{n,q}^{(\ell)}(x) \frac{t^{n+\ell}}{[n]_q!} \\ &= \sum_{n=\ell}^{\infty} E_{n-\ell,q}^{(\ell)}(x) \frac{t^n}{[n-\ell]_q!} \\ &= \sum_{n=0}^{\infty} \frac{[n]_q!}{[n-\ell]_q!} E_{n-\ell,q}^{(\ell)}(x) \frac{t^n}{[n]_q!}, \end{aligned}$$

where we set  $E_{k,q}^{(\ell)}(x) = 0$  for  $k < 0$ . Comparing the coefficients of  $t^n$  provides the result.  $\square$

**Lemma 2.5.** *Each of the following expansion formulas holds true:*

$$(2.8) \quad x^n = \frac{1}{[n+1]_q} \sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_q B_{k,q}(x),$$

$$(2.9) \quad x^n = \frac{1}{2} \left[ E_{n,q}(x) + \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}(x) \right],$$

and

$$(2.10) \quad x^n = \frac{1}{2[n+1]_q} \left[ G_{n+1,q}(x) + \sum_{k=0}^n \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q G_{k+1,q}(x) \right].$$

*Proof.* Form the generating function (??) with  $\alpha = 1$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} &= \frac{e_q(t) - 1}{t} \frac{t}{e_q(t) - 1} e_q(xt) \\ &= \left( \sum_{n=0}^{\infty} \frac{t^n}{[n+1]_q!} \right) \left( \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \frac{[n]_q! B_{k,q}(x)}{[k]_q! [n+1-k]_q!} \right) \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \frac{1}{[n+1]_q} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q B_{k,q}(x) \right) \frac{t^n}{[n]_q!}. \end{aligned}$$

This proves the relation for  $q$ -Bernoulli polynomials. The other results are obtained similarly.

A second proof of (??). It is easy to see from the generating function (??) that

$$G_{n,q}^{(\alpha+\beta)}(x \oplus_q y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}^{(\alpha)}(x) G_{n-k,q}^{(\beta)}(y),$$

which in the special case when  $y = 1$  and  $\beta = 0$ , yields

$$(2.11) \quad G_{n,q}^{(\alpha)}(x \oplus_q 1) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}^{\alpha}(x).$$

Moreover,

$$\begin{aligned} \sum_{n=0}^{\infty} (G_{n,q}(x \oplus_q 1) + G_{n,q}(x)) \frac{t^n}{[n]_q!} &= \frac{2t}{e_q(t) + 1} (e_q((x \oplus_q 1)t) + e_q(xt)) \\ &= 2te_q(xt) = \sum_{n=0}^{\infty} 2[n+1]_q x^n \frac{t^{n+1}}{[n+1]_q!}, \end{aligned}$$

which implies that

$$(2.12) \quad G_{n+1,q}(x \oplus_q 1) + G_{n+1,q}(x) = 2[n+1]_q x^n.$$

Combining (??) with  $\alpha = 1$  and (??) yields the result.

A third proof of (??). Taking  $\ell = 1$  in (??), we obtain

$$E_{n,q}(x) = \frac{1}{[n+1]_q} G_{n+1,q}(x).$$

Equation (??) becomes

$$\begin{aligned} x^n &= \frac{1}{2} \left[ \frac{1}{[n+1]_q} G_{n+1,q}(x) + \sum_{k=0}^n \frac{1}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k+1,q}(x) \right] \\ &= \frac{1}{2} \left[ \frac{1}{[n+1]_q} G_{n+1,q}(x) + \frac{1}{[n+1]_q} \sum_{k=0}^n \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q G_{k+1,q}(x) \right], \end{aligned}$$

which is the required result.  $\square$

**Theorem 2.6.** *Let  $\{f_n^{(\alpha)}(x)\}_{n \in \mathbb{N}_0}$  be a one-parameter sequence of  $q$ -Appell polynomials generated by (??). Then each of the following addition formulas holds true:*

$$\begin{aligned} f_n^{(\alpha)}(x \oplus_q y) &= \sum_{j=0}^n \left[ \sum_{k=j}^n (-1)^{k-j} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{k-j}{2}} f_{n-k}^{(\alpha)}(y) \right] H_j(x; q), \\ f_n^{(\alpha)}(x \oplus_q y) &= \sum_{j=0}^n \left[ \sum_{k=j}^n \frac{1}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k+1 \\ j \end{bmatrix}_q f_{n-k}^{(\alpha)}(y) \right] B_{j,q}(x), \\ f_n^{(\alpha)}(x \oplus_q y) &= \frac{1}{2} \sum_{j=0}^n \left[ \begin{bmatrix} n \\ j \end{bmatrix}_q f_{n-j}^{(\alpha)}(y) + \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q f_{n-k}^{(\alpha)}(y) \right] E_{j,q}(x), \\ f_n^{(\alpha)}(x \oplus_q y) &= \frac{1}{2} \sum_{j=0}^n \frac{1}{[j+1]_q} \left[ \begin{bmatrix} n \\ j \end{bmatrix}_q f_{n-j}^{(\alpha)}(y) + \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q \right] G_{j+1,q}(x), \\ f_n^{(\alpha)}(x \oplus_q y) &= \sum_{j=0}^n \left[ \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q f_{n-k}^{(\alpha)}(y) \sum_{\ell=0}^{k-j} \begin{bmatrix} k-j \\ \ell \end{bmatrix}_q a^\ell \right] U_j^{(a)}(x; q), \end{aligned}$$

and

$$(2.13) \quad f_n^{(\alpha)}(x \oplus_q y) = \sum_{j=0}^n \left[ \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q S_q(k, j) f_{n-k}^{(\alpha)}(y) \right] (x)_{j,q}.$$

*Proof.* The proof of this theorem uses (??), Proposition ??, Lemma ?? and the summation formula

$$(2.14) \quad \sum_{k=0}^n A_k \sum_{j=0}^k B_j = \sum_{j=0}^n \left( \sum_{k=j}^n A_k \right) B_j. \quad \square$$

### 3. $q$ -addition formulas involving $q$ -hypergeometric polynomials

The basic hypergeometric or  $q$ -hypergeometric function  ${}_r\phi_s$  is defined by the series

$${}_r\phi_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) := \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} \left( (-1)^k q^{\binom{k}{2}} \right)^{1+s-r} \frac{z^k}{(q; q)_k},$$

where

$$(a_1, \dots, a_r)_k := (a_1; q)_k \cdots (a_r; q)_k,$$

with

$$(a_i; q)_k = \begin{cases} \prod_{j=0}^{k-1} (1 - a_i q^j) & \text{if } k = 1, 2, 3, \dots \\ 1 & \text{if } k = 0. \end{cases}$$

For two general families of  $q$ -hypergeometric polynomials, Verma [?] derived the following expansion formulas:

$$\begin{aligned} {}_{r+t}\phi_{s+u} \left( \begin{matrix} (a_r), (c_t) \\ (b_s), (d_u) \end{matrix} \middle| q; y\omega \right) &= \sum_{j=0}^{\infty} \frac{((c_t), (e_k); q)_j}{(q, (d_u), \gamma q^j; q)_j} y^j \left[ (-1)^j q^{\binom{j}{2}} \right]^{u+3-t-k} \\ &\quad \cdot {}_{t+k}\phi_{u+1} \left( \begin{matrix} (c_t q^j), (e_k q^j) \\ \gamma q^{2j+1}, (d_u q^j) \end{matrix} \middle| q; y q^{j(u+2-t-k)} \right) \\ (3.1) \quad &\quad \cdot {}_{r+2}\phi_{s+k} \left( \begin{matrix} q^{-j}, \gamma q^j, (a_r) \\ (b_s), (e_k) \end{matrix} \middle| q; \omega q \right) \end{aligned}$$

in powers of  $y\omega$  as given in [?, (3.7.9)] to find the solution of the inversion problem for polynomials of the Askey scheme and its  $q$ -analogue. Here, the notation  $(a_r)$  means  $r$  parameters of the type  $a_1, a_2, \dots, a_r$  and the notation  $(a_r q^j)$  means  $r$  parameters of the form  $a_1 q^j, a_2 q^j, \dots, a_r q^j$ . The method is the following.

We choose  $u = t = 0$ , and  $k = 1$  in (??). Then for  $\omega = x$  and  $\gamma = 0$ , we obtain

$$\begin{aligned} {}_r\phi_s \left( \begin{matrix} (a_r) \\ (b_s) \end{matrix} \middle| q; yx \right) &= \sum_{j=0}^{\infty} \frac{[(-1)^j q^{\binom{j}{2}}]^2}{(q; q)_j} y^j {}_1\phi_1 \left( \begin{matrix} 0 \\ 0 \end{matrix} \middle| q; q^j y \right) \\ &\quad \cdot {}_{r+1}\phi_s \left( \begin{matrix} q^{-j}, (a_r) \\ (b_s) \end{matrix} \middle| q; qx \right). \end{aligned}$$

Expanding the left-hand side, the coefficient of  $y^n$  is

$$(3.2) \quad \frac{((a_r); q)_n}{(q; q)_n (b_s; q)_n} \left[ (-1)^n q^{\binom{n}{2}} \right]^{s-r+1} x^n.$$

Moreover, the right-hand side can be rewritten as

$$\sum_{j=0}^{\infty} \sum_{h=0}^{\infty} \left( \frac{q^{jh}}{(q; q)_j} \left[ (-1)^j q^{\binom{j}{2}} \right]^2 \frac{\left[ (-1)^h q^{\binom{h}{2}} \right]}{(q; q)_h} y^{h+j} \right) {}_{r+1}\phi_s \left( \begin{matrix} q^{-j}, (a_r) \\ (b_s) \end{matrix} \middle| q; qx \right),$$

so that the coefficient of  $y^n$  in this expression is now

$$(3.3) \quad \sum_{\ell=0}^n \frac{(-1)^{n-\ell} q^{2\binom{\ell}{2}} q^{\binom{n-\ell}{2}} q^{(n-\ell)\ell}}{(q; q)_\ell (q; q)_{n-\ell}} {}_{r+1}\phi_s \left( \begin{matrix} q^{-\ell}, (a_r) \\ (b_s) \end{matrix} \middle| q; qx \right).$$

From (??) and (??) we get

$$(3.4) \quad \frac{\left( (-1)^n q^{\binom{n}{2}} \right)^{s-r} (a_2, \dots, a_{r+1}; q)_n}{(b_1, b_2, \dots, b_s; q)_n} x^n \\ = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} {}_{r+1}\phi_s \left( \begin{matrix} q^{-k}, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q; qx \right),$$

already given in [?, (3.3)].

It should be mentioned that until now, the coefficients  $a_R$  and  $b_S$  appearing in (??) are independent of the summation index  $k$ . However, in some families belonging to the Askey scheme and its  $q$ -analogue, one of the numerator parameters depends on  $k$  in the form  $a_2 + k$  (Askey scheme) or  $a_2 q^k$  ( $q$ -analogue). In these situations and in case of polynomials belonging to the  $q$ -analogue of the Askey scheme, the following formula (see [?, (3.5)]) should be used:

$$(3.5) \quad \frac{\left( (-1)^n q^{\binom{n}{2}} \right)^{s-r} (a_3, \dots, a_{r+1})_n}{(b_1, b_2, \dots, b_s; q)_n} x^n \\ = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^k q^{\binom{k}{2}}}{(a_2 q^k, a_2 q^{2k+1}; q)_k} {}_{r+1}\phi_s \left( \begin{matrix} q^{-k}, a_2 q^k, a_3, \dots, a_{r+1} \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q; qx \right).$$

By applying the expansion formula (??) and using relation (??), it is not difficult to prove the following theorem.

**Theorem 3.1.** *Let  $\{f_n^{(\mu)}(x)\}_{n \in \mathbb{N}_0}$  be a one-parameter sequence of  $q$ -Appell polynomials generated by (??) with the parameter  $\alpha$  replaced by  $\mu$ . Then each of the following  $q$ -addition formulas hold true.*

$$(3.6) \quad f_n^{(\mu)}(x \oplus_q y) = \sum_{j=0}^n \left( \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{\left( (-1)^k q^{\binom{k}{2}} \right)^{r-s} \prod_{\ell=1}^s (b_\ell; q)_k}{\prod_{\ell=2}^{r+1} (a_\ell; q)_k} f_{n-k}^{(\mu)}(y) \right) \\ \times (-1)^j q^{\binom{j}{2}} {}_{r+1}\phi_s \left( \begin{matrix} q^{-j}, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q; qx \right)$$

and

$$(3.7) \quad \begin{aligned} f_n^{(\mu)}(x \oplus_q y) &= \sum_{j=0}^n \left( \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{((-1)^k q^{\binom{k}{2}})^{r-s} \prod_{\ell=1}^s (b_\ell; q)_k}{\prod_{\ell=3}^{r+1} (a_\ell; q)_k} f_{n-k}^{(\mu)}(y) \right) \\ &\times \frac{(-1)^j q^{\binom{j}{2}}}{(a_2 q^j, a_2 q^{2j+1}; q)_j} {}_{r+1}\phi_s \left( \begin{matrix} q^{-j}, a_2 q^j, a_3, \dots, a_{r+1} \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q; qx \right). \end{aligned}$$

*Remark 3.2.* Corollary ?? below, which involves such classical  $q$ -orthogonal polynomials as the Little  $q$ -Laguerre, the Little  $q$ -Legendre, the Little  $q$ -Laguerre, the  $q$ -Laguerre, the  $q$ -Bessel and the Stieltjes-Wigert polynomials, can be deduced by suitably specializing Theorem ?? or (alternatively) by directly applying (??) in conjunction with the following known polynomial expansions (see [?]):

- the Little  $q$ -Jacobi polynomials

$$(3.8) \quad x^k = \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}} (aq; q)_k}{(abq^{j+1}; q)_j (abq^{2j+2}; q)_{k-j}} p_j(x; a, b|q).$$

- the Little  $q$ -Legendre polynomials

$$(3.9) \quad x^k = \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}} (q; q)_k}{(q^{j+1}; q)_j (q^{2j+2}; q)_{k-j}} P_j(x|q).$$

- the Little  $q$ -Laguerre polynomials

$$(3.10) \quad x^k = \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}} (aq; q)_k p_j(x; a|q).$$

- the  $q$ -Laguerre polynomials

$$(3.11) \quad x^k = \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{(j-k)(2\alpha+3j+k+1)}{2}} \frac{(q; q)_j}{q^{j(j+\alpha)}} (q^{j+\alpha+1}; q)_{k-j} L_j^{(\alpha)}(x; q).$$

- the  $q$ -Bessel polynomials

$$(3.12) \quad x^k = \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}}}{(-aq^j; q)_j (-aq^{2j+1}; q)_{k-j}} y_j(x; a|q).$$

- the Stieltjes-Wigert polynomials

$$(3.13) \quad x^k = \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{(j-k)(3j+k+1)}{2} - j^2} (q; q)_j S_j(x; q).$$

**Corollary 3.3.** Let  $\{f_n^{(\mu)}(x)\}_{n \in \mathbb{N}_0}$  be a one-parameter sequence of  $q$ -Appell polynomials generated by (??) with the parameter  $\alpha$  replaced by  $\mu$ . Then each

of the following  $q$ -addition formulas hold true.

$$f_n^{(\mu)}(x \oplus_q y) = \sum_{j=0}^n \left( \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}} (aq; q)_k f^{(\mu)}(y)}{(abq^{j+1}; q)_j (abq^{2j+2}; q)_{k-j}} \right) p_j(x; a, b|q),$$

$$f_n^{(\mu)}(x \oplus_q y) = \sum_{j=0}^n \left( \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}} (q; q)_k f^{(\mu)}(y)}{(q^{j+1}; q)_j (q^{2j+2}; q)_{k-j}} \right) P_j(x|q),$$

$$f_n^{(\mu)}(x \oplus_q y) = \sum_{j=0}^n \left( \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j q^{\binom{j}{2}} (aq; q)_k f^{(\mu)}(y) \right) p_j(x; a|q),$$

$$f_n^{(\mu)}(x \oplus_q y) = \sum_{j=0}^n \left( \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{(j-k)(2\alpha+3j+k+1)}{2}} \frac{(q; q)_j}{q^{j(j+\alpha)}} (q^{j+\alpha+1}; q)_{k-j} \right) \cdot L_j^{(\alpha)}(x; q),$$

$$f_n^{(\mu)}(x \oplus_q y) = \sum_{j=0}^n \left( \sum_{k=j}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}}}{(-aq^j; q)_j (-aq^{2j+1}; q)_{k-j}} \right) y_j(x; a|q),$$

and

$$f_n^{(\mu)}(x \oplus_q y) = \sum_{j=0}^n \left( \sum_{k=j}^n (-1)^j \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{(j-k)(3j+k+1)}{2} - j^2} (q; q)_j \right) S_j(x; q).$$

#### 4. A $q$ -umbral-calculus generalization of the addition theorems

In 1978, Roman and Rota voved the classical umbral calculus from a new perspective and proposed an interesting approach based on a simple but innovative indication for effect of linear functional on polynomials, which Roman later called it the modern umbral calculus [?]. Roman, also, proposed a similar approach under the area of nonclassical umbral calculus which is called  $q$ -umbral calculus, [?, ?, ?, ?]. In what follows, we adopt the notations of [?].

Let  $\mathbb{C}$  be the field of complex numbers and  $\mathcal{F}$  the set of all formal power  $q$ -series in the variable  $t$  over  $\mathbb{C}$ . In other words,  $f(t)$  in an element of  $\mathcal{F}$  if

$$(4.1) \quad f(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]_q!} t^k,$$

where  $a_k \in \mathbb{C}$ . Let  $\mathcal{P}$  be the algebra of all polynomials in the variable  $x$  over  $\mathbb{C}$  and  $\mathcal{P}^*$  be the vector space of all linear functionals on  $\mathcal{P}$ . The formal power series (??) defines a linear functional on  $\mathcal{P}^*$  by setting

$$(4.2) \quad \langle f(t) | x^n \rangle = a_n \quad (n \in \mathbb{N}_0).$$

**Lemma 4.1** (see [?]). Let  $\{h_n(x)\}_{n \in \mathbb{N}_0}$  be a  $q$ -Appell sequence for the function  $h(t)$ . Then, for any polynomial  $\mathfrak{p}(x)$ ,

$$(4.3) \quad \mathfrak{p}(x) = \sum_{j \geq 0} \frac{\langle h(t) | \mathfrak{p}^{(j)}(x) \rangle}{[j]_q!} h_j(x),$$

where  $\mathfrak{p}^{(j)}(x)$  denotes the  $q$ -derivative of  $\mathfrak{p}(x)$  of order  $j$ .

**Theorem 4.2.** Let  $\{f_n(x)\}_{n \in \mathbb{N}_0}$  and  $\{g_n(x)\}_{n \in \mathbb{N}}$  be the  $q$ -Appell sequences corresponding to the functions  $f(t)$  and  $g(t)$ , respectively. Then

$$(4.4) \quad f_n(x \oplus_q y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q f_k(y) \sum_{j=0}^{n-k} \frac{\langle g(t) | D_q^j [x^{n-j}] \rangle}{[j]_q!} g_j(x).$$

*Proof.* The proof follows directly from Lemma ?? and Proposition ??.

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