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SINGULAR CLEAN RINGS

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Abstract. In this paper, we define right singular clean rings as rings in which every element can be written as a sum of a right singular element and an idempotent. Several properties of these rings are investigated. It is shown that for a ring R, being singular clean is not left-right symmetric. Also the relations between (nil) clean rings and right singular clean rings are considered. Some examples of right singular clean rings have been constructed by a given one. Finally, uniquely right singular clean rings and weakly right singular clean rings are also studied.

1. Introduction

All rings we consider are associative with identity $1 \neq 0$. The Jacobson radical, the group of units, the set of idempotents and the set of nilpotent elements of a ring R will be denoted by $J(R)$, $U(R)$, $Id(R)$ and $Nil(R)$, respectively. Recall that an element x of a ring R is right (left) singular if $ann_r(x)$ ($ann_l(x)$) is an essential right (left) ideal of R. The set of all right (left) singular elements of R is a two-sided ideal of R which is called the right (left) singular ideal of R and is denoted by $\mathcal{Z}(R_R)$ ($\mathcal{Z}({}_RR)$). A ring R is called right nonsingular if $\mathcal{Z}(R_R) = 0$. Nicholson in [8] introduced clean rings for the first time as rings in which every element is a sum of a unit and an idempotent. After that, some generalizations of clean rings such as strongly clean [9], uniquely clean [10] and weakly clean rings [1], have been considered. Diesel in [4] defined nil clean rings. A ring R is called nil clean, if every element is a sum of a nilpotent element and an idempotent. It was shown that every nil clean ring is clean. In this paper, we study rings in which every element is a sum of a right singular element and an idempotent and we call these rings, right singular clean. We shall give a necessary and sufficient condition for a right singular clean ring to be clean. The direct products and the ring direct summands of right singular clean rings, are also right singular clean. However, the homomorphic images of these rings need not be right singular clean. The $n \times n$ matrix ring, the polynomial ring and the formal power series ring over any ring, are never right singular clean. Similar to [1] and [10], we shall define weakly right singular clean and uniquely

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right singular clean rings. It is proved that a ring R is uniquely right singular clean if and only if R is abelian and right singular clean.

2. Right singular clean rings

In this section we introduce (right) singular clean rings and investigate some basic properties of these rings.

Definition 2.1. We call a ring R *right singular clean*, if every element of R can be expressed as a sum of a right singular element and an idempotent. Left singular clean rings are defined similarly. A ring R is called singular clean, if it is both right and left singular clean.

Example 2.2. (a) Clearly the rings $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are singular clean but the ring $\mathbb{Z}_3$ is not.

(b) The ring $R = \mathbb{Z}_4 \oplus \mathbb{Z}_4 i \oplus \mathbb{Z}_4 j \oplus \mathbb{Z}_4 k$, with multiplication defined by $i^2 = j^2 = -1$ and $ij = -ji = k$ is a noncommutative singular clean ring, since $Id(R) = \{0, 1\}$ and a messy calculation shows that

$$\mathcal{Z}(R_R) = \{a + bi + cj + dk \in R \mid a + b + c + d \in 2\mathbb{Z}_4\}.$$

It is easily seen that a right nonsingular ring is right singular clean if and only if it is Boolean. Now we have the following characterization of right singular clean rings.

Proposition 2.3. *A ring R is right singular clean if and only if $\frac{R}{\mathcal{Z}(R_R)}$ is Boolean and idempotents lift modulo $\mathcal{Z}(R_R)$.*

Recall that a proper ideal P of a ring R is called completely prime, if for any $a, b \in R$ whenever $ab \in P$, then $a \in P$ or $b \in P$.

Corollary 2.4. *Let R be a right singular clean ring. Then*

(1) $2 \in \mathcal{Z}(R_R)$.
(2) *$J(R)$ and $Nil(R)$ are contained in $\mathcal{Z}(R_R)$.*
(3) *If $x \in R$ and $x^n \in \mathcal{Z}(R_R)$ for some $n \in N$, then $x \in \mathcal{Z}(R_R)$.*
(4) *Every prime ideal of R which contains $\mathcal{Z}(R_R)$ is both completely prime and maximal.*
(5) *If R satisfies ACC on right annihilators of elements, then every prime ideal of R is maximal.*
(6) *R is Dedekind-finite.*

Proof. The proofs of parts (1), (2), (3) and (4) follow from the fact that $\frac{R}{\mathcal{Z}(R_R)}$ is a Boolean ring.

(5) By [6, Theorem 7.15], $\mathcal{Z}(R_R) \subseteq Nil_*(R)$, where

$$Nil_*(R) = \cap\{P \mid P \text{ is a prime ideal of } R\}.$$

So $\mathcal{Z}(R_R) \subseteq P$ for every prime ideal P. Now, apply part (4).

(6) Let $a, b \in R$ and $ab = 1$. The element $b(1-ba)$ is nilpotent and so by part (2) belongs to $\mathcal{Z}(R_R)$. Thus $1 - ba = ab(1 - ba) \in Id(R) \cap \mathcal{Z}(R_R)$. Therefore, $1 - ba = 0$. □

Now, we show that for any ring R, the polynomial ring $S = R[x]$ and the formal power series ring $T = R[[x]]$ are neither left nor right singular clean. By [6, Exercise 7.35], $\mathcal{Z}(S_S) = \mathcal{Z}(R_R)[x]$. Therefore, $\frac{S}{\mathcal{Z}(S_S)}$ is not a Boolean ring and so Proposition 2.3 implies that S is not right singular clean. Also, $x \in J(T)$ but $x \notin \mathcal{Z}(_T)$. So by Corollary 2.4 part (2), T is not right singular clean. Moreover, for any ring R and any infinite set Γ, the ring of column (respectively, row) finite $\Gamma \times \Gamma$ matrices over R is not Dedekind-finite and so by Corollary 2.4 part (6), is neither left nor right singular clean.

Proposition 2.5. *If R is a commutative singular clean ring, then every prime ideal P of R is either maximal or an essential ideal.*

Proof. Let P be a prime ideal of R which is not essential and let $x \in R - P$. There exist $s \in \mathcal{Z}(R)$ and $e \in Id(R)$ such that $x = s + e$. Clearly $ann(s) \nsubseteq P$. Choose $r \in ann(s) - P$. So we have $rx = re$ and $rxe = re$. Thus $re(1 - x) = 0$ which implies that $(1 - x) \in P$. Hence $P + \langle x \rangle = R$. So P is a maximal ideal of R. □

Proposition 2.6. *If R is a right singular clean ring which satisfies ACC on right annihilators, then R is a semiprimary ring and*

$$\frac{R}{J(R)} = \frac{R}{\mathcal{Z}(R_R)} \cong \prod_{i=1}^{n} \mathbb{Z}_2$$

for some $n \in \mathbb{N}$.

Proof. By [6, Theorem 7.15], $\mathcal{Z}(R_R)$ is a nilpotent ideal and so Corollary 2.4 implies that $J(R) = \mathcal{Z}(R_R)$. Since R satisfies ACC on right annihilators, by [6, Propositions 6.59 and 6.60], it has no infinite set of orthogonal idempotents. As idempotents lift modulo $\mathcal{Z}(R_R)$= $J(R)$, the ring $\frac{R}{J(R)} = \frac{R}{\mathcal{Z}(R_R)}$ also has no infinite set of orthogonal idempotents. Therefore, the Boolean ring $\frac{R}{J(R)}$ is a semisimple ring [5, Theorem 10.6], and so is isomorphic to $\prod_{i=1}^{n} \mathbb{Z}_2$, for some $n \in \mathbb{N}$. □

Let R and S be rings and M be an (R, S)-bimodule such that $\mathcal{Z}(M_S) \neq M$. We show that the ring $T = \left[\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right]$ is not right singular clean. To see this, let $m \in M - \mathcal{Z}(M_S)$. Then $\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right]$ is a nilpotent element of T which is not in $\mathcal{Z}(T_T)$. Similarly if $\mathcal{Z}(_RM) \neq M$, then T is not left singular clean. With the same technique, we can show that for any $n \geq 2$, the ring of $n \times n$ matrices, the ring of $n \times n$ upper triangular matrices and the ring of $n \times n$ lower triangular matrices over any ring R are neither left nor right singular clean. The following example shows that a right singular clean ring need not be left singular clean.

Example 2.7. Let $R = \left[\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_4 \end{smallmatrix}\right]$. Then

$$Id(R) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

and by some calculations

$$\mathcal{Z}(R_R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right\}.$$

So by a direct check, we can show that R is right singular clean. The element $x = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is nilpotent which is not a left singular element, since $ann_l(x) = \left[\begin{smallmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_4 \end{smallmatrix}\right]$ is a direct summand of ${}_RR$. Thus by Corollary 2.4, R is not left singular clean.

Proposition 2.8. *A ring R is right singular clean if and only if every element of R can be written as a difference of a right singular element and an idempotent.*

Proof. Suppose that R is right singular clean and $x \in R$. Then there exist $s \in \mathcal{Z}(R_R)$ and $e \in Id(R)$, such that $-x = s + e$. So $x = -s - e$. The proof of the converse implication is similar. □

McGovern in [7] introduced the commutative almost clean rings as rings in which every element can be written as a sum of a regular element (an element which is not a zero-divisor) and an idempotent. We call a ring R *right almost clean*, if for every $x \in R$ there exist $a \in R$ and $e \in Id(R)$ such that $x = a + e$, where $ann_r(a) = 0$.

Proposition 2.9. *Every right singular clean ring is right almost clean.*

Proof. Let R be a right singular clean ring and $x \in R$. Then there exist $s \in \mathcal{Z}(R_R)$ and $e \in Id(R)$ such that $x - 1 = s + e$. So $x = (s + 1) + e$ and $ann_r(s + 1) = 0$. Therefore, R is a right almost clean ring. □

We need the following result in the sequel.

Lemma 2.10. *Let R be a ring and $a \in R$. If $a = s + e$, where $s \in \mathcal{Z}(R_R)$, $e \in Id(R)$ and $ann_r(a) = 0$, then $e = 1$.*

Proof. Since $ann_r(e) \cap ann_r(s) = 0$, we have $ann_r(e) = 0$. Thus $e = 1$. □

Proposition 2.11. *Let R be a right singular clean ring. Then for any elements $u, v \in R$ with $ann_r(u) = ann_r(v) = 0$ (in particular, if $u, v \in U(R)$), the element $(u - v) \in \mathcal{Z}(R_R)$.*

Proof. By Lemma 2.10, there exist $s, s' \in \mathcal{Z}(R_R)$ such that $u = s + 1$ and $v = s' + 1$ so that $u - v = s - s' \in \mathcal{Z}(R_R)$. □

In [3], a ring R is called a *UU-ring*, if all units in R are unipotent, i.e., $U(R) \subseteq 1 + Nil(R)$.

Proposition 2.12. *If R is a right singular clean ring and $\mathcal{Z}(R_R)$ is a nil ideal, then R is a UU-ring.*

Proof. Let $u \in U(R)$. By Lemma 2.10, $u = s + 1$, where $s \in \mathcal{Z}(R_R)$. So R is a UU-ring, since $\mathcal{Z}(R_R)$ is nil. □

Proposition 2.13. *Let $\{R_i\}_{i\in I}$ be a family of rings and $R=\prod_{i\in I} R_i$. Then R is right singular clean if and only if each R_i is right singular clean.*

Proof. The proof follows from the fact that $\mathcal{Z}(R_R)=\prod_{i\in I}\mathcal{Z}(R_{i_{R_i}})$. □

If $e\in R$ is a central idempotent, then R is a right singular clean ring if and only if eRe and $(1-e)R(1-e)$ are so. Note that if $e\in Id(R)$ is not central, then the above statement is not valid in general. For example, in the ring $M_2(\mathbb{Z}_2)$, let $e=\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$. Then the rings eRe and $(1-e)R(1-e)$ are isomorphic to $\mathbb{Z}_2$ and so they are singular clean . But the ring R is neither right nor left singular clean.

Proposition 2.14. *Let R be a right singular clean ring and $e\in Id(R)$ such that $eR(1-e)=0$. Then the ring $S=eRe$ is right singular clean.*

Proof. First we show that if $\mathcal{Z}(R_R)\cap S\subseteq\mathcal{Z}(S_S)$. Let $a\in\mathcal{Z}(R_R)\cap S$ and $0\neq s\in S$. There exists $r\in R$ such that $sr\neq 0$ and $asr=0$. Thus $as(ere)=0$ and $s(ere)\neq 0$, since $eR(1-e)=0$. This implies that $a\in\mathcal{Z}(S_S)$. Now, to see that S is right singular clean, let $x\in S$. Then $x=s+f$, where $s\in\mathcal{Z}(R_R)$ and $f\in Id(R)$. Thus $x=ese+efe$. Since $eR(1-e)=0$, $ese\in\mathcal{Z}(S_S)$ and $efe\in Id(S)$. □

Proposition 2.15. *Let R be a local ring. Then R is a right singular clean ring if and only if $\frac{R}{\mathcal{Z}(R_R)}\cong\mathbb{Z}_2$.*

Proof. $\Leftarrow$) The proof follows from Proposition 2.3.
$\Rightarrow$) By Corollary 2.4, $J(R)\subseteq\mathcal{Z}(R_R)$ and hence $\mathcal{Z}(R_R)=J(R)$. So the Boolean ring $\frac{R}{\mathcal{Z}(R_R)}$ is a division ring. Therefore, it is isomorphic to $\mathbb{Z}_2$. □

Corollary 2.16. *Let M be a maximal ideal of a commutative ring R. Then the following are equivalent*:

(1) *The ring $\frac{R}{M}$ is singular clean*;
(2) *The ring $\frac{R}{M^k}$ is singular clean, for every $k\in\mathbb{N}$*;
(3) *The ring $\frac{R}{M^k}$ is singular clean, for some $k\in\mathbb{N}$.*

Proof. The proof is a consequence of the fact that for any $k\geq 1$, $S=\frac{R}{M^k}$ is a local ring with $J(S)=\mathcal{Z}(S)=\frac{M}{M^k}$ and $\frac{S}{\mathcal{Z}(S)}\cong\frac{R}{M}$. □

Example 2.17. The ideal $\langle x\rangle$ is a maximal ideal of $\mathbb{Z}_2[x]$ and the ring $\frac{\mathbb{Z}_2[x]}{\langle x\rangle}\cong\mathbb{Z}_2$ is a singular clean ring. By Corollary 2.16, the ring $\frac{\mathbb{Z}_2[x]}{\langle x\rangle^n}$ is a singular clean ring for every $n\in\mathbb{N}$.

Corollary 2.18. *The ring $\mathbb{Z}_k$, where $k\in\mathbb{N}$ is singular clean if and only if $k=2^n$ for some $n\in\mathbb{N}$.*

Proof. The proof follows from Proposition 2.13 and the fact that for a prime number p, the local ring $\mathbb{Z}_{p^n}$ is singular clean if and only if $p=2$ (Proposition 2.15). □

As we have seen in Corollary 2.4, in a right singular clean ring R, every nilpotent element belongs to $\mathcal{Z}(R_R)$. The following example shows that for a commutative singular clean ring R, $\mathcal{Z}(R)$ need not be a nil ideal. Moreover, the factor ring of a singular clean ring is not necessarily singular clean.

Example 2.19. By Corollary 2.18, for any $n \in \mathbb{N}$ the ring $\mathbb{Z}_{2^n}$ is a singular clean ring and Proposition 2.13 implies that $R = \prod_{n\in\mathbb{N}} \mathbb{Z}_{2^n}$ is a singular clean ring. Note that $\mathcal{Z}(R)$ is not a nil ideal, since the element $(0, 2, 2, \ldots) \in \mathcal{Z}(R) = J(R)$ is not a nilpotent element. Now, since $\mathcal{Z}(R) = J(R)$, there exists a prime ideal P of R which is not maximal. Therefore, the ring $\frac{R}{P}$ is not a singular clean ring, otherwise, the integral domain $\frac{R}{P}$ would be isomorphic to $\mathbb{Z}_2$ and so P is a maximal ideal of R which is a contradiction.

Theorem 2.20. *Let R be a right singular clean ring. Then R is a clean ring if and only if $\mathcal{Z}(R_R)= J(R)$.*

Proof. Suppose that $\mathcal{Z}(R_R)= J(R)$ and let $x \in R$. We have $x - 1 = s + e$ for some $s \in \mathcal{Z}(R_R)$ and $e \in Id(R)$. Thus $x = (s+1)+e$, which is a sum of a unit and an idempotent. Therefore, R is a clean ring. Conversely, assume that R is a clean ring. Now, let $s \in \mathcal{Z}(R_R)$. We have $s = u + e$ for some $u \in U(R)$ and $e \in Id(R)$. By Lemma 2.10, $e = 1$ and hence $s - 1 \in U(R)$. Thus $s \in J(R)$. So $\mathcal{Z}(R_R)\subseteq J(R)$. Since R is right singular clean, by Corollary 2.4, $J(R) \subseteq \mathcal{Z}(R_R)$. □

Note that a clean ring R with $\mathcal{Z}(R_R)= J(R)$, need not be right singular clean. For example, consider the clean ring $\mathbb{Z}_6$ which is not singular clean by Corollary 2.18.

Let R be a commutative ring and M be an R-module. The idealization $S = R(+)M$ with the following addition and multiplication

$$(r, m) + (r', m') = (r + r', m + m') \text{ and } (r, m)(r', m') = (rr', rm' + r'm)$$

is a commutative ring.

Example 2.21. Let $S = \mathbb{Z}(+)\mathbb{Z}_{2^\infty}$, where $\mathbb{Z}_{2^\infty}$ is the Prüfer 2-group. Let $K = \langle \frac{1}{2} + \mathbb{Z}\rangle$ be the minimal subgroup of $\mathbb{Z}_{2^\infty}$. Then $0(+)K$ is an essential ideal of S. Therefore, $2\mathbb{Z}(+)\mathbb{Z}_{2^\infty} = ann(0(+)K) \subseteq \mathcal{Z}(S)$. Since $2\mathbb{Z}(+)\mathbb{Z}_{2^\infty}$ is a maximal ideal of S, we have $\mathcal{Z}(S) = 2\mathbb{Z}(+)\mathbb{Z}_{2^\infty}$. Consequently, $\frac{S}{\mathcal{Z}(S)} \cong \mathbb{Z}_2$ implies that S is a singular clean ring. On the other hand, $J(S) = 0(+)\mathbb{Z}_{2^\infty}$ and according to Theorem 2.20, S is not a clean ring.

Theorem 2.22. *If R is a commutative singular clean ring and M is a nonsingular R-module, then the idealization $S = R(+)M$ is a commutative singular clean ring.*

Proof. Let $(r, m) \in S$. Then there exist $z \in \mathcal{Z}(R)$ and $e \in Id(R)$ such that $r = z + e$. So $(r, m) = (e, 0) + (z, 0) + (0, m)$. Now, $(e, 0) \in Id(S)$ and $(0, m) \in Nil(S) \subseteq \mathcal{Z}(S)$. It is sufficient to show that $(z, 0) \in \mathcal{Z}(S)$. Since

$z \in \mathcal{Z}(R)$, there exists an essential ideal I of R such that $Iz = 0$. So $I(zM) = 0$. Thus $zM = 0$, since M is a nonsingular module. As $ann(z, 0) \supseteq I(+)M$, $ann(z, 0)$ is an essential ideal of S. So $(z, 0) \in \mathcal{Z}(S)$. Therefore, S is a singular clean ring. $\square$

Proposition 2.23. *Let R be a commutative ring and $S = R(+)M$, where M is an ideal of R. Then S is singular clean if and only if R is so.*

Proof. The proof follows from the fact that $\mathcal{Z}(S) = \mathcal{Z}(R)(+)M$. $\square$

Proposition 2.24. *Let R be a ring and M be a two-sided ideal of R such that $M \subseteq \mathcal{Z}(R_R)$. Then the ring R is right singular clean if and only if the ring $S = \left[\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix}\right]$ is so.*

Proof. It is not difficult to check that $\mathcal{Z}(S_S) = \left[\begin{smallmatrix} \mathcal{Z}(R_R) & M \\ 0 & \mathcal{Z}(R_R) \end{smallmatrix}\right]$. Now, the rest of the proof is straightforward. $\square$

Lemma 2.25. *Let R be a ring and M be an ideal of R containing an element m_0 with $ann_l(m_0) = 0$. Let $S = \{\left[\begin{smallmatrix} a & m \\ 0 & a \end{smallmatrix}\right] \mid a \in R,\, m \in M\}$. Then $\mathcal{Z}(S_S) = \{\left[\begin{smallmatrix} a & m \\ 0 & a \end{smallmatrix}\right] \in S \mid a \in \mathcal{Z}(R_R)\}$.*

Proof. First we show that if $a \in \mathcal{Z}(R_R)$, then $\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]$ belongs to $\mathcal{Z}(S_S)$. Consider a nonzero element $\left[\begin{smallmatrix} x & m \\ 0 & x \end{smallmatrix}\right] \in S$. If $x \neq 0$, then there exists $r \in R$ such that $xr \neq 0$ and $axr = 0$. By hypothesis $xrm_0 \neq 0$. Thus

$$\begin{bmatrix} x & m \\ 0 & x \end{bmatrix} \begin{bmatrix} 0 & rm_0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & xrm_0 \\ 0 & 0 \end{bmatrix} \neq 0 \text{ and } \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & xrm_0 \\ 0 & 0 \end{bmatrix} = 0.$$

If $x = 0$, then there exists $r \in R$ such that $mr \neq 0$ *and* $amr = 0$. Thus

$$\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} = \begin{bmatrix} 0 & mr \\ 0 & 0 \end{bmatrix} \neq 0 \text{ and } \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & mr \\ 0 & 0 \end{bmatrix} = 0.$$

Therefore, $\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] \in \mathcal{Z}(S_S)$. Now, we show that for every $m \in M$, $\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{Z}(S_S)$. Let $\left[\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right]$ be a nonzero element of S. If $x \neq 0$, then

$$\begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \begin{bmatrix} 0 & m_0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & xm_0 \\ 0 & 0 \end{bmatrix} \neq 0 \text{ and } \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & xm_0 \\ 0 & 0 \end{bmatrix} = 0.$$

If $x = 0$, then $\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right] = 0$. Thus $\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{Z}(S_S)$. Therefore, for any $a \in \mathcal{Z}(R_R)$ and $m \in M$, $\left[\begin{smallmatrix} a & m \\ 0 & a \end{smallmatrix}\right] \in \mathcal{Z}(S_S)$. Conversely, let $\left[\begin{smallmatrix} a & m \\ 0 & a \end{smallmatrix}\right] \in \mathcal{Z}(S_S)$, it is easy to show that $a \in \mathcal{Z}(R_R)$. $\square$

Proposition 2.26. *Let R be a ring and M be an ideal of R containing an element x with $ann_l(x) = 0$. Then R is a right singular clean ring if and only if the ring $S = \{\left[\begin{smallmatrix} a & m \\ 0 & a \end{smallmatrix}\right] \mid a \in R, m \in M\}$ is so.*

Proof. The proof follows easily from Lemma 2.25. $\square$

In Proposition 2.26 if $M = R$, then the ring S is isomorphic to $\frac{R[x]}{\langle x \rangle^2}$. So we have the following corollary.

Corollary 2.27. *A ring R is right singular clean if and only if the ring $\frac{R[x]}{\langle x\rangle^2}$ is so.*

In general for a ring R, the ring

$$S = \left\{ \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & 0 & a_0 & \dots & a_{n-3} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix} \mid a_i \in R \right\}$$

is isomorphic to the ring $\frac{R[x]}{\langle x\rangle^n}$ and

$$\mathcal{Z}(S_S) = \left\{ \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & 0 & a_0 & \dots & a_{n-3} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix} \in S \mid a_0 \in \mathcal{Z}(R_R) \right\}.$$

Therefore, we have the following proposition:

Proposition 2.28. *For a ring R the following are equivalent:*

1. *The ring R is right singular clean;*
2. *The ring $\frac{R[x]}{\langle x\rangle^n}$ is right singular clean for some $n \in \mathbb{N}$;*
3. *The ring $\frac{R[x]}{\langle x\rangle^n}$ is right singular clean for every $n \in \mathbb{N}$.*

Note that if R is a right singular clean ring with no nontrivial idempotents, then for every element $x \in R$, either $x \in \mathcal{Z}(R_R)$ or $(x-1) \in \mathcal{Z}(R_R)$.

Theorem 2.29. *For a right singular clean ring R, the following are equivalent:*

1. $\frac{R}{\mathcal{Z}(R_R)} \cong \mathbb{Z}_2$;
2. $\mathcal{Z}(R_R)$ *is a maximal ideal;*
3. $\mathcal{Z}(R_R)$ *is a prime ideal;*
4. R *has no nontrivial idempotents.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (4) By Corollary 2.4, $\mathcal{Z}(R_R)$ is a completely prime ideal. If $e \in Id(R)$, then we have $e(1-e) = 0 \in \mathcal{Z}(R_R)$ which implies that $e = 0$ or $e = 1$.

(4) $\Rightarrow$ (1) If $x \notin \mathcal{Z}(R_R)$, then $1 - x \in \mathcal{Z}(R_R)$. So $\frac{R}{\mathcal{Z}(R_R)} \cong \mathbb{Z}_2$. □

It is worthwhile to mention that a right singular clean ring with no nontrivial idempotents is not necessarily local as we saw in Example 2.21.

Proposition 2.30. *Let R be a right singular clean ring with no nontrivial idempotents. Then $\mathcal{Z}(R_R)= \{x \in R \mid ann_r(x) \neq 0\}$.*

Proof. If $x \in R- \mathcal{Z}(R_R)$, then $x - 1 \in \mathcal{Z}(R_R)$ and $ann_r(x-1) \cap ann_r(x) = 0$. So $ann_r(x) = 0$. □

We know that in a commutative ring, all nilpotent elements are singular. Thus every commutative nil clean ring is singular clean. But the converse is not true, (See Example 2.31 below). However, if R is a right singular clean ring and $\mathcal{Z}(R_R)$ is a nil ideal of R (for example, R satisfies ACC on right annihilators of elements), then R is a nil clean ring. In noncommutative case, there exist nil clean rings which are neither right nor left singular clean.

Example 2.31. (a) Let $R = M_2(\mathbb{Z}_2)$. Then R is a nil clean ring but it is neither left nor right singular clean.

(b) Let $R = \begin{bmatrix} \frac{\mathbb{Z}}{4\mathbb{Z}} & \frac{2\mathbb{Z}}{4\mathbb{Z}} \\ 0 & \frac{\mathbb{Z}}{4\mathbb{Z}} \end{bmatrix}$. Then $\mathcal{Z}({}_RR) = \mathcal{Z}(R_R) = J(R)$ is a nilpotent ideal. By Proposition 2.24, R is right singular clean. So it is both nil clean and left singular clean.

(c) The ring $R = \prod_{n=1}^{\infty} \mathbb{Z}_{2^n}$ is singular clean but not nil clean, since the element $(0, 2, 2, 2, \ldots)$ can not be written as a sum of a nilpotent element and an idempotent.

3. Uniquely right singular clean rings

In this section we define uniquely (right) singular clean rings and investigate some of their properties.

Definition 3.1. We call a ring R *uniquely right singular clean*, if every element of R can be written uniquely as the sum of a right singular element and an idempotent. A uniquely left singular clean ring is defined similarly. A ring R is called uniquely singular clean if it is both uniquely left singular clean and uniquely right singular clean.

Example 3.2. (a) If R is a right singular clean ring with no nontrivial idempotents, then R is uniquely right singular clean. In particular, a local right singular clean ring is uniquely right singular clean.

(b) The direct product and ring direct summands of uniquely right singular clean rings are uniquely right singular clean.

Proposition 3.3. *A ring R is uniquely right singular clean if and only if R is abelian and right singular clean.*

Proof. Suppose that R is abelian and right singular clean. If $x \in R$ and $x = s + e = s' + e'$, where $s, s' \in \mathcal{Z}(R_R)$ and $e, e' \in Id(R)$, then $e - e' \in \mathcal{Z}(R_R)$. So $e(e-e') = e(1-e') \in \mathcal{Z}(R_R) \cap Id(R)$. Thus $e = ee'$. Similarly $e' = ee'$ and hence $e = e'$ and $s = s'$. Therefore, R is uniquely right singular clean. Conversely, let R be uniquely right singular clean and $e \in Id(R)$. Then for every $r \in R$, we have $e + er - ere \in Id(R)$ and $(ere - er) \in Nil(R)$. So by Corollary 2.4, $(ere - er) \in \mathcal{Z}(R_R)$. We can write $e = e + 0 = (e + er - ere) + (ere - er)$. Hence $ere - er = 0$, since R is uniquely right singular clean. Similarly $ere - re = 0$. Thus, R is an abelian ring. □

Corollary 3.4. *Let R be a ring and M be an ideal of R containing an element x with $ann_l(x) = 0$. Then R is a uniquely right singular clean ring if and only if the ring $S = \{[\begin{smallmatrix} a & m \\ 0 & a \end{smallmatrix}] \mid a \in R, m \in M\}$ is so.*

Proof. Clearly S is an abelian ring if and only if R is so. Now, apply Propositions 2.26 and 3.3. □

It is worthwhile to mention that Proposition 2.28, is valid if we replace right singular clean by uniquely right singular clean.

Note that a right singular clean ring is not necessarily uniquely right singular clean.

Example 3.5. By Proposition 2.24, $R = \begin{bmatrix} \frac{\mathbb{Z}}{4\mathbb{Z}} & \frac{2\mathbb{Z}}{4\mathbb{Z}} \\ 0 & \frac{\mathbb{Z}}{4\mathbb{Z}} \end{bmatrix}$ is a right singular clean ring. Since the idempotent $[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]$ is not central, R is not uniquely right singular clean.

Recall that for a ring R and an ideal I of R, idempotents lift uniquely modulo I, if for every idempotent $(x + I) \in \frac{R}{I}$, there exists a unique idempotent $e \in R$ such that $(x-e) \in I$. Similar to Propositions 2.3 and 2.8, we have the following characterisations of uniquely right singular clean rings.

Proposition 3.6. *For a ring R, the following are equivalent*:

(1) *R is uniquely right singular clean*;
(2) *$\frac{R}{\mathcal{Z}(R_R)}$ is Boolean and idempotents lift uniquely modulo $\mathcal{Z}(R_R)$*;
(3) *Every element of R can be written uniquely as the difference of a right singular element and an idempotent.*

Proposition 3.7. *Let R be a uniquely right singular clean ring and $e, e' \in Id(R)$. If $e - e'$ or $e + e' \in \mathcal{Z}(R_R)$, then $e = e'$.*

Proof. If $e - e' = s \in \mathcal{Z}(R_R)$, then $e = 0 + e = s + e'$. So $s = 0$ and $e = e'$. Now, let $e + e' = s \in \mathcal{Z}(R_R)$. By Corollary 2.4, we have $2 \in \mathcal{Z}(R_R)$. Thus $e = s - e' = 2e - e$. So by Proposition 3.6 part (3), $s = 2e$ and $e = e'$. □

4. Weakly right singular clean rings

The aim of this section is to introduce and study a generalization of the notion of (right) singular clean rings.

Definition 4.1. We call a ring R *weakly right singular clean*, if for every $x \in R$, there exist $s \in \mathcal{Z}(R_R)$ and $e \in Id(R)$ such that $x = s + e$ or $x = s - e$. Weakly left singular clean rings are defined similarly. A ring R is called weakly singular clean, if R is both weakly left singular clean and weakly right singular clean.

Clearly every right singular clean ring is weakly right singular clean, but the converse is not true. For example, the ring $\mathbb{Z}_3$ is weakly singular clean but is not singular clean.

Proposition 4.2. *A ring R is right singular clean if and only if R is weakly right singular clean and $2 \in \mathcal{Z}(R_R)$.*

Proof. $\Rightarrow$) The proof follows from Corollary 2.4.

$\Leftarrow$) If $x = s - e$, where $s \in \mathcal{Z}(R_R)$and $e \in Id(R)$, then $x = (s - 2e) + e$. $\square$

Proposition 4.3. *If R is a commutative weakly singular clean ring, then every prime ideal of R is either essential or maximal.*

Proof. The proof is similar to the proof of Proposition 2.5. $\square$

Note that the product of weakly right singular clean rings, is not necessarily weakly right singular clean. For example, let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. This ring is nonsingular and $Id(R) = \{(0,0), (1,1), (0,1), (1,0)\}$. Thus the element $(1,2)$ can be written neither as a sum nor as a difference of a singular element and an idempotent.

However, we have the following proposition.

Proposition 4.4. *If $\{R_\alpha\}_{\alpha \in \mathcal{A}}$ is a family of rings such that one of them is weakly right singular clean and the others are singular clean, then $\prod_{\alpha \in \mathcal{A}} R_\alpha$ is a weakly right singular clean ring.*

Proof. By Proposition 2.8, in right singular clean rings, we can write every element both as a sum or as a difference of a right singular element and an idempotent. Now, the proof is easily verified. $\square$

Proposition 4.5. *If $R = R_1 \times R_2$ is a weakly right singular clean ring, then so is each R_i, for $i = 1, 2$.*

Proof. The proof is similar to the proof of Proposition 2.13. $\square$

Definition 4.6. Let R be a ring and I be an ideal of R. We say that idempotents lift weakly modulo I, if for every idempotent $(x + I) \in \frac{R}{I}$, there exists $e \in Id(R)$ such that either $(x - e) \in I$ or $(x + e) \in I$.

Proposition 4.7. *A ring R is weakly right singular clean if and only if for every element $\bar{a} \in \frac{R}{\mathcal{Z}(R_R)}$, we have $\bar{a} = \bar{a}^2$ or $\bar{a} = -\bar{a}^2$ and idempotents lift weakly modulo $\mathcal{Z}(R_R)$.*

Proof. $\Leftarrow$) Let $a \in R$. If $\bar{a} = \bar{a}^2$, then there exists $e \in Id(R)$ such that $a - e \in \mathcal{Z}(R_R)$ or $a + e \in \mathcal{Z}(R_R)$. If $\bar{a} = -\bar{a}^2$, then $-\bar{a} \in Id(\frac{R}{\mathcal{Z}(R_R)})$. So for some $e \in Id(R)$, either $a - e \in \mathcal{Z}(R_R)$ or $a + e \in \mathcal{Z}(R_R)$. Thus R is a weakly right singular clean ring.

$\Rightarrow$) The proof is straightforward. $\square$

Corollary 4.8. *Let R be a weakly right singular clean ring. Then*

(1) *For any $\bar{a} \in \frac{R}{\mathcal{Z}(R_R)}$, $\bar{a} = \bar{a}^3$. In particular, $\frac{R}{\mathcal{Z}(R_R)}$ is a commutative ring.*
(2) $6 \in \mathcal{Z}(R_R)$.
(3) *$J(R)$ and $Nil(R)$ are contained in $\mathcal{Z}(R_R)$.*
(4) *If $x \in R$ and $x^n \in \mathcal{Z}(R_R)$ for some $n \in \mathbb{N}$, then $x \in \mathcal{Z}(R_R)$.*

(5) *Every prime ideal of R containing $\mathcal{Z}(R_R)$ is both completely prime and maximal.*
(6) *If R satisfies ACC on right annihilators of elements, then every prime ideal of R is maximal.*
(7) *R is Dedekind-finite.*

Proof. (1) Let $\bar{a} \in \frac{R}{\mathcal{Z}(R_R)}$. Then $\bar{a}^2 = \bar{a}$ or $\bar{a}^2 = -\bar{a}$ in both cases we have $\bar{a}^3 = \bar{a}$.

(2) In $\frac{R}{\mathcal{Z}(R_R)}$, we have $\bar{2} = \bar{2}^3$. Therefore, $6 \in \mathcal{Z}(R_R)$.

(3) For any $x \in R$, by part (1), $x - x^3 \in \mathcal{Z}(R_R)$. If $x \in J(R)$ or $x \in Nil(R)$, then $1 - x^2 \in U(R)$, so that $x \in \mathcal{Z}(R_R)$.

(4) and (5) follow from part (1).

(6) The proof is similar to the proof of part (5) of Corollary 2.4.

(7) The proof is similar to the proof of part (6) of Corollary 2.4. □

Recall that a ring R is called weakly clean, if every element is a sum or a difference of a unit and an idempotent [1].

Proposition 4.9. *Let R be a weakly right singular clean ring. Then $\mathcal{Z}(R_R) = J(R)$ if and only if R is a weakly clean ring.*

Proof. The proof is similar to the proof of Theorem 2.20. □

Similar to the case of singular clean rings, the polynomial ring, the formal power series ring, the ring of $n \times n$ matrices and the ring of $n \times n$ upper (lower) triangular matrices over any ring R are neither weakly left nor weakly right singular clean. As in Example 2.7, the ring $\left[\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_4 \end{smallmatrix}\right]$ is weakly right singular clean but is not weakly left singular clean.

Proposition 4.10. *Let R be a local ring. Then R is weakly right singular clean if and only if $\frac{R}{\mathcal{Z}(R_R)}$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. The proof is similar to the proof of Proposition 2.15. □

Corollary 4.11. *Let p be a prime number and $n \in \mathbb{N}$.*

(1) *If the ring $\mathbb{Z}_{p^n}$ is weakly singular clean which is not singular clean, then $p = 3$.*
(2) *If $\mathbb{Z}_n$ is a weakly singular clean ring, then $n = 2^k \times 3^l$ for some $k, l \geq 0$.*

Proposition 4.12. *For a weakly right singular clean ring R, the following are equivalent*:

(1) *$\frac{R}{\mathcal{Z}(R_R)}$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$*;
(2) *$\mathcal{Z}(R_R)$ is a maximal ideal*;
(3) *$\mathcal{Z}(R_R)$ is a prime ideal*;
(4) *R has no nontrivial idempotents.*

Proof. The proof is similar to the proof of Theorem 2.29. □

Recall that a ring R is called weakly nil clean, if for every $r \in R$, we can write $r = n + e$ or $r = n - e$ for some $n \in Nil(R)$ and $e \in Id(R)$[2]. Clearly every commutative weakly nil clean ring is weakly singular clean. However, the converse is not true. For example, let $R = \prod_{i=1}^{\infty} R_i$, where $R_1 = \mathbb{Z}_3$ and $R_i = \mathbb{Z}_{2^i}$ for $i \geq 2$. By Proposition 4.4, R is weakly singular clean. But R is not weakly nil clean, since the element $(0, 2, 2, \ldots)$ can not be written as a sum or a difference of a nilpotent element and an idempotent.

Note that if R is a weakly right singular clean ring and $\mathcal{Z}(R_R)$ is a nil ideal of R, then R is a weakly nil clean ring.

Remark. The results of Corollary 2.16, Theorem 2.22, Propositions 2.23, 2.26 and 2.28 are valid if we replace (right) singular clean by weakly (right) singular clean.

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