J. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/JKMS.j170579
pISSN: 0304-9914 / eISSN: 2234-3008

PRIMITIVE IDEALS AND PURE INFINITENESS OF ULTRAGRAPH C^*-ALGEBRAS

Hossein Larki

Abstract. Let $\mathcal{G}$ be an ultragraph and let $C^*(\mathcal{G})$ be the associated C^*-algebra introduced by Tomforde. For any gauge invariant ideal $I_{(H,B)}$ of $C^*(\mathcal{G})$, we approach the quotient C^*-algebra $C^*(\mathcal{G})/I_{(H,B)}$ by the C^*-algebra of finite graphs and prove versions of gauge invariant and Cuntz-Krieger uniqueness theorems for it. We then describe primitive gauge invariant ideals and determine purely infinite ultragraph C^*-algebras (in the sense of Kirchberg-Rørdam) via Fell bundles.

1. Introduction

In order to bring graph C^*-algebras [7] and Exel-Laca algebras [6] together under one theory, Tomforde introduced in [16] the notion of ultragraphs and associated C^*-algebras. An ultragraph is basically a directed graph in which the range of each edge is allowed to be a nonempty set of vertices rather than a single vertex. However, the class of ultragraph C^*-algebras are strictly lager than the graph C^*-algebras as well as the Exel-Laca algebras (see [17, Section 5]). Due to some similarities, some of fundamental results for graph C^*-algebras, such as the Cuntz-Krieger and the gauge invariant uniqueness theorems, simplicity, and K-theory computation have been extended to the setting of ultragraphs [16,17]. In particular, by constructing a specific topological quiver $\mathcal{Q}(\mathcal{G})$ from an ultragraph $\mathcal{G}$, Katsura et al. described some properties of the ultragraph C^*-algebra $C^*(\mathcal{G})$ using those of topological quivers [10]. They showed that every gauge invariant ideal of $C^*(\mathcal{G})$ is of the form $I_{(H,B)}$ corresponding to an admissible pair (H,B) in $\mathcal{G}$.

Recall that for any gauge invariant ideal $I_{(H,B)}$ of a graph C^*-algebra $C^*(E)$, there is a (quotient) graph $E/(H,B)$ such that $C^*(E)/I_{(H,B)} \cong C^*(E/(H,B))$ (see [1, 2]). So, the class of graph C^*-algebras contains such quotients, and results and properties of graph C^*-algebras may be applied for their quotients. For examples, some contexts such as simplicity, K-theory, primitivity, and topological stable rank are directly related to the structure of ideals and quotients.

Received September 5, 2017; Revised July 20, 2018; Accepted September 11, 2018.
2010 *Mathematics Subject Classification.* 46L55.
Key words and phrases. ultragraph C^*-algebra, primitive ideal, pure infiniteness.

Unlike the C^*-algebras of graphs and topological quivers [13], there are no known ways in the literature for describing quotients of an ultragraph C^*-algebra by structure of the initial ultragraph. So, many graph C^*-algebra's techniques could not be applied for the ultragraph setting, causing some obstacles in studying these C^*-algebras. The initial aim of this article is to analyze the structure of the quotient C^*-algebras $C^*(\mathcal{G})/I_{(H,B)}$ for any gauge invariant ideal $I_{(H,B)}$ of $C^*(\mathcal{G})$. For the sake of convenience, we first introduce the notion of quotient ultragraph $\mathcal{G}/(H,B)$ and a relative C^*-algebra $C^*(\mathcal{G}/(H,B))$ such that $C^*(\mathcal{G})/I_{(H,B)} \cong C^*(\mathcal{G}/(H,B))$ and then prove the gauge invariant and the Cuntz-Krieger uniqueness theorems for $C^*(\mathcal{G}/(H,B))$. The uniqueness theorems help us to show when a representation of $C^*(\mathcal{G})/I_{(H,B)}$ is injective. We see that the structure of $C^*(\mathcal{G}/(H,B))$ is close to that of graph C^*-algebras and we can use them to determine primitive gauge invariant ideals. Moreover, in Section 6, we consider the notion of pure infiniteness for ultragraph C^*-algebras in the sense of Kirchberg-Rørdam [11] which is directly related to the structure of quotients. We should note that the initial idea for definition of quotient ultragraphs has been inspired from [9].

The present article is organized as follows. We begin in Section 2 by giving some definitions and preliminaries about the ultragraphs and their C^*-algebras which will be used in the next sections. In Section 3, for any admissible pair (H,B) in an ultragraph $\mathcal{G}$, we introduce the quotient ultragraph $\mathcal{G}/(H,B)$ and an associated C^*-algebra $C^*(\mathcal{G}/(H,B))$. For this, the ultragraph $\mathcal{G}$ is modified by an extended ultragraph $\overline{\mathcal{G}}$ and we define an equivalent relation $\sim$ on $\overline{\mathcal{G}}$. Then $\mathcal{G}/(H,B)$ is the ultragraph $\overline{\mathcal{G}}$ with the equivalent classes $\{[A] : A \in \overline{\mathcal{G}}^0\}$. In Section 4, by approaching with graph C^*-algebras, the gauge invariant and the Cuntz-Krieger uniqueness theorems will be proved for the quotient ultragraphs C^*-algebras. Moreover, we see that $C^*(\mathcal{G}/(H,B))$ is isometrically isomorphic to the quotient C^*-algebra $C^*(\mathcal{G})/I_{(H,B)}$.

In Sections 5 and 6, using quotient ultragraphs, some graph C^*-algebra's techniques will be applied for the ultragraph C^*-algebras. In Section 5, we describe primitive gauge invariant ideals of $C^*(\mathcal{G})$, whereas in Section 6, we characterize purely infinite ultragraph C^*-algebras (in the sense of [11]) via Fell bundles [5, 12].

2. Preliminaries

In this section, we review basic definitions and properties of ultragraph C^*-algebras which will be needed through the paper. For more details, we refer the reader to [10] and [16].

Definition 2.1 ([16])**.** An *ultragraph* is a quadruple $\mathcal{G} = (G^0, \mathcal{G}^1, r_\mathcal{G}, s_\mathcal{G})$ consisting of a countable vertex set G^0, a countable edge set $\mathcal{G}^1$, the source map $s_\mathcal{G} : \mathcal{G}^1 \to G^0$, and the range map $r_\mathcal{G} : \mathcal{G}^1 \to \mathcal{P}(G^0) \setminus \{\emptyset\}$, where $\mathcal{P}(G^0)$ is the collection of all subsets of G^0. If $r_\mathcal{G}(e)$ is a singleton vertex for each edge $e \in \mathcal{G}^1$, then $\mathcal{G}$ is an ordinary (directed) graph.

For our convenience, we use the notation $\mathcal{G}^0$ in the sense of [10] rather than [16, 17]. For any set X, a nonempty subcollection of the power set $\mathcal{P}(X)$ is said to be an *algebra* if it is closed under the set operations $\cap$, $\cup$, and $\setminus$. If $\mathcal{G}$ is an ultragraph, the smallest algebra in $\mathcal{P}(G^0)$ containing $\{\{v\} : v \in G^0\}$ and $\{r_{\mathcal{G}}(e) : e \in \mathcal{G}^1\}$ is denoted by $\mathcal{G}^0$. We simply denote every singleton set $\{v\}$ by v. So, G^0 may be considered as a subset of $\mathcal{G}^0$.

Definition 2.2. For each $n \geq 1$, a *path* α *of length* $|\alpha| = n$ in $\mathcal{G}$ is a sequence $\alpha = e_1 \ldots e_n$ of edges such that $s(e_{i+1}) \in r(e_i)$ for $1 \leq i \leq n-1$. If also $s(e_1) \in r(e_n)$, α is called a *loop* or a *closed path*. We write α^0 for the set $\{s_{\mathcal{G}}(e_i) : 1 \leq i \leq n\}$. The elements of $\mathcal{G}^0$ are considered as the paths of length zero. The set of all paths in $\mathcal{G}$ is denoted by $\mathcal{G}^*$. We may naturally extend the maps $s_{\mathcal{G}}, r_{\mathcal{G}}$ on $\mathcal{G}^*$ by defining $s_{\mathcal{G}}(A) = r_{\mathcal{G}}(A) = A$ for $A \in \mathcal{G}^0$, and $r_{\mathcal{G}}(\alpha) = r_{\mathcal{G}}(e_n)$, $s_{\mathcal{G}}(\alpha) = s_{\mathcal{G}}(e_1)$ for each path $\alpha = e_1 \cdots e_n$.

Definition 2.3 ([16])**.** Let $\mathcal{G}$ be an ultragraph. A *Cuntz-Krieger $\mathcal{G}$-family* is a set of partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal ranges and a set of projections $\{p_A : A \in \mathcal{G}^0\}$ satisfying the following relations:

(UA1) $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$, and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ for all $A, B \in \mathcal{G}^0$,
(UA2) $s_e^* s_e = p_{r_{\mathcal{G}}(e)}$ for $e \in \mathcal{G}^1$,
(UA3) $s_e s_e^* \leq p_{s_{\mathcal{G}}(e)}$ for $e \in \mathcal{G}^1$, and
(UA4) $p_v = \sum_{s_{\mathcal{G}}(e)=v} s_e s_e^*$ whenever $0 < |s_{\mathcal{G}}^{-1}(v)| < \infty$.

The C^*-algebra $C^*(\mathcal{G})$ of $\mathcal{G}$ is the (unique) C^*-algebra generated by a universal Cuntz-Krieger $\mathcal{G}$-family.

By [16, Remark 2.13], we have

$$C^*(\mathcal{G}) = \overline{\operatorname{span}}\left\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{G}^*, A \in \mathcal{G}^0, \text{ and } r_{\mathcal{G}}(\alpha) \cap r_{\mathcal{G}}(\beta) \cap A \neq \emptyset\right\},$$

where $s_\alpha := s_{e_1} \cdots s_{e_n}$ if $\alpha = e_1 \cdots e_n$, and $s_\alpha := p_A$ if $\alpha = A$.

Remark 2.4. As noted in [16, Section 3], every graph C^*-algebra is an ultragraph C^*-algebra. Recall that if $E = (E^0, E^1, r_E, s_E)$ is a directed graph, a collection $\{s_e, p_v : v \in E^0, e \in E^1\}$ containing mutually orthogonal projections p_v and partial isometries s_e is called a *Cuntz-Krieger E-family* if

(GA1) $s_e^* s_e = p_{r_E(e)}$ for all $e \in E^1$,
(GA2) $s_e s_e^* \leq p_{s_E(e)}$ for all $e \in E^1$, and
(GA3) $p_v = \sum_{s_E(e)=v} s_e s_e^*$ for every vertex $v \in E^0$ with $0 < |s_E^{-1}(v)| < \infty$.

We denote by $C^*(E)$ the universal C^*-algebra generated by a Cuntz-Krieger E-family.

By the universal property, $C^*(\mathcal{G})$ admits the *gauge action* of the unit circle $\mathbb{T}$. By an *ideal*, we mean a closed two-sided ideal. Using the properties of quiver C^*-algebras [10], the gauge invariant ideals of $C^*(\mathcal{G})$ were characterized in [10, Theorem 6.12] via a one-to-one correspondence with the admissible pairs of $\mathcal{G}$ as follows.

Definition 2.5. A subset $H \subseteq \mathcal{G}^0$ is said to be *hereditary* if the following properties holds:

(H1) $s_{\mathcal{G}}(e) \in H$ implies $r_{\mathcal{G}}(e) \in H$ for all $e \in \mathcal{G}^1$.
(H2) $A \cup B \in H$ for all $A, B \in H$.
(H3) If $A \in H,\ B \in \mathcal{G}^0$, and $B \subseteq A$, then $B \in H$.

Moreover, a subset $H \subseteq \mathcal{G}^0$ is called *saturated* if for any $v \in G^0$ with $0 < |s_{\mathcal{G}}^{-1}(v)| < \infty$, then $\{r_{\mathcal{G}}(e) : s_{\mathcal{G}}(e) = v\} \subseteq H$ implies $v \in H$. The *saturated hereditary closure* of a subset $H \subseteq \mathcal{G}^0$ is the smallest hereditary and saturated subset $\overline{H}$ of $\mathcal{G}^0$ containing H.

Let H be a saturated hereditary subset of $\mathcal{G}^0$. The set of *breaking vertices of* H is denoted by

$$B_H := \left\{ w \in G^0 : |s_{\mathcal{G}}^{-1}(w)| = \infty \text{ but } 0 < |r_{\mathcal{G}}(s_{\mathcal{G}}^{-1}(w)) \cap (\mathcal{G}^0 \setminus H)| < \infty \right\}.$$

An *admissible pair* (H, B) *in* $\mathcal{G}$ is a saturated hereditary set $H \subseteq \mathcal{G}^0$ together with a subset $B \subseteq B_H$. For any admissible pair (H, B) in $\mathcal{G}$, we define the ideal $I_{(H,B)}$ of $C^*(\mathcal{G})$ generated by

$$\{p_A : A \in \mathcal{G}^0\} \cup \left\{p_w^H : w \in B\right\},$$

where $p_w^H := p_w - \sum_{s_{\mathcal{G}}(e)=w,\ r_{\mathcal{G}}(e)\notin H} s_e s_e^*$. Note that the ideal $I_{(H,B)}$ is gauge invariant and [10, Theoerm 6.12] implies that every gauge invariant ideal I of $C^*(\mathcal{G})$ is of the form $I_{(H,B)}$ by setting

$$H := \{A : p_A \in I\} \text{ and } B := \left\{w \in B_H : p_w^H \in I\right\}.$$

3. Quotient ultragraphs and their C^*-algebras

In this section, for any admissible pair (H, B) in an ultragraph $\mathcal{G}$, we introduce the quotient ultragraph $\mathcal{G}/(H, B)$ and its relative C^*-algebra $C^*(\mathcal{G}/(H, B))$. We will show in Proposition 4.6 that $C^*(\mathcal{G}/(H, B))$ is isomorphic to the quotient C^*-algebra $C^*(\mathcal{G})/I_{(H,B)}$.

Let us fix an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^0, r_{\mathcal{G}}, s_{\mathcal{G}})$ and an admissible pair (H, B) in $\mathcal{G}$. For defining our quotient ultragraph $\mathcal{G}/(H, B)$, we first modify $\mathcal{G}$ by an extended ultragraph $\overline{\mathcal{G}}$ such that their C^*-algebras coincide. For this, add the vertices $\{w' : w \in B_H \setminus B\}$ to G^0 and denote $\overline{A} := A \cup \{w' : w \in A \cap (B_H \setminus B)\}$ for each $A \in \mathcal{G}^0$. We now define the new ultragraph $\overline{\mathcal{G}} = (\overline{G^0}, \overline{\mathcal{G}}^1, \overline{r}_{\mathcal{G}}, \overline{s}_{\mathcal{G}})$ by

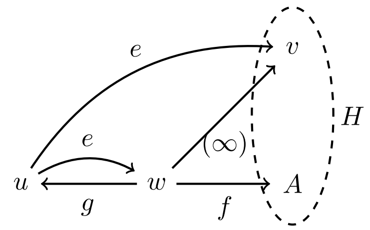
$$\overline{G}^0 := G^0 \cup \{w' : w \in B_H \setminus B\},$$
$$\overline{\mathcal{G}}^1 := \mathcal{G}^1,$$

the source map

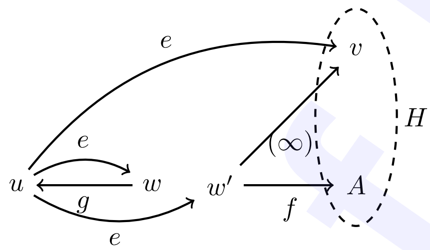
$$\overline{s}_{\mathcal{G}}(e) := \begin{cases} (s_{\mathcal{G}}(e))' & \text{if } s_{\mathcal{G}}(e) \in B_H \setminus B \text{ and } r_{\mathcal{G}}(e) \in H \\ s_{\mathcal{G}}(e) & \text{otherwise,} \end{cases}$$

and the rang map $\overline{r}_{\mathcal{G}}(e) := \overline{r_{\mathcal{G}}(e)}$ for every $e \in \mathcal{G}^1$. In Proposition 3.3 below, we will see that the C^*-algebras of $\mathcal{G}$ and $\overline{\mathcal{G}}$ coincide.

Example 3.1. Suppose $\mathcal{G}$ is the ultragraph



where (∞) indicates infinitely many edges. If H is the saturated hereditary subset of $\mathcal{G}^0$ containing $\{v\}$ and A, then we have $B_H = \{w\}$. For $B := \emptyset$, consider the admissible pair $(H, \emptyset)$ in $\mathcal{G}$. Then the ultragraph $\overline{\mathcal{G}}$ associated to $(H, \emptyset)$ would be



Indeed, since $B_H \setminus B = \{w\}$, for constructing $\overline{\mathcal{G}}$ we first add a vertex w' to $\mathcal{G}$. We then define

$$\begin{aligned} \overline{r}_{\mathcal{G}}(f) &:= \overline{A} = A, \\ \overline{r}_{\mathcal{G}}(e) &:= \overline{\{v, w\}} = \{v, w, w'\}, \text{ and} \\ \overline{r}_{\mathcal{G}}(g) &:= \overline{\{u\}} = \{u\}. \end{aligned}$$

For the source map $\overline{s}_{\mathcal{G}}$, for example, since $s_{\mathcal{G}}(f) \in B_H \setminus B$ and $r_{\mathcal{G}}(f) \in H$, we may define $\overline{s}_{\mathcal{G}}(f) := w'$. Note that the range of each edge emitted by w' belongs to H.

As usual, we write $\overline{\mathcal{G}}^0$ for the algebra generated by the elements of $\overline{G}^0 \cup \{\overline{r}_{\mathcal{G}}(e) : e \in \overline{\mathcal{G}}^1\}$. Note that $\overline{A} = A$ for every $A \in H$, and hence, H would be a saturated hereditary subset of $\overline{\mathcal{G}}^0$ as well. Moreover, the set of breaking vertices of H in $\overline{\mathcal{G}}$ coincides with B (meaning $B_H^{\overline{\mathcal{G}}} = B$).

Remark 3.2. Suppose that $C^*(\mathcal{G})$ is generated by a Cuntz-Krieger $\mathcal{G}$-family $\{s_e, p_A : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$. If a family $M = \{S_e, P_v, P_{\overline{A}} : v \in G^0, A \in \mathcal{G}^0, e \in \overline{\mathcal{G}}^1\}$ in a C^*-algebra X satisfies relations (UA1)-(UA4) in Definition 2.3, we may generate a Cuntz-Krieger $\overline{\mathcal{G}}$-family $N = \{S_e, P_A : A \in \overline{\mathcal{G}}^0, e \in \overline{\mathcal{G}}^1\}$ in X. For this, since $\overline{\mathcal{G}}^0$ is the algebra generated by $\{v, w', \overline{r}_{\mathcal{G}}(e) : v \in G^0, w \in$

$B_H \setminus B, e \in \overline{\mathcal{G}}^1\}$, we may use the definitions

$$\begin{aligned} P_{A\cap C} &:= P_A P_C, \\ P_{A\cup C} &:= P_A + P_C - P_A P_C, \\ P_{A\setminus C} &:= P_A - P_A P_C, \end{aligned}$$

to generate each projection P_A, $A \in \overline{\mathcal{G}}^0$, by finitely many operations. Then N would be a Cuntz-Krieger $\overline{\mathcal{G}}$-family in X, and the C^*-subalgebras generated by M and N coincide.

Proposition 3.3. *Let $\mathcal{G}$ be an ultragraph, and let (H, B) be an admissible pair in $\mathcal{G}$. If $\overline{\mathcal{G}}$ is the extended ultragraph as above, then $C^*(\mathcal{G}) \cong C^*(\overline{\mathcal{G}})$.*

Proof. Suppose that $C^*(\mathcal{G}) = C^*(t_e, q_A)$ and $C^*(\overline{\mathcal{G}}) = C^*(s_e, p_C)$. If we define

$$\begin{array}{ll} P_v := q_v & \text{for } v \in G^0 \setminus (B_H \setminus B), \\ P_w := \sum_{\substack{s_{\mathcal{G}}(e)=w \\ r_{\mathcal{G}}(e)\notin H}} t_e t_e^* & \text{for } w \in B_H \setminus B, \\ P_{w'} := q_w - \sum_{\substack{s_{\mathcal{G}}(e)=w \\ r_{\mathcal{G}}(e)\notin H}} t_e t_e^* & \text{for } w \in B_H \setminus B, \\ P_{\overline{A}} := q_A & \text{for } \overline{A} \in \overline{\mathcal{G}}^0, \\ S_e := t_e & \text{for } e \in \overline{\mathcal{G}}^1, \end{array}$$

then, by Remark 3.2, the family

$$\left\{ P_v, P_w, P_{w'}, P_{\overline{A}}, S_e : v \in G^0 \setminus (B_H \setminus B),\ w \in B_H \setminus B,\ \overline{A} \in \overline{\mathcal{G}}^0,\ e \in \overline{\mathcal{G}}^1 \right\}$$

induces a Cuntz-Krieger $\overline{\mathcal{G}}$-family in $C^*(\mathcal{G})$. Since all vertex projections of this family are nonzero (which follows all set projections P_A are nonzero for $\emptyset \neq A \in \overline{\mathcal{G}}^0$), the gauge-invariant uniqueness theorem [16, Theorem 6.8] implies that the $*$-homomorphism $\phi : C^*(\overline{\mathcal{G}}) \to C^*(\mathcal{G})$ with $\phi(p_*) = P_*$ and $\phi(s_*) = S_*$ is injective. On the other hand, the family generates $C^*(\mathcal{G})$, and hence, ϕ is an isomorphism. □

To define a quotient ultragraph $\mathcal{G}/(H, B)$, we use the following equivalent relation on $\overline{\mathcal{G}}$.

Definition 3.4. Suppose that (H, B) is an admissible pair in $\mathcal{G}$, and that $\overline{\mathcal{G}}$ is the extended ultragraph as above. We define the relation $\sim$ on $\overline{\mathcal{G}}^0$ by

$$A \sim C \iff \exists V \in H \text{ such that } A \cup V = C \cup V.$$

Note that $A \sim C$ if and only if both sets $A \setminus C$ and $C \setminus A$ belong to H.

The following lemma may be proved by a tedious, but straightforward computations.

Lemma 3.5. *The relation $\sim$ is an equivalent relation on $\overline{\mathcal{G}}^0$. Furthermore, the operations*

$$[A] \cup [C] := [A \cup C],\ [A] \cap [C] := [A \cap C],\ \text{and } [A] \setminus [C] := [A \setminus C]$$

are well-defined on the equivalent classes $\{[A] : A \in \overline{\mathcal{G}}^0\}$.

Definition 3.6. Let $\mathcal{G}$ be an ultragraph, let (H, B) be an admissible pair in $\mathcal{G}$, and consider the equivalent relation of Definition 3.4 on the extended ultragraph $\overline{\mathcal{G}} = (\overline{G}^0, \overline{\mathcal{G}}^1, \overline{r}_{\mathcal{G}}, \overline{s}_{\mathcal{G}})$. The *quotient ultragraph of* $\mathcal{G}$ *by* (H, B) is the quintuple $\mathcal{G}/(H, B) = (\Phi(G^0), \Phi(\mathcal{G}^0), \Phi(\mathcal{G}^1), r, s)$, where

$$\Phi(G^0) := \left\{[v] : v \in G^0 \setminus H\right\} \cup \{[w'] : w \in B_H \setminus B\},$$
$$\Phi(\mathcal{G}^0) := \left\{[A] : A \in \overline{\mathcal{G}}^0\right\},$$
$$\Phi(\mathcal{G}^1) := \left\{e \in \overline{\mathcal{G}}^1 : \overline{r}_{\mathcal{G}}(e) \notin H\right\},$$

and $r : \Phi(\mathcal{G}^1) \to \Phi(\mathcal{G}^0)$, $s : \Phi(\mathcal{G}^1) \to \Phi(G^0)$ are the range and source maps defined by

$$r(e) := [\overline{r}_{\mathcal{G}}(e)] \quad \text{and} \quad s(e) := [\overline{s}_{\mathcal{G}}(e)].$$

We refer to $\Phi(G^0)$ as the vertices of $\mathcal{G}/(H, B)$.

Remark 3.7. Lemma 3.5 implies that $\Phi(\mathcal{G}^0)$ is the smallest algebra containing

$$\left\{[v], [w'] : v \in G^0 \setminus H, w \in B_H \setminus B\right\} \cup \left\{[\overline{r}_{\mathcal{G}}(e)] : e \in \overline{\mathcal{G}}^1\right\}.$$

Notation.

(1) For every vertex $v \in \overline{\mathcal{G}}^0 \setminus H$, we usually denote $[v]$ instead of $[\{v\}]$.
(2) For $A, C \in \overline{\mathcal{G}}^0$, we write $[A] \subseteq [C]$ whenever $[A] \cap [C] = [A]$.
(3) Through the paper, we will denote the range and the source maps of $\mathcal{G}$ by $r_{\mathcal{G}}, s_{\mathcal{G}}$, those of $\overline{\mathcal{G}}$ by $\overline{r}_{\mathcal{G}}, \overline{s}_{\mathcal{G}}$, and those of $\mathcal{G}/(H, B)$ by r, s.

Now we introduce representations of quotient ultragraphs and their relative C^*-algebras.

Definition 3.8. Let $\mathcal{G}/(H, B)$ be a quotient ultragraph. A *representation of* $\mathcal{G}/(H, B)$ is a set of partial isometries $\{T_e : e \in \Phi(\mathcal{G}^1)\}$ and a set of projections $\{Q_{[A]} : [A] \in \Phi(\mathcal{G}^0)\}$ which satisfy the following relations:

(QA1) $Q_{[\emptyset]} = 0$, and for $[A], [C] \in \Phi(\mathcal{G}^0)$, $Q_{[A \cap C]} = Q_{[A]} Q_{[C]}$ and $Q_{[A \cup C]} = Q_{[A]} + Q_{[C]} - Q_{[A \cap C]}$.
(QA2) $T_e^* T_f = \delta_{e,f} Q_{r(e)}$ for $e, f \in \Phi(\mathcal{G}^1)$.
(QA3) $T_e T_e^* \leq Q_{s(e)}$ for $e \in \Phi(\mathcal{G}^1)$.
(QA4) $Q_{[v]} = \sum_{s(e)=[v]} T_e T_e^*$, whenever $0 < |s^{-1}([v])| < \infty$.

We denote by $C^*(\mathcal{G}/(H, B))$ the universal C^*-algebra generated by a representation $\{t_e, q_{[A]} : [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$ which exists by Theorem 3.10 below.

Note that if $\alpha = e_1 \cdots e_n$ is a path in $\overline{\mathcal{G}}$ such that $\overline{r}_{\mathcal{G}}(\alpha) \notin H$, then the hereditary property of H yields $\overline{r}_{\mathcal{G}}(e_i) \notin H$, and so $e_i \in \Phi(\mathcal{G}^1)$ for all $1 \leq i \leq n$. In this case, we denote $t_\alpha := t_{e_1} \cdots t_{e_n}$. Moreover, we define

$$(\mathcal{G}/(H, B))^* := \{[A] : [A] \neq [\emptyset]\} \cup \left\{\alpha \in \overline{\mathcal{G}}^* : r(\alpha) \neq [\emptyset]\right\}$$

as the set of finite paths in $\mathcal{G}/(H,B)$ and we can extend the maps s,r on $(\mathcal{G}/(H,B))^*$ by setting

$$s([A]) := r([A]) := [A] \text{ and } s(\alpha) := s(e_1),\ r(\alpha) := r(e_n).$$

The proof of next lemma is similar to the arguments of [16, Lemmas 2.8 and 2.9].

Lemma 3.9. *Let $\mathcal{G}/(H,B)$ be a quotient ultragraph and let $\{T_e, Q_{[A]}\}$ be a representation of $\mathcal{G}/(H,B)$. Then any nonzero word in T_e, $Q_{[A]}$, and T_f^* may be written as a finite linear combination of the forms $T_\alpha Q_{[A]} T_\beta^*$ for $\alpha,\beta \in (\mathcal{G}/(H,B))^*$ and $[A] \in \Phi(\mathcal{G}^0)$ with $[A]\cap r(\alpha)\cap r(\beta) \neq [\emptyset]$.*

Theorem 3.10. *Let $\mathcal{G}/(H,B)$ be a quotient ultragraph. Then there exists a (unique up to isomorphism) C^*-algebra $C^*(\mathcal{G}/(H,B))$ generated by a universal representation $\{t_e, q_{[A]} : [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$ for $\mathcal{G}/(H,B)$. Furthermore, all the t_e's and $q_{[A]}$'s are nonzero for $[\emptyset] \neq [A] \in \Phi(\mathcal{G}^0)$ and $e \in \Phi(\mathcal{G}^1)$.*

Proof. By a standard argument similar to the proof of [16, Theorem 2.11], we may construct such universal C^*-algebra $C^*(\mathcal{G}/(H,B))$. Note that the universality implies that $C^*(\mathcal{G}/(H,B))$ is unique up to isomorphism. To show the last statement, we generate an appropriate representation for $\mathcal{G}/(H,B)$ as follows. Suppose $C^*(\overline{\mathcal{G}}) = C^*(s_e, p_A)$ and consider $I_{(H,B)}$ as an ideal of $C^*(\overline{\mathcal{G}})$ by the isomorphism in Proposition 3.3. If we define

$$\begin{cases} Q_{[A]} := p_A + I_{(H,B)} & \text{for } [A] \in \Phi(\mathcal{G}^0), \\ T_e := s_e + I_{(H,B)} & \text{for } e \in \Phi(\mathcal{G}^1), \end{cases}$$

then the family $\{T_e, Q_{[A]} : [A] \in \Phi(\mathcal{G}^0),\ e \in \Phi(\mathcal{G}^1)\}$ is a representation for $\mathcal{G}/(H,B)$ in the quotient C^*-algebra $C^*(\overline{\mathcal{G}})/I_{(H,B)}$. Note that the definition of $Q_{[A]}$'s is well-defined. Indeed, if $A_1 \cup V = A_2 \cup V$ for some $V \in H$, then $p_{A_1} + p_{V\setminus A_1} = p_{A_2} + p_{V\setminus A_2}$ and hence $p_{A_1} + I_{(H,B)} = p_{A_2} + I_{(H,B)}$ by the facts $V \setminus A_1, V\setminus A_2 \in H$.

Moreover, all elements $Q_{[A]}$ and T_e are nonzero for $[\emptyset] \neq [A] \in \Phi(\mathcal{G}^0)$, $e \in \Phi(\mathcal{G}^1)$. In fact, if $Q_{[A]} = 0$, then $p_A \in I_{(H,B)}$ and we get $A \in H$ by [10, Theorem 6.12]. Also, since $T_e^* T_e = Q_{r(e)} \neq 0$, all partial isometries T_e are nonzero.

Now suppose that $C^*(\mathcal{G}/(H,B))$ is generated by the family $\{t_e, q_{[A]} : [A] \in \Phi(\mathcal{G}^0),\ e \in \Phi(\mathcal{G}^1)\}$. By the universality of $C^*(\mathcal{G}/(H,B))$, there is a $*$-homomorphism $\phi : C^*(\mathcal{G}/(H,B)) \to C^*(\overline{\mathcal{G}})/I_{(H,B)}$ such that $\phi(t_e) = T_e$ and $\phi(q_{[A]}) = Q_{[A]}$, and thus, all elements of $\{t_e, q_{[A]} : [\emptyset] \neq [A] \in \Phi(\mathcal{G}^0),\ e \in \Phi(\mathcal{G}^1)\}$ are nonzero. □

Note that, by a routine argument, one may obtain

$$C^*(\mathcal{G}/(H,B)) = \overline{\operatorname{span}}\big\{t_\alpha q_{[A]} t_\beta^* : \alpha,\beta \in (\mathcal{G}/(H,B))^*,\ r(\alpha)\cap[A]\cap r(\beta) \neq [\emptyset]\big\}.$$

4. Uniqueness theorems

After defining the C^*-algebras of quotient ultragraphs, in this section, we prove the gauge invariant and the Cuntz-Krieger uniqueness theorems for them. To do this, we approach to a quotient ultragraph C^*-algebra by graph C^*-algebras and then apply the corresponding uniqueness theorems for graph C^*-algebras. This approach is a developed version of the dual graph method of [14, Section 2] and [16, Section 5] with more complications. In particular, we show that the C^*-algebra $C^*(\mathcal{G}/(H,B))$ is isomorphic to the quotient $C^*(\mathcal{G})/I_{(H,B)}$, and the uniqueness theorems may applied for such quotients.

We fix again an ultragraph $\mathcal{G}$, an admissible pair (H,B) in $\mathcal{G}$, and the quotient ultragraph $\mathcal{G}/(H,B) = (\Phi(G^0), \Phi(\mathcal{G}^0), \Phi(\mathcal{G}^1), r, s)$.

Definition 4.1. We say that a vertex $[v] \in \Phi(G^0)$ is a *sink* if $s^{-1}([v]) = \emptyset$. If $[v]$ only emits finitely many edges of $\Phi(\mathcal{G}^1)$, $[v]$ is called a *regular vertex*. Any non-regular vertex is called a *singular vertex*. The set of singular vertices in $\Phi(G^0)$ is denoted by

$$\Phi_{\mathrm{sg}}(G^0) := \big\{[v] \in \Phi(G^0) : |s^{-1}([v])| = 0 \text{ or } \infty\big\}.$$

Let F be a finite subset of $\Phi_{\mathrm{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$. Write $F^0 := F \cap \Phi_{\mathrm{sg}}(G^0)$ and $F^1 := F \cap \Phi(\mathcal{G}^1) = \{e_1, \dots, e_n\}$. We want to construct a special graph G_F such that $C^*(G_F)$ is isomorphic to $C^*(t_e, q_{[v]} : [v] \in F^0, e \in F^1)$. For each $\omega = (\omega_1, \dots, \omega_n) \in \{0,1\}^n \setminus \{0^n\}$, we write

$$r(\omega) := \bigcap_{\omega_i = 1} r(e_i) \setminus \bigcup_{\omega_j = 0} r(e_j) \text{ and } R(\omega) := r(\omega) \setminus \bigcup_{[v] \in F^0} [v].$$

Note that $r(\omega) \cap r(\nu) = [\emptyset]$ for distinct $\omega, \nu \in \{0,1\} \setminus \{0^n\}$. If

$$\Gamma_0 := \big\{\omega \in \{0,1\}^n \setminus \{0^n\} : \exists [v_1], \dots, [v_m] \in \Phi(\mathcal{G}^0) \text{ such that}$$
$$R(\omega) = \bigcup_{i=1}^m [v_i] \text{ and } \emptyset \neq s^{-1}([v_i]) \subseteq F^1 \text{ for } 1 \le i \le m\big\},$$

we consider the finite set

$$\Gamma := \{\omega \in \{0,1\}^n \setminus \{0^n\} : R(\omega) \neq [\emptyset] \text{ and } \omega \notin \Gamma_0\}.$$

Now we define the finite graph $G_F = (G_F^0, G_F^1, r_F, s_F)$ containing the vertices $G_F^0 := F^0 \cup F^1 \cup \Gamma$ and the edges

$$\begin{aligned} G_F^1 := &\big\{(e,f) \in F^1 \times F^1 : s(f) \subseteq r(e)\big\} \\ &\cup \big\{(e,[v]) \in F^1 \times F^0 : [v] \subseteq r(e)\big\} \\ &\cup \big\{(e,\omega) \in F^1 \times \Gamma : \omega_i = 1 \text{ when } e = e_i\big\} \end{aligned}$$

with the source map $s_F(e,f) = s_F(e,[v]) = s_F(e,\omega) = e$, and the range map $r_F(e,f) = f$, $r_F(e,[v]) = [v]$, $r_F(e,\omega) = \omega$.

Proposition 4.2. *Let $\mathcal{G}/(H,B)$ be a quotient ultragraph and let F be a finite subset of $\Phi_{\rm sg}(G^0)\cup\Phi(\mathcal{G}^1)$. If $C^*(\mathcal{G}/(H,B))=C^*(t_e,q_{[A]})$, then the elements*

$$\begin{array}{lll} Q_e:=t_et_e^*, & Q_{[v]}:=q_{[v]}(1-\sum_{e\in F^1}t_et_e^*), & Q_\omega:=q_{R(\omega)}(1-\sum_{e\in F^1}t_et_e^*) \\ T_{(e,f)}:=t_eQ_f, & T_{(e,[v])}:=t_eQ_{[v]}, & T_{(e,\omega)}:=t_eQ_\omega \end{array}$$

form a Cuntz-Krieger G_F-family generating the C^*-subalgebra $C^*(t_e,q_{[v]}:[v]\in F^0,e\in F^1)$ of $C^*(\mathcal{G}/(H,B))$. Moreover, all projections Q_* are nonzero.

Proof. We first note that all the projections Q_e, $Q_{[v]}$, and Q_ω are nonzero. Indeed, each $[v]\in F^0$ is a singular vertex in $\mathcal{G}/(H,B)$, so $Q_{[v]}$ is nonzero. Also, by definition, for every $\omega\in\Gamma$ we have $\omega\notin\Gamma_0$ and $R(\omega)\neq[\emptyset]$. Hence, for any $\omega\in\Gamma$, if there is an edge $f\in\Phi(\mathcal{G}^1)\setminus F^1$ with $s(f)\subseteq R(\omega)$, then $0\neq t_ft_f^*\leq Q_\omega$. If there is a sink $[w]$ such that $[w]\subseteq R(\omega)=r(\omega)\setminus\bigcup F^0$, then $0\neq q_{[w]}\leq q_{R(\omega)}(1-\sum_{e\in F^1}t_et_e^*)=Q_\omega$. Thus Q_ω is nonzero in either case. In addition, the projections Q_e, $Q_{[v]}$, and Q_ω are mutually orthogonal because of the factor $1-\sum_{e\in F^1}t_et_e^*$ and the definition of $R(\omega)$.

Now we show the collection $\{T_x,Q_a: a\in G_F^0, x\in G_F^1\}$ is a Cuntz-Krieger G_F-family by checking the relations (GA1)-(GA3) in Remark 2.4.

(GA1): Since $Q_{[v]},Q_\omega\leq q_{r(e)}$ for $(e,[v]),(e,\omega)\in G_F^1$, we have

$$T^*_{(e,f)}T_{(e,f)}=Q_ft_e^*t_eQ_f=t_ft_f^*q_{r(e)}t_ft_f^*=t_fq_{r(f)}t_f^*=Q_f,$$

$$T^*_{(e,[v])}T_{(e,[v])}=Q_{[v]}t_e^*t_eQ_{[v]}=Q_{[v]}q_{r(e)}Q_{[v]}=Q_{[v]},$$

and

$$T^*_{(e,\omega)}T_{(e,\omega)}=Q_\omega t_e^*t_eQ_\omega=Q_\omega q_{r(e)}Q_\omega=Q_\omega.$$

(GA2): This relation may be checked similarly.

(GA3): Note that any element of $F^0\cup\Gamma$ is a sink in G_F. So, fix some $e_i\in F^1$ as a vertex of G_F^0. Write $q_{F^0}:=\sum_{[v]\in F^0}q_{[v]}$. We compute

(i) $$q_{r(e_i)}\sum_{\substack{f\in F^1\\ s(f)\subseteq r(e_i)}}Q_f=q_{r(e_i)}\sum_{\substack{f\in F^1\\ s(f)\subseteq r(e_i)}}t_ft_f^*=q_{r(e_i)}\sum_{f\in F^1}t_ft_f^*;$$

(ii) $$q_{r(e_i)}\sum_{\substack{[v]\in F^0,\\ [v]\subseteq r(e_i)}}Q_{[v]}=q_{r(e_i)}\sum_{[v]\in F^0}q_{[v]}(1-\sum_{e\in F^1}t_et_e^*)$$
$$=q_{r(e_i)}q_{F^0}(1-\sum_{e\in F^1}t_et_e^*);$$

(iii) $$\sum_{\omega\in\Gamma,\omega_i=1}Q_\omega=\sum_{\omega\in\Gamma,\omega_i=1}q_{R(\omega)}(1-\sum_{e\in F^1}t_et_e^*)=\sum_{\omega_i=1}q_{R(\omega)}(1-\sum_{e\in F^1}t_et_e^*),$$
because $\sum_{\omega_i=1}q_{R(\omega)}=q_{r(e_i)}(1-q_{F^0})$.

We can use these relations to get

$$\sum_{s(f)\subseteq r(e_i)}T_{(e_i,f)}+\sum_{[v]\in F^0,\ [v]\subseteq r(e_i)}T_{(e_i,[v])}+\sum_{\omega\in\Gamma,\ \omega_i=1}T_{(e_i,\omega)} \tag{4.1}$$

$$
\begin{aligned}
&= t_{e_i}\left(q_{r(e_i)}\sum_{e\in F^1} t_e t_e^* + q_{r(e_i)}q_{F^0}(\sum_{e\in F^1} t_e t_e^*) + q_{r(e_i)}(1-q_{F^0})(\sum_{e\in F^1} t_e t_e^*)\right)\\
&= t_{e_i}q_{r(e_i)}\left(\sum_{e\in F^1} t_e t_e^* + (q_{F^0}+1-q_{F^0})(1-\sum_{e\in F^1} t_e t_e^*)\right)\\
&= t_{e_i}.
\end{aligned}
$$

Now if e_i is not a sink as a vertex in G_F (i.e., $|\{x\in G_F^1 : s_F(x)=e_i\}|>0$), we conclude that

$$
\begin{aligned}
&\sum_{f\in F^1,\ s(f)\subseteq r(e_i)} T_{(e_i,f)}T^*_{(e_i,f)} + \sum_{[v]\in F^0,\ [v]\subseteq r(e_i)} T_{(e_i,[v])}T^*_{(e_i,[v])}\\
&\quad + \sum_{\omega\in\Gamma,\ \omega_i=1} T_{(e_i,\omega)}T^*_{(e_i,\omega)}\\
&= \sum t_{e_i}Q_f t_{e_i}^* + \sum t_{e_i}Q_{[v]}t_{e_i}^* + \sum t_{e_i}Q_\omega t_{e_i}^*\\
&= t_{e_i}q_{r(e_i)}(\sum Q_f + \sum Q_{[v]} + \sum Q_\omega)t_{e_i}^*\\
&= t_{e_i}t_{e_i}^* = Q_{e_i},
\end{aligned}
$$

which establishes the relation (GA3).

Furthermore, equation (4.1) in above says that $t_{e_i}\in C^*(T_*,Q_*)$ for every $e_i\in F^1$. Also, for each $[v]\in F^0$, we have

$$
\begin{aligned}
Q_{[v]} + \sum_{e\in F^1, s(e)=[v]} Q_e &= t_{[v]}(1-\sum_{e\in F^1} t_e t_e^*) + \sum_{e\in F^1, s(e)=[v]} t_e t_e^*\\
&= t_{[v]} - t_{[v]}\sum_{e\in F^1} t_e t_e^* + t_{[v]}\sum_{e\in F^1} t_e t_e^*\\
&= t_{[v]}.
\end{aligned}
$$

Therefore, the family $\{T_x, Q_a : a\in G_F^0, x\in G_F^1\}$ generates the C^*-subalgebra $C^*(\{t_e, q_{[v]} : e\in F^1, [v]\in F^0\})$ of $C^*(\mathcal{G}/(H,B))$ and the proof is complete. □

Corollary 4.3. *If F is a finite subset of $\Phi_{\rm sg}(G^0)\cup\Phi(\mathcal{G}^1)$, then $C^*(G_F)$ is isometrically isomorphic to the C^*-subalgebra of $C^*(\mathcal{G}/(H,B))$ generated by $\{t_e, q_{[v]} : [v]\in F^0, e\in F^1\}$.*

Proof. Suppose that X is the C^*-subalgebra generated by $\{t_e, q_{[v]} : [v]\in F^0, e\in F^1\}$ and let $\{T_x, Q_a : a\in G_F^0, x\in G_F^1\}$ be the Cuntz-Krieger G_F-family in Proposition 4.2. If $C^*(G_F)=C^*(s_x,p_a)$, then there exists a $*$-homomorphism $\phi : C^*(G_F)\to X$ with $\phi(p_a)=Q_a$ and $\phi(s_x)=T_x$ for every $a\in G_F^0$, $x\in G_F^1$. Since each Q_a is nonzero by Proposition 4.2, the gauge invariant uniqueness theorem implies that ϕ is injective. Moreover, the family $\{T_x, Q_a\}$ generates X, so ϕ is an isomorphism. □

Note that if $F_1 \subseteq F_2$ are two finite subsets of $\Phi_{\rm sg}(G^0)\cup\Phi(\mathcal{G}^1)$ and X_1, X_2 are the C^*-subalgebras of $C^*(\mathcal{G}/(H,B))$ associated to G_{F_1} and G_{F_2}, respectively, we then have $X_1 \subseteq X_2$ by Proposition 4.2.

Remark 4.4. Using relations (QA1)-(QA4) in Definition 3.8, each $q_{[A]}$ for $[A] \in \Phi(G^0)$, can be produced by the elements of

$$\{q_{[v]} : [v] \in \Phi_{\rm sg}(G^0)\} \cup \{t_e : e \in \Phi(\mathcal{G}^1)\}$$

with finitely many operations. So, the $*$-subalgebra of $C^*(\mathcal{G}/(H,B))$ generated by

$$\{q_{[v]} : [v] \in \Phi_{\rm sg}(G^0)\} \cup \{t_e : e \in \Phi(\mathcal{G}^1)\}$$

is dense in $C^*(\mathcal{G}/(H,B))$.

As for graph C^*-algebras, we can apply the universal property to have a strongly continuous *gauge action* $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(\mathcal{G}/(H,B)))$ such that

$$\gamma_z(t_e) = zt_e \text{ and } \gamma_z(q_{[A]}) = q_{[A]}$$

for every $[A] \in \Phi(\mathcal{G}^0)$, $e \in \Phi(\mathcal{G}^1)$, and $z \in \mathbb{T}$. Now we are ready to prove the uniqueness theorems.

Theorem 4.5 (The Gauge Invariant Uniqueness Theorem)**.** *Let $\mathcal{G}/(H,B)$ be a quotient ultragraph and let $\{T_e, Q_{[A]}\}$ be a representation for $\mathcal{G}/(H,B)$ such that $Q_{[A]} \neq 0$ for $[A] \neq [\emptyset]$. If $\pi_{T,Q} : C^*(\mathcal{G}/(H,B)) \to C^*(T_e, Q_{[A]})$ is the $*$-homomorphism satisfying $\pi_{T,Q}(t_e) = T_e$, $\pi_{T,Q}(q_{[A]}) = Q_{[A]}$, and there is a strongly continuous action β of $\mathbb{T}$ on $C^*(T_e, Q_{[A]})$ such that $\beta_z \circ \pi_{T,Q} = \pi_{T,Q} \circ \gamma_z$ for every $z \in \mathbb{T}$, then $\pi_{T,Q}$ is faithful.*

Proof. Select an increasing sequence $\{F_n\}$ of finite subsets of $\Phi_{\rm sg}(G^0) \cup \Phi(\mathcal{G}^1)$ such that $\cup_{n=1}^{\infty} F_n = \Phi_{\rm sg}(G^0) \cup \Phi(\mathcal{G}^1)$. For each n, Corollary 4.3 gives an isomorphism

$$\pi_n : C^*(G_{F_n}) \to C^*(\{t_e, q_{[v]} : [v] \in F^0, e \in F^1\})$$

that respects the generators. We can apply the gauge invariant uniqueness theorem for graph C^*-algebras to see that the homomorphism

$$\pi_{T,Q} \circ \pi_n : C^*(G_{F_n}) \to C^*(T_e, Q_{[A]})$$

is faithful. Hence, for every F_n, the restriction of $\pi_{T,Q}$ on the $*$-subalgebra of $C^*(\mathcal{G}/(H,B))$ generated by $\{t_e, q_{[v]} : [v] \in F_n^0, e \in F_n^1\}$ is faithful. This turns out that $\pi_{T,Q}$ is injective on the $*$-subalgebra $C^*(t_e, q_{[v]} : [v] \in \Phi_{\rm sg}(G^0), e \in \Phi(\mathcal{G}^1))$. Since, this subalgebra is dense in $C^*(\mathcal{G}/(H,B))$, we conclude that $\pi_{T,Q}$ is faithful. □

Proposition 4.6. *Let $\mathcal{G}$ be an ultragraph. If (H,B) is an admissible pair in $\mathcal{G}$, then $C^*(\mathcal{G}/(H,B)) \cong C^*(\mathcal{G})/I_{(H,B)}$.*

Proof. Using Proposition 3.3, we can consider $I_{(H,B)}$ as an ideal of $C^*(\overline{\mathcal{G}})$. Suppose that $C^*(\overline{\mathcal{G}}) = C^*(s_e, p_A)$ and $C^*(\mathcal{G}/(H,B)) = C^*(t_e, q_{[A]})$. If we define

$$T_e := s_e + I_{(H,B)} \text{ and } Q_{[A]} := p_A + I_{(H,B)}$$

for every $[A] \in \Phi(\mathcal{G}^0)$ and $e \in \Phi(\mathcal{G}^1)$, then the family $\{T_e, Q_{[A]}\}$ is a representation for $\mathcal{G}/(H,B)$ in $C^*(\overline{\mathcal{G}})/I_{(H,B)}$. So, there is a $*$-homomorphism $\phi : C^*(\mathcal{G}/(H,B)) \to C^*(\mathcal{G})/I_{(H,B)}$ such that $\phi(t_e) = T_e$ and $\phi(q_{[A]}) = Q_{[A]}$. Moreover, all $Q_{[A]}$ with $[A] \neq [\emptyset]$ are nonzero because $p_A + I_{(H,B)} = I_{(H,B)}$ implies $A \in H$. Then, an application of Theorem 4.5 yields that ϕ is faithful. On the other hand, the family $\{T_e, Q_{[A]} : [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$ generates the quotient $C^*(\mathcal{G})/I_{(H,B)}$, and hence, ϕ is surjective as well. Therefore, ϕ is an isomorphism and the result follows. □

To prove a version of Cuntz-Krieger uniqueness theorem, we extend Condition (L) for quotient ultragraphs.

Definition 4.7. We say that $\mathcal{G}/(H,B)$ satisfies *Condition* (L) if for every loop $\alpha = e_1 \cdots e_n$ in $\mathcal{G}/(H,B)$, at least one of the following conditions holds:

(i) $r(e_i) \neq s(e_{i+1})$ for some $1 \leq i \leq n$, where $e_{i+1} := e_1$ (or equivalently, $r(e_i) \setminus s(e_{i+1}) \neq [\emptyset]$).
(ii) α has an exit; that means, there exists $f \in \Phi(\mathcal{G}^1)$ such that $s(f) \subseteq r(e_i)$ and $f \neq e_{i+1}$ for some $1 \leq i \leq n$.

Lemma 4.8. *Let F be a finite subset of $\Phi_{\mathrm{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$. If $\mathcal{G}/(H,B)$ satisfies Condition (L), then so does the graph G_F.*

Proof. Suppose that $\mathcal{G}/(H,B)$ satisfies Condition (L). As the elements of $F^0 \cup \Gamma$ are sinks in G_F, every loop in G_F is of the form $\widetilde{\alpha} = (e_1, e_2) \cdots (e_n, e_1)$ corresponding with a loop $\alpha = e_1 \cdots e_n$ in $\mathcal{G}/(H,B)$. So, fix a loop $\widetilde{\alpha} = (e_1, e_2) \cdots (e_n, e_1)$ in G_F. Then $\alpha = e_1 \cdots e_n$ is a loop in $\mathcal{G}/(H,B)$ and by Condition (L), one of the following holds:

(i) $r(e_i) \neq s(e_{i+1})$ for some $1 \leq i \leq n$, where $e_{i+1} := e_1$, or
(ii) there exists $f \in \Phi(\mathcal{G}^1)$ such that $s(f) \subseteq r(e_i)$ and $f \neq e_{i+1}$ for some $1 \leq i \leq n$.

We can suppose in the case (i) that $s(e_{i+1}) \subsetneq r(e_i)$ and $r(e_i)$ emits only the edge e_{i+1} in $\mathcal{G}/(H,B)$. Then, by the definition of Γ, there exists either $[v] \in F^0$ with $[v] \subseteq r(e_i) \setminus s(e_{i+1})$, or $\omega \in \Gamma$ with $\omega_i = 1$. Thus, either $(e_i, [v])$ or (e_i, ω) is an exit for the loop $\widetilde{\alpha}$ in G_F, respectively.

Now assume case (ii) holds. If $f \in F^1$, then (e_i, f) is an exit for $\widetilde{\alpha}$. If $f \notin F^1$, for $[v] := s(f)$ we have either $[v] \notin F^0$ or

$$\exists \omega \in \Gamma \text{ with } \omega_i = 1 \text{ such that } [v] \subseteq R(\omega).$$

Hence, $(e_i, [v])$ or (e_i, ω) is an exit for $\widetilde{\alpha}$, respectively. Consequently, in any case, $\widetilde{\alpha}$ has an exit. □

Theorem 4.9 (The Cuntz-Krieger Uniqueness Theorem)**.** *Suppose that $\mathcal{G}/(H,B)$ is a quotient ultragraph satisfying Condition (L). If $\{T_e, Q_A\}$ is a Cuntz-Krieger representation for $\mathcal{G}/(H,B)$ in which all the projection $Q_{[A]}$ are nonzero for $[A] \neq [\emptyset]$, then the $*$-homomorphism $\pi_{T,Q} : C^*(\mathcal{G}/(H,B)) \to C^*(T_e, Q_{[A]})$ with $\pi_{T,Q}(t_e) = T_e$ and $\pi_{T,Q}(q_{[A]}) = Q_{[A]}$ is an isometrically isomorphism.*

Proof. It suffices to show that $\pi_{T,Q}$ is faithful. Similar to Theorem 4.5, choose an increasing sequence $\{F_n\}$ of finite sets such that $\cup_{n=1}^{\infty} F_n = \Phi_{\rm sg}(G^0) \cup \Phi(\mathcal{G}^1)$. By Corollary 4.3, there are isomorphisms $\pi_n : C^*(G_{F_n}) \to C^*(\{t_e, q_{[v]} : [v] \in F_n^0, e \in F_n^1\})$ that respect the generators. Since all the graphs G_{F_n} satisfy Condition (L) by Lemma 4.8, the Cuntz-Krieger uniqueness theorem for graph C^*-algebras implies that the $*$-homomorphisms

$$\pi_{T,Q} \circ \pi_n : C^*(G_{F_n}) \to C^*(T_e, Q_{[A]})$$

are faithful. Therefore, $\pi_{T,Q}$ is faithful on the subalgebra $C^*(t_e, q_{[v]} : [v] \in \Phi_{\rm sg}(G^0),\ e \in \Phi(\mathcal{G}^1))$ of $C^*(\mathcal{G}/(H,B))$. Since this subalgebra is dense in $C^*(\mathcal{G}/(H,B))$, we conclude that $\pi_{T,Q}$ is a faithful homomorphism. □

5. Primitive ideals in $C^*(\mathcal{G})$

In this section, we apply quotient ultragraphs to describe primitive gauge invariant ideals of an ultragraph C^*-algebra. Recall that since every ultragraph C^*-algebra $C^*(\mathcal{G})$ is separable (as assumed $\mathcal{G}^0$ to be countable), a prime ideal of $C^*(\mathcal{G})$ is primitive and vice versa [3, Corollaire 1].

To prove Proposition 5.4 below, we need the following simple lemmas.

Lemma 5.1. *Let $\mathcal{G}/(H,B) = (\Phi(G^0), \Phi(\mathcal{G}^0), \Phi(\mathcal{G}^1), r, s)$ be a quotient ultragraph of $\mathcal{G}$. If $\mathcal{G}/(H,B)$ does not satisfy Condition (L), then $C^*(\mathcal{G}/(H,B))$ contains an ideal Morita-equivalent to $C(\mathbb{T})$.*

Proof. Suppose that $\gamma = e_1 \cdots e_n$ is a loop in $\mathcal{G}/(H,B)$ without exits and $r(e_i) = s(e_{i+1})$ for $1 \le i \le n$. If $C^*(\mathcal{G}/(H,B)) = C^*(t_e, q_{[A]})$, for each i we have

$$t_{e_i}^* t_{e_i} = q_{r(e_i)} = q_{s(e_{i+1})} = t_{e_{i+1}} t_{e_{i+1}}^*.$$

Write $[v] := s(\gamma)$ and let I_γ be the ideal of $C^*(\mathcal{G}/(H,B))$ generated by $q_{[v]}$. Since γ has no exits in $\mathcal{G}/(H,B)$ and we have

$$q_{s(e_i)} = (t_{e_i} \cdots t_{e_n}) q_{[v]} (t_{e_n}^* \cdots t_{e_i}^*) \qquad (1 \le i \le n),$$

an easy argument shows that

$$I_\gamma = \overline{\rm span}\left\{ t_\alpha q_{[v]} t_\beta^* : \alpha, \beta \in (\mathcal{G}/(H,B))^*, [v] \subseteq r(\alpha) \cap r(\beta) \right\}.$$

So, we get

$$q_{[v]} I_\gamma q_{[v]} = \overline{\rm span}\left\{ (t_\gamma)^n q_{[v]} (t_\gamma^*)^m : m, n \ge 0 \right\},$$

where $(t_\gamma)^0 = (t_\gamma^*)^0 := q_{[v]}$. We show that $q_{[v]} I_\gamma q_{[v]}$ is a full corner in I_γ which is isometrically isomorphic to $C(\mathbb{T})$. For this, let E be the graph with one vertex

w and one loop f. If we set $Q_w := q_{[v]}$ and $T_f := t_\gamma$ $(= t_\gamma q_{[v]})$, then $\{T_f, Q_w\}$ is a Cuntz-Krieger E-family in $q_{[v]} I_\gamma q_{[v]}$. Assume $C^*(E) = C^*(s_f, p_w)$. Since $Q_w \neq 0$, the gauge-invariant uniqueness theorem for graph C^*-algebras implies that the $*$-homomorphism $\phi : C^*(E) \to q_{[v]} I_\gamma q_{[v]}$ with $p_w \mapsto Q_w$ and $s_f \mapsto T_f$ is faithful. Moreover, the C^*-algebra $q_{[v]} I_\gamma q_{[v]}$ is generated by $\{T_f, Q_w\}$, and hence ϕ is an isomorphism. As we know $C^*(E) \cong C(\mathbb{T})$, $q_{[v]} I_\gamma q_{[v]}$ is isomorphic to $C(\mathbb{T})$. Moreover, since $q_{[v]}$ generates I_γ, the corner $q_{[v]} I_\gamma q_{[v]}$ is full in I_γ. Thus, I_γ is Morita-equivalent to $q_{[v]} I_\gamma q_{[v]} \cong C(\mathbb{T})$ and the proof is complete. □

Lemma 5.2. *If $\mathcal{G}/(H, B)$ satisfies Condition (L), then any nonzero ideal in $C^*(\mathcal{G}/(H, B))$ contains projection $q_{[A]}$ for some $[A] \neq [\emptyset]$.*

Proof. Take an arbitrary ideal J in $C^*(\mathcal{G}/(H, B))$. If there are no $q_{[A]} \in J$ with $[A] \neq [\emptyset]$, then Theorem 4.9 implies that the quotient homomorphism $\phi : C^*(\mathcal{G}/(H, B)) \to C^*(\mathcal{G}/(H, B))/J$ is injective. Hence, we have $J = \ker \phi = (0)$. □

Definition 5.3. Let $\mathcal{G}$ be an ultragraph. For two sets $A, C \in \mathcal{G}^0$, we write $A \geq C$ if either $A \supseteq C$, or there exists $\alpha \in \mathcal{G}^*$ with $|\alpha| \geq 1$ such that $s(\alpha) \in A$ and $C \subseteq r(\alpha)$. We simply write $A \geq v$, $v \geq C$, and $v \geq w$ if $A \geq \{v\}$, $\{v\} \geq C$, and $\{v\} \geq \{w\}$, respectively. A subset $M \subseteq \mathcal{G}^0$ is said to be *downward directed* whenever for every $A_1, A_2 \in M$, there exists $\emptyset \neq C \in M$ such that $A_1, A_2 \geq C$.

Proposition 5.4. *Let H be a saturated hereditary subset of $\mathcal{G}^0$. Then the ideal $I_{(H,B_H)}$ in $C^*(\mathcal{G})$ is primitive if and only if the quotient ultragraph $\mathcal{G}/(H, B_H)$ satisfies Condition (L) and the collection $\mathcal{G}^0 \setminus H$ is downward directed.*

Proof. Let $I_{(H,B_H)}$ be a primitive ideal of $C^*(\mathcal{G})$. Since $C^*(\mathcal{G})/I_{(H,B_H)} \cong C^*(\mathcal{G}/(H, B_H))$, the zero ideal in $C^*(\mathcal{G}/(H, B_H))$ is primitive. If $\mathcal{G}/(H, B_H)$ does not satisfy Condition (L), then $C^*(\mathcal{G}/(H, B_H))$ contains an ideal J Morita-equivalent to $C(\mathbb{T})$ by Lemma 5.1. Select two ideals I_1, I_2 in $C(\mathbb{T})$ with $I_1 \cap I_2 = (0)$, and let J_1, J_2 be their corresponding ideals in J. Then J_1 and J_2 are two nonzero ideals of $C^*(\mathcal{G}/(H, B_H))$ with $J_1 \cap J_2 = (0)$, contradicting the primness of $C^*(\mathcal{G}/(H, B_H))$. Therefore, $\mathcal{G}/(H, B)$ satisfies Condition (L).

Now we show that $M := \mathcal{G}^0 \setminus H$ is downward directed. For this, we take two arbitrary sets $A_1, A_2 \in M$ and consider the ideals

$$J_1 := C^*(\mathcal{G}/(H, B_H)) q_{[A_1]} C^*(\mathcal{G}/(H, B_H))$$

and

$$J_2 := C^*(\mathcal{G}/(H, B_H)) q_{[A_2]} C^*(\mathcal{G}/(H, B_H))$$

in $C^*(\mathcal{G}/(H, B_H))$ generated by $q_{[A_1]}$ and $q_{[A_2]}$, respectively. Since $A_1, A_2 \notin H$, the projections $q_{[A_1]}, q_{[A_2]}$ are nonzero by Theorem 3.10, and so are the ideals J_1, J_2. The primness of $C^*(\mathcal{G}/(H, B_H))$ implies that the ideal

$$J_1 J_2 = C^* \left(\mathcal{G}/(H, B_H)\right) q_{[A_1]} C^* \left(\mathcal{G}/(H, B_H)\right) q_{[A_2]} C^* \left(\mathcal{G}/(H, B_H)\right)$$

is nonzero, and hence $q_{[A_1]}C^*(\mathcal{G}/(H,B_H))q_{[A_2]} \neq \{0\}$. As the set

$$\operatorname{span}\left\{t_\alpha q_{[D]} t_\beta^* : \alpha,\beta \in (\mathcal{G}/(H,B))^*,\ r(\alpha)\cap[D]\cap r(\beta)\neq[\emptyset]\right\}$$

is dense in $C^*(\mathcal{G}/(H,B_H))$, there exist $\alpha,\beta \in (\mathcal{G}/(H,B_H))^*$ and $[D]\in\Phi(\mathcal{G}^0)$ such that $q_{[A_1]}(t_\alpha q_{[D]} t_\beta^*)q_{[A_2]} \neq 0$. In this case, we must have $s(\alpha)\subseteq[A_1]$ and $s(\beta)\subseteq[A_2]$ and thus, $A_1, A_2 \geq C$ for $C := r_{\mathcal{G}}(\alpha)\cap D\cap r_{\mathcal{G}}(\beta)$. Therefore, $\mathcal{G}^0\setminus H$ is downward directed.

For the converse, we assume that $\mathcal{G}/(H,B_H)$ satisfies Condition (L) and the collection $M=\mathcal{G}^0\setminus H$ is downward directed. Fix two nonzero ideals J_1, J_2 of $C^*(\mathcal{G}/(H,B_H))$. By Lemma 5.2, there are nonzero projections $q_{[A_1]}\in J_1$ and $q_{[A_2]}\in J_2$. Then $A_1,A_2\notin H$ and, since M is downward directed, there exists $C\in M$ such that $A_1,A_2\geq C$. Hence, the ideal $J_1\cap J_2$ contains the nonzero projection $q_{[C]}$. Since J_1 and J_2 were arbitrary, this concludes that the C^*-algebra $C^*(\mathcal{G}/(H,B_H))$ is primitive and $I_{(H,B_H)}$ is a primitive ideal in $C^*(\mathcal{G})$ by Proposition 4.6. □

The following proposition describes another kind of primitive ideals in $C^*(\mathcal{G})$.

Proposition 5.5. *Let (H,B) be an admissible pair in $\mathcal{G}$ and let $B=B_H\setminus\{w\}$. Then the ideal $I_{(H,B)}$ in $C^*(\mathcal{G})$ is primitive if and only if $A\geq w$ for all $A\in\mathcal{G}^0\setminus H$.*

Proof. Suppose that $I_{(H,B)}$ is a primitive ideal and take an arbitrary $A\in\mathcal{G}^0\setminus H$. If $\overline{A} := A\cup\{v' : v\in A\cap(B_H\setminus B)\}$, then $q_{[\overline{A}]}$ and $q_{[w']}$ are two nonzero projections in $C^*(\mathcal{G}/(H,B))$. If we consider ideals $J_{[\overline{A}]} := \langle q_{[\overline{A}]}\rangle$ and $J_{[w']} := \langle q_{[w']}\rangle$ in $C^*(\mathcal{G}/(H,B))$, then the primness of $C^*(\mathcal{G}/(H,B)) \cong C^*(\mathcal{G})/I_{H,B}$ implies that the ideal

$$J_{[\overline{A}]}J_{[w']} = C^*(\mathcal{G}/(H,B))q_{[\overline{A}]}C^*(\mathcal{G}/(H,B))q_{[w']}C^*(\mathcal{G}/(H,B))$$

is nonzero, and hence $q_{[\overline{A}]}C^*(\mathcal{G}/(H,B))q_{[w']}\neq\{0\}$. So, there exist $\alpha,\beta\in(\mathcal{G}/(H,B))^*$ such that $q_{[\overline{A}]}t_\alpha t_\beta^* q_{[w']}\neq 0$. Since $[w']$ is a sink in $\mathcal{G}/(H,B)$, we must have $q_{[\overline{A}]}t_\alpha q_{[w']}\neq 0$. If $|\alpha|=0$, then $[w']\subseteq[\overline{A}]$, $w'\in\overline{A}$ and $w\in A$. If $|\alpha|\geq 1$, then $s(\alpha)\subseteq[\overline{A}]$ and $[w']\subseteq r(\alpha)$, which follow $s_{\mathcal{G}}(\alpha)\in A$ and $w\in r_{\mathcal{G}}(\alpha)$. Therefore, we obtain $A\geq w$ in either case.

Conversely, assume $A\geq w$ for every $A\in\mathcal{G}^0\setminus H$. Then the collection $\mathcal{G}^0\setminus H$ is downward directed. Moreover, for every $[\emptyset]\neq[A]\in\Phi(\mathcal{G}^0)$, there exists $\alpha\in(\mathcal{G}/(H,B))^*$ such that $s(\alpha)\subseteq[A]$ and $[w']\subseteq r(\alpha)$. As $[w']$ is a sink in $\mathcal{G}/(H,B)$, we see that the quotient ultragraph $\mathcal{G}/(H,B)$ satisfies Condition (L). Now similar to the proof of Proposition 5.4, we can show that $I_{(H,B)}$ is a primitive ideal. □

Recall that each loop in $\mathcal{G}/(H,B)$ comes from a loop in the initial ultragraph $\mathcal{G}$. So, to check Condition (L) for a quotient ultragraph $\mathcal{G}/(H,B)$, we can use the following.

Definition 5.6. Let H be a saturated hereditary subset of $\mathcal{G}^0$. For simplicity, we say that a path $\alpha = e_1 \cdots e_n$ lies in $\mathcal{G} \setminus H$ whenever $r_{\mathcal{G}}(\alpha) \in \mathcal{G}^0 \setminus H$. We also say that α has *an exit in* $\mathcal{G} \setminus H$ if either $r_{\mathcal{G}}(e_i) \setminus s_{\mathcal{G}}(e_{i+1}) \in \mathcal{G}^0 \setminus H$ for some i, or there is an edge f with $r_{\mathcal{G}}(f) \in \mathcal{G}^0 \setminus H$ such that $s_{\mathcal{G}}(f) = s_{\mathcal{G}}(e_i)$ and $f \neq e_i$, for some $1 \leq i \leq n$.

It is easy to verify that a quotient ultragraph $\mathcal{G}/(H,B)$ satisfies Condition (L) if and only if every loop in $\mathcal{G} \setminus H$ has an exit in $\mathcal{G} \setminus H$. Hence we have:

Theorem 5.7 (See [1, Theorem 4.7])**.** *Let $\mathcal{G}$ be an ultragraph. A gauge invariant ideal $I_{(H,B)}$ of $C^*(\mathcal{G})$ is primitive if and only if one of the following holds*:

1. *$B = B_H$, $\mathcal{G}^0 \setminus H$ is downward directed, and every loop in $\mathcal{G} \setminus H$ has an exit in $\mathcal{G} \setminus H$.*
2. *$B = B_H \setminus \{w\}$ for some $w \in B_H$, and $A \geq w$ for all $A \in \mathcal{G}^0 \setminus H$.*

Proof. Let $I_{(H,B)}$ be a primitive ideal in $C^*(\mathcal{G})$. Then

$$C^*(\mathcal{G}/(H,B)) \cong C^*(\mathcal{G})/I_{(H,B)}$$

is a primitive C^*-algebra. We claim that $|B_H \setminus B| \leq 1$. Indeed, if w_1, w_2 are two distinct vertices in $B_H \setminus B$, similar to the proof of Propositions 5.4 and 5.5, the primitivity of $C^*(\mathcal{G}/(H,B))$ implies that the corner $q_{[w_1']}C^*(\mathcal{G}/(H,B))q_{[w_2']}$ is nonzero. So, there exist $\alpha, \beta \in (\mathcal{G}/(H,B))^*$ such that $q_{[w_1']}t_\alpha t_\beta^* q_{[w_2']} \neq 0$. But we must have $|\alpha| = |\beta| = 0$ because $[w_1'], [w_2']$ are two sinks in $\mathcal{G}/(H,B)$. Hence, $q_{[w_1']}q_{[w_2']} \neq 0$ which is impossible because $q_{[w_1']}q_{[w_2']} = q_{[\{w_1'\} \cap \{w_2'\}]} = q_{[\emptyset]} = 0$. Thus, the claim holds. Now we may apply Propositions 5.4 and 5.5 to obtain the result. □

Following [10, Definition 7.1], we say that an ultragraph $\mathcal{G}$ satisfies *Condition* (K) if every vertex $v \in G^0$ either is the base of no loops, or there are at least two loops α, β in $\mathcal{G}$ based at v such that neither α nor β is a subpath of the other. In view of [10, Proposition 7.3], if $\mathcal{G}$ satisfies Condition (K), then all ideals of $C^*(\mathcal{G})$ are of the form $I_{(H,B)}$. So, in this case, Theorem 5.7 describes all primitive ideals of $C^*(\mathcal{G})$.

6. Purely infinite ultragraph C^*-algebras via Fell bundles

Mark Tomforde in [17] determined ultragraph C^*-algebras in which every hereditary subalgebra contains infinite projections. Here, we consider the notion of "*pure infiniteness*" in the sense of Kirchberg-Rørdam [11], and generalize [8, Theorem 2.3] to ultragraph setting. In view of Proposition 3.14 and Theorem 4.16 of [11], a (not necessarily simple) C^*-algebra A is *purely infinite* if and only if for every $a \in A^+ \setminus \{0\}$ and closed two-sided ideal $I \trianglelefteq A$, $a + I$ in the quotient A/I is either zero or infinite (in this case, a is called *properly infinite*). Recall from [11, Definition 3.2] that an element $a \in A^+ \setminus \{0\}$ is called *infinite* if there is $b \in A^+ \setminus \{0\}$ such that $a \oplus b \precsim a \oplus 0$ in the matrix algebra $M_2(A)$.

So, the notion of pure infiniteness is directly related to the structure of ideals and quotients. In this section, we use the quotient ultragraphs to characterize purely infinite ultragraph C^*-algebras. Briefly, we consider the natural $\mathbb{Z}$-grading (or Fell bundle) for $C^*(\mathcal{G})$ and then apply the results of [12, Section 4] for pure infiniteness of Fell bundles.

6.1. Condition (K) for $\mathcal{G}$

To prove the main result of this section, Theorem 6.6, we need to show that an ultragraph $\mathcal{G}$ satisfies Condition (K) if and only if every quotient ultragraph $\mathcal{G}/(H,B)$ satisfies Condition (L).

Notation. Let $\alpha = e_1 \cdots e_n$ be a path in an ultragraph $\mathcal{G}$. If $\beta = e_k e_{k+1} \cdots e_l$ is a subpath of α, we simply write $\beta \subseteq \alpha$; otherwise, we write $\beta \nsubseteq \alpha$.

First, we show in the absence of Condition (K) for $\mathcal{G}$ that there is a quotient ultragraph $\mathcal{G}/(H,B)$ which does not satisfy Condition (L). For this, let $\mathcal{G}$ contain a loop $\gamma = e_1 \cdots e_n$ such that there are no loops α with $s(\alpha) = s(\gamma)$, $\alpha \nsubseteq \gamma$, and $\gamma \nsubseteq \alpha$. If $\gamma^0 := \{s_\mathcal{G}(e_1), \ldots, s_\mathcal{G}(e_n)\}$, define

$$X := \left\{ r_\mathcal{G}(\alpha) \setminus \gamma^0 : \alpha \in \mathcal{G}^*, |\alpha| \geq 1, s_\mathcal{G}(\alpha) \in \gamma^0 \right\},$$

$$Y := \left\{ \bigcup_{i=1}^n A_i : A_1, \ldots, A_n \in X, n \in \mathbb{N} \right\},$$

and set

$$H_0 := \left\{ B \in \mathcal{G}^0 : B \subseteq A \text{ for some } A \in Y \right\}.$$

We construct a saturated hereditary subset H of $\mathcal{G}^0$ as follows: for any $n \in \mathbb{N}$ inductively define

$$S_n := \left\{ w \in G^0 : 0 < |s_\mathcal{G}^{-1}(w)| < \infty \text{ and } r_\mathcal{G}(s_\mathcal{G}^{-1}(w)) \subseteq H_{n-1} \right\}$$

and

$$H_n := \{ A \cup F : A \in H_{n-1} \text{ and } F \subseteq S_n \text{ is a finite subset} \}.$$

Then we can see that the subset

$$H = \bigcup_{n=0}^\infty H_n = \left\{ A \cup F : A \in H_0 \text{ and } F \subseteq \bigcup_{n=1}^\infty S_n \text{ is a finite subset} \right\}$$

is hereditary and saturated.

Lemma 6.1. *Suppose that $\gamma = e_1 \cdots e_n$ is a loop in $\mathcal{G}$ such that there are no loops α with $s(\alpha) = s(\gamma)$ and $\alpha \nsubseteq \gamma$, $\gamma \nsubseteq \alpha$. If we construct the set H as above, then H is a saturated hereditary subset of $\mathcal{G}^0$. Moreover, we have $A \cap \gamma^0 = \emptyset$ for every $A \in H$.*

Proof. By induction, we first show that each H_n is a hereditary set in $\mathcal{G}$. For this, we check conditions $(H1)$-$(H3)$ in Definition 2.5. To verify condition $(H1)$ for H_0, let us take $e \in \mathcal{G}^1$ with $s_\mathcal{G}(e) \in H_0$. Then $s_\mathcal{G}(e) \in X$ and there is $\alpha \in \mathcal{G}^*$ such that $s_\mathcal{G}(\alpha) \in \gamma^0$ and $s_\mathcal{G}(e) \in r_\mathcal{G}(\alpha) \setminus \gamma^0$. Hence, $s_\mathcal{G}(\alpha e) = s_\mathcal{G}(\alpha) \in \gamma^0$. Moreover, we have $r_\mathcal{G}(\alpha e) \cap \gamma^0 = \emptyset$ because the otherwise implies the existence of a path $\beta \in \mathcal{G}^*$ with $s_{\mathcal{G}}(\beta) = s_{\mathcal{G}}(\gamma)$ and $\beta \nsubseteq \gamma$, $\gamma \nsubseteq \beta$, contradicting the hypothesis. It turns out

$$r_{\mathcal{G}}(e) = r_{\mathcal{G}}(\alpha e) = r_{\mathcal{G}}(\alpha e) \setminus \gamma^0 \in X \subseteq H_0.$$

Hence, H_0 satisfies condition $(H1)$. We may easily verify conditions $(H2)$ and $(H3)$ for H_0, so H_0 is hereditary. Moreover, for every $w \in S_n$, the range of each edge emitted by w belongs to H_{n-1} by definition. Thus, we can inductively check that each H_n is hereditary, and so is $H = \cup_{n=1}^{\infty} H_n$. The saturation property of H may be verified similar to the proof of [17, Lemma 3.12].

It remains to show $A \cap \gamma^0 = \emptyset$ for every $A \in H$. To do this, note that $A \cap \gamma^0 = \emptyset$ for every $A \in H_0$ because this property holds for all $A \in X$. We claim that $(\cup_{n=1}^{\infty} S_n) \cap \gamma^0 = \emptyset$. Indeed, if $v = s_{\mathcal{G}}(e_i) \in \gamma^0$ for some $e_i \in \gamma$, then $r_{\mathcal{G}}(e_i) \cap \gamma^0 \neq \emptyset$ and $r_{\mathcal{G}}(e_i) \notin H_0$. Hence, $\{r_{\mathcal{G}}(e) : e \in \mathcal{G}^1,\ s_{\mathcal{G}}(e) = v\} \nsubseteq H_0$ that turns out $v \notin S_1$. So, we have $S_1 \cap \gamma^0 = \emptyset$. An inductive argument shows $S_n \cap \gamma^0 = \emptyset$ for $n \geq 1$, and the claim holds. Now since

$$H = \cup_{n=1}^{\infty} H_n = \{A \cup F : A \in H_0 \text{ and } F \subseteq \cup_{n=1}^{\infty} S_n \text{ is a finite subset}\},$$

we conclude that $A \cap \gamma^0 = \emptyset$ for all $A \in H$. □

Proposition 6.2. *An ultragraph $\mathcal{G}$ satisfies Condition (K) if and only if for every admissible pair (H, B) in $\mathcal{G}$, the quotient ultragraph $\mathcal{G}/(H, B)$ satisfies Condition (L).*

Proof. Suppose that $\mathcal{G}$ satisfies Condition (K) and (H, B) is an admissible pair in $\mathcal{G}$. Let $\alpha = e_1 \cdots e_n$ be a loop in $\mathcal{G}/(H, B)$. Since α is also a loop in $\mathcal{G}$, there is a loop $\beta = f_1 \cdots f_m$ in $\mathcal{G}$ with $s_{\mathcal{G}}(\alpha) = s_{\mathcal{G}}(\beta)$, and neither $\alpha \subseteq \beta$ nor $\beta \subseteq \alpha$. Without loos of generality, assume $e_1 \neq f_1$. By the fact $s_{\mathcal{G}}(\alpha) = s_{\mathcal{G}}(\beta) \in r_{\mathcal{G}}(\beta)$, we have $r_{\mathcal{G}}(\beta) \notin H$, and so $r_{\mathcal{G}}(f_1) \notin H$ by the hereditary property of H. Therefore, f_1 is an exit for α in $\mathcal{G}/(H, B)$ and we conclude that $\mathcal{G}/(H, B)$ satisfies Condition (L).

For the converse, suppose on the contrary that $\mathcal{G}$ does not satisfy Condition (K). Then there exists a loop $\gamma = e_1 \cdots e_n$ in $\mathcal{G}$ such that there are no loops α with $s(\alpha) = s(\gamma)$, $\alpha \nsubseteq \gamma$, and $\gamma \nsubseteq \alpha$. As Lemma 6.1, construct a saturated hereditary subset H of $\mathcal{G}^0$ and consider the quotient ultragraph $\mathcal{G}/(H, B_H) = (\Phi(G^0), \Phi(\mathcal{G}^0), \Phi(\mathcal{G}^1), r, s)$. We show that γ as a loop in $\mathcal{G}/(H, B_H)$ has no exits and $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n$. If f is an exit for γ in $\mathcal{G}/(H, B_H)$ such that $s(f) = s(e_j)$ and $f \neq e_j$, then $r_{\mathcal{G}}(f) \notin H$ and $r_{\mathcal{G}}(f) \cap \gamma^0 \neq \emptyset$ (if $r_{\mathcal{G}}(f) \cap \gamma^0 = \emptyset$, then $r_{\mathcal{G}}(f) = r_{\mathcal{G}}(f) \setminus \gamma^0 \in X \subseteq H$, a contradiction). So, there is $e_l \in \gamma$ such that $s_{\mathcal{G}}(e_l) \in r_{\mathcal{G}}(f)$. If we set $\alpha := e_1 \cdots e_{j-1} f e_l \cdots e_n$, then α is a loop in $\mathcal{G}$ with $s_{\mathcal{G}}(\alpha) = s_{\mathcal{G}}(\gamma)$, and $\alpha \nsubseteq \gamma$, $\gamma \nsubseteq \alpha$, that contradicts the hypothesis. Therefore, γ has no exits in $\mathcal{G}/(H, B_H)$. Moreover, we have $r(e_i) \cap [\gamma^0] = s(e_{i+1})$ for each $1 \leq i \leq n$, because the otherwise gives an exit for γ in $\mathcal{G}/(H, B_H)$ by the construction of H. Hence,

$$r(e_i) \setminus s(e_{i+1}) = r(e_i) \setminus [\gamma^0] = [\emptyset]$$

and we get $r(e_i) = s(e_{i+1})$ (note that the fact $r_{\mathcal{G}}(e_i) \setminus \gamma^0 \in H$ implies $r(e_i) \setminus [\gamma^0] = [\overline{r_{\mathcal{G}}(e_i)} \setminus \gamma^0] = [\emptyset]$). Therefore, the quotient ultragraph $\mathcal{G}/(H, B_H)$ does not satisfy Condition (L) as desired. □

6.2. Purely infinite ultragraph C^*-algebras via Fell bundles

Every quotient ultragraph (or ultragraph) C^*-algebra

$$C^*(\mathcal{G}/(H,B)) = C^*(q_{[A]}, t_e)$$

is equipped with a natural $\mathbb{Z}$-grading or Fell bundle $\mathcal{B} = \{B_n : n \in \mathbb{Z}\}$ with the fibers

$$B_n := \overline{\operatorname{span}} \left\{ t_\mu q_{[A]} t_\nu^* : \mu, \nu \in (\mathcal{G}/(H,B))^*, \ |\mu| - |\nu| = n \right\}.$$

These Fell bundles will be considered in this section. The fiber B_0 is the fixed point C^*-subalgebra of $C^*(\mathcal{G}/(H,B))$ for the gauge action which is an AF C^*-algebra. An application of the gauge invariant uniqueness theorem implies that $C^*(\mathcal{G}/(H,B))$ is isomorphic to the cross sectional C^*-algebra $C^*(\mathcal{B})$ (we refer the reader to [5] for details about Fell bundles and their C^*-algebras). Moreover, since $\mathbb{Z}$ is an amenable group, combining Theorem 20.7 and Proposition 20.2 of [5] implies that $C^*(\mathcal{G}/(H,B))$ is also isomorphic to the reduced cross sectional C^*-algebra $C_r^*(\mathcal{B})$.

Following [4, Definition 2.1], an *ideal* in a Fell bundle $\mathcal{B} = \{B_n\}$ is a family $\mathcal{J} = \{J_n\}_{n\in\mathbb{Z}}$ of closed subspaces $J_n \subseteq B_n$, such that $B_m J_n \subseteq J_{mn}$ and $J_n B_m \subseteq J_{nm}$ for all $m, n \in \mathbb{Z}$. If $\mathcal{J}$ is an ideal of $\mathcal{B}$, then the family $\mathcal{B}/\mathcal{J} := \{B_n/J_n\}_{n\in\mathbb{Z}}$ is equipped with a natural Fell bundle structure, which is called a *quotient Fell bundle* of $\mathcal{B}$, cf. [5, Definition 21.14].

Definition 6.3 ([12, Definition 4.1])**.** Let $\mathcal{G}/(H,B)$ be a quotient ultragraph and $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}}$ is the above Fell bundle in $C^*(\mathcal{G}/(H,B))$. We say that $\mathcal{B}$ is *aperiodic* if for each $n \in \mathbb{Z}\setminus\{0\}$, each $b_n \in B_n$, and every hereditary subalgebra A of B_0, we have

$$\inf \left\{ \|a b_n a\| : a \in A^+, \ \|a\| = 1 \right\} = 0.$$

Furthermore, $\mathcal{B}$ is called *residually aperiodic* whenever the quotient Fell bundle $\mathcal{B}/\mathcal{J}$ is aperiodic for every ideal $\mathcal{J}$ of $\mathcal{B}$.

The following lemma is analogous to [12, Proposition 7.3] for quotient ultragraphs.

Lemma 6.4. *Let $\mathcal{G}/(H,B)$ be a quotient ultragraph and let $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}}$ be the Fell bundle associated to $C^*(\mathcal{G}/(H,B))$. Then $\mathcal{B}$ is aperiodic if and only if $\mathcal{G}/(H,B)$ satisfies Condition (L).*

Proof. We may modify the proof of [12, Proposition 7.3] for our case by replacing elements $s_\alpha s_\beta^*$ and $s_\mu s_\mu^*$ with $t_\alpha q_{[A]} t_\beta^*$ and $t_\mu q_{[A]} t_\mu^*$, respectively. Then the proof goes along the same lines as the one in [12, Proposition 7.3]. □

Corollary 6.5. *Let $\mathcal{G}$ be an ultragraph and let $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}}$ be the described Fell bundle of $C^*(\mathcal{G})$. If $\mathcal{G}$ satisfies Condition (K), then $\mathcal{B}$ is residually aperiodic.*

Proof. Suppose that $\mathcal{G}$ satisfies Condition (K). In view of [10, Proposition 7.3], we know that all ideals of $C^*(\mathcal{G})$ are graded and of the form $I_{(H,B)}$. So, each ideal $\mathcal{J} = \{J_n\}_{n\in\mathbb{Z}}$ of $\mathcal{B}$ is corresponding with an ideal $I_{(H,B)}$ with the homogenous components $J_n := I_{(H,B)} \cap B_n$. Moreover, the quotient Fell bundle $\mathcal{B}/\mathcal{J} := \{B_n/J_n : n \in \mathbb{Z}\}$ is a grading (or a Fell bundle) for $C^*(\mathcal{G})/I_{(H,B)} \cong C^*(\mathcal{G}/(H,B))$. Therefore, quotient Fell bundles $\mathcal{B}/\mathcal{J}$ are corresponding with quotient ultragraphs $\mathcal{G}/(H,B)$. Since such quotient ultragraphs satisfy Condition (L) by Proposition 6.2, Lemma 6.4 follows the result. □

Theorem 6.6. *Let $\mathcal{G}$ be an ultragraph. Then $C^*(\mathcal{G})$ is purely infinite (in the sense of [11]) if and only if $\mathcal{G}$ satisfies Condition (K), and for every saturated hereditary subset H of $\mathcal{G}^0$, we have*

(1) *$B_H = \emptyset$, and*
(2) *every $A \in \mathcal{G}^0 \setminus H$ connects to a loop α in $\mathcal{G} \setminus H$, which means $A \geq s_{\mathcal{G}}(\alpha)$ (see Definition 5.3).*

Proof. First, suppose that $C^*(\mathcal{G})$ is purely infinite. If $\mathcal{G}$ does not satisfy Condition (K), by the second paragraph in the proof of Proposition 6.2, there is a quotient ultragraph $\mathcal{G}/(H,B)$ containing a loop $\alpha \in (\mathcal{G}/(H,B))^*$ with no exits in $\mathcal{G}/(H,B)$. The argument of Lemma 5.1 follows that the ideal $J := \langle q_{s(\alpha)} \rangle \trianglelefteq C^*(\mathcal{G}/(H,B))$ is Morita-equivalent to $C(\mathbb{T})$. Hence, the projection $p_{s(\alpha)}$ is not properly infinite which contradicts [11, Theorem 4.16].

Now assume that H is a saturated hereditary subset of $\mathcal{G}^0$. We consider the quotient ultragraph $\mathcal{G}/(H,\emptyset)$ and take an arbitrary $[A] \in \Phi(\mathcal{G}^0) \setminus \{[\emptyset]\}$. If there is no loops $\alpha \in r_{\mathcal{G}}^{-1}(\mathcal{G}^0 \setminus H)$ with $A \geq s_{\mathcal{G}}(\alpha)$, then the ideal $I_{[A]} := \langle q_{[A]} \rangle \trianglelefteq C^*(\mathcal{G}/(H,\emptyset))$ is AF. Thus $q_{[A]}$ is not infinite and $C^*(\mathcal{G})$ contains a non-properly infinite projection, contradicting [11, Theorem 4.16]. Moreover, we notice that for any $w \in B_H$, $[w']$ is a sink in $\mathcal{G}/(H,\emptyset)$ and the projection $q_{[w']}$ is not infinite, which is impossible.

Conversely, suppose that $\mathcal{G}$ satisfies Condition (K) and the asserted properties hold for any saturated hereditary set H. To show that $C^*(\mathcal{G})$ is purely infinite we apply [12, Theorem 5.12] for the pure infiniteness of Fell bundles. Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}}$ be the natural Fell bundle in $C^*(\mathcal{G})$. Corollary 6.5 says that $\mathcal{B}$ is residually aperiodic. Moreover, every projection in B_0 is Murray-von Neumann equivalent to a finite sum $\sum_{i=1}^n r_i s_{\alpha_i} p_{B_i} s_{\beta_i}^*$ of mutually orthogonal projections such that $|\alpha_i| = |\beta_i|$ for $1 \leq i \leq n$. Note that each projection $s_{\alpha_i} p_{B_i} s_{\beta_i}^*$ is Murray-von Neumann equivalent to $(s_{\alpha_i} p_{B_i})^* (p_{B_i} s_{\beta_i})$ which equals to zero unless $\alpha_i = \beta_i$. Hence, in view of [12, Lemma 5.13], it suffices to show that every nonzero projection of the form $s_\mu p_B s_\mu^*$ is properly infinite.

Let $I_{(H,\emptyset)}$ be an ideal in $C^*(\mathcal{G})$ such that $s_\mu p_B s_\mu^* \notin I_{(H,\emptyset)}$. Then $B \cap r_{\mathcal{G}}(\mu) \in \mathcal{G}^0 \setminus H$. Assume $C^*(\mathcal{G}/(H,\emptyset)) = C^*(t_e, q_{[A]})$ and let $q : C^*(\mathcal{G}) \to C^*(\mathcal{G}/(H,\emptyset))$

be the canonical quotient map by Proposition 4.6. Then $q(s_\mu p_B s_\mu^*) = t_\mu q_{[B]} t_\mu^* \neq 0$. By hypothesis, there are a path λ and a loop $\alpha \in r_{\mathcal{G}}^{-1}(\mathcal{G}^0 \setminus H)$ such that $s_{\mathcal{G}}(\lambda) \in B \cap r_{\mathcal{G}}(\mu)$ and $s_{\mathcal{G}}(\alpha) \in r_{\mathcal{G}}(\lambda)$. Since $\mathcal{G}$ satisfies Condition (K), α has an exit f in $r^{-1}(\mathcal{G}^0 \setminus H)$. Thus we have

$$\left(t_\alpha q_{s(\alpha)}\right)\left(t_\alpha q_{s(\alpha)}\right)^* + t_f t_f^* \leq q_{s(\alpha)},$$

and since

$$\left(t_\alpha q_{s(\alpha)}\right)\left(t_\alpha q_{s(\alpha)}\right)^* \sim \left(t_\alpha q_{s(\alpha)}\right)^*\left(t_\alpha q_{s(\alpha)}\right) = q_{s(\alpha)},$$

it turns out that $q_{s(\alpha)}$ is an infinite projection in $C^*(\mathcal{G}/(H,\emptyset)) \cong C^*(\mathcal{G})/I_{(H,\emptyset)}$. On the other hand, the fact

$$\left(t_{\mu\lambda} q_{s(\alpha)}\right)^* t_\mu q_{[B]} t_\mu^* \left(t_{\mu\lambda} q_{s(\alpha)}\right) = q_{s(\alpha)}$$

says that $q_{s(\alpha)} \precsim t_\mu q_{[B]} t_\mu^*$ (see [15, Proposition 2.4]), and thus $t_\mu q_{[B]} t_\mu^*$ is infinite by [11, Lemma 3.17]. It follows that $s_\mu p_B s_\mu^*$ is a properly infinite projection. Now apply [12, Theorem 5.11(ii)] to conclude that the C^*-algebra $C^*(\mathcal{G}) \cong C_r^*(\mathcal{B})$ is purely infinite. □

References

[1] T. Bates, J. H. Hong, I. Raeburn, and W. Szymański, *The ideal structure of the C^*-algebras of infinite graphs*, Illinois J. Math. **46** (2002), no. 4, 1159–1176.

[2] T. Bates, D. Pask, I. Raeburn, and W. Szymański, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324.

[3] J. Dixmier, *Sur les C^*-algèbres*, Bull. Soc. Math. France **88** (1960), 95–112.

[4] R. Exel, *Exact groups and Fell bundles*, Math. Ann. **323** (2002), no. 2, 259–266.

[5] ———, *Partial dynamical systems, Fell bundles and applications*, Mathematical Surveys and Monographs, **224**, American Mathematical Society, Providence, RI, 2017.

[6] R. Exel and M. Laca, *Cuntz-Krieger algebras for infinite matrices*, J. Reine Angew. Math. **512** (1999), 119–172.

[7] N. J. Fowler, M. Laca, and I. Raeburn, *The C^*-algebras of infinite graphs*, Proc. Amer. Math. Soc. **128** (2000), no. 8, 2319–2327.

[8] J. H. Hong and W. Szymański, *Purely infinite Cuntz-Krieger algebras of directed graphs*, Bull. London Math. Soc. **35** (2003), no. 5, 689–696.

[9] J. A. Jeong, S. H. Kim, and G. H. Park, *The structure of gauge-invariant ideals of labelled graph C^*-algebras*, J. Funct. Anal. **262** (2012), no. 4, 1759–1780.

[10] T. Katsura, P. S. Muhly, A. Sims, and M. Tomforde, *Ultragraph C^*-algebras via topological quivers*, Studia Math. **187** (2008), no. 2, 137–155.

[11] E. Kirchberg and M. Rørdam, *Non-simple purely infinite C^*-algebras*, Amer. J. Math. **122** (2000), no. 3, 637–666.

[12] B. K. Kwaśniewski and W. Szymański, *Pure infiniteness and ideal structure of C^*-algebras associated to Fell bundles*, J. Math. Anal. Appl. **445** (2017), no. 1, 898–943.

[13] P. S. Muhly and M. Tomforde, *Topological quivers*, Internat. J. Math. **16** (2005), no. 7, 693–755.

[14] I. Raeburn and W. Szymański, *Cuntz-Krieger algebras of infinite graphs and matrices*, Trans. Amer. Math. Soc. **356** (2004), no. 1, 39–59.

[15] M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra. II*, J. Funct. Anal. **107** (1992), no. 2, 255–269.

[16] M. Tomforde, *A unified approach to Exel-Laca algebras and C^*-algebras associated to graphs*, J. Operator Theory **50** (2003), no. 2, 345–368.

[17] ———, *Simplicity of ultragraph algebras*, Indiana Univ. Math. J. **52** (2003), no. 4, 901–925.

Hossein Larki
Department of Mathematics
Faculty of Mathematical Sciences and Computer
Shahid Chamran University of Ahvaz
P.O. Box: 83151-61357
Ahvaz, Iran
Email address: h.larki@scu.ac.ir