

## BOUNDEDNESS OF THE COMMUTATOR OF THE INTRINSIC SQUARE FUNCTION IN VARIABLE EXPONENT SPACES

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**ABSTRACT.** In this paper, we show that the commutator of the intrinsic square function with BMO symbols is bounded on the variable exponent Lebesgue spaces  $L^{p(\cdot)}(\mathbb{R}^n)$  applying a generalization of the classical Rubio de Francia extrapolation. As a consequence we further establish its boundedness on the variable exponent Morrey spaces  $\mathcal{M}_{p(\cdot),u}$ , Morrey-Herz spaces  $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$  and Herz type Hardy spaces  $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ , where the exponents  $\alpha(\cdot)$  and  $p(\cdot)$  are variable. Observe that, even when  $\alpha(\cdot) \equiv \alpha$  is constant, the corresponding main results are completely new.

### 1. Introduction

We begin with the definition of intrinsic square function  $S_\beta$  from Wilson [38, Page 773]. For  $0 < \beta \leq 1$ , let  $\mathcal{C}_\beta$  be the family of functions  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\phi$ 's support is contained in  $\{x \in \mathbb{R}^n : |x| \leq 1\}$ ,  $\int_{\mathbb{R}^n} \phi(x) dx = 0$  and for all  $x_1, x_2 \in \mathbb{R}^n$ ,

$$(1) \quad |\phi(x_1) - \phi(x_2)| \leq |x_1 - x_2|^\beta.$$

For  $(y, t) \in \mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$  and  $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ , set

$$(2) \quad A_\beta(f)(y, t) = \sup_{\phi \in \mathcal{C}_\beta} |f * \phi_t(y)| = \sup_{\phi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} \phi_t(y - z) f(z) dz \right|.$$

We define the intrinsic square function of  $f$  (of order  $\beta$ ) by

$$(3) \quad S_\beta(f)(x) = \left( \int \int_{\Gamma(x)} (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

where  $\phi_t(x) = \frac{1}{t^n} \phi(\frac{x}{t})$  and  $\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$ .

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The intrinsic square function  $S_\beta$  was first introduced by Wilson in [38] to solve a conjecture raised by Fefferman and Stein [11] on the boundedness of the classical Lusin area function  $S$  from the weighted Lebesgue space  $L^2_{M(\nu)}(\mathbb{R}^n)$  to  $L^2_\nu(\mathbb{R}^n)$ , where  $\nu$  is a non-negative, locally integrable function and  $M$  denotes the Hardy-Littlewood maximal operator. Lerner [23] proved sharp  $L^p(\omega)$  norm inequalities for the intrinsic square function in terms of the  $A_p$  (the class of Muckenhoupt weights) characteristic of  $\omega$  for all  $1 < p < \infty$ . Liang and Yang [24] obtained the intrinsic square function characterizations of Musielak-Orlicz Hardy spaces. More applications of such intrinsic square function were also given by Wilson [39] and Wang and Liu [36].

Let  $B$  be any ball centered at  $x \in \mathbb{R}^n$  and radius of  $r > 0$ . A locally integrable function  $b$  is said to be a BMO function, if it satisfies

$$\|b\|_* := \sup_{x \in \mathbb{R}^n, r > 0} |B|^{-1} \int_B |b(y) - b_B| dy < \infty,$$

where  $b_B := |B|^{-1} \int_B b(t) dt$  and  $\|b\|_*$  is the norm in  $\text{BMO}(\mathbb{R}^n)$ . For  $b \in \text{BMO}(\mathbb{R}^n)$ ,  $[b, S_\beta]$ , the commutator of the intrinsic square function  $S_\beta(f)$ , is then defined by

$$(4) \quad [b, S_\beta](f)(x) = \left( \iint_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x) - b(z)) \phi_t(y-z) f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Recently, Wang [33] proved that if  $\omega \in A_p$  for some  $1 < p < \infty$ , then the commutator  $[b, S_\beta]$  is bounded on the weighted Lebesgue space  $L^p(\omega)$ , and is also bounded on the weighted Morrey space  $L^{p,\kappa}(\omega)$  for  $0 < \kappa < 1$ . Hu and Wang [19] established its boundedness on the classical weighted Herz spaces. Guliyev et al. [13] considered the intrinsic square function and its commutator on generalized weighted Orlicz-Morrey spaces. There are many other interesting works on this operator, among them we refer to [34] and references therein.

On the other hand, following the fundamental work of Kováčik and Rákosník [22], the variable exponent Lebesgue and some other function spaces arising in analysis such as Besov spaces and Triebel-Lizorkin spaces etc. have been intensively studied in the recent years, see [4, 6, 29, 31, 35, 42, 43] and their references. These spaces are of interest in their own right, and also have applications to fluid dynamics [30], image restoration [2] and PDE with non-standard growth conditions [15].

In many applications, a crucial step has been to prove that the classical operators, such as maximal operators, singular integrals and their commutators, are bounded in variable exponent Lebesgue spaces  $L^{p(\cdot)}(\mathbb{R}^n)$ . As shown in [5], one of the important methods used for extending the boundedness results from the weighted Lebesgue spaces  $L^p(\omega)$  to variable  $L^{p(\cdot)}(\mathbb{R}^n)$  spaces is the method of Rudio de Francia extrapolation. In Section 2 we show that the commutator  $[b, S_\beta]$  is bounded on variable  $L^{p(\cdot)}(\mathbb{R}^n)$  spaces applying a generalization of the

classical Rubio de Francia extrapolation, which generalizes the corresponding statement in [33, Theorem 3.1] to the variable exponent case.

The variable exponent Morrey spaces  $\mathcal{M}_{p(\cdot),u}$  are introduced and studied in Ho [16]. Simultaneously, he has given some sufficient conditions on  $u$  for the boundedness of fractional integrals and fractional maximal operator on such spaces. In 2017, Ho [18] established the boundedness of the vector-valued intrinsic square function on variable Morrey spaces  $\mathcal{M}_{p(\cdot),u}$ , where  $u \in \mathbb{W}_{p(\cdot)}$  (see Definition 9 below) is a Morrey weight function for  $L^{p(\cdot)}(\mathbb{R}^n)$ . Hence, it is natural to ask whether the variable exponent Morrey spaces estimates for the commutator  $[b, S_\beta]$  are still true if  $b \in \text{BMO}(\mathbb{R}^n)$ ? The main result in Section 3 is to give an affirmative answer to this question.

The variable exponent Herz spaces  $\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$  and Morrey-Herz spaces  $M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$  were recently studied by Izuki [20,21], and he also obtained some basic lemmas on generalization of the BMO norms. Under natural regularity assumptions on the exponent  $\alpha$  and  $p$ , either at the origin or at infinity, Almeida and Direhem in [1] established the boundedness of a wide class of sublinear operators including maximal, potential and Calderón-Zygmund operators on the variable exponent Herz spaces  $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ , where the exponent  $\alpha$  is variable as well. Lu and Zhu [27] further generalized some results in [1] and obtained the boundedness for such sublinear operators and their commutators on the Morrey-Herz spaces  $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$  with variable exponent  $\alpha(\cdot)$  and  $p(\cdot)$ . Motivated by the above results, in Section 4 we consider the boundedness properties of the commutator  $[b, S_\beta]$  on the variable Morrey-Herz spaces  $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ . It is worth pointing out that even in the particular case  $\alpha(\cdot) \equiv \alpha$  is constant, the main results are also new.

There are several versions of Herz-type Hardy spaces with variable exponents, see [9, 10, 37]. The variable Herz type Hardy spaces  $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ , as well as their atomic decomposition characterizations, have been studied by Dong and Xu in [9]. Using these decompositions, they proved some boundedness results for singular integral operators on these spaces. Following the idea of [9], in Section 5 we further obtain the boundedness of the commutator  $[b, S_\beta]$  from the variable Herz type Hardy spaces  $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$  to the variable Herz spaces  $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ .

In general, we define  $B := B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ .  $f_B$  denotes the integral average of  $f$  on  $B$ , namely,  $f_B = |B|^{-1} \int_B f(x) dx$ .  $p'(\cdot)$  means the conjugate exponent defined by  $1/p(\cdot) + 1/p'(\cdot) = 1$ . By  $\mathcal{S}(\mathbb{R}^n)$  and  $\mathcal{S}'(\mathbb{R}^n)$  we denote the Schwartz class and the space of tempered distributions, respectively. For  $x \in \mathbb{R}$ , we denote by  $[x]$  the largest integer less than or equal to  $x$ . The symbol  $C$  stands for a positive constant, which may vary from line to line. The expression  $f \lesssim g$  means that  $f \leq Cg$ , and  $f \approx g$  means  $f \lesssim g \lesssim f$ .

## 2. Preliminaries and lemmas

We first recall some basic lemmas and definitions on the variable exponent Lebesgue spaces, see [3, 8] for more information.

Given a measurable set  $E \subset \mathbb{R}^n$ , and a measurable function  $p(\cdot) : E \rightarrow [1, \infty)$ , let  $L^{p(\cdot)}(E)$  denote the space of all measurable functions  $f$  on  $E$  such that

$$I_{p(\cdot)}(f) := \int_E |f(x)|^{p(x)} dx < \infty.$$

This is a Banach space with respect to the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\{\mu > 0 : I_{p(\cdot)}(f/\mu) \leq 1\}.$$

The space  $L_{\text{loc}}^{p(\cdot)}(E)$  is defined by

$$L_{\text{loc}}^{p(\cdot)}(E) = \{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset E\}.$$

For brevity, we denote

$$p_E^- := \text{ess inf}_{x \in E} p(x), \quad p_E^+ := \text{ess sup}_{x \in E} p(x), \quad p_- := p_{\mathbb{R}^n}^-, \quad p_+ := p_{\mathbb{R}^n}^+.$$

Using this notation, we define

$$\mathcal{P}(E) := \{p(\cdot) : E \rightarrow [1, \infty) : p_E^- > 1 \text{ and } p_E^+ < \infty\}$$

and

$$\mathcal{B}(E) := \{p(\cdot) \in \mathcal{P}(E) : M \text{ is bounded on } L^{p(\cdot)}(E)\},$$

where  $M$  is the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r) \cap E} |f(y)| dy.$$

When  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ ,  $f \in L^{p(\cdot)}(\mathbb{R}^n)$  and  $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ , the generalized Hölder inequality holds in the form

$$(5) \quad \int_{\mathbb{R}^n} |f(x)g(x)| dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

with  $r_p = 1 + 1/p_- - 1/p_+$ , see [22, Theorem 2.1].

Diening's characterization of variable  $L^{p(\cdot)}(\mathbb{R}^n)$  spaces on which the maximal operator is bounded has the following important consequence, see [7, Theorem 8.1].

**Lemma 1.** *Let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ . Then the following conditions are equivalent:*

- (a)  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ .
- (b)  $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ .
- (c)  $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$  for some  $1 < q < p_-$ .
- (d)  $(p(\cdot)/q)'$   $\in \mathcal{B}(\mathbb{R}^n)$  for some  $1 < q < p_-$ .

Let  $\omega$  be a weight function on  $\mathbb{R}^n$ , that is,  $\omega$  is real-valued, non-negative and locally integrable. For  $1 < p < \infty$ , we say that  $\omega$  is an  $A_p$  weight if

$$\sup_B \left( \frac{1}{|B|} \int_B \omega(x) dx \right) \left( \frac{1}{|B|} \int_B \omega(x)^{1-p'} dx \right)^{p-1} < \infty.$$

The following is a direct generalization of the classical Rubio de Francia extrapolation theorem, see [5, Corollary 1.11].

**Lemma 2.** *Given a family  $\mathcal{F}$  and an open set  $\Omega \subset \mathbb{R}^n$ , assume that for some  $1 < p_0 < \infty$ , and for every  $\omega \in A_{p_0}$ ,*

$$\int_{\Omega} f(x)^{p_0} \omega(x) dx \leq C_0 \int_{\Omega} g(x)^{p_0} \omega(x) dx, \quad (f, g) \in \mathcal{F}.$$

*Let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  be such that there exists  $1 < p_1 < p_-$  with  $(p(\cdot)/p_1)' \in \mathcal{B}(\Omega)$ . Then for all  $(f, g) \in \mathcal{F}$  such that  $f \in L^{p(\cdot)}(\Omega)$*

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leq C \|g\|_{L^{p(\cdot)}(\Omega)}.$$

In [33, Theorem 3.1] it was shown that for all  $1 < p < \infty$  and all  $\omega \in A_p$ ,

$$\int_{\mathbb{R}^n} |[b, S_{\beta}]f(x)|^p \omega(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx.$$

Thus, we can apply Lemmas 1 and 2 to get the following.

**Theorem 3.** *Let  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . Then we have for all  $f \in L^{p(\cdot)}(\mathbb{R}^n)$  and  $b \in \text{BMO}(\mathbb{R}^n)$ ,*

$$\|[b, S_{\beta}]f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

The following Lemmas 4, 5 and 7 are due to Izuki [20] (see also Diening [7]), and Lemma 8 comes from Almeida and Drihem [1, Lemma 2.1].

**Lemma 4.** *Let  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . Then we have*

$$|B|^{-1} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

**Lemma 5.** *Let  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . Then we have for all measurable subsets  $E \subset B$ ,*

$$\frac{\|\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|E|}{|B|} \right)^{\delta_1}, \quad \frac{\|\chi_E\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|E|}{|B|} \right)^{\delta_2},$$

where  $\delta_1, \delta_2$  are constants with  $0 < \delta_1, \delta_2 < 1$ .

*Remark 6.* We would like to stress that everywhere below, the constants  $\delta_1$  and  $\delta_2$  are always the same as in Lemma 5.

**Lemma 7.** *If  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ ,  $b \in \text{BMO}(\mathbb{R}^n)$ ,  $k > j$  ( $k, j \in \mathbb{N}$ ), then we have*

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \|b\|_*,$$

$$\|(b - b_{B_j})\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(k - j) \|b\|_* \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where  $B_k = B(0, 2^k)$  and  $\chi_{B_k}$  is the characteristic function of  $B_k$  for  $k \in \mathbb{Z}$ .

A function  $\alpha(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$  is called log-Hölder continuous at the origin (or has a log decay at the origin), if there exists a constant  $C_{\log} > 0$  such that

$$|\alpha(x) - \alpha(0)| \leq \frac{C_{\log}}{\log(e + 1/|x|)}, \quad x \in \mathbb{R}^n.$$

If, for some  $\alpha_{\infty} \in \mathbb{R}$  and  $C_{\log} > 0$ , there holds

$$|\alpha(x) - \alpha_{\infty}| \leq \frac{C_{\log}}{\log(e + |x|)}, \quad x \in \mathbb{R}^n,$$

then  $\alpha(\cdot)$  is called log-Hölder continuous at infinity (or has a log decay at the infinity).

**Lemma 8.** *Let  $r_1, r_2 > 0$  and  $\alpha(\cdot) \in L^{\infty}(\mathbb{R}^n)$  be log-Hölder continuous both at origin and at infinity. Then we have*

$$r_1^{\alpha(x)} \leq Cr_2^{\alpha(y)} \times \begin{cases} \left(\frac{r_1}{r_2}\right)^{\alpha_+}, & 0 < r_2 \leq r_1/2, \\ 1, & r_1/2 < r_2 \leq 2r_1, \\ \left(\frac{r_1}{r_2}\right)^{\alpha_-}, & r_2 > 2r_1, \end{cases}$$

for any  $x \in B(0, r_1) \setminus B(0, r_1/2)$  and  $y \in B(0, r_2) \setminus B(0, r_2/2)$ .

### 3. Boundedness on variable exponent Morrey spaces

We first recall the following definitions given by Ho in [16].

**Definition 9.** Let  $p(\cdot) : \mathbb{R}^n \rightarrow (1, \infty)$  with  $1 < p_- \leq p_+ < \infty$ . A Lebesgue measurable function  $u(x, r) : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+$  is said to be a Morrey weight function for  $L^{p(\cdot)}(\mathbb{R}^n)$  if, for any  $x \in \mathbb{R}^n$  and  $r > 0$ ,

$$(6) \quad \sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(x, 2^{j+1}r) < Cu(x, r).$$

We denote the class of Morrey weight functions by  $\mathbb{W}_{p(\cdot)}$ .

**Definition 10.** Let  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  and  $u \in \mathbb{W}_{p(\cdot)}$ . The variable exponent Morrey space  $\mathcal{M}_{p(\cdot), u}$  is the collection of all Lebesgue measurable functions  $f$  satisfying

$$\|f\|_{\mathcal{M}_{p(\cdot), u}} = \sup_{x \in \mathbb{R}^n, R > 0} \frac{1}{u(x, R)} \|f\chi_{B(x,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

We note that condition (6) is introduced by Ho in [16] to generalize the well known Hardy-Littlewood-Sobolev theorem to the case of variable exponent Morrey spaces on unbounded domains, and is also used to show the Fefferman-Stein vector-valued maximal inequalities for weighted Morrey spaces, see [17].

For any  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ , let  $\mathcal{K}_{p(\cdot)}$  denote the supremum of those  $q > 1$  such that  $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$  and  $\mathcal{E}_{p(\cdot)}$  be the conjugate of  $\mathcal{K}_{p(\cdot)}$ . The following result can be seen as a special case of the general result in [17] for Banach function spaces.

**Proposition 11.** *Let  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . For any  $1 < q < \mathcal{K}_{p(\cdot)}$  and  $1 < \tau < \mathcal{K}_{p'(\cdot)}$ , there exist constants  $C_1, C_2 > 0$  such that for any  $x \in \mathbb{R}^n$  and  $r > 0$ ,*

$$C_1 2^{jn(1-\frac{1}{\tau})} \leq \frac{\|\chi_{B(x,2^j r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C_2 2^{\frac{jn}{q}}, \quad \forall j \in \mathbb{N}.$$

Now, let us state the main result in this section.

**Theorem 12.** *Suppose  $0 < \beta \leq 1$  and  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . If there exists a constant  $C > 0$  such that for any  $x \in \mathbb{R}^n$  and  $r > 0$ ,  $u$  fulfills*

$$(7) \quad \sum_{j=0}^{\infty} (j+1) \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(x, 2^{j+1}r) < Cu(x, r),$$

then we have for all  $f \in \mathcal{M}_{p(\cdot),u}$  and  $b \in \text{BMO}(\mathbb{R}^n)$ ,

$$\|[b, S_\beta](f)\|_{\mathcal{M}_{p(\cdot),u}} \leq C \|b\|_* \|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

*Remark 13.* Note that if  $u \equiv 1$ , then the variable exponent Morrey spaces  $\mathcal{M}_{p(\cdot),u}$  reduce to the variable exponent Lebesgue spaces  $L^{p(\cdot)}(\mathbb{R}^n)$ . Therefore, the above theorem is a generalization of Theorem 3.

*Remark 14.* Condition (7) is satisfied by a number of Lebesgue measurable functions. For instance, if for any  $0 \leq \gamma < 1/\mathcal{E}_{p(\cdot)}$ , a weight function  $u$  satisfies  $u(x, 2r) \leq 2^{n\gamma}u(x, r)$  for any  $x \in \mathbb{R}^n$  and  $r > 0$ , then (7) holds. In fact, for any  $\gamma < 1/\mathcal{E}_{p(\cdot)}$ , there always exists a  $\tau < 1/\mathcal{K}_{p'(\cdot)}$  such that  $\gamma < 1 - 1/\tau < 1 - 1/\mathcal{K}_{p'(\cdot)} = 1/\mathcal{E}_{p(\cdot)}$ . An application of Proposition 11 gives

$$\sum_{j=0}^{\infty} (j+1) \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{u(x, 2^{j+1}r)}{u(x, r)} < C \sum_{j=0}^{\infty} (j+1) 2^{jn(\frac{1}{\tau} + \gamma - 1)} < C.$$

To be more precise, the weight function  $u(x, r) = r^{\gamma(x)}$ ,  $0 \leq \gamma(x) \leq \gamma_+ < 1/\mathcal{E}_{p(\cdot)}$ , satisfies condition (7). In addition, it is easy to check that condition (7) together with Proposition 11 yields  $u(x, 2r) \leq Cu(x, r)$  for any  $x \in \mathbb{R}^n$  and  $r > 0$ .

*Proof of Theorem 12.* Let  $f \in \mathcal{M}_{p(\cdot),u}$ . For any  $x_0 \in \mathbb{R}^n$  and  $r > 0$ , we decompose  $f = g + h$ , where  $g = f\chi_{B(x_0, 2r)}$  and  $h = \sum_{j=1}^{\infty} f\chi_{B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)}$ . Then we have

$$\begin{aligned} & \frac{1}{u(x_0, r)} \|\chi_{B(x_0, r)} [b, S_\beta](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ & \leq \frac{1}{u(x_0, r)} \|\chi_{B(x_0, r)} [b, S_\beta](g)\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \frac{1}{u(x_0, r)} \|\chi_{B(x_0, r)} [b, S_\beta](h)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ & := U + V. \end{aligned}$$

For  $U$ , using  $u(x_0, 2r) \leq Cu(x_0, r)$  and the  $L^{p(\cdot)}(\mathbb{R}^n)$ -boundedness of  $[b, S_\beta]$ , we obtain

$$U \leq C \|b\|_* \frac{1}{u(x_0, 2r)} \|f\chi_{B(x_0, 2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\begin{aligned} &\leq C \|b\|_* \sup_{x_0 \in \mathbb{R}^n, R > 0} \frac{1}{u(x_0, R)} \|f \chi_{B(x_0, R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C \|b\|_* \|f\|_{\mathcal{M}_{p(\cdot), u}}. \end{aligned}$$

We now turn to estimate  $V$ . For any  $0 < \beta \leq 1$ ,  $\phi \in \mathcal{C}_\beta$  and  $(y, t) \in \Gamma(x)$ , we have

$$(8) \quad \left| \int_{\mathbb{R}^n} (b(x) - b(z)) \phi_t(y - z) h(z) dz \right| \leq Ct^{-n} \sum_{j=1}^{\infty} \int_{\widetilde{R}_j \cap \{z: |y-z| \leq t\}} |b(x) - b(z)| |f(z)| dz,$$

where  $\widetilde{R}_j := B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)$ .

Note that if  $x \in B(x_0, r)$ ,  $(y, t) \in \Gamma(x)$  and  $z \in \widetilde{R}_j \cap \{z: |y-z| \leq t\}$ , then

$$(9) \quad t \geq \frac{1}{2}(|x-y| + |y-z|) \geq \frac{1}{2}|x-z| \geq \frac{1}{2}(|z-x_0| - |x-x_0|) > 2^{j-2}r.$$

Thus, from (8), (9) and the Minkowski inequality, we get

$$\begin{aligned} &|[b, S_\beta](h)(x)| \\ &= \left( \int_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x) - b(z)) \phi_t(y - z) h(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} \\ &\leq C \left( \int_{2^{j-2}r}^{\infty} \int_{|x-y| < t} \left| t^{-n} \sum_{j=1}^{\infty} \int_{\widetilde{R}_j \cap \{z: |y-z| \leq t\}} |b(x) - b(z)| |f(z)| dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} \\ &\leq C \sum_{j=1}^{\infty} \frac{1}{(2^{j+1}r)^n} \int_{B(x_0, 2^{j+1}r)} |b(x) - b(z)| |f(z)| dz \\ &\leq C \sum_{j=1}^{\infty} \frac{1}{(2^{j+1}r)^n} |b(x) - b_{B(x_0, r)}| \int_{B(x_0, 2^{j+1}r)} |f(z)| dz \\ &\quad + C \sum_{j=1}^{\infty} \frac{1}{(2^{j+1}r)^n} \int_{B(x_0, 2^{j+1}r)} |b(z) - b_{B(x_0, r)}| |f(z)| dz \\ &:= V_1 + V_2. \end{aligned}$$

For  $V_1$ , the generalized Hölder inequality implies that

$$V_1 \leq C \sum_{j=1}^{\infty} \frac{1}{(2^{j+1}r)^n} |b(x) - b_{B(x_0, r)}| \|f \chi_{B(x_0, 2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B(x_0, 2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

For  $V_2$ , noting that  $|b_{B(x_0, 2^{j+1}r)} - b_{B(x_0, r)}| \leq C(j+1)\|b\|_*$  (see [32, Page 206]), we apply Lemma 7 with  $B = B(x_0, 2^{j+1}r)$  and obtain

$$V_2 \leq C \sum_{j=1}^{\infty} \frac{1}{(2^{j+1}r)^n} |b_{B(x_0, 2^{j+1}r)} - b_{B(x_0, r)}| \int_{B(x_0, 2^{j+1}r)} |f(z)| dz$$



$$\begin{aligned}
& + C \sum_{j=1}^{\infty} \frac{1}{(2^{j+1}r)^n} \int_{B(x_0, 2^{j+1}r)} |b(z) - b_{B(x_0, 2^{j+1}r)}| |f(z)| dz \\
& \leq C \|b\|_* \sum_{j=1}^{\infty} (j+1) \frac{1}{(2^{j+1}r)^n} \|f \chi_{B(x_0, 2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B(x_0, 2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
& \quad + C \sum_{j=1}^{\infty} \frac{1}{(2^{j+1}r)^n} \|f \chi_{B(x_0, 2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|(b - b_{B(x_0, 2^{j+1}r)}) \chi_{B(x_0, 2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
& \leq C \|b\|_* \sum_{j=1}^{\infty} (j+1) \frac{1}{(2^{j+1}r)^n} \|f \chi_{B(x_0, 2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B(x_0, 2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Combining the estimates of  $V_1$  and  $V_2$ , by Lemma 4, we get

$$\begin{aligned}
& \|\chi_{B(x_0, r)} [b, S_\beta](h)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& \leq C \|b\|_* \sum_{j=1}^{\infty} (j+1) \frac{1}{(2^{j+1}r)^n} \|\chi_{B(x_0, r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|f \chi_{B(x_0, 2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& \quad \|\chi_{B(x_0, 2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
& \leq C \|b\|_* \sum_{j=1}^{\infty} (j+1) \frac{\|\chi_{B(x_0, r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x_0, 2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(x_0, 2^{j+1}r) \\
& \quad \times \sup_{x_0 \in \mathbb{R}^n, R > 0} \frac{1}{u(x_0, R)} \|f \chi_{B(x_0, R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& \leq C \|b\|_* \|f\|_{\mathcal{M}_{p(\cdot), u}} \sum_{j=1}^{\infty} (j+1) \frac{\|\chi_{B(x_0, r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x_0, 2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(x_0, 2^{j+1}r).
\end{aligned}$$

Thus, in view of the condition (7), we arrive at the estimate

$$\begin{aligned}
V & = \frac{1}{u(x_0, r)} \|\chi_{B(x_0, r)} [b, S_\beta](h)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& \leq C \|b\|_* \|f\|_{\mathcal{M}_{p(\cdot), u}} \sum_{j=1}^{\infty} (j+1) \frac{\|\chi_{B(x_0, r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x_0, 2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{u(x_0, 2^{j+1}r)}{u(x_0, r)} \\
& \leq C \|b\|_* \|f\|_{\mathcal{M}_{p(\cdot), u}}.
\end{aligned}$$

This completes the proof of Theorem 12.  $\square$

#### 4. Boundedness on variable exponent Morrey-Herz spaces

Let  $B_l = \{x \in \mathbb{R}^n : |x| \leq 2^l\}$ ,  $R_l = B_l \setminus B_{l-1}$  and  $\chi_l = \chi_{R_l}$  be the characteristic function of the set  $R_l$  for  $l \in \mathbb{Z}$ .

**Definition 15.** Let  $0 < q \leq \infty$ ,  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $\alpha(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ . The homogeneous Herz space  $\dot{K}_{p(\cdot)}^{\alpha(\cdot), q}(\mathbb{R}^n)$  is defined as the

class of all  $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$  such that

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} := \left( \sum_{k \in \mathbb{Z}} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} < \infty$$

with the usual modification when  $q = \infty$ .

The next proposition was initially proved by Almeida and Drihem in [1, Page 785].

**Proposition 16.** *Let  $0 < q \leq \infty$ ,  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ . If  $\alpha(\cdot)$  is log-Hölder continuous both at the origin and at infinity, then*

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} \approx \left( \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} + \left( \sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \|f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{\frac{1}{q}}.$$

**Definition 17.** Let  $0 \leq \lambda < \infty$ ,  $0 < q \leq \infty$ ,  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $\alpha(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ . The homogeneous Morrey-Herz space  $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$  is defined as the class of all  $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$  such that

$$\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} := \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} < \infty,$$

with the usual modification when  $q = \infty$ .

*Remark 18.* It obviously follows that  $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),0}(\mathbb{R}^n) = \dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ . If  $\alpha(\cdot) \equiv \alpha$ ,  $p(\cdot) \equiv p$  are constant, then  $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$  coincides with the classical Morrey-Herz spaces  $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$  defined by Lu and Xu in [25].

Similar to Proposition 16, Lu and Zhu [27] obtained the following result.

**Proposition 19.** *Let  $0 \leq \lambda < \infty$ ,  $0 < q \leq \infty$ ,  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ . If  $\alpha(\cdot)$  is log-Hölder continuous both at origin and at infinity, then*

$$\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \approx \max \left\{ \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \|f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q}, \right. \\ \left. \sup_{k_0 \geq 0, k_0 \in \mathbb{Z}} \left[ 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} \right. \right. \\ \left. \left. + 2^{-k_0\lambda} \left( \sum_{k=0}^{k_0} 2^{k\alpha_\infty q} \|f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} \right] \right\}.$$

The main results in this section can be summarized as follows.

**Theorem 20.** *Suppose  $0 < \beta \leq 1$  and  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . Let  $\lambda > 0$ ,  $0 < q \leq \infty$  and  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$  be log-Hölder continuous both at the origin and at infinity, such that*

$$\lambda - n\delta_1 < \alpha_- \leq \alpha_+ < n\delta_2,$$

where  $0 < \delta_1, \delta_2 < 1$  are the constants appearing in Lemma 5. Then we have for all  $f \in MK_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$  and  $b \in \text{BMO}(\mathbb{R}^n)$ ,

$$\|[b, S_\beta]f\|_{MK_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \leq C \|b\|_* \|f\|_{MK_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}.$$

In fact, Theorem 20 remains valid for  $\lambda = 0$ , namely, in the framework of variable exponent Herz spaces. Using the same method of proving Theorem 20, we get the following.

**Corollary 21.** *Suppose  $0 < \beta \leq 1$  and  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . Let  $0 < q \leq \infty$  and  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$  be log-Hölder continuous both at the origin and at infinity, such that*

$$-n\delta_1 < \alpha_- \leq \alpha_+ < n\delta_2,$$

where  $0 < \delta_1, \delta_2 < 1$  are the constants appearing in Lemma 5. Then we have for all  $f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$  and  $b \in \text{BMO}(\mathbb{R}^n)$ ,

$$\|[b, S_\beta]f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} \leq C \|b\|_* \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}.$$

*Proof of Theorem 20.* Without loss of generality, we may assume that  $\|b\|_* = 1$ .

Let  $f \in MK_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ . We decompose

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

The Minkowski inequality implies that

$$\begin{aligned} & \|[b, S_\beta](f)\|_{MK_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q \\ &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} |[b, S_\beta](f)| \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left( \sum_{j=-\infty}^{k-2} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ &\quad + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left( \sum_{j=k-1}^{k+1} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ &\quad + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left( \sum_{j=k+2}^{\infty} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ &:= E + F + G. \end{aligned}$$

We treat  $F$  first. By Proposition 19 and the  $L^{p(\cdot)}(\mathbb{R}^n)$ -boundedness of the commutator  $[b, S_\beta]$ , we get

$$F \approx \max \left\{ \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left\| \left( \sum_{j=k-1}^{k+1} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q, \right.$$

$$\begin{aligned}
& \sup_{k_0 \geq 0, k_0 \in \mathbb{Z}} \left[ 2^{-k_0 \lambda q} \sum_{k=-\infty}^{-1} 2^{k \alpha(0) q} \left\| \left( \sum_{j=k-1}^{k+1} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right. \\
& \quad \left. + 2^{-k_0 \lambda q} \sum_{k=0}^{k_0} 2^{k \alpha_\infty q} \left\| \left( \sum_{j=k-1}^{k+1} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right] \\
& \leq C \max \left\{ \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k \alpha(0)} |f \chi_k| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q, \right. \\
& \quad \left. \sup_{k_0 \geq 0, k_0 \in \mathbb{Z}} \left[ 2^{-k_0 \lambda q} \sum_{k=-\infty}^{-1} \left\| 2^{k \alpha(0)} |f \chi_k| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right. \right. \\
& \quad \left. \left. + 2^{-k_0 \lambda q} \sum_{k=0}^{k_0} \left\| 2^{k \alpha_\infty} |f \chi_k| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right] \right\} \\
& \leq C \|f\|_{MK_{q, p(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^q.
\end{aligned}$$

For  $E$ , noticing that  $x \in R_k$ ,  $(y, t) \in \Gamma(x)$  and  $z \in R_j \cap \{z : |y - z| \leq t\}$  with  $j \leq k - 2$ , then we have

$$(10) \quad t \geq \frac{1}{2}(|x - y| + |y - z|) \geq \frac{1}{2}|x - z| \geq \frac{1}{2}(|x| - |z|) \geq \frac{1}{4}|x|.$$

Thus, we obtain

$$\begin{aligned}
(11) \quad & |[b, S_\beta](f_j)(x)| \\
& = \left( \int \int_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x) - b(z)) \phi_t(y - z) f_j(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} \\
& \leq C \left( \int_{\frac{|x|}{4}}^\infty \int_{|x-y| < t} \left| t^{-n} \int_{R_j \cap \{z : |y-z| \leq t\}} (b(x) - b(z)) f_j(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} \\
& \leq C \left( \int_{R_j} |b(x) - b(z)| |f_j(z)| dz \right) \left( \int_{\frac{|x|}{4}}^\infty \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} \\
& \leq C 2^{-kn} \int_{R_j} |b(x) - b(z)| |f_j(z)| dz.
\end{aligned}$$

This together with Lemma 8 yields

$$\begin{aligned}
(12) \quad & 2^{k \alpha(x)} \sum_{j=-\infty}^{k-2} |[b, S_\beta](f_j)(x)| \chi_k(x) \\
& \leq C \sum_{j=-\infty}^{k-2} 2^{-kn} \int_{R_j} 2^{k \alpha(x)} |b(x) - b(z)| |f_j(z)| dz \cdot \chi_k(x)
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha_+ - kn} \int_{R_j} 2^{j\alpha(z)} |b(x) - b(z)| |f_j(z)| dz \cdot \chi_k(x) \\
&\leq C \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha_+ - kn} \left( |b(x) - b_{B_j}| \int_{R_j} 2^{j\alpha(z)} |f_j(z)| dz \right. \\
&\quad \left. + \int_{R_j} 2^{j\alpha(z)} |b_{B_j} - b(z)| |f_j(z)| dz \right) \cdot \chi_k(x).
\end{aligned}$$

Then, from (11), (5), Lemmas 4, 5 and 7, we deduce that

$$\begin{aligned}
&\left\| 2^{k\alpha(\cdot)} \sum_{j=-\infty}^{k-2} |[b, S_\beta](f_j)| \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq C \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha_+ - kn} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left( \|(b - b_{B_j})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \right. \\
&\quad \left. + \|(b - b_{B_j})\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right) \\
&\leq C \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha_+ - kn} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \times \left( (k-j) \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right) \\
&\leq C \sum_{j=-\infty}^{k-2} (k-j) 2^{(k-j)\alpha_+ - kn} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq C \sum_{j=-\infty}^{k-2} (k-j) 2^{(k-j)\alpha_+} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \\
&\leq C \sum_{j=-\infty}^{k-2} (k-j) 2^{(j-k)(n\delta_2 - \alpha_+)} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, we arrive at the estimate

$$\begin{aligned}
E &\approx \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left( \sum_{j=-\infty}^{k-2} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\
&\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \left( \sum_{j=-\infty}^{k-2} (k-j) 2^{(j-k)(n\delta_2 - \alpha_+)} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q.
\end{aligned}$$

Now we can distinguish two cases as follows:

Case 1°: If  $0 < q \leq 1$ , using the well-known inequality

$$(13) \quad \left( \sum_{j=1}^{\infty} \theta_j \right)^q \leq \sum_{j=1}^{\infty} \theta_j^q, \quad (\theta_j > 0, j = 1, 2, \dots),$$

we obtain

$$\begin{aligned} E &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \sum_{j=-\infty}^{k-2} (k-j)^q 2^{(j-k)(n\delta_2 - \alpha_+)q} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{j=-\infty}^{k_0-2} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=j+2}^{k_0} (k-j)^q 2^{(j-k)(n\delta_2 - \alpha_+)q} \\ &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{j=-\infty}^{k_0-2} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ &\leq C \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q. \end{aligned}$$

Case 2°: If  $1 < q < \infty$ , the Hölder inequality implies that

$$\begin{aligned} E &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \left( \sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2 - \alpha_+)q/2} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right) \\ &\quad \times \left( \sum_{j=-\infty}^{k-2} (k-j)^{q'} 2^{(j-k)(n\delta_2 - \alpha_+)q'/2} \right)^{q/q'} \\ &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{j=-\infty}^{k_0-2} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=j+2}^{k_0} 2^{(j-k)(n\delta_2 - \alpha_+)q/2} \\ &\leq C \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q. \end{aligned}$$

We proceed now to estimate  $G$ . Once again by Proposition 19, we have

$$G \approx \max\{H, J\},$$

where

$$\begin{aligned} H &= \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left\| \left( \sum_{j=k+2}^{\infty} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q, \\ J &= \sup_{k_0 \geq 0, k_0 \in \mathbb{Z}} \left\{ 2^{-k_0 \lambda q} \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left\| \left( \sum_{j=k+2}^{\infty} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right. \\ &\quad \left. + 2^{-k_0 \lambda q} \sum_{k=0}^{k_0} 2^{k\alpha_\infty q} \left\| \left( \sum_{j=k+2}^{\infty} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}. \end{aligned}$$

For  $H$ , noticing that  $x \in R_k$ ,  $(y, t) \in \Gamma(x)$  and  $z \in R_j \cap \{z : |y - z| \leq t\}$  with  $j \geq k + 2$ , then we have

$$(14) \quad t \geq \frac{1}{2}(|x - y| + |y - z|) \geq \frac{1}{2}|x - z| \geq \frac{1}{2}(|z| - |x|) \geq \frac{1}{4}|z|.$$

Thus, arguing as in (11), we get

$$(15) \quad \begin{aligned} & |[b, S_\beta](f_j)(x)| \\ & \leq C \left( \int_{\frac{|z|}{4}}^{\infty} \int_{|x-y|<t} \left| t^{-n} \int_{R_j \cap \{z:|y-z|\leq t\}} (b(x) - b(z)) f_j(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} \\ & \leq C \left( \int_{R_j} |b(x) - b(z)| |f_j(z)| dz \right) \left( \int_{\frac{|z|}{4}}^{\infty} \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} \\ & \leq C 2^{-jn} \int_{R_j} |b(x) - b(z)| |f_j(z)| dz \\ & \leq C 2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left( \|b(x) - b_{B_k}\| \|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} + \|(b - b_{B_k})\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \right). \end{aligned}$$

We now obtain from (15), Lemmas 4, 5 and 7 that

$$\begin{aligned} & \|[b, S_\beta](f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ & \leq C 2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left( \|(b - b_{B_k})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \right. \\ & \quad \left. + \|(b - b_{B_k})\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right) \\ & \leq C(j - k) 2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ & \leq C(j - k) \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\ & \leq C(j - k) 2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} H & = \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left\| \left( \sum_{j=k+2}^{\infty} |[b, S_\beta](f_j)| \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ & \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left( \sum_{j=k+2}^{\infty} (j - k) 2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q. \end{aligned}$$

If  $0 < q \leq 1$ , by the inequality (13), we get

$$H \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \sum_{j=k+2}^{k_0-1} (j - k)^q 2^{(k-j)n\delta_1 q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q$$

$$\begin{aligned}
& + \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} 2^{k \alpha(0) q} \sum_{j=k_0}^{\infty} (j-k)^q 2^{(k-j)n\delta_1 q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\
& := H_1 + H_2.
\end{aligned}$$

For  $H_1$ , noting that  $n\delta_1 + \alpha(0) > n\delta_1 + \alpha_- > 0$ , hence we obtain

$$\begin{aligned}
H_1 & \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{j=-\infty}^{k_0-1} 2^{j \alpha(0) q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=-\infty}^{j-2} (j-k)^q 2^{(k-j)(n\delta_1 + \alpha(0)) q} \\
& \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{j=-\infty}^{k_0-1} 2^{j \alpha(0) q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \leq C \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^q.
\end{aligned}$$

For  $H_2$ , in view of  $\alpha(0) + n\delta_1 - \lambda > \alpha_- + n\delta_1 - \lambda > 0$ , we get

$$\begin{aligned}
H_2 & \approx \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} (j-k)^q 2^{(k-j)(n\delta_1 + \alpha(0)) q} 2^{j \alpha(0) q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\
& \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} (j-k)^q 2^{(k-j)(n\delta_1 + \alpha(0)) q} 2^{j \lambda q} \\
& \quad \times 2^{-j \lambda q} \sum_{l=-\infty}^j 2^{l \alpha(0) q} \|f_l\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\
& \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} 2^{k \lambda q} \sum_{j=k_0}^{\infty} (j-k)^q 2^{(k-j)(n\delta_1 + \alpha(0) - \lambda) q} \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^q \\
& \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \left( \sum_{k=-\infty}^{k_0} 2^{k \lambda q} \right) \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^q \leq C \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^q.
\end{aligned}$$

If  $1 < q < \infty$ , we have

$$\begin{aligned}
H & \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} 2^{k \alpha(0) q} \left( \sum_{j=k+2}^{k_0-1} (j-k) 2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
& \quad + \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} 2^{k \alpha(0) q} \left( \sum_{j=k_0}^{\infty} (j-k) 2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
& := H_3 + H_4.
\end{aligned}$$

For  $H_3$ , an application of Hölder's inequality gives

$$\begin{aligned}
H_3 & \approx \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \left( \sum_{j=k+2}^{k_0-1} (j-k) 2^{(k-j)(n\delta_1 + \alpha(0))} 2^{j \alpha(0)} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
& \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \left( \sum_{j=k+2}^{k_0-1} 2^{(k-j)(n\delta_1 + \alpha(0)) q} 2^{j \alpha(0) q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)
\end{aligned}$$



$$\begin{aligned}
& \times \left( \sum_{j=k+2}^{k_0-1} (j-k)^{q'} 2^{(k-j)(n\delta_1+\alpha(0))q'/2} \right)^{q/q'} \\
& \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\alpha(0))q/2} \\
& \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \leq C \|f\|_{MK_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q.
\end{aligned}$$

For  $H_4$ , as argued for  $H_2$ , we conclude that

$$\begin{aligned}
H_4 & \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left( \sum_{j=k_0}^{\infty} 2^{(k-j)(n\delta_1+\alpha(0)+\lambda)q/2} 2^{j\alpha(0)q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right) \\
& \quad \times \left( \sum_{j=k_0}^{\infty} (j-k)^{q'} 2^{(k-j)(n\delta_1+\alpha(0)-\lambda)q'/2} \right)^{q/q'} \\
& \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left( \sum_{j=k_0}^{\infty} 2^{(k-j)(n\delta_1+\alpha(0)+\lambda)q/2} 2^{j\lambda q} 2^{-j\lambda q} \sum_{l=-\infty}^j 2^{l\alpha(0)q} \|f_l\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right) \\
& \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\lambda q} \left( \sum_{j=k_0}^{\infty} 2^{(k-j)(n\delta_1+\alpha(0)-\lambda)q/2} \right) \|f\|_{MK_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q \\
& \leq C \sup_{k_0 < 0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda q} \left( \sum_{k=-\infty}^{k_0} 2^{k\lambda q} \right) \|f\|_{MK_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q \leq C \|f\|_{MK_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q.
\end{aligned}$$

We omit the estimation of  $J$  since it is essentially similar to that of  $H$ . Consequently, the proof of Theorem 20 is complete.  $\square$

## 5. Boundedness on variable exponent Herz-type Hardy spaces

Note that in Corollary 21 for the range of  $\alpha$ , we have the restriction  $\alpha < n\delta_2$ . It is of interest to ask what will happen if  $\alpha \geq n\delta_2$ . The main aim of this section is to further consider the mapping properties of the commutator  $[b, S_\beta]$  in this situation.

Let  $G_N f$  be the grand maximal function of  $f$  defined by

$$G_N(f)(x) := \sup_{\phi \in \mathcal{A}_N} |\phi_{\nabla}^*(f)(x)|, \quad \forall x \in \mathbb{R}^n,$$

where

$$\mathcal{A}_N = \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha| \leq N, |\beta| \leq N} |x^\alpha D^\beta \phi(x)| \leq 1 \right\}, \quad N > n + 1,$$

and  $\phi_{\nabla}^*(f)(x) = \sup_{|y-x| < t} |\phi_t * f(y)|$  with  $\phi_t(x) = t^{-n} \phi(\frac{x}{t})$ .

The grand maximal operator  $G_N$  was first introduced by Fefferman and Stein in [12] to study the classical Hardy spaces. For an extensive study of the Hardy

spaces and the variable exponent cases, see, for example, [14, 28, 40, 41, 44, 45] and their references.

**Definition 22.** Let  $0 < q \leq \infty$ ,  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ ,  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$  and  $N > n + 1$ . The homogeneous Herz-type Hardy spaces  $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$  is defined by

$$H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n) \right\}$$

and  $\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} = \|G_N f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}$ .

**Definition 23.** Let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$  be log-Hölder continuous both at the origin and at infinity. Suppose  $b \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n)$  and non-negative integer  $s \geq [\alpha_r - n\delta_2]$ , where  $n\delta_2 \leq \alpha_r < \infty$  and  $\delta_2$  is the constant as in Lemma 5. Here  $\alpha_r = \alpha(0)$  if  $0 < r < 1$  and  $\alpha_r = \alpha_\infty$  if  $r \geq 1$ . A function  $a(\cdot)$  is said to be a central  $(\alpha(\cdot), p(\cdot), s; b)$ -atom if

- (1)  $\text{supp} a \subset B(0, r) := \{x \in \mathbb{R}^n : |x| < r\}$ .
- (2)  $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq |B(0, r)|^{-\frac{\alpha(0)}{n}}$ ,  $0 < r < 1$ .
- (3)  $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq |B(0, r)|^{-\frac{\alpha_\infty}{n}}$ ,  $r \geq 1$ .
- (4)  $\int_{\mathbb{R}^n} a(x)x^\beta dx = \int_{\mathbb{R}^n} a(x)b(x)x^\beta dx = 0$ ,  $|\beta| \leq s$ .

Obviously, if  $p(\cdot) \equiv p$ ,  $\alpha(\cdot) \equiv \alpha$  are constant, then taking  $\delta_2 = 1 - \frac{1}{p}$  we can get the classical case, see [26]. Similar to [9, Theorem 2.1], we have the following characterizations of the spaces  $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$  in terms of central atomic decompositions.

**Proposition 24.** Let  $0 < q < \infty$ ,  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ ,  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$  be log-Hölder continuous both at the origin and at infinity. Suppose  $n\delta_2 \leq \alpha(0)$ ,  $\alpha_\infty < \infty$ , where  $\delta_2$  is the constant appearing in Lemma 5. Then  $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$  if and only if

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \text{ in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each  $a_k$  is a central  $(\alpha(\cdot), p(\cdot), s; b)$ -atom with  $\text{supp} a_k \subset B_k$  and  $\sum_{k=-\infty}^{\infty} |\lambda_k|^q < \infty$ . Moreover,

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} \approx \inf \left( \sum_{k=-\infty}^{\infty} |\lambda_k|^q \right)^{\frac{1}{q}},$$

where the infimum is taken over all above decompositions of  $f$ .

The main result in this section can be stated as follows.

**Theorem 25.** Suppose  $0 < \beta \leq 1$  and  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . Let  $0 < q < \infty$  and  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$  be log-Hölder continuous both at the origin and at infinity. If

$n\delta_2 \leq \alpha(0), \alpha_\infty < n\delta_2 + \beta$ , where  $\delta_2$  is the constant appearing in Lemma 5. Then we have for all  $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$  and  $b \in \text{BMO}(\mathbb{R}^n)$ ,

$$\|[b, S_\beta]f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} \leq C\|b\|_*\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}.$$

*Proof of Theorem 25.* Without loss of generality, we may assume that  $\|b\|_* = 1$ . Let  $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ , by Proposition 24, we have

$$f = \sum_{j=-\infty}^{\infty} \lambda_j a_j, \text{ in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where  $\lambda_j \geq 0$  and each  $a_j$  is a central  $(\alpha(\cdot), p(\cdot), 0; b)$ -atom with  $\text{supp} a_j \subset B_j$  and

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} \approx \inf \left( \sum_{j=-\infty}^{\infty} |\lambda_j|^q \right)^{\frac{1}{q}}.$$

Therefore, we get

$$\begin{aligned} & \|[b, S_\beta](f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} \\ & \approx \left( \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|[b, S_\beta](f)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \\ & \quad + \left( \sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \|[b, S_\beta](f)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \\ & \leq C \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| \|[b, S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{\frac{1}{q}} \\ & \quad + C \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left( \sum_{j=k-1}^{\infty} |\lambda_j| \|[b, S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{\frac{1}{q}} \\ & \quad + C \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| \|[b, S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{\frac{1}{q}} \\ & \quad + C \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \left( \sum_{j=k-1}^{\infty} |\lambda_j| \|[b, S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{\frac{1}{q}} \\ & := P_1 + P_2 + P_3 + P_4. \end{aligned}$$

For  $P_1$ , noticing that if  $x \in R_k$ ,  $(y, t) \in \Gamma(x)$  and  $z \in B_j \cap \{z : |y - z| \leq t\}$  with  $j \leq k - 2$ , then we have  $t \geq \frac{1}{4}|x|$ . Using the vanishing moments of  $a_j$ , we deduce that

$$(16) \quad |[b, S_\beta](a_j)(x)|$$

$$\begin{aligned}
&= \left( \int \int_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x) - b(z)) \phi_t(y - z) a_j(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&= \left( \int \int_{\Gamma(x)} \sup_{\phi \in \tilde{\mathcal{C}}_\beta} \left| \int_{\mathbb{R}^n} (b(x) - b(z)) (\phi_t(y - z) - \phi_t(y)) a_j(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&\leq C 2^{j\beta} \left( \int_{B_j} |b(x) - b(z)| |a_j(z)| dz \right) \left( \int_{\frac{|x|}{4}}^\infty \int_{|x-y|<t} \frac{dy dt}{t^{3n+1+2\beta}} \right)^{\frac{1}{2}} \\
&\leq C 2^{j\beta} |x|^{-(n+\beta)} \int_{B_j} |b(x) - b(z)| |a_j(z)| dz \\
&\leq C 2^{(j-k)\beta - kn} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \left( \|b(x) - b_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} + \|(b - b_{B_j})\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \right).
\end{aligned}$$

From (16), Lemmas 4, 5 and 7, it follows that

$$\begin{aligned}
(17) \quad &\|[b, S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq C 2^{(j-k)\beta - kn} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left( \|(b - b_{B_j})\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \right. \\
&\quad \left. + \|(b - b_{B_j})\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right) \\
&\leq C(k-j) 2^{(j-k)\beta - kn} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leq C(k-j) 2^{(j-k)\beta} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \\
&\leq C(k-j) 2^{(j-k)(\beta+n\delta_2)} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For convenience below we put  $\xi = \beta + n\delta_2 - \alpha(0) > 0$  and  $\eta = \beta + n\delta_2 - \alpha_\infty > 0$ . If  $0 < q \leq 1$ , noting that  $\|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 2^{-j\alpha(0)}$  with  $j \leq k-2$  and  $k < 0$ , by (17) and  $n\delta_2 \leq \alpha(0) < n\delta_2 + \beta$ , then we get

$$\begin{aligned}
P_1^q &= C \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| \|[b, S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\leq C \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| (k-j) 2^{(j-k)(\beta+n\delta_2)-j\alpha(0)} \right)^q \\
&\leq C \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left( \sum_{j=-\infty}^{k-2} |\lambda_j|^q (k-j)^q 2^{(j-k)(\beta+n\delta_2)q-j\alpha(0)q} \right) \\
&= C \sum_{j=-\infty}^{-3} |\lambda_j|^q \sum_{k=j+2}^{-1} (k-j)^q 2^{(j-k)\xi q} \leq C \sum_{j=-\infty}^{-1} |\lambda_j|^q.
\end{aligned}$$

If  $1 < q < \infty$ , by the Hölder inequality, we have

$$\begin{aligned}
P_1^q &= C \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| \| [b, S_\beta](a_j) \chi_k \|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\leq C \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| (k-j) 2^{(j-k)(\beta+n\delta_2)-j\alpha(0)} \right)^q \\
&\leq C \sum_{k=-\infty}^{-1} \left( \sum_{j=-\infty}^{k-2} |\lambda_j|^q 2^{(j-k)\xi q/2} \right) \left( \sum_{j=-\infty}^{k-2} (k-j)^{q'} 2^{(j-k)\xi q'/2} \right)^{q/q'} \\
&\leq C \sum_{j=-\infty}^{-3} |\lambda_j|^q \sum_{k=j+2}^{-1} 2^{(j-k)\xi q/2} \leq C \sum_{j=-\infty}^{-1} |\lambda_j|^q.
\end{aligned}$$

For  $P_3$ , if  $0 < q \leq 1$ , from (17) and the size conditions of  $a_j$ , it follows that

$$\begin{aligned}
P_3^q &= C \sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| \| [b, S_\beta](a_j) \chi_k \|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\leq C \sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \left( \sum_{j=-\infty}^{-1} |\lambda_j|^q (k-j)^q 2^{(j-k)(\beta+n\delta_2)q-j\alpha(0)q} \right) \\
&\quad + C \sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \left( \sum_{j=0}^{k-2} |\lambda_j|^q (k-j)^q 2^{(j-k)(\beta+n\delta_2)q-j\alpha_\infty q} \right) \\
&\leq C \sum_{k=0}^{\infty} 2^{k(\alpha_\infty-\beta-n\delta_2)q} \sum_{j=-\infty}^{-1} |\lambda_j|^q (k-j)^q 2^{j\xi q} \\
&\quad + C \sum_{j=0}^{\infty} |\lambda_j|^q \sum_{k=j+2}^{\infty} (k-j)^q 2^{(j-k)\eta q} \\
&\leq C \left( \sum_{j=-\infty}^{-1} |\lambda_j|^q + \sum_{j=0}^{\infty} |\lambda_j|^q \right) = C \sum_{j=-\infty}^{\infty} |\lambda_j|^q.
\end{aligned}$$

If  $1 < q < \infty$ , by the Hölder inequality, we have

$$\begin{aligned}
P_3^q &= C \sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| \| [b, S_\beta](a_j) \chi_k \|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\leq C \sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \left( \sum_{j=-\infty}^{-1} |\lambda_j| (k-j) 2^{(j-k)(\beta+n\delta_2)-j\alpha(0)} \right)^q \\
&\quad + C \sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \left( \sum_{j=0}^{k-2} |\lambda_j| (k-j) 2^{(j-k)(\beta+n\delta_2)-j\alpha_\infty} \right)^q
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k=0}^{\infty} 2^{k(\alpha_{\infty}-\beta-n\delta_2)q} \left( \sum_{j=-\infty}^{-1} |\lambda_j|(k-j)2^{j\xi} \right)^q + C \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k-2} |\lambda_j|(k-j)2^{(j-k)\eta} \right)^q \\
&\leq C \left( \sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{j\xi q/2} \right) \left( \sum_{j=-\infty}^{-1} (k-j)^{q'} 2^{j\xi q'/2} \right)^{q/q'} \\
&\quad + C \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k-2} |\lambda_j|^q 2^{(j-k)\eta q/2} \right)^q \left( \sum_{j=0}^{k-2} (k-j)^{q'} 2^{(j-k)\eta q'/2} \right)^{q/q'} \\
&\leq C \sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{j\xi q/2} + C \sum_{j=0}^{\infty} |\lambda_j|^q \sum_{k=j+2}^{\infty} 2^{(j-k)\eta q/2} \leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^q.
\end{aligned}$$

For  $P_2$  and  $P_4$ , using a combination of the  $L^{p(\cdot)}(\mathbb{R}^n)$ -boundedness of  $[b, S_{\beta}]$  and the arguments used in the estimation of  $I_1$  and  $I_3$  in [9, Page 338], we can easily get

$$P_2^q \leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^q \quad \text{and} \quad P_4^q \leq C \sum_{j=0}^{\infty} |\lambda_j|^q.$$

Hence, combining the estimates above, the proof of Theorem 25 is complete.  $\square$

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