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CHARACTERIZATION OF RELATIVELY DEMICOMPACT OPERATORS BY MEANS OF MEASURES OF NONCOMPACTNESS

Aref Jeribi, Bilel Krichen, and Makrem Salhi

Abstract. In this paper, we show that an unbounded S_0-demicompact linear operator T with respect to a bounded linear operator S_0, acting on a Banach space, can be characterized by the Kuratowskii measure of noncompactness. Moreover, some other quantities related to this measure provide sufficient conditions to the operator T to be S_0-demicompact. The obtained results are used to discuss the connection with Fredholm and upper Semi-Fredholm operators.

1. Introduction

In 1966, W. V. Petryshyn [16] has developed the concept of demicompactness for nonlinear operator. Several applications of this concept were provided, especially on fixed point theory. In other direction, the demicompactness concept was used to provide several results on Fredholm and Spectral theories (see [2, 7, 12, 17]). Obviously, the class of demicompact operators acting on a Banach space contains the class of compact operators. Hence, it plays an important role when studying perturbations of Fredholm operators. Recently, W. Chaker, A. Jeribi and B. Krichen [7] have utilized demicompact operators in order to investigate the essential spectra of closed linear operators. In 2014, B. Krichen [13], introduced the relative demicompactness class with respect to a given closed linear operator as a generalization of the demicompactness notion. This definition asserts that if X is a Banach space, $T : \mathcal{D}(T) \subset X \longrightarrow X$, and $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ are two linear operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$, then T is said to be S_0-demicompact (or relative demicompact with respect to S_0), if every bounded sequence (x_n) in $\mathcal{D}(T)$ such that $(S_0x_n - Tx_n)$ converges in X, have a convergent subsequence. In 2016, B. Krichen and D. O'Regan [14] discussed some topological properties of the set $\mathcal{F}(S_0, T, z) := \{x \in X : S_0x \in Tx + z\}$, where T is a nonlinear multi-valued

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mapping and S_0 is a single-valued mapping acting on a Banach space X. Their study was based on a new concept, the so called weakly relative demicompactness for nonlinear operators. Recently, the same authors developed in [14] some Fredholm and perturbation results involving the class of weakly demicompact linear operators. Moreover, they studied the relationship between this class and measures of weak noncompactness of linear operator with respect to an axiomatic one. The central aim of this work is to show that an unbounded S_0-demicompact linear operator T with respect to a bounded linear operator S_0 can be characterized by the Kuratowskii's measure of noncompactness. The obtained results will be used to discuss the relation with Fredholm and upper Semi-Fredholm operators.

Now let us recall some standard tools from Fredholm theory needed in this work. Let X and Y be two Banach spaces. By an operator T from X into Y, we mean a linear operator with domain $\mathcal{D}(T) \subseteq X$ and range $\mathcal{R}(T) \subseteq Y$. By $\mathcal{C}(X,Y)$ we mean the set of all closed densely defined operators from X into Y, by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operator from X into Y, and $\mathcal{K}(X,Y)$ the subspace of all compact operators of $\mathcal{L}(X,Y)$. If $T \in \mathcal{C}(X,Y)$, then we denote by $\alpha(T)$ the dimension of the Kernel $\mathcal{N}(T)$ and $\beta(T)$ the codimension of $\mathcal{R}(T)$ in Y. The classes of upper semi-Fredholm, bounded upper semi-Fredholm and lower semi-Fredholm from X into Y are defined respectively by

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\}$$

$$\Phi_+^b(X,Y) = \{T \in \mathcal{L}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\}$$

and

$$\Phi_-(X,Y) := \{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_+(X,Y)$ is the set of Fredholm operators from X into Y. The index of an operator $T \in \Phi_+(X) \cup \Phi_-(X)$ is $ind(T) = \alpha(T) - \beta(T)$. If $X = Y$, the sets $\mathcal{L}(X,Y), \Phi(X,Y), \Phi_+(X,Y), \Phi_-(X,Y)$, and $\Phi_+^b(X,Y)$ are replaced by $\mathcal{L}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$, and $\Phi_+^b(X)$ respectively. If $x \in X$ and $r > 0$, then $B(x,r)$ will denote the closed ball of X with center at x and radius r. We denote by B_X the closed unit ball of X and $\mathcal{M}_X$ the set of all bounded subset in X. Finally, we write $\overline{D}$, $conv(D)$ to denote respectively the closure and the convex hull of a subset D of X.

Definition 1.1. Let X be a Banach space and $T \in \mathcal{L}(X)$. T is said to have a right Fredholm inverse if there exists $T_r \in \mathcal{L}(X)$ such that $I - TT_r \in \mathcal{K}(X)$. We denote by

$$\Phi^r(X) := \{T \in \mathcal{L}(X) \text{ such that } T \text{ has a right Fredholm inverse}\}.$$

$\mathcal{G}^r(T)$ will denote the set of right Fredholm inverses of T.

Definition 1.2 ([15, 18]). Let X be a Banach space. For any $D \in \mathcal{M}_X$, the Kuratowskii's measure of noncompactness of D, denoted by $\mu(D)$, is the infimum of the set of real $\varepsilon > 0$ such that D can be covered by a finite number of sets of diameter less than or equal to ε.

The following proposition gives some properties of the Kuratowskii's measure of noncompactness which are frequently used.

Proposition 1.1 ([4,15])**.** *Let X be a Banach space, D and D' be two bounded subsets of X. Then, we have the following properties.*

(1) *If $D \subset D'$, then $\mu(D) \leq \mu(D')$.*
(2) $\mu(\overline{D}) = \mu(D)$.
(3) $\mu(\overline{conv(D)}) = \mu(D)$.
(4) $\mu(tD + (1-t)D') \leq t\mu(D) + (1-t)\mu(D'), \forall t \in [0,1]$.
(5) $\mu(D) = 0$ *if and only if D is relatively compact.*
(6) *For every $\lambda \in \mathbb{C}$, we have $\mu(\lambda D) = |\lambda|\mu(D)$.*
(7) $\mu(D + D') \leq \mu(D) + \mu(D')$.
(8) $\mu(D \cup D') = \max(\mu(D), \mu(D'))$.
(9) *If $T \in \mathcal{L}(X)$, then $\mu(T(D)) \leq \|T\|\mu(D)$.*

Remark 1.1. We notice that the definition of the Kuratowskii's measure, which is restricted to bounded subsets, can be extended to all subsets by the following definition (see [20]).

$$\gamma(D) := \begin{cases} \mu(D) & \text{if D is bounded} \\ \infty & \text{if D is unbounded.} \end{cases}$$

Definition 1.3. Given an operator $T \in \mathcal{L}(X, Y)$, we define its Kuratowskii's measure $\overline{\gamma}(T)$ by

$$\overline{\gamma}(T) = \sup\{\frac{\gamma(T(D))}{\gamma(D)},\ D \in M_X,\ \gamma(D) > 0\}. \tag{1.1}$$

In the next proposition, we recall some properties of the Kuratowskii's measure of a bounded linear operator.

Proposition 1.2 ([6,10])**.** *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then, we have the following properties*:

(i) $\overline{\gamma}(T) = 0$ *if, and only if, $T \in \mathcal{K}(X)$.*
(ii) *If $S \in \mathcal{L}(X)$, then $\overline{\gamma}(ST) \leq \overline{\gamma}(S)\overline{\gamma}(T)$.*
(iii) $\overline{\gamma}(T + K) = \overline{\gamma}(K)$, $\forall K \in \mathcal{K}(X)$.
(iv) $\overline{\gamma}(T + S) \leq \overline{\gamma}(T) + \overline{\gamma}(S), \forall S \in \mathcal{L}(X)$.
(v) $\overline{\gamma}(T) \leq \|T\|$.

The paper is organized in the following way: In Section 2, we prove that an upper semi-Fredholm operator can be characterized by means of relatively demicompactness concept. Furthermore, we give a characterization of relatively demicompact operators by means of Kuratowskii's measure of noncompactness. Section 3 is devoted to investigate the results obtained in Section 2 to give some properties of demicompact operators involving the first and the second adjoint of a bounded operator acting on a Banach space.

2. Characterization of relatively demicompact operators

We start this section by the following definition of relative demicompactness.

Definition 2.1 ([13]). Let X be a Banach space, S_0, $T \in \mathcal{C}(X)$ such that $\mathcal{D}(T) \subset \mathcal{D}(S_0)$. We say that T is S_0-demicompact if every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $(S_0x_n - Tx_n)_n$ converges on X has a convergent subsequence.

We denote by

$$\mathcal{DC}(S_0)(X) := \{T \in \mathcal{C}(X) \text{ such that } T \text{ is } S_0\text{-demicompact}\},$$

and

$$\mathcal{DC}^b(S_0)(X) := \{T \in \mathcal{L}(X) \text{ such that } T \text{ is } S_0\text{-demicompact}\}.$$

Note that if we put $S_0 = I$, then we recover the usual definition of demicompact operator. In this case the previous sets will simply be denoted by $\mathcal{DC}(X)$ and $\mathcal{DC}^b(X)$ respectively. It was shown in [13] that if X is a Banach space such that X is finite dimensional, then $\mathcal{L}(X) = \mathcal{DC}^b(S_0)(X)$, where S_0 is any bounded operator from X into itself.

Remark 2.1. (i) It should be noticed that if we assume that T is S_0-demicompact for some $S_0 \in \mathcal{K}(X)$, then T is S-demicompact for all $S \in \mathcal{K}(X)$. Indeed, let $(x_n)_n$ be a bounded sequence on X, S_0 a compact operator such that $(S_0x_n - Tx_n)_n$ converges on X. Given an operator $S \in \mathcal{K}(X)$ we can write

$$Sx_n - Tx_n = (S - S_0)x_n + S_0x_n - Tx_n.$$

Since $S - S_0 \in \mathcal{K}(X)$, then $(S_0x_n - Tx_n)_n$ has a convergent subsequence. By using the fact that T is S_0-demicompact, we deduce that $(x_n)_n$ has a convergent subsequence. It follows that T is S-demicompact.

(ii) If X is a Banach space and $T \in \mathcal{DC}(X)$, then for every bounded, invertible operator S defined on X, STS^{-1} is demicompact.

We start this section by showing that a compact operator may be "moved" to a demicompact operator under a small perturbation. More precisely, we have the following result.

Proposition 2.1. *Let T and T_0 be two bounded operators on a Banach space X. Assume that $T_0 \in \mathcal{K}(X)$, and there exist nonnegative constants a and b such that $b < 1$ and, for every $x \in X$,*

$$\|Tx\| \leq a\|T_0x\| + b\|x\|.$$

Then, $T \in \mathcal{DC}^b(X)$.

Proof. For all $x \in X$,

$$\|Tx\| \leq a\|T_0x\| + b\|x\|.$$

Hence,

$$\forall x \in X, \|(I - T)x\| \geq -a\|T_0x\| + (1 - b)\|x\|.$$

Thus,

$$\|x\| \leq \frac{1}{1-b}\|(I-T)x\| + \frac{a}{1-b}\|T_0 x\|.$$

Now, take $(x_n)_n$ a bounded sequence of X such that

$$x_n - Tx_n \longrightarrow x \in X.$$

Since T_0 is compact, it follows that $(T_0 x_n)_n$ has a convergent subsequence $((T_0 x_{\varphi(n)})_n)$, hence $(x_{\varphi(n)})_n$ is a cauchy subsequence and so it is convergent on X. We conclude that $T \in \mathcal{DC}^b(X)$. □

Through the next proposition, we will give a perturbation result for demicompact operators.

Proposition 2.2. (i) *Let T and T_0 be two linear operators with the same domain $\mathcal{D}$ acting on a Banach space X. Suppose that $T_0 \in \mathcal{DC}(X)$ and there exist nonnegative constants a and b such that $b < 1$, $a < 1$ and for all $x \in \mathcal{D}$,*

$$\|Tx - T_0 x\| \leq a\|x - Tx\| + b\|x - T_0 x\|.$$

Then, $T \in \mathcal{DC}(X)$.

(ii) *Let X be a Banach space, and $(T, T_0) \in \mathcal{L}(X) \times \mathcal{L}(X)$. Assume that $T_0 \in \mathcal{DC}^b(X)$ and there exist nonnegative constants a and b such that $b < 1$, and for all $x \in X$,*

$$\|Tx - T_0 x\| \leq a\|x - Tx\| + b\|x - T_0 x\|.$$

Then, $T \in \mathcal{DC}^b(X)$.

Proof. (i) For every $x \in \mathcal{D}$ we have,

$$\|Tx - T_0 x\| \geq |\, \|x - T_0 x\| - \|x - Tx\| \,|.$$

Hence, for all $x \in \mathcal{D}$,

$$\|x - T_0 x\| \leq \frac{a+1}{1-b}\|x - Tx\|$$

and

$$\|x - Tx\| \leq \frac{b+1}{1-a}\|x - T_0 x\|.$$

Now, let $(x_n)_n \subseteq \mathcal{D}$ such that $x_n \longrightarrow x \in X$ and $Tx_n \longrightarrow y \in X$, it follows from the last inequalities that $(x_n - T_0 x_n)_n$ is a Cauchy sequence so it converges, put z its limit. Since T_0 is closed then, $I - T_0 \in \mathcal{C}(X)$ and so $x \in \mathcal{D}$ and $z = (I - T_0)x$. Hence,

$$\|(I-T)(x_n - x)\| \leq \frac{b+1}{1-a}\|(I - T_0)(x_n - x)\|.$$

Thus,

$$(I-T)(x_n - x) \longrightarrow 0.$$

It follows that

$$Tx_n \longrightarrow Tx.$$

Thus, $T \in \mathcal{C}(X)$. Now, take a bounded sequence $(x_n)_n \subseteq \mathcal{D}$ such as

$$x_n - Tx_n \longrightarrow x \in X.$$

We infer that $((I-T_0)x_n)_n$ is a Cauchy sequence, so it is a convergent sequence. Since T_0 is demicompact, we conclude that $(x_n)_n$ has a convergent subsequence. Hence, $T \in \mathcal{DC}(X)$.

(ii) The proof is the same as in (i). □

The following result shows the link between relative demicompact operators and upper semi-Fredholm operators. Before that let us recall the following key lemma.

Lemma 2.1 ([20]). *An operator A is in $\Phi_+(X)$ with $ind(A) \leq 0$ if and only if there exist two operators A_0 and K such that A_0 is in $\Phi_+(X)$ and is injective, and K is a finite rank operator such that $A = A_0 + K$.*

Now, we are in position to state the following result.

Theorem 2.1. *Let T and S_0 be two operators acting on a Banach space X. Suppose that $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ and $S_0 - T$ is closed. Then,*

$$T \text{ is } S_0\text{-demicompact if, and only if, } S_0 - T \in \Phi_+(X).$$

Proof. Suppose that $S_0 - T$ is closed and is demicompact with respect to S_0. Let $(x_n) \subseteq \mathcal{N}(S_0-T)\cap B_X$, then $(x_n)_n$ is bounded and $(S_0x_n - Tx_n)_n$ converges to zero. Since T is S_0-demicompact, then $(x_n)_n$ has a convergent subsequence $(x_{\varphi(n)})_n$, say a its limit. Obviously, for all $n \in \mathbb{N}$, $\|x_{\varphi(n)}\| \leq 1$ and so we get $\|a\| \leq 1$. Moreover, for all $n \in \mathbb{N}, (S_0 - T)(x_{\varphi(n)}) = 0$. Since $S_0 - T$ is closed, then $(S_0 - T)(a) = 0$. Hence, $a \in \mathcal{N}(S_0 - T) \cap B_X$. We conclude that the unit ball of $\mathcal{N}(S_0 - T)$ is compact and so $\alpha(S_0 - T) < \infty$. Therefore,

$$\mathcal{D}(T) = \mathcal{N}(S_0 - T) \oplus \mathcal{D}(T) \cap M,$$

where M is a closed subspace of X with finite codimension. Now, we have to prove that $\mathcal{R}(S_0 - T)$ is closed. Note that the closed subspace $X_0 = \mathcal{D}(T) \cap M$ endowed with the graph norm, associated to the closed operator $S_0 - T$, is a Banach space. Then, by using Theorem 3.12 in [19], it suffices to prove that there exists a positive constant λ such that for every $x \in X_0$,

$$\|(S_0 - T)x\| \geq \lambda \|x\|_{S_0 - T},$$

where $\|x\|_{S_0-T} := \|x\| + \|(S_0 - T)x\|$. If that is not the case, then there exists a sequence $(x_n)_n$ of X_0 such that for all $n \in \mathbb{N}$,

$$\|x_n\|_{S_0-T} = 1 \text{ and } \|(S_0 - T)x_n\| \to 0.$$

Since T is S_0-demicompact and $(x_n)_n$ is bounded, there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ which converges to $x \in X_0$. By using the fact that $S_0 - T$ is closed, we get $(S_0 - T)x = 0$ and then $x = 0$. This is absurd because of the continuity of the norm which leads to $\|x\| = 1$. Now, we suppose that $S_0 - T \in \Phi_+(X)$. There are two cases:

First case: If $ind(S_0 - T) > 0$, then $S_0 - T \in \Phi(X)$. By using Theorem 7.2 in [19], there exist $A \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that

$$A(S_0 - T) = I + K.$$

Let $(x_n)_n$ be a bounded sequence of $\mathcal{D}(T)$ such that

$$(S_0 - T)x_n \longrightarrow x \in X.$$

Then, $(A(S_0 - T)x_n)_n$ converges to Ax. Hence, $(x_n + Kx_n)_n$ converges to Ax. Since K is compact, then $(Kx_n)_n$ has a convergent subsequence and so $(x_n)_n$ has also a convergent subsequence.

Second case: If $ind(S_0 - T) \leq 0$. Then, by using Lemma 2.1, there exist a bounded below operator A_0 and $K \in \mathcal{K}(X)$ such that

$$S_0 - T = A_0 + K.$$

Let $(x_n)_n$ be a bounded sequence in $\mathcal{D}(T)$ such that

$$(S_0 - T)x_n \longrightarrow x \in X.$$

Then,

$$(A_0 + K)x_n \longrightarrow x \in X.$$

Since K is compact, then $(Kx_n)_n$ has a convergent subsequence $(Kx_{\varphi(n)})_n$. Consequently, $(A_0 x_{\varphi(n)})_n$ is a convergent sequence and so it is a Cauchy sequence. By using the fact that A_0 is bounded below, we deduce that $(x_n)_n$ has a convergent subsequence. It follows that T is S_0-demicompact. □

Now, we will give a characterization of demicompact bounded projections.

Corollary 2.1. *Let X be a Banach space and P a bounded projection on X. Then, the following assertions are equivalent*:

(i) $P \in \mathcal{DC}^b(X)$.
(ii) $P \in \mathcal{K}(X)$.
(iii) $I - P \in \Phi^b(X)$ *and* $ind(I - P) = 0$.

Proof. (i)⇒(ii) Suppose that P is demicompact. Then, by using Theorem 2.1, we infer that $I - P \in \Phi_+(X)$. Hence, $\mathcal{R}(P) = \mathcal{N}(I - P)$ is finite dimensional space. This shows that P is a finite rank operator. Consequently, $P \in \mathcal{K}(X)$.

(ii)⇒(iii) This is a well known result (see [1, 19]).

(iii)⇒(i) Since $I - P \in \Phi^b(X)$, then $I - P \in \Phi^b_+(X)$. By using Theorem 2.1, we obtain the desired result. □

As a consequence of the last theorem, we have the following result.

Corollary 2.2. *Let X be a Banach space and $(T, S) \in \mathcal{L}(X) \times \mathcal{L}(X)$. Let $n \geq 2$ and $X_1, \ldots, X_n$ be n subspaces of X. Assume that*

(i) $X = \oplus_{k=1}^n X_k$,
(ii) *For every* $k \in \{1, \ldots, n\}$, $T_{|X_k} := T_k \in \mathcal{L}(X_k)$, *and*
(iii) *For every* $k \in \{1, \ldots, n\}$, $S_{|X_k} := S_k \in \mathcal{L}(X_k)$.

Then, T is S-demicompact if and only if for every $k \in \{1, \ldots, n\}$, T_k is S_k-demicompact.

Proof. Hypothesis (ii) let us write

$$T = T_1 \oplus ... \oplus T_n.$$

Similarly, by hypothesis (iii) we have

$$S = S_1 \oplus \cdots \oplus S_n.$$

Hence,

$$T - S = (T_1 - S_1) \oplus \cdots \oplus (T_n - S_n).$$

Now when applying Theorem 2.1, we get

$$\begin{aligned} T \text{ is } S\text{-demicompact} &\Longleftrightarrow T - S \in \Phi_+^b(X) \\ &\Longleftrightarrow T_k - S_k \in \Phi_+^b(X_k) \text{ for every } k \in \{1, \ldots, n\} \\ &\Longleftrightarrow T_k \text{ is } S_k\text{-demicompact for every } k \in \{1, \ldots, n\}. \end{aligned}$$ □

Some topological properties of $\mathcal{DC}^b(S_0)(X)$ are provided in the following proposition.

Proposition 2.3. *Let X be a Banach space, $S_0 \in \mathcal{L}(X)$. Then, $\mathcal{DC}^b(S_0)(X)$ is an open subset of $\mathcal{L}(X)$ and the index is constant on every component of $\mathcal{DC}^b(S_0)(X)$. Moreover, when X is a Hilbert space, $\mathcal{DC}^b(S_0)(X)$ has an infinite disjoint arcwise connected components. Precisely, we have $\mathcal{DC}^b(S_0)(X) = F_{-\infty} \bigcup (\bigcup_{n\in\mathbb{Z}} F_n)$, where*

$$F_n = \{T \in \mathcal{DC}^b(S_0)(X) : \ ind(S_0 - T) = n, \ n \in \mathbb{Z}\}.$$

Proof. We can easily show that the application $\Psi : T \longmapsto S_0 - T$ is an homeomorphism of $\mathcal{L}(X)$. By using Theorem 2.1, we get $\mathcal{DC}^b(S_0)(X) = \Psi^{-1}(\Phi_+^b(X))$. Since $\Phi_+^b(X)$ is an open set of $\mathcal{L}(X)$, then $\mathcal{DC}^b(S_0)(X)$ is an open set.

Now, suppose that X is a Hilbert space. By using Theorem 5.1 in [8], we deduce that the components of $\Phi_+^b(X)$ are

$$H_n = \{T \in \Phi_+^b(X) : \ ind(T) = n, \ n \in \mathbb{Z}\}.$$

Moreover, the components $H_n, n \in \mathbb{Z}$ are arcwise connected (for more details see [9]). Now, by using the homeomorphism Ψ of $\mathcal{L}(X)$, we deduce that the components of $\mathcal{DC}^b(S_0)(X)$ are the sets: F_n, $n \in \mathbb{Z}$. Furthermore, by using again the homeomorphism Ψ, we deduce that the components of $\mathcal{DC}^b(S_0)(X)$ are arcwise connected. □

The following lemma will be useful for the proof of the next theorem.

Lemma 2.2. *Let X and Y be two infinite dimensional Banach spaces and $A \in \mathcal{C}(X, Y)$. Suppose that there exists a positive constant C such that for every $x \in \mathcal{D}(A)$*

$$\|x\| \leq C\|Ax\|.$$

Then, for every bounded set $D \subseteq \mathcal{D}(A)$ we have

$$\gamma(D) \leq 2C\gamma(A(D)).$$

Proof. Let D be a bounded set of $\mathcal{D}(A)$. If $\gamma(A(D)) = +\infty$, then the inequality $\gamma(D) \leq 2C\gamma(A(D))$ is trivial. Suppose that $\gamma(A(D)) < +\infty$. Take $r > \gamma(A(D))$, then there exists a finite number of sets $V_1, \dots, V_n$; $n \geq 1$ with diameter less than or equal r such that

$$A(D) \subseteq \bigcup_{k=1}^{n} V_k.$$

Note that we can suppose that

$$A^{-1}(V_j) \neq \emptyset;\ j = 1, \dots, n.$$

Obviously, we have

$$D \subseteq \bigcup_{k=1}^{n} A^{-1}(V_k).$$

Fix arbitrarily $x_k \in A^{-1}(V_k)$; $k = 1, \dots, n$. Now, take $z \in D$. Then, there exists $k \in \{1, \dots, n\}$ such as $A(z) \in V_k$. Hence,

$$\begin{aligned}\|z - x_k\| &\leq C\|A(z) - A(x_k)\| \\ &\leq Cr.\end{aligned}$$

Thus,

$$D \subseteq \bigcup_{k=1}^{n} B(x_k, Cr).$$

Now, using the properties of the Kuratowskii's measure we get

$$\gamma(D) \leq Cr\gamma(B_X).$$

By using the fact that $\gamma(B_X) = 2$ (see [3]), we conclude that

$$\gamma(D) \leq 2Cr.$$

Letting $r \longrightarrow \gamma(A(D))$, we deduce that

$$\gamma(D) \leq 2C\gamma(A(D)). \qquad \square$$

Now, we give a characterization of relatively demicompact operators by means of Kuratowskii's measure of noncompactness.

Theorem 2.2. *Let X be a Banach space, T and S_0 be two operators on X such that $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ and $S_0 - T$ is closed. The following statements are equivalent*:

(i) *T is S_0-demicompact.*

(ii) *There exists a positive constant C such that for all $D \subseteq \mathcal{D}(T)$,*

$$\gamma(D) \leq C\gamma(S_0 - T)(D).$$

Proof. (i)$\Longrightarrow$(ii) Suppose first that T is S_0-demicompact. Then, by using Theorem 2.1, $S_0-T \in \Phi_+(X)$. Now, if $ind(S_0-T) > 0$, then there exist a bounded operator A and a compact operator K such that

$$A(S_0 - T) = I + K.$$

Let D be a bounded set of X.
Hence,

$$\begin{aligned}\gamma(D) &\leq \gamma(A(S_0 - T)(D)) \\ &\leq \|A\|\gamma((S_0 - T)(D)).\end{aligned}$$

In the case where $ind(S_0 - T) \leq 0$, then, by using Lemma 2.1, there exist a compact operator K and a bounded below operator A_0 such that $S_0 - T = K + A_0$. Let β be a positive constant such that for all $x \in \mathcal{D}(T)$,

$$\|x\| \leq \beta\|A_0x\|.$$

Taking into account Lemma 2.2, we deduce that for any bounded subset D of $\mathcal{D}(T)$,

$$\begin{aligned}\gamma(D) &\leq 2\beta\gamma(A_0(D)) \\ &\leq 2\beta\gamma((S_0 - T)(D)).\end{aligned}$$

Choose $C = \max(\|A\|, 2\beta)$, then for any bounded subset D of $\mathcal{D}(T)$ we have

$$\gamma(D) \leq C\gamma(S_0 - T)(D).$$

(ii)$\Longrightarrow$(i) Suppose that there exists a positive constant C such that for every bounded set D of X,

$$\gamma(D) \leq C\gamma(S_0 - T)(D).$$

Let $(x_n)_n$ be a bounded sequence of $\mathcal{D}(T)$ such that

$$(S_0 - T)x_n \longrightarrow x \in X.$$

Choose $D = \{x_n : \ n \in \mathbb{N}\}$. It is clear that D is a bounded set of $\mathcal{D}(T)$ such that $\gamma(S_0 - T)(D) = 0$. Hence $\gamma(D) = 0$, so that $(x_n)_n$ has a convergent subsequence. It follows that T is S_0-demicompact. $\square$

The characterization of relatively demicompact operators proved in the last theorem will be not valid if we replace the Kuratowskii's measure of an operator by its norm. More precisely we have the following remark.

Remark 2.2. Let X be a Banach space, T and S_0 be two operators on X such that $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ and $S_0 - T$ is closed. Assume that there exists a positive constant C such that for every $x \in \mathcal{D}(T)$

$$\|x\| \leq C\|S_0x - Tx\|.$$

Then, T is S_0-demicompact. The reciprocal is not true. Indeed, suppose that there exists a positive constant C such that for every $x \in \mathcal{D}(T)$,

$$\|x\| \leq C\|S_0x - Tx\|.$$

Then, $S_0 - T$ is bounded below. Hence $S_0 - T \in \Phi_+(X)$. In view of Theorem 2.1 we infer that T is S_0-demicompact. Now, suppose that X is finite dimensional. Let $T \in \mathcal{L}(X)$ such that $\mathcal{N}(T) \neq \{0\}$. Put $S_0 = 2T$, then T is S_0-demicompact. However, $S_0 - T = T$ is not bounded below because it is not injective.

Corollary 2.3. *Let T be a closed operator with domain $\mathcal{D}(T)$ on a Banach space X. If $I - T$ is bounded below, then T is demicompact.*

Proof. Let $T \in \mathcal{C}(X)$ with domain $\mathcal{D}(T)$ such that $I - T$ is bounded below. It follows that, there exists a positive constant C such as, for every $x \in \mathcal{D}(T)$,

$$\|(I-T)(x)\| \geq C\|x\|.$$

Let D be a bounded subset of $\mathcal{D}(T)$, then

$$\gamma(D) \leq \frac{1}{C}\gamma(I-T)(D).$$

We notice that the last inequality is true even if $\gamma(I-T)(D)$ is infinite. We conclude, by using Theorem 2.2, that T is demicompact. □

Corollary 2.4. *Let T and S_0 be two bounded, commuting operators acting on a Banach Space X. Suppose that there exists a complex polynomial P such that*

(i) *$P(1) = 1$, and*

(ii) *$P(T)$ is $P(S_0)$-demicompact.*

Then, T is S_0-demicompact.

Proof. Let $P = \sum_{k=0}^{n} a_k X^k \in \mathbb{C}[X]$. Suppose that $P(1) = 1$. Since $TS_0 = S_0 T$, we have

$$\begin{aligned} P(S_0) - P(T) &= \sum_{k=0}^{n} a_k^k = \sum_{k=0}^{n} a_k S_o^k - \sum_{k=0}^{n} a_k T^k \\ &= \sum_{k=0}^{n} a_k (S_0^k - T^k) \\ &= (S_0 - T)\sum_{k=1}^{n} a_k \sum_{j=1}^{k} S_0^{k-j} T^j \\ &= (S_0 - T)Q(S_0, T). \end{aligned}$$

Q is clearly a polynomial in two variables, precisely,

$$Q(X,Y) = \sum_{k=1}^{n} a_k \sum_{j=1}^{k} X^{k-j} Y^j.$$

Since T and S_0 are bounded, then $Q(S_0, T) \in \mathcal{L}(X)$. Now, by hypothesis (ii), $P(T)$ is $P(S_0)$-demicompact. By using Theorem 2.2, there exists a positive constant C such as, for every bounded subset $D \subseteq X$,

$$\gamma(D) \leq C\gamma(P(S_0) - P(T))(D).$$

Thus, for every bounded subset $D \subseteq X$,

$$\begin{aligned}\gamma(D) &\leq C\gamma(P(S_0) - P(T))(D)\\ &\leq C\gamma((S_0 - T)Q(S_0, T))(D)\\ &\leq C\|Q(S_0, T)\|\gamma(S_0 - T)(D).\end{aligned}$$

Hence, by applying Theorem 2.2, we conclude that T is S_0-demicompact. □

Remark 2.3. Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that T^n is demicompact for some positive integer n. Then, T is demicompact. Indeed, it suffices to apply Corollary 2.4 with $P = X^n$ and $S_0 = I$.

Corollary 2.5. *Let X be a Banach space, $(S, T) \in \mathcal{L}(X) \times \mathcal{L}(X)$. Assume that $ST - TS \in \mathcal{K}(X)$. Then, for every integer $n \geq 1$,*

$$(ST)^n \in \mathcal{DC}^b(X) \text{ if and only if } (TS)^n \in \mathcal{DC}^b(X).$$

Proof. Let n be a positive integer. Suppose that $(ST)^n \in \mathcal{DC}^b(X)$. In view of Theorem 2.2, there exists a positive constant C_n such as for all $D \in \mathcal{M}_X$,

$$\gamma(D) \leq C_n\gamma(I - (ST)^n)(D).$$

Hence, for every bounded set D,

$$\gamma(D) \leq C_n\gamma(I - (TS)^n)(D) + C_n\gamma((TS)^n - (ST)^n)(D).$$

Using the following identity

$$(TS)^n - (ST)^n = \sum_{k=0}^{n-1}(TS)^k(TS - ST)(ST)^{n-1-k},$$

we infer that $(TS)^n - (ST)^n \in \mathcal{K}(X)$. Hence, for all $D \in \mathcal{M}_X$,

$$\gamma((TS)^n - (ST)^n)(D) = 0.$$

Accordingly, for all $D \in \mathcal{M}_X$,

$$\gamma(D) \leq C_n\gamma(I - (TS)^n)(D).$$

In view of Theorem 2.2, we conclude that $(TS)^n \in \mathcal{DC}^b(X)$. The reciprocal is proved similarly. □

Theorem 2.3. *Let X be a Banach space, $T \in \mathcal{L}(X)$ and let P be a complex polynomial. Assume that*

(i) $P(1) = 1$, *and*

(ii) $\lim_{p \to +\infty} (\overline{\gamma}(P(T^p)))^{\frac{1}{p}} = 0$.

Then, $T \in \mathcal{DC}^b(X)$.

Proof. (i) Since $\lim_{p \to +\infty} (\overline{\gamma}(P(T^p)))^{\frac{1}{p}} = 0$, there exists $p_0 \in \mathbb{N}^*$ such that

$$\overline{\gamma}(P(T^{p_0})) < 1.$$

Now, take a bounded sequence $(x_n)_n$ in X such that $y_n := x_n - P(T^{p_0})x_n$ converges to some element $x \in X$. Then,

$$\{x_n\} \subset \{y_n\} + P(T^{p_0})\{x_n\}.$$

Since the operator $P(T^{p_0})$ is bounded, then

$$(1 - \overline{\gamma}(P(T^{p_0})))\gamma\{x_n\} = 0.$$

Thus, $\gamma\{x_n\} = 0$ and hence there exists a convergent subsequence of $(x_n)_n$. It follows that $P(T^{p_0})$ is demicompact. By using Corollary 2.4 we deduce that T^{p_0} is demicompact. Taking into account Remark 2.3, we conclude that T is demicompact. □

As a consequence, we can give the following result.

Corollary 2.6. *Let X be a Banach space, $T \in \mathcal{L}(X)$. Assume that*

$$\lim_{p \to +\infty} (\overline{\gamma}(T^p))^{\frac{1}{p}} = 0.$$

Then,
(i) $T \in \mathcal{DC}^b(X)$.
(ii) *$I - T$ is a Fredholm operator. Moreover, $ind(I - T) = 0$.*

Proof. (i) Let $P = X$, then by applying Theorem 2.3, we deduce that $T \in \mathcal{DC}^b(X)$.
(ii) Let $\lambda \in [0, 1]$. Then,

$$\overline{\gamma}(((\lambda T)^p))^{\frac{1}{p}} = |\lambda| \overline{\gamma}(T^p)^{\frac{1}{p}}.$$

Thus, $\lim_{p \to +\infty} (\overline{\gamma}((\lambda T)^p))^{\frac{1}{p}} = 0$. By applying (i), we get for every $\lambda \in [0, 1]$, $\lambda T \in \mathcal{DC}^b(X)$. Hence, by using Theorem 2.1, we infer that for every $\lambda \in [0, 1]$, $I - \lambda T \in \Phi^b_+(X)$. Since The index is continuous on $\Phi^b_+(X)$ and constant on every component of $\Phi^b_+(X)$, then

$$\begin{aligned} ind(I - \lambda T) &= ind(I - T) \\ &= ind(I) \\ &= 0. \end{aligned}$$

It follows that $I - T$ is a Fredholm operator. □

Proposition 2.4. *Let T and S_0 be two bounded operators of a Banach space X. Suppose that $\overline{\gamma}(I - (S_0 - T)) < 1$, then T is S_0-demicompact. In particular, if $\|I - (S_0 - T)\| < 1$, then T is S_0-demicompact.*

Proof. Let $(x_n)_n$ be a bounded sequence of X such that $(S_0 x_n - T x_n)_n$ converges to an element $x \in X$. Since

$$x_n = (I - (S_0 - T))x_n + (S_0 - T)x_n,$$

it follows by using the properties of the Kuratowskii's measure, that

$$\gamma\{x_n\} \leq \gamma\{(I - S_0 - T)x_n\}$$

$$\leq \overline{\gamma}(I - S_0 - T)\gamma\{x_n\}.$$

Therefore, $(1 - \overline{\gamma}(I - S_0 - T))\gamma\{x_n\} \leq 0$ and so $\gamma\{x_n\} = 0$. Consequently, $(x_n)_n$ has a convergent subsequence. We conclude that T is S_0-demicompact. Now, suppose that

$$\|I - S_0 - T\| < 1.$$

Then,

$$\begin{aligned} \overline{\gamma}(I - S_0 - T) &\leq \overline{\gamma}(I - S_0 - T) \\ &< 1. \end{aligned}$$

Hence, T is S_0-demicompact. □

Now, a generalization of the preceding result will be given in the following proposition.

Proposition 2.5. *Let T and S_0 be two bounded operators of a Banach space X. We suppose that there exists $n \in \mathbb{N} \setminus \{0\}$ such that $\overline{\gamma}(I - (S_0 - T)^n) < 1$, then T is S_0-demicompact. In particular, if $\|I - (S_0 - T)^n\| < 1$ for some $n \in \mathbb{N} \setminus \{0\}$, then T is S_0-demicompact.*

Proof. Let $n \in \mathbb{N}\setminus\{0\}$ such that $\overline{\gamma}(I-(S_0-T)^n) < 1$. Take a bounded sequence $(x_k)_k$ such that $((S_0-T)x_k)_k$ converges to $x \in X$, then $((S_0-T)^n x_k)_k$ converges to $y = (S_0 - T)^{n-1}x \in X$. Since

$$x_k = (I - (S_0 - T)^n)x_k + (S_0 - T)^n x_k.$$

Then,

$$\begin{aligned} \gamma\{x_k\} &\leq \gamma\{(I - (S_0 - T)^n)x_k\} \\ &\leq \overline{\gamma}(I - (S_0 - T)^n)\gamma\{x_k\}. \end{aligned}$$

We conclude, as in the previous proposition, that $\gamma\{x_k\} = 0$. Thus, T is S_0-demicompact. For the rest of the proof, we use the fact that $\overline{\gamma}(I-(S_0-T)^n) \leq \|I - (S_0 - T)^n\|$. □

The following result shows the connection between demicompact operators and relatively demicompact ones.

Proposition 2.6. *Let X be a Banach space, T and S_0 two bounded operators on X. Then, we have*

$$T \text{ is } S_0\text{-demicompact} \implies \mathcal{G}^r(I - S_0 + T) \subseteq \mathcal{DC}^b(X).$$

Proof. Let $A \in \mathcal{G}_r(I - S_0 + T)$, then there exists $K \in \mathcal{K}(X)$ such that

$$(I - S_0 + T)A = I + K.$$

Now, let $(x_n)_n$ be a bounded sequence of X such that

$$x_n - Ax_n \longrightarrow x \in X.$$

Then,

$$x_n - (I + K - TA + S_0A)x_n \longrightarrow x.$$

It follows that

$$-Kx_n + (T - S_0)Ax_n \longrightarrow x.$$

Since K is compact, then $(Kx_n)_n$ has a convergent subsequence. Hence, $((S_0 - T)Ax_n)_n$ has a convergent subsequence. By using the fact that T is S_0-demicompact, we conclude that $(Ax_n)_n$ has a convergent subsequence, then $(x_n)_n$ has a convergent subsequence. It follows that $A \in \mathcal{DC}^b(X)$. □

Now, let us recall the quantity $\Gamma(T)$ of a bounded linear operator T.

Definition 2.2. Let X, Y be two infinite dimensional Banach spaces and $T \in \mathcal{L}(X, Y)$. We define the quantity $\Gamma(T)$ by

$$\Gamma(T) := \inf\{\overline{\gamma}(T_{|M}) : \; M \subset X\}.$$

The subset M designs a closed subset of X with finite codimension.

The following lemma is very useful for the next results.

Lemma 2.3 ([19])**.** *Let X, Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$. Suppose that $T \notin \Phi_+(X, Y)$, then for every $\varepsilon > 0$ there is an infinite dimensional closed subspace $M \subset X$ and $K \in \mathcal{K}(X, Y)$ such that $T = K$ on M and $\|K_{|M}\| \leq \varepsilon$.*

Theorem 2.4. *Let X be an infinite dimensional Banach space, $(T, S_0) \in \mathcal{L}(X) \times \mathcal{L}(X)$. Assume that $\|T\| < \Gamma(S_0)$, then T is S_0-demicompact.*

Proof. Suppose that $\|T\| < \Gamma(S_0)$. If T is not S_0-demicompact, then by using Theorem 2.1, we get $S_0 - T \notin \Phi_+(X)$. Now, by using Lemma 2.3, we deduce that for every $\varepsilon > 0$, there is an infinite dimensional closed subspace $M \subset X$ with finite codimension and $K \in \mathcal{K}(X, Y)$ such that $S_0 - T = K$ on M and $\|K_{|M}\| \leq \varepsilon$. Therefore, for all $x \in M$,

$$\|S_0 x\| \leq \varepsilon\|x\| + \|Tx|.$$

It follows that for every bounded set $D \subset M$,

$$\gamma(S_0(D)) \leq \varepsilon\gamma(D) + \gamma(T(D)).$$

Thus,

$$\begin{aligned}\overline{\gamma}(S_{0|M}) &\leq \varepsilon + \overline{\gamma}(T_{|M}) \\ &\leq \varepsilon + \overline{\gamma}(T).\end{aligned}$$

We conclude that

$$\begin{aligned}\Gamma(S_0) &\leq \varepsilon + \overline{\gamma}(T) \\ &\leq \varepsilon + \|T\|.\end{aligned}$$

Letting $\varepsilon \to 0^+$, we get $\Gamma(S_0) \leq \|T\|$. This is absurd, then T is S_0-demicompact. □

We notice that there are several measures of operators, like essential norm, that can be used to characterize demicompact operator, among this operator quantities we introduce the following measure.

Definition 2.3 ([11,20])**.** Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$. We define

$$\Delta(T) = \inf_{M} \|T_{|M}\|,$$

where M represents an infinite dimensional, closed subspace of X, and $T_{|M}$ denotes the restriction of T to the subspace M.

Now, we give a characterization of a demicompact operator T using the quantity $\Delta(T)$.

Theorem 2.5. *Let X an infinite dimensional Banach space and $T \in \mathcal{L}(X)$, if $\Delta(T) > 1$, then T is demicompact.*

Proof. Suppose that $\Delta(T) > 1$ but T is not demicompact, then by Theorem 2.1, $I - T \notin \Phi_+(X)$. By using Theorem 2.3 combined with Theorem 14.29 in [19], we infer that for every $\varepsilon > 0$, there exist a closed infinite-dimensional subspace M with finite codimension and a compact operator K such that $I - T = K$ on M, with $\|K_{|M}\| \leq \varepsilon$. Hence,

$$\|T_{|M}\| \leq 1 + \varepsilon.$$

It follows that, for every $\varepsilon > 0$,

$$\Delta(T) \leq 1 + \varepsilon.$$

Next, letting $\varepsilon \to 0^+$, we get $\Delta(T) \leq 1$ which is absurd. We conclude that T is demicompact. □

3. Duality and demicompactness

In this section, we show that the demicompactness of a bounded linear operator, as well as, of its first and second duals, can be ensured by means of its essential norm.

Definition 3.1 ([5,11])**.** Let X be a Banach space, and $T \in \mathcal{L}(X)$. We define the essential norm of T by

$$\|T\|_e = \inf\{\|T - K\| : \ K \in \mathcal{K}(X)\}.$$

Let X be a Banach space. In what follows we denote by X^* and X^{**} its first and second dual respectively.

Let us recall some properties of the essential norm through the following proposition.

Proposition 3.1 ([5])**.** *Let X be a Banach space, and $T \in \mathcal{L}(X)$. Then we have*

(i) $\|T^*\|_e \leq \|T\|_e$,
(ii) $\|T^*\|_e = \|T^{**}\|_e$,

where T^* and T^{**} are respectively the first and the second adjoint of T.

Remark 3.1. It was shown in [5] that, if X is a reflexive space then, for all $T \in \mathcal{L}(X)$ we have $\|T\|_e = \|T^*\|_e$. But, for almost common Banach spaces we have

$$\begin{aligned}\frac{1}{2}\|T\|_e &\leq \|T^*\|_e \\ &\leq \|T\|_e.\end{aligned}$$

Theorem 3.1. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that $\|T\|_e < 1$, then*
(i) *T, T^* and T^{**} are demicompact operators.*
(ii) *$I - T \in \Phi(X)$, $T^* \in \Phi_+^b(X^*)$ and $T^{**} \in \Phi_+^b(X^{**})$.*
(iii) *$ind(I - T) = ind(I - T^*) = ind(I - T^{**}) = 0$.*

Proof. We suppose that $\|T\|_e < 1$ then, there exists $K \in \mathcal{K}(X)$ such that

$$\begin{aligned}\|T\|_e &\leq \|T - K\| \\ &< 1.\end{aligned}$$

Now, let $(x_n)_n$ be a bounded sequence on X such that

$$y_n := x_n - Tx_n \longrightarrow x \in X.$$

Then, for all $(n, m) \in \mathbb{N}^2$ we have

$$\begin{aligned}\|x_n - x_m\| &= \|y_n - y_m + K(x_n - x_m) + (T - K)(x_n - x_m)\| \\ &\leq \|y_n - y_m\| + \|Kx_n - Kx_m\| + \|T - K\|(\|x_n - x_m\|).\end{aligned}$$

Hence,

$$\|x_n - x_m\| \leq \frac{1}{1 - \|T - K\|}(\|y_n - y_m\| + \|Kx_n - Kx_m\|).$$

Since K is compact and $(x_n)_n$ is bounded then, $(Kx_n)_n$ has a convergent subsequence which we note again $(Kx_n)_n$. Noticing that $(y_n)_n$ is a converging sequence, then $(x_n)_n$ is a Cauchy sequence and so it has a convergent subsequence. We conclude that T is demicompact. Now, if $\|T\|_e < 1$ then, for all $\lambda \in [0, 1]$ we have $\|\lambda T\|_e < 1$, and so λT is demicompact. By using Theorem 2.1, we get $I - \lambda T \in \Phi_+^b(X)$. Combining the fact that the index is continuous on $\Phi_+^b(X)$ and it is constant on every component of $\Phi_+(X)$, we deduce that

$$\begin{aligned}ind(I - \lambda T) &= ind(I - T) \\ &= ind(I) \\ &= 0.\end{aligned}$$

Hence, $I - T \in \Phi(X)$. Now, suppose that $\|T\|_e < 1$, then by using Proposition 3.1, we get

$$\begin{aligned}\|T^{**}\|_e &= \|T^*\|_e \\ &\leq \|T\|_e \\ &< 1.\end{aligned}$$

Therefore, T^* and T^{**} are demicompact operators. Furthermore, we have

$$\begin{aligned} ind(I-T) &= ind(I-T^*) \\ &= ind(I-T^{**}) \\ &= 0. \end{aligned}$$

Thus, $(I-T^*, I-T^{**}) \in \Phi(X^*) \times \Phi(X^{**})$. □

As a consequence of this Theorem, we have the following corollary.

Corollary 3.1. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that there exists a complex polynomial P such that the following conditions hold*
(i) $P(1) = 1$, *and*
(ii) $\|P(T)\|_e < 1$.
Then, T is demicompact.
In particular, if there exists $n \in \mathbb{N}$ such that $\|T^n\|_e < 1$, then
(i) *T, T^* and T^{**} are demicompact operators.*
(ii) $ind(I-T) = ind(I-T^*) = ind(I-T^{**}) = 0$.
(iii) $I-T \in \Phi(X), I-T^* \in \Phi(X^*)$ *and* $I-T^{**} \in \Phi(X^{**})$.

Proof. If $\|P(T)\|_e < 1$, then in view of Theorem 3.1, $P(T)$ is demicompact. By using Corollary 2.4, we infer that T is demicompact. Now, if $\|T^n\|_e < 1$ for some $n \in \mathbb{N} \setminus \{0\}$, then for all $\lambda \in [0,1]$, $\|(\lambda T)^n\|_e < 1$ so, λT is demicompact for all $\lambda \in [0,1]$. Hence, $I - \lambda T \in \Phi_+(X)$. The continuity of the index ensure that

$$\begin{aligned} ind(I-T) &= ind(I-\lambda T) \\ &= ind(I) \\ &= 0. \end{aligned}$$

Hence, $T \in \Phi(X)$. Since

$$\begin{aligned} \|(T^*)^n\|_e &= \|(T^{**})^n\|_e \\ &\leq \|T^n\|_e \\ &< 1, \end{aligned}$$

then, we may complete the proof with no difficulty. □

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Aref Jeribi
Department of Mathematics
Faculty of Sciences of Sfax
University of Sfax
Road Soukra Km 3.5, B. P. 1171, 3000, Sfax Tunisia
Email address: Aref.Jeribi@fss.rnu.tn

Bilel Krichen
Department of Mathematics
Faculty of Sciences of Sfax
University of Sfax
Road Soukra Km 3.5, B. P. 1171, 3000, Sfax Tunisia
Email address: krichen_bilel@yahoo.fr

Makrem Salhi
Department of Mathematics
Faculty of Sciences of Sfax
University of Sfax
Road Soukra Km 3.5, B. P. 1171, 3000, Sfax Tunisia
Email address: makremessalhi@gmail.com