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ESTIMATION OF DRIFT PARAMETER AND CHANGE POINT VIA KALMAN-BUCY FILTER FOR LINEAR SYSTEMS WITH SIGNAL DRIVEN BY A FRACTIONAL BROWNIAN MOTION AND OBSERVATION DRIVEN BY A BROWNIAN MOTION

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Abstract. We study the estimation of the drift parameter and the change point obtained through a Kalman-Bucy filter for linear systems with signal driven by a fractional Brownian motion and the observation driven by a Brownian motion.

1. Introduction

Change-point problems or disorder problems have been of interest to statisticians for their applications and for probabilists for their challenging problems. Our aim in this paper is to consider estimation of the change point τ and the drift parameter θ for a linear system when the signal is driven by fractional Brownian motion and the observation is driven by a Brownian motion with a small diffusion coefficient. We assume that $\tau \in [t_1, t_2]$ and $\theta \in \Theta$ compact in R. Consider the linear system

$$\begin{aligned} dX_t &= \theta X_t dt + \epsilon\, dV_t^H,\ X_0 = x_0 \neq 0,\ 0 \leq t \leq T \\ dY_t &= f_t(\tau) X_t dt + \epsilon\, dW_t,\ Y_0 = y_0,\ 0 \leq t \leq T, \end{aligned} \tag{1.1}$$

where $\{V_t^H, 0 \leq t \leq T\}$ is the standard fractional Brownian motion with Hurst parameter $H \in \left[\frac{1}{2}, 1\right]$ and $\{W_t, 0 \leq t \leq T\}$ is the standard Brownian motion independent of each other. Suppose that the function $f_t(\tau) = h$ if $t \in (0, \tau]$, and $f_t(\tau) = g$ if $t \in [\tau, T]$ where h and g are known constants with $h \neq g$. We assume that the process $\{Y_t, 0 \leq t \leq T\}$ is observable but the state $\{X_t, 0 \leq t \leq T\}$ of the system is unobservable.

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We now estimate the change point τ by $\hat{\tau}_\epsilon$ and θ by $\hat{\theta}_\epsilon$ based on the observation $\{Y_t, 0 \le t \le T\}$ by the maximum likelihood method and study the asymptotic properties following the methods in Ibragimov and Has'minskii [2] and Prakasa Rao [12]. Kutoyants [6] investigated a similar problem for linear systems driven by independent Brownian motions. We show that the normalized sequence

$$(\epsilon^{-2}(\hat{\tau}_\epsilon - \tau), \epsilon^{-1}(\hat{\theta}_\epsilon - \theta))$$

has a limiting distribution as $\epsilon \to 0$.

Proofs of results presented in this paper are akin to those discussed in Mishra and Prakasa Rao [8, 9]. In view of the fact that the observation process Y is driven by a Brownian motion in the model studied here, the arguments get much simplified and hence we consider this important special case. Detailed proofs are presented in Mishra and Prakasa Rao [10].

2. Signal and observation

Let us consider the model

$$\begin{aligned} dX_t &= \theta X_t dt + \epsilon\, dV_t^H,\ X_0 = x_0 \ne 0,\ 0 \le t \le T, \\ dY_t &= f_t(\tau) X_t dt + \epsilon\, dW_t,\ Y_0 = y_0,\ 0 \le t \le T, \end{aligned} \tag{2.1}$$

where $\{V_t^H(t), 0 \le t \le T\}$ is the standard fractional Brownian motions with Hurst parameter $H \in \left[\frac{1}{2}, 1\right]$ and $\{W_t, 0 \le t \le T\}$ is the standard Brownian motion independent of each other. Let $f_t(\tau) = h$ if $t \in [0, \tau]$ and $f_t(\tau) = g$ if $t \in (\tau, T]$, where h and g are known constants with $h \ne g$. Here τ is the change point and θ is called the drift parameter. We assume that the process $\{Y_t, 0 \le t \le T\}$ is observable but the state $\{X_t, 0 \le t \le T\}$ of the system is unobservable. The problem is to estimate the drift parameter θ and the change point τ based on the observation $Y = \{Y_t, 0 \le t \le T\}$ and study the asymptotic properties as $\epsilon \to 0$. The system (2.1) has a unique solution (X, Y) which is a Gaussian process. Suppose that we observe the process Y alone but would like to have information about the process X at time t. This problem is known as *filtering* the signal X at time t from the observation of Y up to time t. The optimal solution to this problem is the conditional expectation of X_t given the σ-algebra generated by the process $\{Y_s, 0 \le s \le t\}$. Since the processes (X, Y) is jointly Gaussian, the conditional expectation of X_t given $\{Y_s, 0 \le s \le t\}$ is linear function of the observation $\{Y_s, 0 \le s \le t\}$. It is also the *optimal filter* in the sense of minimizing the mean square error. Let $P_{\theta,\tau}$ denote the probability measure induced by the process $\{(X_t, Y_t), 0 \le t \le T\}$ when (θ, τ) is the true parameter vector and $E_{\theta,\tau}$ is the expectation under the probability measure $P_{\theta,\tau}$. The problem of finding the optimal filter reduces to finding the conditional mean $\pi_t(\theta, \tau, X) = E_{\theta,\tau}(X_t | Y_s, 0 \le s \le t)$. This problem leads to Kalman-Bucy filter if $H = \frac{1}{2}$. Le Breton [7] investigated this problem for a simple linear model driven by a fractional Brownian motion. Kutoyants [6, p. 206] considered the problem of estimation of the drift parameter and change point applying Kalman filtering technique using the equations for the optimal filter when the signal and the observation system are driven by independent Brownian motions. Kleptsyna et al. [5] presented a general approach to filtering for linear systems driven by independent fractional Brownian motions when the drift functions are bounded and smooth. A close look at the proofs in Kleptsyna et al. [5] indicate that their results continue to hold for the drift function of the type $f_t(\tau)$ when the observation process Y is considered as two distinct processes on the intervals $(0,\tau)$ and $[\tau,T]$ as was the case in the optimal filter derivation in the equations (6.27) to (6.30) in Kutoyants [6, p. 206] for linear systems driven by independent Brownian motions. Observe that, as $\epsilon \to 0$, the processes converge to the non-random functions

$$x_t = x_0 e^{\theta t},\ 0 \le t \le T$$

and

$$y_t(\tau) = y_0 + \int_0^t f_s(\tau) x_s ds,\ 0 \le t \le T.$$

Define

$$k_H = 2H\,\Gamma(\frac{3}{2} - H)\,\Gamma(H + \frac{1}{2}),$$

$$\kappa_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H},\ 0 < s < t$$

and

$$K_H(t,s) = H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}} dr. \tag{2.2}$$

Let us consider the transformed process

$$Z_t = \int_0^t \kappa_H(t,s) dX_s,\ 0 \le t \le T, \tag{2.3}$$

and let

$$M_t^H = \int_0^t \kappa_H(t,s) dV_s^H,\ t \ge 0. \tag{2.4}$$

Let

$$\lambda_H = \frac{2H\,\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}$$

and

$$w_t^H = \lambda_H^{-1} t^{2-2H}.$$

The process M^H is a Gaussian fundamental martingale associated with the fBm V^H with the quadratic variation w^H. Furthermore the semimartingale Z can be called the signal fundamental semimartingale (cf. Kleptsyna and Le Breton [3]). The natural filtrations of the processes X and Z coincide. In addition, it is known that

$$X_t = x_0 + \int_0^t K_H(t,s) dZ_s,\ 0 \le t \le T.$$

Suppose that $\{\eta_t, 0 \leq t \leq T\}$ is a random process adopted to the filtration $(\mathcal{F}_t)$ such that $E_{\theta,\tau}(|\eta_t|) < \infty$ on the underlying probability space $(\Omega, \mathcal{F}, P)$. Let $\pi_t(\theta, \tau, \eta)$ denote the conditional expectation of η_t given the observation $\{Y_s, 0 \leq s \leq t\}$. Let $\{\mathcal{Y}_t\}$ denote the filtration generated by the process Y. Let

$$\epsilon\, \nu_t = Y_t - \int_0^t \pi_s(\theta, \tau, X) ds,\ 0 \leq t \leq T,$$

where $\pi_t(\theta, \tau, X) = E_{\theta,\tau}[X_t | Y_s,\ 0 \leq s \leq t]$. The process $\nu = \{\nu_t, 0 \leq t \leq T\}$ is called the *innovation type process*. Kleptsyna et al. [4] proved that the process $\{\nu_t\}$ is a continuous Gaussian $(\mathcal{Y}_t)$-martingale with the quadratic variation function t and hence a Wiener process. Furthermore, if $\zeta = \{\zeta_t, 0 \leq t \leq T\}$ is a square integrable $(\mathcal{Y}_t)$–martingale, $\zeta_0 = 0$, then there exists a $(\mathcal{Y}_t)$-adapted process $\alpha = \{\alpha_t, 0 \leq t \leq T\}$ such that

$$E_{\theta,\tau}(\int_0^T \alpha_t^2 dt) < \infty$$

and

$$\zeta_t = \int_0^t \alpha_s d\nu_s,\ 0 \leq t \leq T.$$

3. Auxiliary results

Consider the linear system described by (2.1). We have the following representation for the process X from the discussion given above:

$$X_t = x_0 + \theta \int_0^t X_s ds + \epsilon \int_0^t K_H(t,s) dM_s^H,\ 0 \leq t \leq T.$$

An application of Theorem 4 in Kleptsyna et al. [5] to the process X leads to the equation

$$\pi_t(\theta, \tau, X) = x_0 + \int_0^t \pi_s(\theta, \tau, X) ds + \epsilon \int_0^t c(t,s) d\nu_s,\ 0 \leq t \leq T, \tag{3.1}$$

where $c(t,s)$ is a non-random function and $\{\nu(t), 0 \leq t \leq T\}$ is the innovation process.

In particular, by considering the special case $\epsilon = 0$ in the equation given above, we obtain the integral equations

$$\pi_t(\theta, \tau, x) = x_0 + \int_0^t \pi_s(\theta, \tau, x) ds,\ 0 \leq t \leq T.$$

Combining the above equations, it follows that there exist a non-random function $c(t,s)$, $0 \leq s \leq T$ such that

$$\begin{aligned} \pi_t(\theta, \tau, X) - \pi_t(\theta, \tau, x) = & \int_0^t (\pi_s(\theta, \tau, X) - \pi_s(\theta, \tau, x)) ds \\ & + \epsilon \int_0^t c(t,s) d\nu_s,\ 0 \leq t \leq T. \end{aligned} \tag{3.2}$$

Lemma 3.1. *Let* $\theta_v = \theta + \epsilon v$ *and* $\tau_u = \tau + \epsilon^2 u$. *Under the conditions stated above, there exist positive constants* c_t *and* c_{1t} *such that*

(i) $\sup_{0\le s\le t} E_{\theta,\tau}|X_s - x_s|^2 \le c_t\epsilon^2$,

(ii) $\sup_{0\le s\le t} E_{\theta,\tau}|\pi_s(\theta_v,\tau_u,X) - \pi_s(\theta_v,\tau_u,x)|^2 \le c_{1t}\epsilon^2 t$,

The bounds in (i) *and* (ii) *hold uniformly for* (θ_v,τ_u) *in a neighbourhood of* (θ,τ).

Proof. An application of the Grownwall's inequality implies (i) (cf. Prakasa Rao [15, p. 131]. Another application of Grownwall's inequality using the equation (3.2) shows that

$$|\pi_s(\theta_v,\tau_u,X) - \pi_s(\theta_v,\tau_u,x)| \le c_{1t}\epsilon \sup_{0\le s\le t}|\nu_s|,\ 0\le s\le t$$

and hence

$$\sup_{0\le s\le t} E_{\theta,\tau}|\pi_s(\theta_v,\tau_u,X) - \pi_s(\theta_v,\tau_u,x)|^2 \le c_{1t}\epsilon^2 t.$$ □

4. Main results

Fix θ,τ and define $\theta_v = \theta + \epsilon v$ and $\tau_u = \tau + \epsilon^2 u$. Suppose $u, v > 0$. Let

$$\Delta_t = f_t(\tau_u)\pi_t(\theta_v,\tau_u,X) - f_t(\tau)\pi_t(\theta,\tau,X)$$

and

$$\overline{\Delta}_t = f_t(\tau_u)\pi_t(\theta_v,\tau_u,x) - f_t(\tau)\pi_t(\theta,\tau,x)$$

We now consider the problem of estimation of the change point τ and the drift parameter θ based on the observation $\{Y_t, 0\le t\le T\}$ by the method of maximum likelihood. Let $P_{\theta,\tau}$ be the probability measure generated by the process Y on the space $C[0,T]$ associated with the uniform topology when τ is the change point and θ is the drift parameter. Let θ_0 be the true drift parameter and τ_0 be the true change point. The maximum likelihood estimator $(\hat\theta_\epsilon,\hat\tau_\epsilon)$, based on the observation $\{Y_t,\ 0\le t\le T\}$, is a random vector at which the likelihood function

$$\frac{dP_{\theta,\tau}}{dP_{\theta_0,\tau_0}}$$

is supremum over the interval $[t_1,t_2]\times\Theta$. We assume that there exists a measurable maximum likelihood estimator $(\hat\theta_\epsilon,\hat\tau_\epsilon)$. Sufficient conditions for the existence of a measurable maximum likelihood estimator are given in Prakasa Rao [14]. Note that

$$J_\tau^2 = \lim_{\epsilon\to 0}\frac{1}{\epsilon^2 u}\int_\tau^{\tau+\epsilon^2 u}(g-h)^2 x_t^2\,dt = (g-h)^2 x_\tau^2$$

exists. Define

$$L_0(u,v) = u\xi - \frac{1}{2}u^2\sigma^2(\theta,\tau) + J_\tau W_1(v) - \frac{1}{2}|v|J_\tau^2 \quad \text{for } v\ge 0,\ u\in\mathbb{R},$$

where $\{W_1(v), v \geq 0\}$ is a standard Wiener process and ξ is an independent Gaussian random variable with mean zero and some variance $\sigma^2(\theta, \tau)$ to be specified later. Similarly, for $v < 0$ and $u \in \mathbb{R}$, let

$$L_0(u, v) = u\xi - \frac{1}{2}u^2\sigma^2(\theta, \tau) + J_\tau W_2(-v) - \frac{1}{2}|v|J_\tau^2,$$

where $\{W_2(-v), v < 0\}$ is another standard Wiener process. Here W_1 and W_2 are independent standard Wiener processes.

We now state the main result.

Theorem 4.1. *Let τ denote the true change point and θ be the true drift parameter. Let $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ denote the maximum likelihood estimator of (θ, τ) based on the observation of the process Y satisfying the linear system defined by* (2.1)*. Then the normalized random variable*

$$(\epsilon^{-2}(\hat{\tau}_\epsilon - \tau), \epsilon^{-1}(\hat{\theta}_\epsilon - \theta))$$

converges in law, as $\epsilon \to 0$, to a random vector whose distribution is the bivariate distribution of location of the maximum of the random field $\{L_0(u, v), -\infty < u, v < \infty\}$ as defined above.

Since the proof of this theorem is similar to the results proved in Mishra and Prakasa Rao [8, 9] for the estimation of drift parameter and the change point, we will only sketch the proof. For detailed proofs, see Mishra and Prakasa Rao [10]. Before we give a proof of this main result, we derive some related results.

Consider the log-likelihood ratio random field

$$\begin{aligned} L_\epsilon(u, v) =& \log \frac{dP_{\theta_v, \tau_u}}{dP_{\theta, \tau}} \\ =& \frac{1}{\epsilon} \int_0^T [f_t(\tau_u)\pi_t(\theta_v, \tau_u, X) - f_t(\tau)\pi_t(\theta, \tau, X)]\, d\nu_t \\ &- \frac{1}{2\epsilon^2} \int_0^T [f_t(\tau_u)\pi_t(\theta_v, \tau_u, X) - f_t(\tau)\pi_t(\theta, \tau, X)]^2 dt \\ =& \frac{1}{\epsilon} \int_0^T \Delta_t d\nu_t - \frac{1}{2\epsilon^2} \int_0^T \Delta_t^2 dt \end{aligned}$$

for fixed $u > 0$ and $v > 0$ such that $0 \leq \tau, \tau + \epsilon^2 u \leq T$ and $\theta, \theta_v \in \Theta$. Let $C[K]$ denote the space of continuous functions defined on a compact set $K \subset \mathbb{R}^2$.

Theorem 4.2 (Local asymptotic normality)**.** *Let $K \subset \mathbb{R}^2$ be compact. The probability measure generated by the log-likelihood ratio random fields $\{L_\epsilon(u, v), (u, v) \in K\}$ on $C[K]$ converges weakly to the probability measure generated by the random field $\{L_0(u, v), (u, v) \in K\}$ on $C[K]$ associated with the uniform norm topology as $\epsilon \to 0$.*

In order to prove Theorem 4.2, it is sufficient to prove that the finite dimensional distributions of the random field $\{L_\epsilon(u, v), (u, v) \in K\}$ converge weakly to the corresponding finite dimensional distributions of the random field $\{L_0(u,v),(u,v)\in K\}$ and the family of measures generated by the random fields $\{L_\epsilon(u,v),(u,v)\in K\}$ for different ϵ is tight (cf. Prakasa Rao [13]).

5. Proofs of Theorems 4.1 and 4.2

We now state some lemmas and sketch the proofs. Detailed proofs of the lemmas are given in the authors research report (Mishra and Prakasa Rao [10]).

Lemma 5.1. *There exists a constant $c>0$ possibly depending on H and T such that*

$$\sup_{0\le t_1\le\tau\le t_2\le T,\theta\in\Theta}\ \sup_{0\le t\le T} E_{\theta,\tau}[\Delta_t-\bar{\Delta}_t]^2\le c\epsilon^2. \tag{5.1}$$

This lemma follows as application of Lemma 3.1. We omit the proof.

Following the arguments in Kutoyants [6, pp. 168–169], it can be shown that

$$\epsilon^{-2}E_{\theta,\tau}[||f_t(\tau_u)\pi(\theta_v,\tau_u,X)-f_t(\tau)\pi(\theta,\tau,X)||^2]\to u^2\sigma^2(\theta,\tau)+vJ_\tau^2$$

as $\epsilon\to 0$ where

$$\sigma^2(\theta,\tau)=g^2\int_0^\tau x_t^2dt+h^2\int_\tau^T x_t^2dt. \tag{5.2}$$

Lemma 5.2. *The finite-dimensional distributions of the random field*

$$\{L_\epsilon(u,v),(u,v)\in K\}$$

converge to the corresponding finite dimensional distributions of the random field

$$\{L_0(u,v),(u,v)\in K\}$$

as $\epsilon\to 0$.

Proof. We will first investigate the convergence of the marginal distributions of the random field $L_\epsilon(u,v)$ as $\epsilon\to 0$. The convergence of other classes of finite-dimensional distributions follow from the Cramer-Wold device. Note that

$$L_\epsilon(u,v)=\frac{1}{\epsilon}\int_0^T\Delta_t d\nu_t-\frac{1}{2\epsilon^2}\int_0^T\Delta_t^2dt. \tag{5.3}$$

Consider

$$\begin{aligned}\frac{1}{\epsilon}\int_0^T\Delta_t d\nu_t&=\frac{1}{\epsilon}\int_0^\tau\Delta_t d\nu_t+\frac{1}{\epsilon}\int_\tau^{\tau+\epsilon^2u}\Delta_t d\nu_t+\frac{1}{\epsilon}\int_{\tau+\epsilon^2u}^T\Delta_t d\nu_t\\&=I_1+I_2+I_3\ \ \text{(say)}.\end{aligned}$$

Note that

$$I_1=\frac{1}{\epsilon}\int_0^\tau\Delta_t d\nu_t=\frac{1}{\epsilon}\int_0^\tau(\Delta_t-\overline{\Delta}_t)d\nu_t+\frac{1}{\epsilon}\int_0^\tau\overline{\Delta}_t d\nu_t.$$

The first integral converges to zero in probability and the second integral is

$$\frac{1}{\epsilon}\int_0^\tau\bar{\Delta}_t d\nu_t=vh\int_0^\tau x_t d\nu_t+o_p(1).$$

Similarly

$$I_3 = \frac{1}{\epsilon}\int_{\tau+\epsilon^2 u}^{T} \Delta_t d\nu_t = vg\int_{\tau}^{T} x_t d\nu_t + o_p(1).$$

Observe that

$$I_2 = \frac{1}{\epsilon}\int_{\tau}^{\tau+\epsilon^2 u} \Delta_t d\nu_t$$

which is Gaussian with mean zero and variance

$$\frac{1}{\epsilon^2}\int_{\tau}^{\tau+\epsilon^2 u} E_{\theta,\tau}(\Delta_t^2)dt = J_\tau^2 + o(1).$$

From the above computations, observe that,as $\epsilon \to 0$,

$$\frac{1}{\epsilon^2}\int_0^{\tau} \Delta_t^2 dt = v^2h^2\int_0^{\tau} x_t^2 dt + o_p(1)$$

and similarly

$$\frac{1}{\epsilon^2}\int_{\tau+\epsilon^2 u}^{T} \Delta_t^2 dt = v^2g^2\int_{\tau}^{T} x_t^2 dt + o_p(1).$$

Furthermore

$$\frac{1}{\epsilon^2}\int_{\tau}^{\tau+\epsilon^2 u} \Delta_t^2 dt = (gx_t - hx_t)^2 u + o_p(1) = uJ_\tau^2 + o_p(1).$$

As a consequence of the above computations, we get that the random variable $L_0(u,v)$ is asymptotically Gaussian with the mean

$$-\frac{1}{2}v^2\sigma^2(\theta,\tau) - \frac{1}{2}J_\tau^2 u$$

and the variance

$$v^2\sigma^2(\theta,\tau) + J_\tau^2 u$$

for $u > 0$ and $v \in \mathbb{R}$. Similar results hold for $u < 0$ and $v \in \mathbb{R}$.

We have proved the convergence of the univariate distributions of the random field $\{L_\epsilon(u,v), (u,v) \in K\}$, as $\epsilon \to 0$, after proper scaling of the process. Convergence of all the other finite-dimensional distributions of the random field $\{L_\epsilon(u,v), (u,v) \in K\}$ as $\epsilon \to 0$, after proper scaling, follows by an application of the Cramer-Wold device. □

Lemma 5.3. *Let* $\Gamma_\epsilon(u,v) = \exp\{L_\epsilon(u,v)\}$. *Then, for any compact set* $K \subset \mathbb{R}^2$, *there exists a constant* $C > 0$ *such that*

$$\sup_{(u_i,v_i)\in K, i=1,2} E_{\theta,\tau}\left|\Gamma_\epsilon^{\frac{1}{4}}(u_2,v_2) - \Gamma_\epsilon^{\frac{1}{4}}(u_1,v_1)\right|^4 \le C[(u_1-u_2)^2 + (v_1-v_2)^4].$$

For detailed proof of this lemma, see Lemma 6.5 in Mishra and Prakasa Rao [10].

Proof of Theorem 4.2. As a consequence of Lemma 5.3, it follows that the family of probability measures generated by the random fields $\{\Gamma_\epsilon^{\frac{1}{4}}(u,v),(u,v)\in K\}$ on $C[K]$ with uniform topology is tight from the results in Billingsley [1] (cf. Prakasa Rao [14] and hence the family of probability measures generated by the random fields $\{L_\epsilon(u,v),(u,v)\in K\}$ on $C[K]$ is tight.

As a consequence of lemmas derived above, it follows that the family of probability measures generated by the random fields $\{L_\epsilon(u,v),(u,v)\in K\}$ on $C[K]$ converge weakly to the probability measure generated by the random field $\{L_0(u,v),(u,v)\in K\}$ on $C[K]$ from the general theory of weak convergence of probability measures on complete separable metric spaces (cf. Billingsley [1], Parthasarathy [11], Prakasa Rao [14] and Ibragimov and Has'minskii [2]). This completes the proof of Theorem 4.2. □

It remains to show that the maximum likelihood estimator $(\hat\theta_\epsilon,\hat\tau_\epsilon)$ will lie in a compact set K with probability tending to one as $\epsilon\to 0$ after suitable normalizations of the components.

Lemma 5.4. *Let $\Gamma_\epsilon(u,v)=\exp\{L_\epsilon(u,v)\}, u,v\in\mathbb{R}$. Then, for any compact set $K\subset\mathbb{R}^2$ and for any $0<p<1$, there exists a positive constant C such that*

$$\sup_{(u,v)\in K} E_{\theta,\tau}[(\Gamma_\epsilon(u,v))^p]\le e^{-C\,g(u,v)}, \tag{5.4}$$

where $g(u,v)=k_1|u|^2+k_2|v|^2$ for some $k_1>0$ and $k_2>0$.

Proof of this lemma ia analogous to the proof of Lemma 6.7 in Mishra and Prakasa Rao [9]. For details, see Lemma 6.7 in Mishra and Prakasa Rao [10].

Proof of Theorem 4.1. Let $C[K]$ denote the family of continuous functions defined on a compact set K in $\mathbb{R}$. In view of Theorem 4.2, it follows that the family of probability measures generated by the random fields $\{L_\epsilon(u,v),(u,v)\in K\},\epsilon>0$ on $C[K]$ converge weakly to the probability measure generated by the random field $\{L_0(u,v),(u,v)\in K\}$ on $C[K]$ as $\epsilon\to 0$. Let $(\hat u_\epsilon,\hat v_\epsilon)$ denote a point at which the random field $\{L_\epsilon(u,v),(u,v)\in K\}$ is maximum. Let (u_0,v_0) denote the location of the maxima of the process $\{L_0(u,v),(u,v)\in K\}$ on $C[K]$. The location (u_0,v_0) of the maxima is unique almost surely by the property of Gaussian random fields. Since the random fields $\{L_\epsilon(u,v),(u,v)\in K\},\epsilon>0$ on $C[K]$ converge weakly to the random field $\{L_0(u,v),(u,v)\in K\}$ on $C[K]$ as $\epsilon\to 0$, by the continuous mapping theorem, it follows that the distribution of $(\hat\tau_\epsilon,\hat\theta_\epsilon))$ appropriately normalized converges in law to the distribution of (u_0,v_0) by the continuous mapping theorem (cf. Billingsley [1]). Lemma 5.4 implies that the random variable $(\hat u_\epsilon,\hat v_\epsilon)=(\epsilon^{-2}(\hat\tau_\epsilon-\tau),\epsilon^{-1}(\hat\theta_\epsilon-\theta))\in K$ with probability tending to one as $\epsilon\to 0$. Applying arguments similar to those in Theorem 10.1 in Chapter II, p. 103 of Ibragimov and Has'minskii [2] (cf. Prakasa Rao [12]), we obtain the following result. Let (θ,τ) be the true parameter. As a consequence of the arguments and the discussion given above, it follows that the random variable $(\epsilon^{-2}(\hat\tau_\epsilon-\tau),\epsilon^{-1}(\hat\theta_\epsilon-\theta))$ converges in law to the distribution of the random variable (u_0, v_0) which is the location of the maximum of the random field $\{L_0(u,v),(u,v) \in R\}$, as $\epsilon \to 0$. $\square$

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