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WEIGHTED VECTOR-VALUED BOUNDS FOR A CLASS OF MULTILINEAR SINGULAR INTEGRAL OPERATORS AND APPLICATIONS

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Abstract. In this paper, we investigate the weighted vector-valued bounds for a class of multilinear singular integral operators, and its commutators, from $L^{p_1}(l^{q_1};\mathbb{R}^n,w_1)\times\cdots\times L^{p_m}(l^{q_m};\mathbb{R}^n,w_m)$ to $L^p(l^q;\mathbb{R}^n,\nu_{\vec{w}})$, with $p_1,\dots,p_m,\ q_1,\,\dots,\,q_m\in(1,\,\infty)$, $1/p=1/p_1+\cdots+1/p_m$, $1/q=1/q_1+\cdots+1/q_m$ and $\vec{w}=(w_1,\,\dots,\,w_m)$ a multiple $A_{\vec{P}}$ weights. Our argument also leads to the weighted weak type endpoint estimates for the commutators. As applications, we obtain some new weighted estimates for the Calderón commutator.

1. Introduction

In his remarkable work [31], Muckenhoupt characterized the class of weights w such that M, the Hardy-Littlewood maximal operator, satisfies the weighted L^p $(p\in(1,\,\infty))$ estimate

$$\|Mf\|_{L^{p,\,\infty}(\mathbb{R}^n,\,w)}\lesssim\|f\|_{L^p(\mathbb{R}^n,\,w)}. \tag{1.1}$$

The inequality (1.1) holds if and only if w satisfies the $A_p(\mathbb{R}^n)$ condition, that is,

$$[w]_{A_p}:=\sup_Q\Big(\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}(x)\mathrm{d}x\Big)^{p-1}<\infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$, $[w]_{A_p}$ is called the A_p constant of w. Also, Muckenhoupt proved that M is bounded on $L^p(\mathbb{R}^n,\,w)$ if and only if w satisfies the $A_p(\mathbb{R}^n)$ condition. Since then, considerable attention has been paid to the theory of $A_p(\mathbb{R}^n)$ and the weighted norm inequalities with $A_p(\mathbb{R}^n)$ weights for main operators in Harmonic Analysis, see [15, Chapter 9] and related references therein.

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However, the classical results on the weighted norm inequalities with $A_p(\mathbb{R}^n)$ weights did not reflect the quantitative dependence of the $L^p(\mathbb{R}^n,\, w)$ operator norm in terms of the relevant constant involving the weights. The question of the sharp dependence of the weighted estimates in terms of the $A_p(\mathbb{R}^n)$ constant specifically raised by Buckley [1], who proved that if $p \in (1,\, \infty)$ and $w \in A_p(\mathbb{R}^n)$, then

$$\|Mf\|_{L^p(\mathbb{R}^n,\, w)} \lesssim_{n,\, p} [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(\mathbb{R}^n,\, w)}. \tag{1.2}$$

Moreover, the estimate (1.2) is sharp in the sense the exponent $1/(p-1)$ can not be replaced by a smaller one. Hytönen and Pérez [23] improved the estimate (1.2), and showed that

$$\|Mf\|_{L^p(\mathbb{R}^n,\, w)} \lesssim_{n,\, p} \left([w]_{A_p}[w^{-\frac{1}{p-1}}]_{A_\infty}\right)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n,\, w)}, \tag{1.3}$$

where and in the following, for a weight u, $[u]_{A_\infty}$ is defined by

$$[u]_{A_\infty} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{u(Q)} \int_Q M(u\chi_Q)(x)\mathrm{d}x.$$

It is well known that for $w \in A_p(\mathbb{R}^n)$, $[w^{-\frac{1}{p-1}}]_{A_\infty} \lesssim [w]_{A_p}^{\frac{1}{p-1}}$. Thus, (1.3) is more subtle than (1.2).

The sharp dependence of the weighted estimates of singular integral operators in terms of the $A_p(\mathbb{R}^n)$ constant was much more complicated. Petermichl [33,34] solved this question for Hilbert transform and Riesz transform. Hytönen [21] proved that for a Calderón-Zygmund operator T and $w \in A_2(\mathbb{R}^n)$,

$$\|Tf\|_{L^2(\mathbb{R}^n,\, w)} \lesssim_n [w]_{A_2} \|f\|_{L^2(\mathbb{R}^n,\, w)}. \tag{1.4}$$

This solved the so-called A_2 conjecture. Combining the estimate (1.4) and the extrapolation theorem in [11], we know that for a Calderón-Zygmund operator T, $p \in (1,\, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\|Tf\|_{L^p(\mathbb{R}^n,\, w)} \lesssim_{n,\, p} [w]_{A_p}^{\max\{1,\, \frac{1}{p-1}\}} \|f\|_{L^p(\mathbb{R}^n,\, w)}. \tag{1.5}$$

In [24], Lerner gave a much simplier proof of (1.5) by directly controlling the Calderón-Zygmund operator using sparse operators.

Let $K(x;\, y_1,\, \dots,\, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $\mathbb{R}^{mn}$. An operator T defined on $\mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)$ (Schwartz space) and taking values in $\mathcal{S}'(\mathbb{R}^n)$, is said to be an m-multilinear singular integral operator with kernel K, if T is m-multilinear, and satisfies that

$$T(f_1,\, \dots,\, f_m)(x) = \int_{\mathbb{R}^{mn}} K(x;\, y_1,\, \dots,\, y_m) \prod_{j=1}^m f_j(y_j)\mathrm{d}y_1 \dots \mathrm{d}y_m \tag{1.6}$$

for bounded functions $f_1, \dots, f_m$ with compact supports, and $x \in \mathbb{R}^n \setminus \cap_{j=1}^m \operatorname{supp} f_j$. Operators of this type were originated in the remarkable works of Coifman and Meyer [6], [7], and are useful in multilinear analysis. We say that T is an m-linear Calderón-Zygmund operator with kernel K, if T is bounded from $L^{r_1}(\mathbb{R}^n)\times\cdots\times L^{r_m}(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ for some $r_1,\,\dots,\,r_m\in(1,\,\infty)$ and $r\in(1/m,\,\infty)$ with $1/r=1/r_1+\cdots+1/r_m$, and K is a multilinear Calderón-Zygmund kernel, that is, K satisfies the size condition that for all $(x,\,y_1,\dots,y_m)\in\mathbb{R}^{(m+1)n}$ with $x\neq y_j$ for some $1\le j\le m$,

$$|K(x;\,y_1,\dots,y_m)|\lesssim\frac{1}{\Big(\sum_{j=1}^m|x-y_j|\Big)^{mn}},\tag{1.7}$$

and satisfies the regularity condition that for some $\alpha\in(0,\,1]$,

$$\begin{aligned}&\|K(x;\,y_1,\dots,y_m)-K(x';\,y_1,\dots,y_m)|\\&\lesssim\frac{|x-x'|^{\alpha}}{\Big(\sum_{j=1}^m|x-y_j|\Big)^{mn+\alpha}},\ \text{if}\ \max_{1\le k\le m}|x-y_k|\ge2|x-x'|,\end{aligned}\tag{1.8}$$

and for all $1\le j\le m$,

$$\big|K(x;\,y_1,\dots,y_j\dots,y_m)-K(x;\,y_1,\dots,y_j',\dots,y_m)\big|\lesssim\frac{|y_j-y_j'|^{\alpha}}{(\sum_{i=1}^m|x-y_i|)^{mn+\alpha}}$$

whenever $\max_{1\le k\le m}|x-y_k|\ge2|y_j-y_j'|$. Grafakos and Torres [17] considered the behavior of multilinear Calderón-Zygmund operators on $L^1(\mathbb{R}^n)\times\cdots\times L^1(\mathbb{R}^n)$, and established a $T1$ type theorem for the operator T. To consider the weighted estimates for the multilinear Calderón-Zygmund operators, Lerner, Ombrossi, Pérez, Torres and Trojillo-Gonzalez [26] introduced the following definition.

Definition 1.1. Let $m\in\mathbb{N}$, $w_1,\dots,w_m$ be weights, $p_1,\dots,p_m\in[1,\,\infty)$, $p\in[1/m,\,\infty)$ with $1/p=1/p_1+\cdots+1/p_m$. Set $\vec{w}=(w_1,\,\dots,\,w_m)$, $\vec{P}=(p_1,\,...,\,p_m)$ and $\nu_{\vec{w}}=\prod_{k=1}^m w_k^{p/p_k}$. We say that $\vec{w}\in A_{\vec{P}}(\mathbb{R}^{mn})$ if

$$[\vec{w}]_{A_{\vec{P}}}=\sup_{Q\subset\mathbb{R}^n}\Big(\frac{1}{|Q|}\int_Q\nu_{\vec{w}}(x)\,\mathrm{d}x\Big)\prod_{k=1}^m\Big(\frac{1}{|Q|}\int_Q w_k^{-\frac{1}{p_k-1}}(x)\,\mathrm{d}x\Big)^{p/p_k'}<\infty,$$

here and in the following, for $r\in[1,\infty)$, $r'=\frac{r}{r-1}$; when $p_k=1$, $\Big(\frac{1}{|Q|}\int_Q w_k^{-\frac{1}{p_k-1}}\Big)^{\frac{1}{p_k'}}$ is understood as $\big(\inf_Q w_k\big)^{-1}$.

Lerner et al. [26] proved that if $p_1,\,\dots,\,p_m\in[1,\,\infty)$ and $p\in[1/m,\,\infty)$ with $1/p=1/p_1+\cdots+1/p_m$, and $\vec{w}=(w_1,\,\dots,\,w_m)\in A_{\vec{P}}(\mathbb{R}^{mn})$, then an m-linear Calderón-Zygmund operator T is bounded from $L^{p_1}(\mathbb{R}^n,\,w_1)\times\cdots\times L^{p_m}(\mathbb{R}^n,\,w_m)$ to $L^{p,\,\infty}(\mathbb{R}^n,\nu_{\vec{w}})$, and when $\min_{1\le j\le m}p_j>1$, T is bounded from $L^{p_1}(\mathbb{R}^n,w_1)\times\cdots\times L^{p_m}(\mathbb{R}^n,w_m)$ to $L^p(\mathbb{R}^n,\,\nu_{\vec{w}})$. Li, Moen and Sun [29] considered the sharp dependence of the weighted estimates of multilinear Calderón-Zygmund operators in terms of the $A_{\vec{P}}(\mathbb{R}^{mn})$ constant, and proved the following theorem.

Theorem 1.2. *Let T be an m-linear Calderón-Zygmund operator, $p_1, \dots, p_m \in (1, \infty)$, $p \in [1, \infty)$ such that $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}(\mathbb{R}^{mn})$. Then*

$$\|T(f_1, \dots, f_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p_1'}{p}, \cdots, \frac{p_m'}{p}\}} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}.$$

Moreover, the exponent on $[\vec{w}]_{A_{\vec{P}}}$ is sharp.

Conde-Alonso and Rey [8], Lerner and Nazarov [25] proved that the conclusion in Theorem 1.2 is still true for the case $p \in (1/m, 1)$. For other works about the weighted estimates of multilinear Calderón-Zygmund operators, see [2, 10, 30] and references therein.

To consider the mapping properties for the commutator of Calderón, Duong, Grafakos and Yan [13] introduced a class of multilinear singular integral operators via the following generalized approximation to the identity.

Definition 1.3. A family of operators $\{A_t\}_{t>0}$ is said to be an approximation to the identity, if for every $t > 0$, A_t can be represented by the kernel at in the following sense: for every function $u \in L^p(\mathbb{R}^n)$ with $p \in [1, \infty]$ and a.e. $x \in \mathbb{R}^n$,

$$A_t u(x) = \int_{\mathbb{R}^n} a_t(x, y) u(y) \mathrm{d}y,$$

and the kernel a_t satisfies that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$|a_t(x, y)| \le h_t(x, y) = t^{-n/s} h\Big(\frac{|x-y|}{t^{1/s}}\Big), \tag{1.9}$$

where $s > 0$ is a constant and h is a positive, bounded and decreasing function such that for some constant $\eta > 0$,

$$\lim_{r \to \infty} r^{n+\eta} h(r) = 0. \tag{1.10}$$

Assumption 1.4. For each fixed j with $1 \le j \le m$, there exists an approximation to the identity $\{A_t^j\}_{t>0}$ with kernels $\{a_t^j(x, y)\}_{t>0}$, and there exist kernels $K_t^j(x; y_1, \dots, y_m)$, such that for bounded functions $f_1, \dots, f_m$ with compact supports, and $x \in \mathbb{R}^n \setminus \cap_{k=1}^m \operatorname{supp} f_k$,

$$T(f_1, \dots, f_{j-1}, A_t^j f_j, f_{j+1} \dots, f_m)(x) = \int_{\mathbb{R}^{nm}} K_t^j(x; y_1, \dots, y_m) \prod_{k=1}^{m} f_k(y_k) \mathrm{d}\vec{y},$$

and there exists a function $\phi \in C(\mathbb{R})$ with $\operatorname{supp} \phi \subset [-1, 1]$, and a constant $\varepsilon \in (0, 1]$, such that for all $x, y_1, \dots, y_m \in \mathbb{R}^n$ and all $t > 0$ with $2t^{1/s} \le |x - y_j|$,

$$\begin{aligned} &|K(x; y_1, \dots, y_m) - K_t^j(x; y_1, \dots, y_m)| \\ &\lesssim \frac{t^{\varepsilon/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\varepsilon}} + \frac{1}{(\sum_{k=1}^m |x-y_k|)^{mn}} \sum_{1 \le i \le m,\, i \ne j} \phi\Big(\frac{|y_i - y_j|}{t^{1/s}}\Big). \end{aligned}$$

As it was pointed out in [13], an operator with such a kernel is called a multilinear singular integral operator with non-smooth kernel, since the kernel K may enjoy no smoothness in the variables $y_1 \dots, y_m$. Also, it was pointed out in [13] that if T is an m-linear Calderón-Zygmund operator, then T also satisfies Assumption 1.4. Duong, Grafakos and Yan [13] proved that if T satisfies Assumption 1.4, and is bounded from $L^{r_1}(\mathbb{R}^n)\times\cdots\times L^{r_m}(\mathbb{R}^n)$ to $L^{r,\infty}(\mathbb{R}^n)$ for some $r_1, \dots, r_m \in (1, \infty)$ and $r \in (1/m, \infty)$ with $1/r = 1/r_1+\cdots+1/r_m$, then T is also bounded from $L^1(\mathbb{R}^n)\times\cdots\times L^1(\mathbb{R}^n)$ to $L^{1/m,\infty}(\mathbb{R}^n)$. Recently, Hu and Li [19] considered the mapping properties from $L^1(l^{q_1}; \mathbb{R}^n)\times\cdots\times L^1(l^{q_m}; \mathbb{R}^n)$ to $L^{1/m,\infty}(l^q; \mathbb{R}^n)$ for the multilinear operator which satisfies Assumption 1.4.

To consider the weighted estimates with $A_p(\mathbb{R}^n)$ weights for multilinear singular integral operators with nonsmooth kernels, Duong et al. [12] introduced the following two assumptions.

Assumption 1.5. There exists an approximation to the identity $\{B_t\}_{t>0}$ with kernels $\{b_t(x, y)\}_{t>0}$, and there exist kernels $\{K_t^0(x; y_1, \dots, y_m)\}_{t>0}$ such that

$$K_t^0(x; y_1, \dots, y_m) = \int_{\mathbb{R}^n} K(z; y_1, \dots, y_m) b_t(x, z)\mathrm{d}z,$$

and there exists a function $\psi \in C(\mathbb{R})$ with $\operatorname{supp}\psi \subset [-1, 1]$, and a constant $\gamma \in (0, 1]$, such that for all $x, y_1, \dots, y_m \in \mathbb{R}^n$ and all $t > 0$ with $2t^{1/s} \le \max_{1\le k\le m}|x - y_k|$,

$$\begin{aligned}&|K(x; y_1, \dots, y_m) - K_t^0(x; y_1, \dots, y_m)|\\ \lesssim\ & \frac{t^{\gamma/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\gamma}} + \frac{1}{(\sum_{k=1}^m |x-y_k|)^{mn}} \sum_{1\le j\le m} \psi\Big(\frac{|x-y_j|}{t^{1/s}}\Big).\end{aligned}$$

Assumption 1.6. The kernel $K_t^0(x; y_1, \dots, y_m)$ in Assumption 1.5 satisfies the size condition that

$$|K_t^0(x; y_1, \dots, y_m)| \lesssim \frac{1}{(\sum_{j=1}^m |x-y_j|)^{mn}}$$

whenever $2t^{1/s} \le \min_{1\le j\le m}|x - y_j|$, and the regularity condition that

$$|K_t^0(x; y_1, \dots, y_m) - K_t^0(x'; y_1, \dots, y_m)| \lesssim \frac{t^{\gamma/s}}{(\sum_{j=1}^m |x-y_j|)^{mn+\gamma}}$$

whenever $2|x - x'| \le t^{1/s}$ and $2t^{1/s} \le \min_{1\le j\le m}|x - y_j|$.

Duong et al. [12] proved that if T satisfies Assumption 1.4, Assumption 1.5 and Assumption 1.6, and is bounded from $L^{q_1}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for some $q_1, \dots, q_m \in (1, \infty)$ and $q \in (0, \infty)$ with $1/q = 1/q_1 + \cdots + 1/q_m$, then for $p_1, \dots, p_m \in [1, \infty)$ and $p \in (0, \infty)$ with $1/p = 1/p_1 + \cdots + 1/p_m$, and $w \in A_{\min_{1\le j\le m} p_j}(\mathbb{R}^n)$, both T and T^* are bounded from $L^{p_1}(\mathbb{R}^n, w) \times \cdots \times L^{p_m}(\mathbb{R}^n, w)$ to $L^{p,\infty}(\mathbb{R}^n, w)$, and when $\min_{1\le j\le m} p_j > 1$, T is bounded from $L^{p_1}(\mathbb{R}^n, w) \times \cdots \times L^{p_m}(\mathbb{R}^n, w)$ to $L^p(\mathbb{R}^n, w)$. Grafakos, Liu and Yang

[16] proved that if T satisfies Assumption 1.4, Assumption 1.5 and Assumption 1.6, and is bounded from $L^{q_1}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for some $q_1,\dots,q_m\in(1,\infty)$ and $q\in(0,\infty)$ with $1/q=1/q_1+\cdots+1/q_m$, then T enjoy the same weighted estimates with $A_{\vec{P}}(\mathbb{R}^{mn})$ as the multilinear Calderón-Zygmund operators.

The first purpose of this paper is to give an extension of Theorem 1.2 to the operators satisfying Assumption 1.4. We further assume the kernel K satisfies the following regularity condition: for $x,x',y_1,\dots,y_m\in\mathbb{R}^n$ with

$$\begin{aligned}(1.11)\qquad &|K(x;y_1,\dots,y_m)-K(x';y_1,\dots,y_m)|\\ &\lesssim\frac{|x-x'|^{\gamma}}{\big(\sum_{j=1}^m|x-y_j|\big)^{nm+\gamma}},\ \text{if } 12|x-x'|<\min_{1\le j\le m}|x-y_j|.\end{aligned}$$

We remark that the regularity condition (1.11) was introduced in [20] in order to established certain weighted estimates for the Calderón commutators. The condition (1.11) is different from, and weaker than the condition (1.8). As in the proof of Theorem 1.1 in [20], we can proved that if K satisfies the Assumption 1.5 and Assumption 1.6, it then also satisfies (1.11). On the other hand, it is obvious that if T is an m-linear Calderón-Zygmund operator, then T also satisfies (1.11). Thus, the operators we consider here contain multilinear Calderón-Zygmund operators and multilinear singular integral operators with non-smooth kernels. To state our results, we first recall some notations.

Let $p,r\in(0,\infty]$ and w be a weight. As usual, for a sequence of numbers $\{a_k\}_{k=1}^{\infty}$, we denote $\|\{a_k\}\|_{l^r}=\big(\sum_k|a_k|^r\big)^{1/r}$. The space $L^p(l^r;\mathbb{R}^n,w)$ is defined as

$$L^p(l^r;\mathbb{R}^n,w)=\big\{\{f_k\}_{k=1}^{\infty}:\ \|\{f_k\}\|_{L^p(l^r;\mathbb{R}^n,w)}<\infty\big\},$$

where

$$\|\{f_k\}\|_{L^p(l^r;\mathbb{R}^n,w)}=\Big(\int_{\mathbb{R}^n}\|\{f_k(x)\}\|_{l^r}^p w(x)\,\mathrm{d}x\Big)^{1/p}.$$

When $w\equiv1$, we denote $\|\{f_k\}\|_{L^p(l^r;\mathbb{R}^n,w)}$ by $\|\{f_k\}\|_{L^p(l^r;\mathbb{R}^n)}$ for simplicity.

Our first result can be stated as follows.

Theorem 1.7. *Let $m\ge2$, T be an m-linear operator with kernel K in the sense of* (1.6), *$r_1,\dots r_m\in(1,\infty)$, $r\in(1/m,\infty)$ such that $1/r=1/r_1+\cdots+1/r_m$. Suppose that*

(i) *T is bounded from $L^{r_1}(\mathbb{R}^n)\times\cdots\times L^{r_m}(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$;*
(ii) *The kernel K satisfies size condition* (1.7) *and regular condition* (1.11);
(iii) *T satisfies the Assumption* 1.4.

Let $p_1,\dots,p_m$, $q_1,\dots,q_m\in(1,\infty)$, $p,q\in(\frac{1}{m},\infty)$ such that $1/p=1/p_1+\cdots+1/p_m$, $1/q=1/q_1+\cdots+1/q_m$, $\vec{w}=(w_1,\dots,w_m)\in A_{\vec{P}}(\mathbb{R}^{mn})$. Then

$$(1.12)\qquad \|\{T(f_1^k,\dots,f_m^k)\}\|_{L^p(l^q;\mathbb{R}^n,\nu_{\vec{w}})}\lesssim[\vec{w}]_{A_{\vec{P}}}^{\max\{1,\frac{p_1'}{p},\cdots,\frac{p_m'}{p}\}}\prod_{j=1}^m\|\{f_j^k\}\|_{L^{p_j}(l^{q_j};\mathbb{R}^n,w_j)}.$$

Remark 1.8. As we pointed out, operators in Theorem 1.7 contain multilinear Calderón-Zygmund operators as examples. This, together with the examples in [29], shows that the estimate (1.12) is sharp. On the other hand, Theorem 1.7 gives an vector-valued analogy of the weighted bounds for the multilinear Calderón-Zygmund operators obtained in [8, 25, 29].

Now let b be a locally integrable function. For $1 \le j \le m$, define the commutator $[b,\, T]_j$ by

$$[b,\, T]_j(f_1,\, \dots,\, f_m)(x) = b(x)T(f_1,\, \dots,\, f_m)(x) \\ - T(f_1,\, \dots,\, f_{j-1}, bf_j,\, f_{j+1},\, \dots,\, f_m)(x).$$

Let $b_1,\, \dots,\, b_m$ be locally integrable functions and $\vec{b} = (b_1,\, \dots,\, b_m)$. The multilinear commutator of T and $\vec{b}$ is defined by

$$T_{\vec{b}}(f_1, \dots, f_m)(x) = \sum_{j=1}^{m} [b_j,\, T]_j(f_1, \dots, f_m)(x). \tag{1.13}$$

As it was showed in [5, 10, 23], by the conclusion (1.12), we can prove that, for $p_1,\, \dots,\, p_m,\, p \in (1,\, \infty)$ and $\vec{w} \in A_{\vec{P}}(\mathbb{R}^{mn})$,

$$\|T_{\vec{b}}(f_1, \dots, f_m)\|_{L^p(\mathbb{R}^n,\, \nu_{\vec{w}})} \lesssim [\vec{w}]_{A_p}^{\max\{1, \frac{p_1'}{p}, \dots, \frac{p_m'}{p}\}} \big([\nu_{\vec{w}}]_{A_\infty} + \sum_{j=1}^{m} [\sigma_j]_{A_\infty}\big) \\ \times \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n,\, w_j)},$$

where and in the following, for $\vec{P} = (p, \dots, p_m)$ and $\vec{w} \in A_{\vec{P}}(\mathbb{R}^{mn})$, $\sigma_j = w_j^{-\frac{1}{p_j - 1}}$.

However, the argument in [5, 10, 23] does not apply to the case $p \in (0,\, 1)$.

Our result concerning the weighted bound of $T_{\vec{b}}$ can be stated as follows.

Theorem 1.9. *Let T be an m-linear operator in Theorem 1.7, $b_1,\, \dots,\, b_m \in \mathrm{BMO}(\mathbb{R}^n)$, and $T_{\vec{b}}$ the commutator defined by (1.13). Then for $p_1, \dots, p_m$, $q_1, \dots, q_m \in (1,\, \infty)$, $p,\, q \in (1/m,\, \infty)$ with $1/p = 1/p_1 + \dots + 1/p_m$ and $1/q = 1/q_1 + \dots + 1/q_m$, and $\vec{w} \in A_{\vec{P}}(\mathbb{R}^{mn})$,*

$$\|\{T_{\vec{b}}(f_1^k, \dots, f_m^k)\}\|_{L^p(l^q;\, \mathbb{R}^n, \nu_{\vec{w}})} \\ \lesssim \Big([\nu_{\vec{w}}]_{A_\infty} + \sum_{i=1}^{m} [\sigma_i]_{A_\infty}\Big) [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p_1'}{p}, \cdots, \frac{p_m'}{p}\}} \prod_{j=1}^{m} \|\{f_j^k\}\|_{L^{p_j}(l^{q_j};\, \mathbb{R}^n, w_j)}. \tag{1.14}$$

Our argument in the proof of Theorems 1.7 and 1.9 also leads to the following weighted weak type endpoint estimate of $T_{\vec{b}}$.

Theorem 1.10. *Let T be an m-linear operator in Theorem 1.7, $b_1,\, \dots,\, b_m \in \mathrm{BMO}(\mathbb{R}^n)$ and $T_{\vec{b}}$ be the commutator defined by* (1.13). *Then for $q_1,\, \dots,\, q_m \in$*

$(1,\infty)$, $q\in(1/m,\infty)$ *with* $1/q=1/q_1+\cdots+1/q_m$, $\vec{w}\in A_{1,\ldots,1}(\mathbb{R}^{mn})$ *and* $\lambda>0$,

$$\begin{aligned}(1.15)\qquad &\nu_{\vec{w}}(\{x\in\mathbb{R}^n:\,\|\{T_{\vec{b}}(f_1^k,\ldots,f_m^k)(x)\}\|_{l^q}>\lambda\})\\ &\lesssim_{\vec{w}}\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^n}\frac{\|\{f_j^k(y_j)\}\|_{l^{q_j}}}{\lambda^{\frac{1}{m}}}\log\Big(1+\frac{\|\{f_j^k(y_j)\}\|_{l^{q_j}}}{\lambda^{\frac{1}{m}}}\Big)w_j(y_j)\mathrm{d}y_j\Big)^{\frac{1}{m}}.\end{aligned}$$

Remark 1.11. For the case that T is multilinear Calderón-Zygmund operator and $b_1,\ldots,b_m\in\mathrm{BMO}(\mathbb{R}^n)$, (1.15) (the case $\{f_j^k\}=\{f_j\}$) was proved in [26]. As far as we know, there has been no result about the endpoint estimate of the commutators of multilinear singular integral operators with nonsmooth kernels, namely, even for the unweighted case and $\{f_j^k\}=\{f_j\}$, the endpoint estimate (1.15) is new.

In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A\lesssim B$ to denote that there exists a positive constant C such that $A\le CB$. Constant with subscript such as C_1, does not change in different occurrences. For any set $E\subset\mathbb{R}^n$, χ_E denotes its characteristic function. For a cube $Q\subset\mathbb{R}^n$ and $\lambda\in(0,\infty)$, we use $\ell(Q)$ $(\mathrm{diam}Q)$ to denote the side length (diamter) of Q, and λQ to denote the cube with the same center as Q and whose side length is λ times that of Q. For $x\in\mathbb{R}^n$ and $r>0$, $B(x,r)$ denotes the ball centered at x and having radius r.

2. Estimates for sparse operators

This section is devoted to some weighted estimates for multilinear sparse operators, which will be used in the proof of our theorems and are of independent interest.

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of all cubes of the form

$$2^{-k}([0,1)^n+j),\ k\in\mathbb{Z},\ j\in\mathbb{Z}^n.$$

Denote the standard dyadic grid by $\mathcal{D}$. For a fixed cube Q, denote by $\mathcal{D}(Q)$ the set of dyadic cubes with respect to Q, that is, the cubes from $\mathcal{D}(Q)$ are formed by repeating subdivision of Q and each of descendants into 2^n congruent subcubes.

As usual, by a general dyadic grid $\mathscr{D}$, we mean a collection of cube with the following properties: (i) for any cube $Q\in\mathscr{D}$, its side length $\ell(Q)$ is of the form 2^k for some $k\in\mathbb{Z}$; (ii) for any cubes $Q_1,Q_2\in\mathscr{D}$, $Q_1\cap Q_2\in\{Q_1,Q_2,\emptyset\}$; (iii) for each $k\in\mathbb{Z}$, the cubes of side length 2^k form a partition of $\mathbb{R}^n$.

Let $\mathcal{S}$ be a family of cubes and $\eta\in(0,1)$. We say that $\mathcal{S}$ is η-sparse, if, for each fixed $Q\in\mathcal{S}$, there exists a measurable subset $E_Q\subset Q$, such that $|E_Q|\ge\eta|Q|$ and $\{E_Q\}$ are pairwise disjoint. A family is called simply sparse if $\eta=1/2$.

For constants $\beta_1, \dots, \beta_m \in [0, \infty)$, let $\vec{\beta} = (\beta_1, \dots, \beta_m)$. Associated with the sparse family $\mathcal{S}$ and $\vec{\beta}$, we define the sparse operator $\mathcal{A}_{m;\,\mathcal{S},L(\log L)^{\vec{\beta}}}$ by

$$\mathcal{A}_{m;\,\mathcal{S},\,L(\log L)^{\vec{\beta}}}(f_1,\dots,f_m)(x)=\sum_{Q\in\mathcal{S}}\prod_{j=1}^{m}\|f_j\|_{L(\log L)^{\beta_j},\,Q}\chi_Q(x), \tag{2.1}$$

with

$$\|f_j\|_{L(\log L)^{\beta_j},\,Q}=\inf\Big\{\lambda>0:\,\frac{1}{|Q|}\int_Q\frac{|f(y)|}{\lambda}\log^{\beta_j}\Big(1+\frac{|f(y)|}{\lambda}\Big)\mathrm{d}y\le 1\Big\}.$$

For locally integrable functions $b\in \mathrm{BMO}(\mathbb{R}^n)$ and a sparse family $\mathcal{S}$, let

$$\mathcal{A}_{m;\,\mathcal{S},\,b}(f_1,\,\dots,\,f_m)(x)=\sum_{Q\in\mathcal{S}}\big(|b(x)-\langle b\rangle_Q|\big)\prod_{j=1}^{m}\langle f_j\rangle_Q\chi_Q(x). \tag{2.2}$$

For the case of $\vec{\beta}=(0,\dots,0)$, we denote $\mathcal{A}_{m;\,\mathcal{S},\,L(\log L)^{\vec{\beta}}}$ by $\mathcal{A}_{m;\,\mathcal{S}}$ for simplicity. Also, we denote $\mathcal{A}_{1;\,\mathcal{S},\,L(\log L)^{\beta}}$ ($\mathcal{A}_{1;\,\mathcal{S}}$) by $\mathcal{A}_{\mathcal{S},\,L(\log L)^{\beta}}$ ($\mathcal{A}_{\mathcal{S}}$). For a weight u, let

$$\langle h\rangle_Q^u=\frac{1}{u(Q)}\int_Q h(y)u(y)\mathrm{d}y,$$

and

$$\widetilde{\mathcal{A}}_{m;\mathcal{S}}(f_1,\dots,f_m)(x)=\mathcal{A}_{m;\mathcal{S}}(f_1\sigma_1,\dots,f_m\sigma_m)(x). \tag{2.3}$$

For a dyadic grid $\mathscr{D}$ and sparse family $\mathcal{S}\subset\mathscr{D}$, it was proved in [29] that for $p_1,\,\dots,\,p_m\in(1,\infty)$, $p\in(0,\infty)$ such that $1/p=1/p_1+\dots+1/p_m$, $\vec{w}=(w_1,\,\dots,\,w_m)\in A_{\vec{P}}(\mathbb{R}^{mn})$,

$$\|\widetilde{\mathcal{A}}_{m;\,\mathcal{S}}(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n,\,\nu_{\vec{w}})}\lesssim[\vec{w}]_{A_p}^{\max\{1,\,\frac{p_1'}{p},\,\dots,\,\frac{p_m'}{p}\}}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n,\,\sigma_j)}, \tag{2.4}$$

and so

$$\|\mathcal{A}_{m;\,\mathcal{S}}(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n,\,\nu_{\vec{w}})}\lesssim[\vec{w}]_{A_p}^{\max\{1,\,\frac{p_1'}{p},\,\dots,\,\frac{p_m'}{p}\}}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n,\,w_j)}. \tag{2.5}$$

Theorem 2.1. *Let $p_1,\,\dots,\,p_m\in(1,\infty)$, $p\in(0,\infty)$ such that $1/p=1/p_1+\dots+1/p_m$, and $\vec{w}=(w_1,\,\dots,\,w_m)\in A_{\vec{P}}(\mathbb{R}^{mn})$. Let $\mathcal{S}$ be a sparse family. Then for $\beta_1,\,\dots,\,\beta_m\in[0,\infty)$,*

$$\begin{aligned}&\|\mathcal{A}_{m;\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})}\\&\quad\lesssim[\vec{w}]_{A_p}^{\max\{1,\frac{p_1'}{p},\dots,\frac{p_m'}{p}\}}\prod_{j=1}^{m}[\sigma_j]_{A_\infty}^{\beta_j}\|f_j\|_{L^{p_j}(\mathbb{R}^n,w_j)}.\end{aligned} \tag{2.6}$$

Proof. We employ the ideas used in the proof of Theorem 3.2 in [29], in which Theorem 2.1 was proved for the case of $\beta_1=\dots=\beta_m=0$, see also the proof of Theorem B in [2]. By the well known one-third trick (see [22, Lemma 2.5]), we know that if $\mathcal{S}$ is a sparse family, then there exist general dyadic grids $\mathscr{D}_1, \dots, \mathscr{D}_{3^n}$, and sparse families $\mathcal{S}_i \subset \mathscr{D}_i$ with $i = 1, \dots, 3^n$, such that

$$\mathcal{A}_{m;\,\mathcal{S},\,L(\log L)^{\vec{\beta}}}(f_1, \dots, f_m)(x) \lesssim_n \sum_{i=1}^{3^n} \mathcal{A}_{m;\,\mathcal{S}_i,\,L(\log L)^{\vec{\beta}}}(f_1, \dots, f_m)(x).$$

Thus, it suffices to assume that $\mathcal{S} \subset \mathscr{D}$ with $\mathscr{D}$ a dyadic grid. As it is well known, $\vec{w} \in A_{\vec{P}}(\mathbb{R}^{mn})$ implies $\sigma_j = w_j^{-\frac{1}{p_j-1}} \in A_{mp_j'}(\mathbb{R}^n)$ (see [26]). Also, it was pointed out in [23] that for the constant $r_{\sigma_j} = 1 + \frac{1}{\tau_n [\sigma_j]_{A_\infty}}$ with $\tau_n = 2^{11+n}$,

$$\Big(\frac{1}{|Q|}\int_Q \sigma_j^{r_{\sigma_j}}(x)\mathrm{d}x\Big)^{\frac{1}{r_{\sigma_j}}} \le 2\frac{1}{|Q|}\int_Q \sigma_j(x)\mathrm{d}x. \tag{2.7}$$

Let $\varrho_j = (1+p_j)/2$. We can verify that

$$\|\sigma_j^{\frac{1}{\varrho_j'}}\|_{L^{\varrho_j'}(\log L)^{\varrho_j'\beta_j},\,Q} \lesssim \|\sigma_j\|^{\frac{1}{\varrho_j'}}_{L(\log L)^{\varrho_j'\beta_j},\,Q}.$$

Recall that

$$\|h\|_{L(\log L)^{\varrho},\,Q} \lesssim \max\big\{1,\, \frac{1}{(\delta-1)^{\varrho}}\big\}\Big(\frac{1}{|Q|}\int_Q |h(y)|^{\delta}\mathrm{d}y\Big)^{\frac{1}{\delta}}. \tag{2.8}$$

It then follows that

$$\begin{aligned}\|\sigma_j^{\frac{1}{\varrho_j'}}\|_{L^{\varrho_j'}(\log L)^{\varrho_j'\beta_j},\,Q} &\lesssim \frac{1}{(r_{\sigma_j}-1)^{\beta_j}}\Big(\frac{1}{|Q|}\int_Q \sigma_j^{r_{\sigma_j}}(y)\mathrm{d}y\Big)^{\frac{1}{\varrho_j' r_{\sigma_j}}}\\ &\lesssim [\sigma_j]_{A_\infty}^{\beta_j}\Big(\frac{1}{|Q|}\int_Q \sigma_j(y)\mathrm{d}y\Big)^{\frac{1}{\varrho_j'}}.\end{aligned}$$

Applying the generalization of Hölder's inequality (see [35]), we deduce that

$$\begin{aligned}\|f_j\sigma_j\|_{L(\log L)^{\beta_j},\,Q} &\lesssim \Big(\frac{1}{|Q|}\int_Q |f_j|^{\varrho_j}\sigma_j\Big)^{\frac{1}{\varrho_j}}\|\sigma_j^{\frac{1}{\varrho_j'}}\|_{L^{\varrho_j'}(\log L)^{\varrho_j'\beta_j},\,Q}\\ &\lesssim [\sigma_j]_{A_\infty}^{\beta_j}\Big(\frac{1}{|Q|}\int_Q |f_j|^{\varrho_j}\sigma_j\Big)^{\frac{1}{\varrho_j}}\Big(\frac{1}{|Q|}\int_Q \sigma_j\Big)^{\frac{1}{\varrho_j'}}\\ &= [\sigma_j]_{A_\infty}^{\beta_j}\Big(\frac{1}{\sigma_j(Q)}\int_Q |f_j|^{\varrho_j}\sigma_j\Big)^{\frac{1}{\varrho_j}}\frac{\sigma_j(Q)}{|Q|}\\ &\lesssim [\sigma_j]_{A_\infty}^{\beta_j}\langle M^{\mathscr{D}}_{\sigma_j,\,\varrho_j}f_j\rangle_Q^{\sigma_j}\langle\sigma_j\rangle_Q,\end{aligned} \tag{2.9}$$

here and in the following, $M^{\mathscr{D}}_{\sigma_j,\,\varrho_j}$ is the maximal operator defined by

$$M^{\mathscr{D}}_{\sigma_j,\,\varrho_j}f_j(x) = \sup_{I\ni x,\, I\in\mathscr{D}}\Big(\frac{1}{\sigma_j(I)}\int_I |f_j(y)|^{\varrho_j}\sigma_j(y)\mathrm{d}y\Big)^{\frac{1}{\varrho_j}}.$$

We then deduce that

$$\prod_{j=1}^m \|f_j\sigma_j\|_{L(\log L)^{\beta_j},\,Q} \lesssim \prod_{i=1}^m [\sigma_i]_{A_\infty}^{\beta_i}\sum_{Q\in\mathcal{S}}\prod_{j=1}^m \langle M^{\mathscr{D}}_{\sigma_j,\,\varrho_j}f_j\rangle_Q^{\sigma_j}\langle\sigma_j\rangle_Q.$$

This, via the estimate (2.4) and the fact that $M^{\mathscr{D}}_{\sigma_j,\,\varrho_j}$ is bounded on $L^{p_j}(\mathbb{R}^n,\,\sigma_j)$ with bounds independent of σ_j, yields

$$\begin{aligned}\Big\|\sum_{Q\in\mathcal{S}}\prod_{j=1}^{m}\|f_j\sigma_j\|_{L(\log L)^{\beta_j},\,Q}\chi_Q\Big\|_{L^p(\mathbb{R}^n,\,\nu_{\vec{w}})}&\\ \lesssim\prod_{i=1}^{m}[\sigma_i]_{A_\infty}^{\beta_i}\Big\|\sum_{Q\in\mathcal{S}}\prod_{j=1}^{m}\langle M_{\sigma_j,\,\varrho_j}f_j\rangle_Q^{\sigma_j}\langle\sigma_j\rangle_Q\chi_Q\Big\|_{L^p(\mathbb{R}^n,\,\nu_{\vec{w}})}&\\ \lesssim[\vec{w}]_{A_p}^{\max\{1,\,\frac{p_1'}{p},\,\dots,\,\frac{p_m'}{p}\}}\prod_{i=1}^{m}[\sigma_i]_{A_\infty}^{\beta_i}\prod_{j=1}^{m}\|M^{\mathscr{D}}_{\sigma_j,\,\varrho_j}f_j\|_{L^{p_j}(\mathbb{R}^n,\,\sigma_j)},&\end{aligned}$$

and then completes the proof of Theorem 2.1. □

Theorem 2.2. *Let* $p_1,\,\dots,\,p_m\in(1,\,\infty)$, $p\in(0,\,\infty)$ *such that* $1/p=1/p_1+\dots+1/p_m$, *and* $\vec{w}=(w_1,\,\dots,\,w_m)\in A_{\vec{P}}(\mathbb{R}^{mn})$. *Let* $\mathcal{S}$ *be a sparse family,* $b\in\mathrm{BMO}(\mathbb{R}^n)$ *with* $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}=1$. *Then*

$$\begin{aligned}&\|\mathcal{A}_{m;\mathcal{S},b}(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})}\\ &\lesssim[\vec{w}]_{A_p}^{\max\{1,\frac{p_1'}{p},\dots,\frac{p_m'}{p}\}}[\nu_{\vec{w}}]_{A_\infty}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n,w_j)}.\end{aligned}\tag{2.10}$$

Proof. Again we assume that $\mathcal{S}\subset\mathscr{D}$ with $\mathscr{D}$ a dyadic grid. We first consider the case of $p\in(0,\,1]$. Write

$$\begin{aligned}\int_{\mathbb{R}^n}\big(\mathcal{A}_{m,\,\mathcal{S},\,b}\vec{f}(x)\big)^p\nu_{\vec{w}}(x)\mathrm{d}x&\le\sum_{Q\in\mathcal{S}}\prod_{j=1}^{m}\langle|f_j|\rangle_Q^p\int_{\mathbb{R}^n}|b(x)-\langle b\rangle_Q|^p\nu_{\vec{w}}(x)\mathrm{d}x\\ &\le\sum_{Q\in\mathcal{S}}\prod_{j=1}^{m}\langle|f_j|\rangle_Q^p|Q|\|\nu_{\vec{w}}\|_{L(\log L)^p,\,Q}\\ &\le[\nu_{\vec{w}}]_{A_\infty}^p\sum_{Q\in\mathcal{S}}\prod_{j=1}^{m}\langle|f_j|\rangle_Q^p\nu_{\vec{w}}(Q),\end{aligned}$$

where in the last inequality, we have invoked the estimates (2.8) and (2.7) for $\nu_{\vec{w}}$. It was proved in [29, pp. 757–758] that

$$\sum_{Q\in\mathcal{S}}\prod_{j=1}^{m}\langle|f_j|\rangle_Q^p\nu_{\vec{w}}(Q)\lesssim[\vec{w}]_{A_p}^{\max\{p_1',\,\dots,\,p_m'\}}\prod_{j=1}^{m}\|f_j\|^p_{L^{p_j}(\mathbb{R}^n,\,w_j)}.$$

The inequality (2.10) then follows in this case.

To consider the case of $p\in(1,\,\infty)$, let $\varrho=\frac{1+p'}{2}$. Observe that by (2.8),

$$\begin{aligned}\|g\nu_{\vec{w}}\|_{L(\log L),\,Q}&\lesssim\Big(\frac{1}{|Q|}\int_Q|g(x)|^{\varrho}\nu_{\vec{w}}(x)\mathrm{d}x\Big)^{\frac{1}{\varrho}}\|\nu_{\vec{w}}^{\frac{1}{\varrho'}}\|_{L^{\varrho'}(\log L)^{\varrho'},\,Q}\\ &\lesssim\Big(\frac{1}{|Q|}\int_Q|g(x)|^{\varrho}\nu_{\vec{w}}(x)\mathrm{d}x\Big)^{\frac{1}{\varrho}}\|\nu_{\vec{w}}\|^{\frac{1}{\varrho'}}_{L(\log L)^{\varrho'},\,Q}\end{aligned}$$

$$\lesssim [w]_{A_\infty}^{\frac{1}{s}}\Big(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q |g(x)|^{\varrho}\nu_{\vec{w}}(x)\mathrm{d}x\Big)^{\frac{1}{\varrho}}\frac{\nu_{\vec{w}}(Q)}{|Q|}.$$

Therefore, by the generalization of Hölder's inequality (see [35]),

$$\begin{aligned}
&\int_{\mathbb{R}^n}\mathcal{A}_{m,\mathcal{S},b}(f_1,\dots,f_m)(x)g(x)\nu_{\vec{w}}(x)\mathrm{d}x\\
&=\sum_{Q\in\mathcal{S}}\prod_{j=1}^m\langle|f_j|\rangle_Q\int_{\mathbb{R}^n}|b(x)-\langle b\rangle_Q|g(x)\nu_{\vec{w}}(x)\mathrm{d}x\\
&\le\sum_{Q\in\mathcal{S}}\prod_{j=1}^m\langle|f_j|\rangle_Q|Q|\|g\nu_{\vec{w}}\|_{L\log L,\,Q}\\
&\le[\nu_{\vec{w}}]_{A_\infty}\sum_{Q\in\mathcal{S}}\prod_{j=1}^m\langle|f_j|\rangle_Q\inf_{x\in Q}M^{\mathscr{D}}_{\nu_{\vec{w}},\,\varrho}g(x)\nu_{\vec{w}}(Q)\\
&\le[\nu_{\vec{w}}]_{A_\infty}\|\mathcal{A}_{\mathcal{S}}(f_1,\,\dots,\,f_m)\|_{L^p(\mathbb{R}^n,\,\nu_{\vec{w}})}\|M^{\mathscr{D}}_{\nu_{\vec{w}},\,\varrho}g\|_{L^{p'}(\mathbb{R}^n,\,\nu_{\vec{w}})}.
\end{aligned}$$

Our desired conclusion then follows from (2.5) and the fact that $M^{\mathscr{D}}_{\nu_{\vec{w}},\,\varrho}$ is bounded on $L^{p'}(\mathbb{R}^n,\,\nu_{\vec{w}})$ with bound independent of $\nu_{\vec{w}}$. □

For $\beta_1,\,\dots,\,\beta_m\in[0,\,\infty)$, let $\mathcal{M}_{L(\log L)^{\vec{\beta}}}$ be the maximal operator defined by

$$\mathcal{M}_{L(\log L)^{\vec{\beta}}}(f_1,\,\dots,\,f_m)(x)=\sup_{Q\ni x}\prod_{j=1}^m\|f_j\|_{L(\log L)^{\beta_j},\,Q}.$$

For the case of $\vec{\beta}=(0,\,\dots,\,0)$, we denote $\mathcal{M}_{L(\log L)^{\vec{\beta}}}$ by $\mathcal{M}$. As in [26] and [32], we can prove the following lemma.

Lemma 2.3. *Let $\beta_1,\,\dots,\,\beta_m\in[0,\,\infty)$, $|\beta|=\beta_1+\cdots+\beta_m$ and $\vec{w}=(w_1,\,\dots,\,w_m)\in A_{1,\dots,\,1}(\mathbb{R}^{mn})$. Then for each $\lambda>0$,*

$$\begin{aligned}
&\nu_{\vec{w}}\big(\{x\in\mathbb{R}^n:\,\mathcal{M}_{L(\log L)^{\vec{\beta}}}(f_1,\,\dots,\,f_m)(x)>\lambda\}\big)\\
&\lesssim\prod_{j=1}^m\Big(\int_{\mathbb{R}^n}\frac{|f_j(x)|}{\lambda^{\frac{1}{m}}}\log^{|\beta|}\Big(1+\frac{|f_j(x)|}{\lambda^{\frac{1}{m}}}\Big)w_j(x)\mathrm{d}x\Big)^{\frac{1}{m}}.
\end{aligned}$$

The following conclusion was established by Lerner et al. in [27].

Lemma 2.4. *Let $\beta\in[0,\,\infty)$ and $\mathcal{S}$ be a sparse family of cubes. Then for each fixed $\lambda>0$,*

$$|\{x\in\mathbb{R}^n:\,\mathcal{A}_{\mathcal{S},\,L(\log L)^\beta}f(x)>\lambda\}|\lesssim\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log^{\beta}\Big(1+\frac{|f(x)|}{\lambda}\Big)\mathrm{d}x,$$

and for $b\in\mathrm{BMO}(\mathbb{R}^n)$,

$$|\{x\in\mathbb{R}^n:\,\mathcal{A}_{\mathcal{S},\,b}f(x)>\lambda\}|\lesssim\lambda^{-1}\int_{\mathbb{R}^n}|f(x)|\mathrm{d}x.$$

Lemma 2.5. *Let $\varrho \in [0, \infty)$ and $\delta \in (0, 1)$, T be a sublinear operator which satisfies the weak type estimate that*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^{\varrho}\Big(1 + \frac{|f(x)|}{\lambda}\Big) \mathrm{d}x.$$

Then for any cube I and appropriate function f with $\operatorname{supp} f \subset I$,

$$\Big(\frac{1}{|I|}\int_I |Tf(x)|^{\delta} \mathrm{d}x\Big)^{\frac{1}{\delta}} \lesssim \|f\|_{L(\log L)^{\varrho},\, I}.$$

For the proof of Lemma 2.5, see [18].

Lemma 2.6. *Let $\tau \in (0, 1)$ and M_τ be the maximal operator defined by*

$$M_\tau f(x) = \big(M(|f|^{\tau})(x)\big)^{\frac{1}{\tau}}.$$

Then for any $p \in (\tau, \infty)$ and $u \in A_{p/\tau}(\mathbb{R}^n)$,

$$\begin{aligned} &u(\{x \in \mathbb{R}^n : \|\{M_\tau f_k(x)\}\|_{l^q} > \lambda\}) \\ &\lesssim_{u,p} \lambda^{-p} \sup_{t \ge C\lambda} t^p u(\{x \in \mathbb{R}^n : \|\{f_k(x)\}\|_{l^q} > t\}). \end{aligned}$$

Proof. For each fixed $\lambda > 0$, decompose f_k as

$$f_k(y) = f_k(y)\chi_{\{\|\{f_k(y)\}\|_{l^q} \le \lambda\}}(y) + f_k(y)\chi_{\{\|\{f_k(y)\}\|_{l^q} > \lambda\}}(y) := f_k^1(y) + f_k^2(y).$$

It then follows that

$$\begin{aligned} &u(\{x \in \mathbb{R}^n : \|\{M_\tau f_k(x)\}\|_{l^q} > 2^{\frac{1}{\tau}}\lambda\}) \\ &\le u(\{x \in \mathbb{R}^n : \|\{M(|f_k^2|^{\tau})(x)\}\|_{l^{\frac{q}{\tau}}} > \lambda^{\tau}\}). \end{aligned}$$

Recall that $u \in A_{p/\tau}(\mathbb{R}^n)$ implies that $u \in A_{\frac{p-\epsilon}{\tau}}(\mathbb{R}^n)$ for some $\epsilon \in (0, p - \tau)$, and that M is bounded on $L^{\frac{p-\epsilon}{\tau}}(l^q; \mathbb{R}^n, u)$ (see [14]). Therefore,

$$\begin{aligned} &u(\{x \in \mathbb{R}^n : \|\{M(|f_k^2|^{\tau})(x)\}\|_{l^{\frac{q}{\tau}}} > \lambda^{\tau}\}) \\ &\lesssim \lambda^{-p+\epsilon} \int_{\mathbb{R}^n} \|\{f_k^2(x)\}\|_{l^q}^{p-\epsilon} u(x)\mathrm{d}x \\ &\lesssim u(\{x \in \mathbb{R}^n : \|\{f_k\}\|_{l^q} > \lambda\}) \\ &\quad + \lambda^{-p+\epsilon} \int_{\lambda}^{\infty} u(\{x \in \mathbb{R}^n : \|\{f_k^2(x)\}\|_{l^q} > t\}) t^{p-\epsilon-1} \mathrm{d}t \\ &\lesssim \lambda^{-p} \sup_{t \ge \lambda} t^p u(\{x \in \mathbb{R}^n : \|\{f_k(x)\}\|_{l^q} > t\}). \end{aligned}$$

This yields our desired conclusion. □

Let $\mathscr{D}$ be a dyadic grid. Associated with $\mathscr{D}$, define the sharp maximal function $M_{\mathscr{D}}^{\sharp}$ as

$$M_{\mathscr{D}}^{\sharp} f(x) = \sup_{\substack{Q \ni x \\ Q \in \mathscr{D}}} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| \mathrm{d}y.$$

For $\delta\in(0,\,1)$, let $M^{\sharp}_{\mathscr{D},\,\delta}f(x)=\big[M^{\sharp}_{\mathscr{D}}(|f|^{\delta})(x)\big]^{1/\delta}$. Repeating the argument in [36, p. 153], we can verify that if $u\in A_{\infty}(\mathbb{R}^n)$ and Φ is a increasing function on $[0,\,\infty)$ which satisfies that

$$\Phi(2t)\leq C\Phi(t),\ t\in[0,\,\infty),$$

then

$$\begin{aligned}\sup_{\lambda>0}\Phi(\lambda)u(\{x\in\mathbb{R}^n:|h(x)|>\lambda\})\\ \lesssim\sup_{\lambda>0}\Phi(\lambda)u(\{x\in\mathbb{R}^n:M^{\sharp}_{\mathscr{D},\delta}h(x)>\lambda\}),\end{aligned}\tag{2.11}$$

provided that $\sup_{\lambda>0}\Phi(\lambda)u(\{x\in\mathbb{R}^n:\,M_{\delta}h(x)>\lambda\})<\infty$, here and in the following, $M_{\delta}f(x)=\big(M(|f|^{\delta})(x)\big)^{1/\delta}$.

Theorem 2.7. *Let $m\geq 2$ be an integer, $\beta_1,\,\dots,\,\beta_m\in[0,\,\infty)$, $\mathcal{S}$ be a finite sparse family. Then for $\vec{w}=(w_1,\,\dots,\,w_m)\in A_{1,\,\dots,\,m}(\mathbb{R}^n)$,*

$$\begin{aligned}&\nu_{\vec{w}}(\{x\in\mathbb{R}^n:\,\mathcal{A}_{m;\,\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1,\,\dots,\,f_m)(x)>\lambda\})\\ &\lesssim\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^n}\frac{|f_j(y_j)|}{\lambda^{\frac{1}{m}}}\log^{|\vec{\beta}|}\Big(1+\frac{|f_j(y_j)|}{\lambda^{\frac{1}{m}}}\Big)w_j(y_j)\mathrm{d}y_j\Big)^{\frac{1}{m}},\end{aligned}\tag{2.12}$$

and for $b\in\mathrm{BMO}(\mathbb{R}^n)$,

$$\nu_{\vec{w}}(\{x\in\mathbb{R}^n:\,\mathcal{A}_{m;\,\mathcal{S},\,b}(f_1,\,\dots,\,f_m)(x)>1\})\lesssim\prod_{j=1}^{m}\|f_j\|^{\frac{1}{m}}_{L^1(\mathbb{R}^n,\,w_j)}.\tag{2.13}$$

Proof. Again by the one-third trick (see [22, Lemma 2.5]), we assume that $\mathcal{S}\subset\mathscr{D}$ with $\mathscr{D}$ a dyadic grid, and the family is finite. Without loss of generality, we may assume that the functions $f_1,\,\dots,\,f_m$ are nonnegative, bounded and have compact support. We claim that for $\delta\in(0,\,\frac{1}{m})$,

$$M^{\sharp}_{\mathscr{D},\,\delta}\big(\mathcal{A}_{m;\,\mathcal{S},L(\log L)^{\vec{\beta}_i}}(f_1,\,\dots,\,f_m)\big)(x)\lesssim\mathcal{M}_{L(\log L)^{\vec{\beta}}}(f_1,\,\dots,\,f_m)(x).\tag{2.14}$$

To see this, for fixed $I\in\mathscr{D}$, let $c_0=\sum_{Q\supset I}\prod_{j=1}^{m}\|f_j\|_{L(\log L)^{\beta_j},\,Q}$. As in [9], it follows that

$$\begin{aligned}&\int_I|\mathcal{A}_{m;\,\mathcal{S},\,L(\log L)^{\vec{\beta}}}(f_1,\dots,f_m)(y)-c_0|^{\delta}\mathrm{d}y\\ &\lesssim\int_I\Big|\sum_{Q\in\mathcal{S},\,Q\subset I}\|f_j\|_{L(\log L)^{\beta_j},\,Q}\chi_Q(y)\Big|^{\delta}\mathrm{d}y\\ &\lesssim\int_I\Big|\mathcal{A}_{m;\,\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1\chi_I,\dots,\,f_m\chi_I)(y)\Big|^{\delta}\mathrm{d}y.\end{aligned}$$

On the other hand, by Lemma 2.4 and Lemma 2.5, we know that

$$\Big(\frac{1}{|I|}\int_I\Big|\mathcal{A}_{\mathcal{S},L(\log L)^{\beta_j}}(f_j\chi_I)(y)\Big|^{m\delta}\mathrm{d}y\Big)^{\frac{1}{m\delta}}\lesssim\|f_j\|_{L(\log L)^{\beta_j},\,I}.$$

This, together with Hölder's inequality, leads to that

$$\begin{aligned}&\Big(\frac{1}{|I|}\int_I\Big|\mathcal{A}_{m;\,\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1\chi_I,\dots,\,f_m\chi_I)(y)\Big|^{\delta}{\rm d}y\Big)^{\frac{1}{\delta}}\\ &\lesssim\prod_{j=1}^m\Big(\frac{1}{|I|}\int_I\Big|\mathcal{A}_{\mathcal{S},L(\log L)^{\beta_j}}(f_j\chi_I)(y)\Big|^{m\delta}{\rm d}y\Big)^{\frac{1}{m\delta}}\\ &\lesssim\prod_{j=1}^m\|f_j\|_{L(\log L)^{\beta_j},\,I},\end{aligned}$$

and then establishes (2.14).

We can now prove (2.12). Let $\psi(t)=t^{\frac{1}{m}}\log^{-|\vec{\beta}|}(1+t^{-\frac{1}{m}})$. By a standard limit argument, it suffices to consider the case that the sparse family $\mathcal{S}$ is finite. Then for bounded functions $f_1,\,\dots,\,f_m$ with compact supports, and $\vec{w}=(w_1,\,\dots,\,w_m)\in A_{1,\,\dots,1}(\mathbb{R}^n)$,

$$\big\|\mathcal{A}_{m;\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1,\dots,f_m)\big\|_{L^{\frac{1}{m}}(\mathbb{R}^n,\nu_{\vec{w}})}<\infty.$$

This, together with the fact that M_{δ} is bounded from $L^{\frac{1}{m}}(\mathbb{R}^n)$ to $L^{\frac{1}{m},\infty}(\mathbb{R}^n)$, implies that

$$\begin{aligned}&\sup_{\lambda>0}\psi(\lambda)\nu_{\vec{w}}(\{x\in\mathbb{R}^n:\,M_{\delta}\big(\mathcal{A}_{m;\,\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1,\,\dots,\,f_m)\big)(x)>\lambda\})\\ &\lesssim\sup_{\lambda>0}\psi(\lambda)\lambda^{-\frac{1}{m}}\big\|\mathcal{A}_{m;\mathcal{S},L(\log L)^{\vec{\beta}_i}}(f_1,\dots,f_m)\|_{L^{\frac{1}{m}}(\mathbb{R}^n,\nu_{\vec{w}})}<\infty.\end{aligned}$$

Therefore, by Lemma 2.3, inequalities (2.11) and (2.14), we deduced that

$$\begin{aligned}&\nu_{\vec{w}}(\{x\in\mathbb{R}^n:\,\mathcal{A}_{m;\,\mathcal{S},\,L(\log L)^{\vec{\beta}}}(f_1,\dots,f_m)(x)>1\})\\ &\lesssim\sup_{t>0}\psi(t)\nu_{\vec{w}}\big(\{x\in\mathbb{R}^n:\,M^{\sharp}_{\mathscr{D},\delta}\big(\mathcal{A}_{m;\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1,\dots,f_m)\big)(x)>t\}\big)\\ &\lesssim\prod_{j=1}^m\Big(\int_{\mathbb{R}^n}|f_j(y_j)|\log\big(1+|f_j(y_j)|\big)w_j(y_j){\rm d}y_j\Big)^{\frac{1}{m}}.\end{aligned}$$

We turn our attention to the estimate (2.13). We claim that for $\delta\in(0,\,1/m)$ and $\gamma\in(\delta,\,1/m)$,

$$\begin{aligned}M^{\sharp}_{\delta}\big(\mathcal{A}_{m;\,\mathcal{S},\,b}(f_1,\dots,f_m)\big)(x)\lesssim\,&M_{\gamma}\big(\mathcal{A}_{m;\mathcal{S}}(f_1,\dots,f_m)\big)(x)\\ &+\mathcal{M}(f_1,\,\dots,\,f_m)(x).\end{aligned}\tag{2.15}$$

To prove this, we first observe that, for each constant $c\in\mathbb{C}$ and a cube $I\subset\mathscr{D}$,

$$\begin{aligned}\big|\mathcal{A}_{m;\,\mathcal{S},\,b}(f_1,\dots,f_m)(y)-c\big|\le\,&|b(y)-\langle b\rangle_I|\mathcal{A}_{m;\,\mathcal{S}}(f_1,\,\dots,\,f_m)(y)\\ &+\Big|\sum_{Q\in\mathcal{S}}\big(|\langle b\rangle_I-\langle b\rangle_Q|\big)\prod_{j=1}^m\langle f_j\rangle_Q\chi_Q(y)-c\Big|.\end{aligned}$$

Let $c_1=\sum_{Q\in\mathcal{S},\,Q\supset I}\big|\langle b\rangle_I-\langle b\rangle_Q\big|\prod_{j=1}^m\langle f_j\rangle_Q$, we thus have that

$$\begin{aligned}&\inf_{c\in\mathbb{C}}\Big(\frac{1}{|I|}\int_I|\mathcal{A}_{m;\,\mathcal{S},\,b}(f_1,\dots,f_m)(y)-c|^{\delta}\mathrm{d}y\Big)^{\frac{1}{\delta}}\\ \lesssim&\Big(\frac{1}{|I|}\int_I\Big||b(y)-\langle b\rangle_I|\mathcal{A}_{m;\,\mathcal{S}}(f_1,\,\dots,\,f_m)(y)\Big|^{\delta}\mathrm{d}y\Big)^{\frac{1}{\delta}}\\ &+\Big(\frac{1}{|I|}\int_I\Big|\sum_{Q\in\mathcal{S}}|\langle b\rangle_I-\langle b\rangle_Q|\prod_{j=1}^m\langle f_j\rangle_Q\chi_Q(y)-c_1\Big|^{\delta}\mathrm{d}y\Big)^{\frac{1}{\delta}}.\end{aligned}$$

It follows from Hölder's inequality that

$$\begin{aligned}&\Big(\frac{1}{|I|}\int_I\Big|\big(b(y)-\langle b\rangle_I\big)\mathcal{A}_{m;\,\mathcal{S}}(f_1,\,\dots,\,f_m)(y)\Big|^{\delta}\mathrm{d}y\Big)^{\frac{1}{\delta}}\\ \lesssim&\inf_{y\in I}M_{\gamma}\big(\mathcal{A}_{m;\,\mathcal{S}}(f_1,\,\dots,\,f_m)\big)(y).\end{aligned}$$

On the other hand, we deduce from Hölder's inequality, Lemma 2.4 and Lemma 2.5, that

$$\begin{aligned}&\Big(\frac{1}{|I|}\int_I\Big|\sum_{Q\in\mathcal{S},\,Q\subset I}|\langle b\rangle_I-\langle b\rangle_Q|\prod_{j=1}^m\langle f_j\chi_I\rangle_Q\chi_Q(y)\Big|^{\delta}\mathrm{d}y\Big)^{\frac{1}{\delta}}\\ \lesssim&\Big(\frac{1}{|I|}\int_I\big(\mathcal{A}_{m;\,\mathcal{S},\,b}(f_1\chi_I,\dots,\,f_m\chi_I)(y)\big)^{\delta}\mathrm{d}y\Big)^{\frac{1}{\delta}}\\ &+\Big(\frac{1}{|I|}\int_I\Big(|b(y)-\langle b\rangle_I|\Big)^{\delta}\big(\mathcal{A}_{m;\,\mathcal{S}}(f_1\chi_I,\dots,\,f_m\chi_I)(y)\big)^{\delta}\mathrm{d}y\Big)^{\frac{1}{\delta}}\\ \lesssim&\prod_{j=1}^m\langle|f_j|\rangle_I.\end{aligned}$$

Combining the estimates above leads to (2.15).

Recalling that $\nu_{\vec{w}}\in A_{\infty}(\mathbb{R}^{mn})$, we can choose δ and γ in (2.15) small enough such that $\nu_{\vec{w}}\in A_{\frac{1}{m\gamma}}(\mathbb{R}^{mn})$. It then follows from Lemma 2.6, the inequality (2.12) and Lemma 2.3 that

$$\begin{aligned}&\lambda^{\frac{1}{m}}\nu_{\vec{w}}(\{x\in\mathbb{R}^n:\,M_{\gamma}\big(\mathcal{A}_{m;\,\mathcal{S}}(f_1,\,\dots,\,f_m)\big)(x)>\lambda\})\\ \lesssim&\sup_{t>0}t^{\frac{1}{m}}\nu_{\vec{w}}(\{x\in\mathbb{R}^n:\,\mathcal{A}_{m;\,\mathcal{S}}(f_1,\,\dots,\,f_m)(x)>t\})\\ \lesssim&\sup_{t>0}t^{\frac{1}{m}}\nu_{\vec{w}}(\{x\in\mathbb{R}^n:\,M^{\sharp}_{\mathscr{D},\,\delta}\big(\mathcal{A}_{m;\,\mathcal{S}}(f_1,\,\dots,\,f_m)\big)(x)>t\})\\ \lesssim&\prod_{j=1}^m\|f_j\|^{\frac{1}{m}}_{L^1(\mathbb{R}^n,\,w_j)}.\end{aligned}$$

Note that if the sparse family $\mathcal{S}$ is finite, and $f_1, \dots, f_m$ are bounded with compact supports, then for $\vec{w} = (w_1, \dots, w_m) \in A_{1,\dots,1}(\mathbb{R}^{mn})$,

$$\sup_{\lambda>0} \lambda^{\frac{1}{m}} \nu_{\vec{w}}\big(\{x \in \mathbb{R}^n : M_\delta\big(\mathcal{A}_{m;\mathcal{S},b}(f_1, \dots, f_m)\big)(x) > \lambda\}\big) < \infty.$$

This, together with inequalities (2.11) and (2.15), leads to that

$$\begin{aligned}
&\nu_{\vec{w}}(\{x \in \mathbb{R}^n : \mathcal{A}_{m;\mathcal{S},b}(f_1, \dots, f_m)(x) > 1\}) \\
\lesssim &\sup_{t>0} t^{\frac{1}{m}} \nu_{\vec{w}}\big(\{x \in \mathbb{R}^n : M^{\sharp}_{\mathscr{D},\delta}\big(\mathcal{A}_{m;\mathcal{S},b}(f_1, \dots, f_m)\big)(x) > t\}\big) \\
\lesssim &\prod_{j=1}^{m} \|f_j\|^{\frac{1}{m}}_{L^1(\mathbb{R}^n, w_j)},
\end{aligned}$$

and then establishes the inequality (2.13). □

3. Proof of theorems

Let T be an m-sublinear operator. Associated with T, let

$$\mathcal{M}_T(f_1, \dots, f_m)(x) = \sup_{Q \ni x} \big\|T(f_1, \dots, f_m)(\xi) - T(f_1\chi_{3Q}, \dots, f_m\chi_{3Q})(\xi)\big\|_{L^\infty(Q)}.$$

Following the argument in [24], we have

Lemma 3.1. *Let $q_1, \dots, q_m \in (1, \infty)$, $q \in (1/m, \infty)$ such that $1/q = 1/q_1 + \dots + 1/q_m$, T be an m-sublinear operator which is bounded from $L^1(l^{q_1}; \mathbb{R}^n) \times \dots \times L^1(l^{q_m}; \mathbb{R}^n)$ to $L^{\frac{1}{m},\infty}(l^q; \mathbb{R}^n)$. Then for any cube Q_0 and a. e. $x \in Q_0$, we have that*

$$\begin{aligned}
\|\{T(f_1^k\chi_{3Q_0}, \dots, f_m^k\chi_{3Q_0})(x)\}\|_{l^q} \le C_1 &\prod_{j=1}^{m} \|\{f_j^k(x)\}\|_{l^{q_j}} \\
&+ \big\|\{\mathcal{M}_T(f_1^k\chi_{3Q_0}, \dots, f_m^k\chi_{3Q_0})(x)\}\big\|_{l^q},
\end{aligned}$$

provided that $\|\{f_1^k\}\|_{l^{q_1}}, \dots, \|\{f_m^k\}\|_{l^{q_m}} \in L^1_{loc}(\mathbb{R}^n)$.

Proof. We follow the line in [27]. Let $x \in \text{int}Q_0$ be a point of approximation continuity of $\|\{T(f_1\chi_{3Q_0}, \dots, f_m\chi_{3Q_0})\}\|_{l^q}$. For $r, \epsilon > 0$, the set

$$\begin{aligned}
E_r(x) = \{y \in B(x, r) : \Big|&\|\{T(f_1^k\chi_{3Q_0}, \dots, f_m^k\chi_{3Q_0})(x)\}\|_{l^q} \\
&- \|\{T(f_1^k\chi_{3Q_0}, \dots, f_m^k\chi_{3Q_0})(y)\}\|_{l^q}\Big| < \epsilon\}
\end{aligned}$$

satisfies that $\lim_{r\to 0} \frac{|E_r(x)|}{|B(x,r)|} = 1$. Denote by $Q(x, r)$ the smallest cube centered at x and containing $B(x, r)$. Let $r > 0$ small enough such that $Q(x, r) \subset Q_0$. Then for $y \in E_r(x)$,

$$\begin{aligned}
&\|\{T(f_1^k\chi_{3Q_0}, \dots, f_m^k\chi_{3Q_0})(x)\}\|_{l^q} \\
\le &\|\{T(f_1^k\chi_{3Q(x,r)}, \dots, f_m^k\chi_{3Q(x,r)})(y)\}\|_{l^q} \\
&+ \big\|\{\mathcal{M}_T(f_1^k\chi_{3Q_0}, \dots, f_m^k\chi_{3Q_0})(x)\}\big\|_{l^q} + \epsilon.
\end{aligned}$$

Thus, for $\varsigma \in (0, 1/m)$,

$$\begin{aligned}
&\big\|\{T(f_1^k\chi_{3Q_0}, \dots, f_m^k\chi_{3Q_0})(x)\}\big\|_{l^q}\\
&\le \Big(\frac{1}{|E_s(x)|}\int_{E_s(x)}\|\{T(f_1^k\chi_{3Q(x,r)}, \dots, f_m^k\chi_{3Q(x,r)})(y)\}\|_{l^q}^{\varsigma}\mathrm{d}y\Big)^{\frac{1}{\varsigma}}\\
&\quad+\big\|\{\mathcal{M}_T(f_1^k\chi_{3Q_0}, \dots, f_m^k\chi_{3Q_0})(x)\}\big\|_{l^q}+\epsilon\\
&\le C\prod_{j=1}^m\langle\|\{f_j^k\}\|_{l^{q_j}}\rangle_{3Q(x,\,r)}+\big\|\{\mathcal{M}_T(f_1^k\chi_{3Q_0}, \dots, f_m^k\chi_{3Q_0})(x)\}\big\|_{l^q}+\epsilon,
\end{aligned}$$

since T is bounded from $L^1(l^{q_1}; \mathbb{R}^n)\times\cdots\times L^1(l^{q_m}; \mathbb{R}^n)$ to $L^{\frac{1}{m},\infty}(l^q; \mathbb{R}^n)$. Taking $r\to 0$ then leads to the desired conclusion. $\square$

Theorem 3.2. *Let $q_1, \dots, q_m \in (1, \infty)$ and $q \in (1/m, \infty)$ with $1/q = 1/q_1 + \cdots + 1/q_m$. Suppose that both the operators T and $\mathcal{M}_T$ are bounded from $L^1(l^{q_1}; \mathbb{R}^n)\times\cdots\times L^1(l^{q_m}; \mathbb{R}^n)$ to $L^{1/m,\infty}(l^q; \mathbb{R}^n)$. Then for $N \in \mathbb{N}$ and bounded functions $\{f_1^k\}_{1\le k\le N}, \dots, \{f_m^k\}_{1\le k\le N}$ with compact supports, there exists a $\frac{1}{2}\frac{1}{3^n}$-sparse of family $\mathcal{S}$ such that for a.e. $x \in \mathbb{R}^n$,*

$$\|\{T(f_1^k, \dots, f_m^k)(x)\}\|_{l^q} \lesssim \mathcal{A}_{m;\,\mathcal{S}}(\|\{f_1^k\}\|_{l^{q_1}}, \dots, \|\{f_m^k\}\|_{l^{q_m}})(x). \tag{3.1}$$

Proof. Again, we only consider the case $m = 2$. We follow the argument used in [27]. As it was pointed out in [27], it suffices to prove that for each cube $Q_0 \subset \mathbb{R}^n$, there exist pairwise disjoint cubes $\{P_j\} \subset \mathcal{D}(Q_0)$, such that $\sum_j |P_j| \le \frac{1}{2}|Q_0|$ and for a.e. $x \in Q_0$,

$$\begin{aligned}
&\big\|\{T(f_1^k\chi_{3Q_0}, f_2^k\chi_{3Q_0})(x)\}\big\|_{l^q}\chi_{Q_0}(x)\\
&\quad\le C\prod_{i=1}^2\langle\|\{f_i^k\}\|_{l^{q_i}}\rangle_{3Q_0}+\sum_j\|\{T(f_1^k\chi_{3P_j}, f_2^k\chi_{3P_j})(x)\}\|_{l^q}\chi_{P_j}(x).
\end{aligned}\tag{3.2}$$

To prove (3.2), let $C_2 > 0$ which will be chosen later and

$$\begin{aligned}
E = &\big\{x \in Q_0 : \|\{f_1^k(x)\}\|_{l^{q_1}}\|\{f_2^k(x)\}\|_{l^{q_2}} > C_2\prod_{i=1}^2\langle\|\{f_i^k\}\|_{l^{q_i}}\rangle_{3Q_0}\big\}\\
&\cup\big\{x \in Q_0 : \|\{\mathcal{M}_T(f_1^k\chi_{3Q_0}, f_2^k\chi_{3Q_0})(x)\}\|_{l^q} > C_2\langle\prod_{i=1}^2\langle\|\{f_i^k\}\|_{l^{q_i}}\rangle_{3Q_0}\big\}.
\end{aligned}$$

If we choose C_2 large enough, our assumption then says that $|E| \le \frac{1}{2^{n+2}}|Q_0|$. Now applying the Calderón-Zygmund decomposition to χ_E on Q_0 at level $\frac{1}{2^{n+1}}$, we then obtain a family of pairwise disjoint cubes $\{P_j\}$ such that

$$\frac{1}{2^{n+1}}|P_j| \le |P_j \cap E| \le \frac{1}{2}|P_j|,$$

and $|E\setminus\cup_j P_j|=0$. It then follows that $\sum_j |P_j|\le \frac{1}{2}|E|$, and $P_j\cap E^c\ne\emptyset$. Therefore, for some $\xi_0\in P_j\cap E^c$,

$$\Big\|\Big\{\|T(f_1^k\chi_{3Q_0},\, f_2^k\chi_{3Q_0})-T(f_1^k\chi_{3P_j},\, f_2^k\chi_{3P_j})\|_{l^\infty(P_j)}\Big\}\Big\|_{l^q} \tag{3.3}$$

$$\le \|\{\mathcal{M}_T(f_1^k\chi_{3Q_0},\, f_2^k\chi_{3Q_0})(\xi_0)\}\|_{l^q}\le C_2\prod_{i=1}^{2}\langle\|\{f_i^k\}\|_{l^{q_i}}\rangle_{3Q_0}.$$

Note that

$$\begin{aligned}(3.4)\quad &\|\{T(f_1^k\chi_{3Q_0}, f_2^k\chi_{3Q_0})(x)\}\|_{l^q}\chi_{Q_0}(x)\\ &\le \|\{T(f_1^k\chi_{3Q_0}, f_2^k\chi_{3Q_0})(x)\}\|_{l^q}\chi_{Q_0\setminus\cup_j P_j}(x)\\ &\quad+\sum_j\|\{T(f_1^k\chi_{3P_j},\, f_2^k\chi_{3P_j})(x)\}\|_{l^q}\chi_{P_j}(x)\\ &\quad+\sum_j\Big\|\Big\{T(f_1^k\chi_{3Q_0},\, f_2^k\chi_{3Q_0})(x)-T(f_1^k\chi_{3P_j},\, f_2^k\chi_{3P_j})(x)\Big\}\Big\|_{l^q}\chi_{P_j}(x).\end{aligned}$$

(3.2) now follows from (3.3), (3.4) and Lemma 3.1. □

Similar to the proof of Theorem 3.2, by mimicking the proof of Theorem 1.1 in [27], we can prove the following theorem.

Theorem 3.3. *Let $q_1,\dots,q_m\in(1,\infty)$ and $q\in(1/m,\infty)$ with $1/q=1/q_1+\cdots+1/q_m$, $b\in L^1_{\rm loc}(\mathbb{R}^n)$. Suppose that both the operators T and $\mathcal{M}_T$ are bounded from $L^1(l^{q_1};\mathbb{R}^n)\times\cdots\times L^1(l^{q_m};\mathbb{R}^n)$ to $L^{1/m,\infty}(l^q;\mathbb{R}^n)$. Then for $N\in\mathbb{N}$ and bounded functions $\{f_1^k\}_{1\le k\le N},\dots,\{f_m^k\}_{1\le k\le N}$ with compact supports, there exists a $\frac{1}{2}\frac{1}{3^n}$-sparse of family $\mathcal{S}$ such that for a.e. $x\in\mathbb{R}^n$,*

$$\begin{aligned}&\big\|\{[b,T]_i(f_1^k,\dots,f_m^k)(x)\}\big\|_{l^q}\\ \lesssim&\sum_{Q\in\mathcal{S}}|b(x)-\langle b\rangle_Q|\prod_{j=1}^m\langle\|\{f_j^k\}\|_{l^{q_j}}\rangle_Q\chi_Q(x)\\ &+\sum_{Q\in\mathcal{S}}\langle|b-\langle b\rangle_Q|\|\{f_i^k\}\|_{l^{q_i}}\rangle_Q\prod_{j\ne i}\langle\|\{f_j^k\}\|_{l^{q_j}}\rangle_Q\chi_Q(x).\end{aligned}$$

Lemma 3.4. *Let $q_1,\dots,q_m\in(1,\infty)$, $q\in(1/m,\infty)$ such that $1/q=1/q_1+\cdots+1/q_m$. Under the hypothesis of Theorem 1.7, the operator $\mathcal{M}_T$ is bounded from $L^1(l^{q_1};\mathbb{R}^n)\times\cdots L^1(l^{q_m};\mathbb{R}^n)$ to $L^{\frac{1}{m},\infty}(l^q;\mathbb{R}^n)$.*

Proof. For simplicity, we only consider the case $m=2$. For $\epsilon>0$, let

$$T_\epsilon(f_1,\, f_2)(x)=\int_{\min_j|x-y_j|>\epsilon}K(x;\,y_1,\,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2.$$

We claim that for each $\tau\in(0,\,1/2)$,

$$\sup_{\epsilon>0}|T_\epsilon(f_1,\,f_2)(x)|\lesssim M_\tau(T(f_1,\,f_2))(x)+Mf_1(x)Mf_2(x). \tag{3.5}$$

To prove this, let

$$f_j^{(0)}(y) = f_j(y)\chi_{B(x,\,\epsilon)}(y),\ \ f_j^{\infty} = f_j(y)\chi_{\mathbb{R}^n\setminus B(x,\,\epsilon)}(y).$$

For each $z \in B(x,\, \epsilon/12)$, it follows from the regularity condition (1.11) that

$$\begin{aligned}|T_\epsilon(f_1,\, f_2)(x)| &= |T(f_1^{\infty},\, f_2^{\infty})(x) - T(f_1^{\infty},\, f_2^{\infty})(z)| + |T(f_1^{\infty},\, f_2^{\infty})(z)| \\ &\lesssim Mf_1(x)Mf_2(x) + |T(f_1,\, f_2)(z)| + |T(f_1^0, f_2^0)(z)| \\ &\quad + |T(f_1^0,\, f_2^{\infty})(z)| + |T(f_1^{\infty},\, f_2^0)(z)|.\end{aligned}$$

Again by the size condition (1.7), we can verify that

$$|T(f_1^0,\, f_2^{\infty})(z)| + |T(f_1^{\infty},\, f_2^0)(z)| \lesssim Mf_1(x)Mf_2(x).$$

Therefore, for any $z \in B(x,\, \frac{\epsilon}{12})$,

$$|T_\epsilon(f_1,\, f_2)(x)| \le |T(f_1,\, f_2)(z)| + |T(f_1^{(0)},\, f_2^{(0)})(z)| + Mf_1(x)Mf_2(x).$$

This, together with the fact that T is bounded from $L^1(\mathbb{R}^n)\times\cdots\times L^1(\mathbb{R}^n)$ to $L^{1/m,\,\infty}(\mathbb{R}^n)$, leads to (3.5).

Let $Q \subset \mathbb{R}^n$ be a cube and $x, \xi \in Q$. Denote by B_x the ball centered at x and having diameter $12n\mathrm{diam}\, Q$. Then $3Q \subset B_x$. As in [27], we write

$$\begin{aligned}&\big|T(f_1\chi_{\mathbb{R}^n\setminus 3Q}, f_2\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\big| \\ &\le \big|T(f_1\chi_{\mathbb{R}^n\setminus B_x}, f_2\chi_{\mathbb{R}^n\setminus B_x})(\xi) - T(f_1\chi_{\mathbb{R}^n\setminus B_x}, f_2\chi_{\mathbb{R}^n\setminus B_x})(x)\big| \\ &\quad + \sup_{\epsilon>0}|T_\epsilon(f_1, f_2)(x)| + \big|T(f_1\chi_{\mathbb{R}^n\setminus B_x}, f_2\chi_{B_x\setminus 3Q})(\xi)\big| \\ &\quad + \big|T(f_1\chi_{B_x\setminus 3Q}, f_2\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\big|.\end{aligned}$$

It follows from the regularity condition (1.11) that

$$\big|T(f_1\chi_{\mathbb{R}^n\setminus B_x},\, f_2\chi_{\mathbb{R}^n\setminus B_x})(\xi) - T(f_1\chi_{\mathbb{R}^n\setminus B_x},\, f_2\chi_{\mathbb{R}^n\setminus B_x})(x)\big| \lesssim Mf_1(x)Mf_2(x).$$

On the other hand, by the size condition (1.7), we have

$$\begin{aligned}\big|T(f_1\chi_{B_x\setminus 3Q}, f_2\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\big| &\lesssim \int_{B_x}|f_1(y_1)|\mathrm{d}y_1\int_{\mathbb{R}^n\setminus 3Q}\frac{|f_2(y_2)|}{|x-y_2|^{2n}}\mathrm{d}y_2 \\ &\lesssim Mf_1(x)Mf_2(x).\end{aligned}$$

Similarly,

$$\big|T(f_1\chi_{\mathbb{R}^n\setminus B_x}, f_2\chi_{B_x\setminus 3Q})(\xi)\big| \lesssim Mf_1(x)Mf_2(x),$$

and

$$\big|T(f_1\chi_{\mathbb{R}^n\setminus 3Q},\, f_2\chi_{3Q})(\xi) + T(f_1\chi_{3Q},\, f_2\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\big| \lesssim Mf_1(x)Mf_2(x).$$

Combining the estimates above leads to that

$$\mathcal{M}_T(f_1,\, f_2)(x) \lesssim M_\tau(T(f_1,\, f_2))(x) + Mf_1(x)Mf_2(x). \tag{3.6}$$

Recall that T is bounded from $L^1(l^{q_1};\mathbb{R}^n)\times L^1(l^{q_2};\mathbb{R}^n)$ to $L^{\frac{1}{2},\,\infty}(l^q;\mathbb{R}^n)$ (see [19]), and M is bounded from $L^1(l^{q_j};\mathbb{R}^n)$ to $L^{1,\,\infty}(l^{q_j};\mathbb{R}^n)$. Now we choose

$\tau \in (0,\,1/2)$ in (3.6), our desired conclusion now follows from (3.6) and Lemma 2.6 immediately. □

We are now ready to prove our theorems.

Proof of Theorem 1.7. Obviously, it suffices to prove (1.12) for the case that $\{f_1^k\},\ldots,\{f_m^k\}$ are finite sequences. But this follows from Theorem 3.2, Lemma 3.4 and the estimate (2.5) directly. □

Proof of Theorem 1.9. Let $b_1,\,\ldots,\,b_m \in \mathrm{BMO}(\mathbb{R}^n)$. By the generalization of Hölder's inequality (see [35]), we know that

$$\left\langle |b_i(x)-\langle b_i\rangle_Q| \|\{f_i^k\}\|_{l^{q_i}}\right\rangle_Q \lesssim \left\| \|\{f_i^k\}\|_{l^{q_i}}\right\|_{L\log L,\,Q}.$$

As in the proof of Theorem 1.7, Theorem 1.9 now follows from Theorem 3.3, Lemma 3.4 and Theorem 2.1 and Theorem 2.2. □

Proof of Theorem 1.10. Theorem 1.10 now follows from Theorem 3.3, Lemma 3.4 and Theorem 2.7 immediately. □

4. Applications to the commutators of Calderón

Let us consider the m-th commutator of Calderón, which is defined by

$$\mathcal{C}_{m+1}(a_1,\,\ldots,\,a_m,\,f)(x) = \mathrm{p.\,v.}\int_{\mathbb{R}^n} \frac{\prod_{j=1}^m (A_j(x)-A_j(y))}{(x-y)^{m+1}} f(y)\mathrm{d}y,$$

where $a_j = A_j'$. This operator first appeared in the study of the Cauchy integrals along Lipschitz curves and, in fact, led to the first proof of the L^2 boundedness of the latter.

When $m=1$, it is well known that $\mathcal{C}_2$ is bounded from $L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1<p_1,\,p_2\le\infty$ and $\frac{1}{2}<p\le\infty$ satisfying $1/p=1/p_1+1/p_2$; and moreover, it is bounded from $L^{p_1}(R)\times L^{p_2}(\mathbb{R})$ to $L^{p,\,\infty}(\mathbb{R})$ if $\min\{p_1,\,p_2\}=1$ and in particular it is bounded from $L^1(\mathbb{R})\times L^1(\mathbb{R})$ to $L^{\frac{1}{2},\,\infty}(\mathbb{R})$; see [3,4]. The corresponding result that $\mathcal{C}_3$ maps $L^1(\mathbb{R})\times L^1(\mathbb{R})\times L^1(\mathbb{R})$ to $L^{\frac{1}{3},\,\infty}(\mathbb{R})$ was proved by Coifman and Meyer; see [7], while the analogous result for $\mathcal{C}_{m+1}$, $m\ge 3$, was established by Duong, Grafakos, and Yan [13]. As it was proved in [13], $\mathcal{C}_{m+1}$ can be rewritten as the following multilinear singular integral operator

$$\mathcal{C}_{m+1}(a_1,\,\ldots,\,a_m,\,f)(x) = \int_{\mathbb{R}^{m+1}} K(x;y_1,\ldots,y_{m+1}) \prod_{j=1}^m a_j(y_j) f(y_{m+1})\mathrm{d}y_1\ldots\mathrm{d}y_{m+1}, \tag{4.1}$$

with

$$K(x;y_1,\ldots,y_{m+1}) = \frac{(-1)^{me(y_{m+1}-x)}}{(x-y_{m+1})^{m+1}} \prod_{j=1}^m \chi_{(\min\{x,y_{m+1}\},\max\{x,y_{m+1}\})}(y_j),$$

and e is the characteristic function of $[0, \infty)$. It was pointed out in [20] that $\mathcal{C}_{m+1}$ satisfies Assumption 1.4 and (1.11). Thus by Theorems 1.7, 1.9 and 1.10, we have the following conclusions.

Corollary 4.1. *Let* $m \geq 1$, $p_1, \dots, p_{m+1} \in (1, \infty)$, $q_1, \dots, q_{m+1} \in (1, \infty)$, $p, q \in (\frac{1}{m+1}, \infty)$ *with* $1/p = 1/p_1 + \cdots + 1/p_{m+1}$, $1/q = 1/q_1 + \cdots + 1/q_{m+1}$, $\vec{w} = (w_1, \dots, w_{m+1}) \in A_{\vec{P}}(\mathbb{R}^{m+1})$. *Then*

$$\begin{aligned}&\|\{\mathcal{C}_{m+1}(a_1^k, \dots, a_m^k,\, f^k)\}\|_{L^p(l^q;\mathbb{R}^n,\nu_{\vec{w}})}\\ &\lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p_1'}{p}, \cdots, \frac{p_{m+1}'}{p}\}} \prod_{j=1}^m \|\{a_j^k\}\|_{L^{p_j}(l^{q_j};\,\mathbb{R},w_j)} \|\{f^k\}\|_{L^{p_{m+1}}(l^{q_{m+1}};\,\mathbb{R},w_{m+1})}.\end{aligned}$$

Corollary 4.2. *Let* $m \geq 1$, $p_1, \dots, p_{m+1} \in (1, \infty)$, $q_1, \dots, q_{m+1} \in (1, \infty)$, $p, q \in (\frac{1}{m+1}, \infty)$ *with* $1/p = 1/p_1 + \cdots + 1/p_{m+1}$, $1/q = 1/q_1 + \cdots + 1/q_{m+1}$, $\vec{w} = (w_1, \dots, w_{m+1}) \in A_{\vec{P}}(\mathbb{R}^{m+1})$. *Let* $b_1, \dots, b_m \in \mathrm{BMO}(\mathbb{R})$. *Then* $\mathcal{C}_{m+1,\,\vec{b}}$, *the commutator of* $\mathcal{C}_{m+1}$ *defined as in* (1.13), *satisfies the weighted estimate that*

$$\begin{aligned}&\|\{\mathcal{C}_{m+1,\,\vec{b}}(a_1^k, \dots, a_m^k,\, f^k)\}\|_{L^p(l^q;\mathbb{R}^n,\nu_{\vec{w}})}\\ &\lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p_1'}{p}, \cdots, \frac{p_{m+1}'}{p}\}}\\ &\quad\times \Big([\nu_{\vec{w}}]_{A_\infty} + \sum_{i=1}^m [\sigma_i]_{A_\infty}\Big) \prod_{j=1}^m \|\{a_j^k\}\|_{L^{p_j}(l^{q_j};\,\mathbb{R},w_j)} \|\{f^k\}\|_{L^{p_{m+1}}(l^{q_{m+1}};\,\mathbb{R},w_{m+1})}.\end{aligned}$$

Moreover, if $\vec{w} = (w_1, \dots, w_{m+1}) \in A_{1,\dots,1}(\mathbb{R}^{m+1})$, *then for each* $\lambda > 0$,

$$\begin{aligned}&\nu_{\vec{w}}(\{x \in \mathbb{R}^n : \|\{\mathcal{C}_{m+1,\,\vec{b}}(a_1^k, \dots, a_m^k,\, f^k)(x)\}\|_{l^q} > \lambda\})\\ &\lesssim \prod_{j=1}^m \Big(\int_{\mathbb{R}^n} \frac{\|\{a_j^k(y_j)\}\|_{l^{q_j}}}{\lambda^{\frac{1}{m+1}}} \log\Big(1 + \frac{\|\{a_j^k(y_j)\}\|_{l^{q_j}}}{\lambda^{\frac{1}{m+1}}}\Big) w_j(y_j)\mathrm{d}y_j\Big)^{\frac{1}{m+1}}\\ &\quad\times \Big(\int_{\mathbb{R}^n} \frac{\|\{f^k(y)\}\|_{l^{q_j}}}{\lambda^{\frac{1}{m+1}}} \log\Big(1 + \frac{\|\{f^k(y)\}\|_{l^{q_j}}}{\lambda^{\frac{1}{m+1}}}\Big) w_{m+1}(y)\mathrm{d}y\Big)^{\frac{1}{m}}.\end{aligned}$$

Added in Proof. After this paper was prepared, we learned that Dr. Kangwei Li [28] also observed that, Lerner's idea in [24] applies to the multilinear singular integral operators. We remark that our argument in the proof of Theorem 3.2 also based on this observation. Li [28] proved that the multilinear singular integral operators whose kernels satisfy L^r- Hörmander condition can be dominated by multilinear sparse operators. The main results in [28] are different from the results in this paper and are of independent interest.

References

[1] S. M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. **340** (1993), no. 1, 253–272.

[2] A. Bui, J. Conde-Alonso, X. T. Duong, and M. Hormozi, *Weighted bounds for multilinear operators with non-smooth kernels*, arxiv:1506.07752.

[3] A.-P. Calderón, *Commutators of singular integral operators*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1092–1099.

[4] C. P. Calderón, *On commutators of singular integrals*, Studia Math. **53** (1975), no. 2, 139–174.

[5] D. Chung, M. C. Pereyra, and C. Pérez, *Sharp bounds for general commutators on weighted Lebesgue spaces*, Trans. Amer. Math. Soc. **364** (2012), no. 3, 1163–1177.

[6] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[7] ———, *Au delà des opérateurs pseudo-différentiels*, Astérisque, 57, Société Mathématique de France, Paris, 1978.

[8] J. M. Conde-Alonso and G. Rey, *A pointwise estimate for positive dyadic shifts and some applications*, Math. Ann. **365** (2016), no. 3-4, 1111–1135.

[9] D. Cruz-Uribe, J. M. Martell, and C. Pérez, *Sharp weighted estimates for classical operators*, Adv. Math. **229** (2012), no. 1, 408–441.

[10] W. Damián, M. Hormozi, and K. Li, *New bounds for bilinear Calderón-Zygmund operators and applications*, arxiv:1512.02400.

[11] O. Dragicević, L. Grafakos, M. Pereyra, and S. Petermichl, *Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces*, Publ. Mat. **49** (2005), 73–91.

[12] X. Duong, R. Gong, L. Grafakos, J. Li, and L. Yan,, *Maximal operator for multilinear singular integrals with non-smooth kernels*, Indiana Univ. Math. J. **58** (2009), no. 6, 2517–2541.

[13] X. T. Duong, L. Grafakos, and L. Yan, *Multilinear operators with non-smooth kernels and commutators of singular integrals*, Trans. Amer. Math. Soc. **362** (2010), no. 4, 2089–2113.

[14] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.

[15] L. Grafakos, *Modern Fourier Analysis*, second edition, Graduate Texts in Mathematics, **250**, Springer, New York, 2009.

[16] L. Grafakos, L. Liu, and D. Yang, *Multiple-weighted norm inequalities for maximal multi-linear singular integrals with non-smooth kernels*, Proc. Roy. Soc. Edinburgh Sect. A **141** (2011), no. 4, 755–775.

[17] L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), no. 1, 124–164.

[18] G. Hu and D. Li, *A Cotlar type inequality for the multilinear singular integral operators and its applications*, J. Math. Anal. Appl. **290** (2004), no. 2, 639–653.

[19] G. Hu and K. Li, *Weighted vector-valued inequalities for a class of multilinear singular integral operators*, Proc. Edinb. Math. Soc., to appear.

[20] G. E. Hu and Y. P. Zhu, *Weighted norm inequalities with general weights for the commutator of Calderón*, Acta Math. Sin. (Engl. Ser.) **29** (2013), no. 3, 505–514.

[21] T. P. Hytönen, *The sharp weighted bound for general Calderón-Zygmund operators*, Ann. of Math. (2) **175** (2012), no. 3, 1473–1506.

[22] T. P. Hytönen, M. T. Lacey, and C. Pérez, *Sharp weighted bounds for the q-variation of singular integrals*, Bull. Lond. Math. Soc. **45** (2013), no. 3, 529–540.

[23] T. Hytönen and C. Pérez, *Sharp weighted bounds involving A_∞*, Anal. PDE **6** (2013), no. 4, 777–818.

[24] A. K. Lerner, *On pointwise estimates involving sparse operators*, New York J. Math. **22** (2016), 341–349.

[25] K. Lerner and F. Nazarov, *Intuitive dyadic calculus: the basics*, arXiv:1508.05639.

[26] A. Lerner, S. Ombrossi, C. Pérez, R. H. Torres, and R. Trojillo-Gonzalez, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math. **220** (2009), no. 4, 1222–1264.
[27] A. K. Lerner, S. Ombrosi, and I. P. Rivera-RRRos, *On pointwise and weighted estimates for commutators of Calderón–Zygmund operators*, Adv. Math. **319** (2017), 153–181.
[28] K. Li, *Sparse domination theorem for mltilinear singular integral operators with L^r-Hörmander condition*, arxiv:1606.03952.
[29] K. Li, K. Moen, and W. Sun, *The sharp weighted bound for multilinear maximal functions and Calderón-Zygmund operators*, J. Fourier Anal. Appl. **20** (2014), no. 4, 751–765.
[30] K. Li and W. Sun, *Weak and strong type weighted estimates for multilinear Calderón-Zygmund operators*, Adv. Math. **254** (2014), 736–771.
[31] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
[32] C. Pérez, G. Pradolini, R. H. Torres, and R. Trujillo-González, *End-point estimates for iterated commutators of multilinear singular integrals*, Bull. Lond. Math. Soc. **46** (2014), no. 1, 26–42.
[33] S. Petermichl, *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic*, Amer. J. Math. **129** (2007), no. 5, 1355–1375.
[34] ———, *The sharp weighted bound for the Riesz transforms*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1237–1249.
[35] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, **146**, Marcel Dekker, Inc., New York, 1991.
[36] E. M. Stein, *Harmonic Analysis*: *Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, **43**, Princeton University Press, Princeton, NJ, 1993.

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