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EXTENSIONS OF NAGATA'S THEOREM

Ahmed Hamed

Abstract. In [1], the authors generalize the concept of the class group of an integral domain D ($Cl_t(D)$) by introducing the notion of the S-class group of an integral domain where S is a multiplicative subset of D. The S-class group of D, S-$Cl_t(D)$, is the group of fractional t-invertible t-ideals of D under the t-multiplication modulo its subgroup of S-principal t-invertible t-ideals of D. In this paper we study when S-$Cl_t(D) \simeq S$-$Cl_t(D_T)$, where T is a multiplicative subset generated by prime elements of D. We show that if D is a Mori domain, T a multiplicative subset generated by prime elements of D and S a multiplicative subset of D, then the natural homomorphism S-$Cl_t(D) \to S$-$Cl_t(D_T)$ is an isomorphism. In particular, we give an S-version of Nagata's Theorem [13]: Let D be a Krull domain, T a multiplicative subset generated by prime elements of D and S another multiplicative subset of D. If D_T is an S-factorial domain, then D is an S-factorial domain.

1. Introduction

Let D be an integral domain with quotient field K. Let $\mathcal{F}(D)$ be the set of nonzero fractional ideals of D. For an $I \in \mathcal{F}(D)$, set $I^{-1} = \{x \in K/\ xI \subseteq D\}$. The mapping on $\mathcal{F}(D)$ defined by $I \mapsto I_v = (I^{-1})^{-1}$ is called the v-operation on D. A nonzero fractional ideal I is said to be a v-ideal or divisorial if $I = I_v$, and I is said to be v-invertible if $(II^{-1})_v = D$. For properties of the v-operation the reader is referred to [11, Section 34]. However, we will be mostly interested in the t-operation defined on $\mathcal{F}(D)$ by $I \mapsto I_t = \bigcup\{J_v,\ J$ is a nonzero finitely generated fractional subideal of $I\}$. (For properties of the t-operation the reader may consult [2].) A fractional ideal I is called a t-ideal if $I = I_t$. A t-ideal (respectively, v-ideal) I has t- (respectively, v-) finite type if $I = J_t$ (respectively, $I = J_v$) for some finitely generated fractional ideal J of D. The set of v-ideals may be a proper subset of the set of t-ideals. A fractional ideal I is said to be t-invertible if $(II^{-1})_t = D$. If I is t-invertible, then I_t and I^{-1} are v-ideals of finite type. The set $T(D)$ of t-invertible fractional t-ideals of D is a group under the t-multiplication $I \star J := (IJ)_t$, and the set $P(D)$ of nonzero principal fractional ideals of D is a subgroup of $T(D)$.

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Following [5], we define the class group of D, denoted by $Cl_t(D)$, to be the group of t-invertible fractional t-ideals of D under the t-multiplication modulo its subgroup of principal fractional ideals, that is, $Cl_t(D) = T(D)/P(D)$. The t-class group of an integral domain was studied by many authors ([2], [5] and [6]).

Let D be an integral domain and S a multiplicative subset of D generated by prime elements of D. Many authors have studied when the natural homomorphism $Cl_t(D) \to Cl_t(D_S)$ induced by $[I] \to [I_S]$ for $I \in T(D)$ is an isomorphism ([7], [10], and [13]). In [13], Nagata show that if D is a Krull domain and S a multiplicatively closed subset of D generated by principal prime elements of D, then $Cl_t(D) \to Cl_t(D_S)$ is an isomorphism. So by Nagata's Theorem we have the following result: Let D be a Krull domain and S a multiplicatively closed subset of D generated by principal primes of D. If D_S is a factorial domain, then D is a factorial domain [9, Corollary 7.3]. Later, S. Gabelli and M. Roitman generalize the Nagata's Theorem by relaxing the Krull assumption, they showed that, if D satisfies the ACCP (ascending chain condition on principal ideals) and T a multiplicatively closed subset of D generated by principal primes of D, then $Cl_t(D) \to Cl_t(D_T)$ is an isomorphism [10]. Also, in [7], El Abidine gave another class of domains D such that the natural homomorphism $Cl_t(D) \to Cl_t(D_T)$ is an isomorphism. First let us recall that an integral domain D is said to be a Prufer v-multiplication domain (PVMD) if every finitely generated $I \in \mathcal{F}(D)$ is t -invertible. According to [7], an integral domain D satisfies $(*)$ if for any finitely generated ideal I of D, I^{-1} is of v-finite type. For examples, Mori domains, PVMD's satisfy $(*)$. In [7], the author showed that if D is an integral domain satisfying $(*)$ and T a multiplicative subset generated by prime elements of D, then the homomorphism $Cl_t(D) \to Cl_t(D_T)$ is an isomorphism.

On the other hand, in [1], the authors generalize the concept of the class group of an integral domain $(Cl_t(D))$ by introducing the notion of the S-class group of an integral domain (S-$Cl_t(D)$) where S is a multiplicative subset of D. First, recall that from [3], an ideal I of D is said S-principal if $sI \subseteq J \subseteq I$, for some principal ideal J of D and some $s \in S$. Set $S\text{-}P(D) = S\text{-}Prin(D) \cap T(D)$, where $S\text{-}Prin(D)$ is the set of S-principal fractional ideals of D. Then $S\text{-}P(D)$ is a subgroup of $T(D)$. The S-class group of D, $S\text{-}Cl_t(D)$, is the group of fractional t-invertible t-ideals of D, under the t-multiplication modulo its subgroup of S-principal t-invertible t-ideals of D, that is, $S\text{-}Cl_t(D) = T(D)/S\text{-}P(D)$. Note that if the multiplicative subset S is included in the set of units of D, then $S\text{-}Cl_t(D) = Cl_t(D)$. In [1], the authors showed that if $D \subseteq L$ is an extension of integral domains such that L is a flat D-module and S a multiplicative subset of D, then the canonical mapping $\varphi : S\text{-}Cl_t(D) \to S\text{-}Cl_t(L)$, $[I]^S \mapsto [IL]^S$ is well-defined and it is an homomorphism ([1, Theorem 4.3]). Note that if T is a multiplicative subset of D, then D_T is a flat D-module. It is then natural to try to study when the homomorphism $S\text{-}Cl_t(D) \to S\text{-}Cl_t(D_T)$ is an isomorphism.

In particular we give an S-version of Nagata's Theorem and generalize some known results about the class group of an integral domain ([7], [13]).

In this paper we prove several versions of Nagata's Theorem and we investigate when I_f being an S-principal ideal of D_f implies that I is an S-principal ideal of D, for a principal prime f of D, a divisorial ideal I of D, and a multiplicative subset S of D. Also we study some conditions to put on f or S to have the same result. This gives us two generalizations of the main Theorems of [2], each one is useful to use for particular domains. In this article we show that if D is an integral domain, T a multiplicative subset generated by prime elements of D and S a saturated multiplicative subset of D, then the homomorphism S-$Cl_t(D) \to S$-$Cl_t(D_T)$ is injective. So in the particular case when S consists of units of D, we prove the result of D. D. Anderson and D. F. Anderson ([2, Theorem 2.3]). Also we prove that if D is a Krull domain, T a multiplicative subset generated by prime elements of D and S a saturated multiplicative subset of D. Then the homomorphism $\varphi : S$-$Cl_t(D) \to S$-$Cl_t(D_T)$ is an isomorphism. So we give an S-version of Nagata's Theorem. Moreover, we generalize the result of El Abidine [7], we show that if D is an integral domain satisfying $(*)$, T a multiplicative subset generated by prime elements of D and S a saturated multiplicative subset of D. Then the homomorphism S-$Cl_t(D) \to S$-$Cl_t(D_T)$ is an isomorphism. Also we prove another version of Nagata's Theorem when the multiplicative set S is not necessarily saturated. We show that if D is a Mori domain, T a multiplicative subset generated by prime elements of D and S another multiplicative subset of D. Then the homomorphism S-$Cl_t(D) \to S$-$Cl_t(D_T)$ is an isomorphism. As an application of these results we have the following characterizations of S-factorial and S-GCD properties. First let us recall that the mapping on $\mathcal{F}(D)$ defined by $I \mapsto I_w = \{x \in K,\ xJ \subseteq I$ for some finitely generated ideal J of D such that $J_v = D\}$ is a star operation on D called the w-operation on D. Let D be an integral domain, S a multiplicative subset of D and I a nonzero ideal of D. We say that I is an S-w-principal ideal of D, if there exist an $s \in S$ and a principal ideal J of D such that $sI \subseteq J \subseteq I_w$. We also define D to be an S-factorial domain if each nonzero ideal of D is S-w-principal [12]. We show that if D is a Krull domain, T a multiplicative subset generated by prime elements of D and S another multiplicative subset of D, then D is an S-factorial domain if and only if D_T is an S-factorial domain. Also if D is a PvMD, T a multiplicative subset generated by prime elements of D and S a saturated multiplicative subset of D, then D is an S-GCD domain if and only if D_T is an S-GCD domain.

2. S-principal ideals

Let D be an integral domain and S a multiplicative subset of D. Recall from [1] that, the S-class group of D, S-$Cl_t(D)$, is the group of fractional t-invertible t-ideals of D under the t-multiplication modulo its subgroup of S-principal t-invertible t-ideals of D, that is, S-$Cl_t(D) = T(D)/S$-$P(D)$. Note that if the multiplicative subset S is included in the set of units of D, then $S\text{-}Cl_t(D) = Cl_t(D)$. We denote by $[I]^S$ the equivalence class of an ideal I of D. We start this section by the following Proposition:

Proposition 2.1. *Let D be an integral domain and S a multiplicative subset of D. If $Cl_t(D) = 0$, then $S\text{-}Cl_t(D) = 0$.*

Proof. Let I be a t-invertible t-ideal of D. Since $Cl_t(D) = 0$, then $[I] = 0$. So I is a principal ideal, which implies that I is S-principal. Therefore $[I]^S = 0$.

Let D be an integral domain and S a multiplicative subset of D. Recall from [3] that, an ideal I of D is S-principal, if $sI \subseteq J \subseteq I$ for some $s \in S$ and some principal ideal J of D. Also we define D to be an S-Principal Ideal Domain (S-PID), if every ideal of D is S-principal. □

Remark 2.2. The converse of Proposition 2.1 is false in general. Indeed, let D be a Krull domain which is not factorial (For example, $D = \mathbb{Z}[i\sqrt{5}]$) and let $S = D\backslash\{0\}$. Then D is an S-PID, which implies that $S\text{-}Cl_t(D) = 0$. But D is a Krull domain which is not factorial, then by [5, Proposition 2], $Cl_t(D) \neq 0$. In particular by [1, Theorem 4.1], D is an S-factorial domain which is not factorial.

Let D be an integral domain, I an ideal of D and f an element of D. We denote by I_f, the localization of the ideal I of D by the multiplicative subset $S = \{f^n,\ n \in \mathbb{N}\}$ of D. Then Theorem 1.3 of [2] can be rewritten as follows, the proof remains practically the same.

Lemma 2.3. *Let D be an integral domain with $*$ a star operation and nonzero $x_1, \ldots, x_n \in D$. Then the following statements are equivalent:*

1. $(x_1, \ldots, x_n)^* = D$.
2. *For ideals I,J of D, if $I_{x_i} \subseteq J_{x_i}$, $i = 1, \ldots, n$, then $I^* \subseteq J^*$.*
3. *For *-ideals I,J of D, if $I_{x_i} \subseteq J_{x_i}$, $i = 1, \ldots, n$, then $I \subseteq J$.*
4. *For finitely generated ideals I and J of D, if $I_{x_i} \subseteq J_{x_i}$, $i = 1, \ldots, n$, then $I^* \subseteq J^*$.*
5. *For finite type $*$-ideals I and J of D, if $I_{x_i} \subseteq J_{x_i}$, $i = 1, \ldots, n$, then $I \subseteq J$.*

Proof. (1) $\Longrightarrow$ (2). Let $c \in I$. Then $c \in I_{x_i} \subseteq J_{x_i}$, so $x_i^{N_i} c \in J$ for some N_i. Thus for some N, $(x_1, \ldots, x_n)^N c \subseteq J$. Hence $c \in ((x_1, \ldots, x_n)^N)^* c \subseteq ((x_1, \ldots, x_n)^N c)^* \subseteq J^*$. Thus $I^* \subseteq J^*$.

As (2) $\Longrightarrow$ (3), (2) $\Longrightarrow$ (4), (3) $\Longrightarrow$ (5) and (4) $\Longrightarrow$ (5) are each immediate, we need only to prove (5) $\Longrightarrow$ (1). Since $x_1, \ldots, x_n \in D$, then $(x_1, \ldots, x_n)^* \subseteq D$. Conversely, as $(x_1, \ldots, x_n)^*$ and D are each finite-type *-ideals and $D_{x_i} \subseteq {(x_1, \ldots, x_n)^*}_{x_i}$ for each $i = 1, \ldots, n$, then by (5), $D \subseteq (x_1, \ldots, x_n)^*$. □

Proposition 2.4. *Let D be an integral domain, S a multiplicative subset of D and I a divisorial ideal of D. Then I is an S-principal ideal of D if and only if I^{-1} is an S-principal ideal of D.*

Proof. If I is S-principal, then there exist an $s \in S$ and an $a \in I$ such that $sI \subseteq aD \subseteq I$. Thus $I^{-1} \subseteq \frac{1}{a}D \subseteq \frac{1}{s}I^{-1}$, furthermore $sI^{-1} \subseteq \frac{s}{a}D \subseteq I^{-1}$. Conversely, if there exist an $s \in S$ and an $\alpha \in I^{-1}$ such that $sI^{-1} \subseteq \alpha D \subseteq I^{-1}$, then $(I^{-1})^{-1} \subseteq \frac{1}{\alpha}D \subseteq \frac{1}{s}(I^{-1})^{-1}$, so $s(I^{-1})^{-1} \subseteq \frac{s}{\alpha}D \subseteq (I^{-1})^{-1}$. Since I is a divisorial ideal, then $sI \subseteq \frac{s}{\alpha}D \subseteq I$, hence I is an S-principal ideal of D. □

Recall that an element f of D is said to be prime, if fD is a prime ideal of D. In [2], the authors determine when the condition that the localization I_f of a divisorial ideal I by a principal prime f is principal implies that I is also principal. Our next Theorem give an S-version of this result [2, Theorem 2.1].

Theorem 2.5. *Let D be an integral domain, S a saturated multiplicative subset of D, I a divisorial ideal of D and f a prime element of D. Then the following statements hold.*

(1) *If I is an integral ideal of D and $I_f \cap S \neq \emptyset$, then I is an S-principal ideal of D.*
(2) *If I_f is an S-principal ideal of D_f and I has v-finite type, then I is an S-principal ideal of D.*

Proof. (1) Since $I_f \cap S \neq \emptyset$, then there exist an $n \in \mathbb{N}$ and an $s \in S$ such that $sf^n \in I$. If $I \not\subseteq fD$, let $i \in I \setminus fD$. Since fD is a maximal divisorial ideal ([9, Lemma 3.7]), then $(i, f)_v = D$. We have $(sD)_f \subseteq I_f$ and $(sD)_i = sD_i \subseteq D_i = I_i$. Then by Lemma 2.3, $sD = (sD)_v \subseteq I_v$. But by hypothesis I is divisorial, then $sD \subseteq I$. So $sI \subseteq sD \subseteq I$, and hence I is S-principal.

Now if $I \subseteq fD$. Set $F = \{m \in \mathbb{N},\ I \subseteq f^m D\}$, F is nonempty because $1 \in F$. If F is bounded, it has a maximum $N \in \mathbb{N}$ such that $I \subseteq f^N D$ and $I \not\subseteq f^{N+1}D$. Then $f^{-N}I \subseteq D$ and $f^{-N}I \not\subseteq fD$. Set $I' = f^{-N}I$. Since I is divisorial and $I' \subseteq D$, then I' is a divisorial integral ideal of D. Since $I_f \cap S \neq \emptyset$, then $(I')_f \cap S \neq \emptyset$. So by the first case applied on I', there exists a $t \in S$ such that $tI' \subseteq tD \subseteq I'$. This implies that $tI \subseteq tf^N D \subseteq I$, and hence I is S-principal.

If F is not bounded, then we can find $k \in \mathbb{N}\setminus\{0\}$ such that $I \subseteq f^{n+k}D$. Since $sf^n \in I$, then $sf^n \in f^{n+k}D$, which implies that $s \in f^k D$. As S is saturated, then $f^k \in S$. So $f \in S$ and $f^n \in S$. Hence $sf^n \in S \cap I$ and consequently I is S-principal.

(2) If I is of v-finite type, then $(D : I)_f = (D_f : I_f)$, ($I = J_v$ where J is finitely generated, we use the fact that the extension $D \subseteq D_f$ is flat and so $J_f^{-1} = (J_f)^{-1}$). Since I_f is S-principal, then there exist an $s \in S$ and an $a \in I$ such that $sI_f \subseteq aD_f \subseteq I_f$, thus $I_f^{-1} \subseteq \frac{1}{a}D_f \subseteq \frac{1}{s}I_f^{-1}$. Set $J = aI^{-1}$, then J is a divisorial integral ideal of D and $J_f = aI_f^{-1}$, so $sJ_f \subseteq sD_f \subseteq J_f$. By (1), J is S-principal, thus I^{-1} is S-principal, so by Proposition 2.4, I is S-principal. □

Let D be an integral domain and T a multiplicative subset generated by prime elements of D. In [2], the authors showed that the natural homomorphism $Cl_t(D) \longrightarrow Cl_t(D_T)$ is injective. Our next Theorem generalize this result. Let us first recall the following fact: Hamed and Hizem in [1], showed that if D

$\subseteq L$ is an extension of integral domains such that L is a flat D-module and S a multiplicative subset of D, then the canonical mapping $\varphi : S\text{-}Cl_t(D) \to S\text{-}Cl_t(L)$, $[I]^S \mapsto [IL]^S$ is well-defined and it is an homomorphism [1, Theorem 4.3]. Note that if T is a multiplicative subset of D, then D_T is a flat D-module.

Theorem 2.6. *Let D be an integral domain, T a multiplicative subset generated by prime elements of D and S a saturated multiplicative subset of D. Then the homomorphism $S\text{-}Cl_t(D) \to S\text{-}Cl_t(D_T)$ is injective.*

Proof. We show that for $I \in T(D)$ if I_T is an S-principal ideal of D_T, then I is an S-principal ideal of D. Let $I \in T(D)$ such that I_T is S-principal. Since I is of v-finite type, then $(D : I)_T = (D_T : I_T)$ ($I = J_v$ where J is finitely generated, we use the fact that the extension $D \subseteq D_T$ is flat and so $J_T^{-1} = (J_T)^{-1}$). Since I_T is S-principal, then there exist an $s \in S$ and an $a \in I$ such that $sI_T \subseteq aD_T \subseteq I_T$. Thus $I_T^{-1} \subseteq \frac{1}{a}D_T \subseteq \frac{1}{s}I_T^{-1}$. Set $J = aI^{-1}$. Then J is a divisorial integral ideal of D, $J_T = aI_T^{-1}$ and $sJ_T \subseteq sD_T \subseteq J_T$. So there exists an $h \in T$ such that $sh \in J$. Write $h = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ for some prime elements $p_1, \dots, p_n$ of D such that $p_i \neq p_j$ for all $i \neq j$. Let $f = p_1 \cdots p_n$ and let $m = max\{\alpha_i\ ,\ 1 \leq i \leq n\ \}$. Then $sf^m \in J$. Thus $J_f \cap S \neq \emptyset$. We proceed then by induction on n:

For $n = 1$, we have $J_{p_1} \cap S = J_f \cap S \neq \emptyset$. Then by the Theorem 2.5(1), J is an S-principal ideal of D. But $J^{-1} = \frac{1}{a}I_v = \frac{1}{a}I$, so by Proposition 2.4, I is an S-principal ideal of D.

Suppose that it remains true until the order n, we show that it holds for $n+1$: Let $f = p_1 \cdots p_n p_{n+1}$, $f_1 = p_1 \cdots p_n$ and $R = D_{f_1}$. Then $D_f = R_{p_{n+1}}$. It is easy to show that J_{f_1} is an integral divisorial ideal of R and p_{n+1} is a prime element of R. Moreover $(J_{f_1})_{p_{n+1}} \cap S = J_f \cap S \neq \emptyset$. Then by Theorem 2.5(1), J_{f_1} is an S-principal ideal of R. So by the induction hypothesis J is an S-principal ideal of D. But $J^{-1} = \frac{1}{a}I_v = \frac{1}{a}I$, so by Proposition 2.4, I is an S-principal ideal of D. □

Let S be a multiplicative subset of D. If $I \in T(D)$, then $I_S \in T(D_S)$ [6, Lemma 2.8]. Thus there is a natural homomorphism $Cl_t(D) \to Cl_t(D_T)$ induced by $[I] \to [I_T]$ for $I \in T(D)$. In Theorem 2.6 if S consists of units of D, then we can recover the result of D. D. Anderson and D. F. Anderson [2, Theorem 2.3].

Corollary 2.7 ([2])**.** *Let D be an integral domain, T a multiplicative subset generated by prime elements of D. Then the homomorphism $Cl_t(D) \to Cl_t(D_T)$ is injective.*

3. On S-Nagata's Theorem

In [13], Nagata showed that if D is a Krull domain and S a multiplicatively closed subset of D generated by prime elements of D, then the natural homomorphism $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ is an isomorphism. In this section we give an S-version of Nagata's Theorem [13].

Theorem 3.1. *Let D be a Krull domain, T a multiplicative subset generated by prime elements of D and S a saturated multiplicative subset of D. Then the homomorphism $\varphi : S\text{-}Cl_t(D) \to S\text{-}Cl_t(D_T)$ is an isomorphism.*

Proof. Since the extension $D \subseteq D_T$ is flat, then φ is an homomorphism [1, Theorem 4.3]. We show that φ is surjective.

Let

$$\begin{array}{rlcl} \Psi : & Cl_t(D) & \longrightarrow & Cl_t(D_T) \\ & [I] & \longrightarrow & [I_T] \end{array}$$

By Nagata's Theorem [9, Corollary 7.3], the mapping Ψ is surjective. So the mapping $\varphi : S\text{-}Cl_t(D) \to S\text{-}Cl_t(D_T)$ is surjective. Indeed, let $[J]^S \in S\text{-}\mathrm{Cl}_t(D_T)$. Since $J \in T(D_T)$ and Ψ is surjective, there exist an $I \in T(D)$ such that $[I_T] = [J]$. This implies that $(I_T J^{-1})_t$ is a principal ideal of D_T, in particular $(I_T J^{-1})_t$ is an S-principal ideal of D_T. So $[I_T]^S = [J]^S$, and hence φ is surjective. Moreover by Theorem 2.6, the mapping φ is injective. Hence φ is an isomorphism. □

Our next result relaxes the Krull assumption in Theorem 3.1. First, let us recall from [14] that a domain D is said to be a Mori domain if it satisfies the ascending chain condition on integral divisorial ideals. Also, according to [7], D is said to satisfy the property $(*)$ if for any finitely generated ideal I of D, I^{-1} is of v-finite type. For examples, Mori domains, PVMD's satisfy $(*)$. In [7], El Abidine generalized Nagata's Theorem: Let D be an integral domain satisfying $(*)$ and T a multiplicative subset generated by prime elements of D. Then the homomorphism $Cl_t(D) \to Cl_t(D_T)$ is an isomorphism. Our next Theorem gives an S-version of this result. So we generalize both Nagata's Theorem and the result of El Abidine [7, Theorem 1].

Theorem 3.2. *Let D be an integral domain satisfying $(*)$, T a multiplicative subset generated by prime elements of D and S a saturated multiplicative subset of D. Then the homomorphism $\varphi : S\text{-}Cl_t(D) \to S\text{-}Cl_t(D_T)$ is an isomorphism.*

Proof. Since the extension $D \subseteq D_T$ is flat, then φ is an homomorphism [1, Theorem 4.3].

The injectivity of φ follows from Theorem 2.6.
We show that φ is surjective. Let

$$\begin{array}{rlcl} \Psi : & Cl_t(D) & \longrightarrow & Cl_t(D_T) \\ & [I] & \longrightarrow & [I_T] \end{array}$$

Since D satisfies $(*)$, then by [7, Theorem 1], Ψ is an isomorphism. So by the proof of Theorem 3.1, φ is surjective. Hence φ is an isomorphism. □

Let D be an integral domain and d an element of D. Recall from [4] that d is said to be Archimedean (or bounded), if $\bigcap_{n\geq 0} d^n D = 0$. We say that D is Archimedean, if all element of D are Archimedean.

Example 3.3. (1) Completely integrally closed domains and domains that satisfies the ACCP condition (Mori domains and Noetherian domains) are Archimedeans domains.

(2) Let D be an integral domain and X, Y two indeterminates over D. Then it is easy to see that $X \in D[X, Y]$ is an Archimedean prime element.

(3) There exists a prime element which is not Archimedean. Indeed, let (D, M) be a rank-two discrete valuation domain. Then by [8, Proposition 5.3.1. Page 145], $M = pD$ where p is a prime element of D. Let Q be a height-one prime ideal of D. Since D is a valuation domain, then for all $n \in \mathbb{N}$, $Q \subseteq p^n D$. Which implies that $Q \subseteq \bigcap_{n\in\mathbb{N}} p^n D$. So $\bigcap_{n\in\mathbb{N}} p^n D \neq (0)$, and hence p is a prime element of D which is not Archimedean.

If we want to avoid the condition on S (saturated) in Theorem 2.5, we can take f to be a prime Archimedean element of D. The following Lemma prove this result.

Lemma 3.4. *Let D be an integral domain, S a multiplicative subset of D, I a divisorial ideal of D and f a prime Archimedean element of D.*

1. *If I is an integral ideal of D and $I_f \cap S \neq \emptyset$, then I is S-principal.*
2. *If I_f is an S-principal ideal of D_f and I has v-finite type, then I is an S-principal ideal of D.*

Proof. (1) Since $I_f \cap S \neq \emptyset$, then there exist an $n \in \mathbb{N}$ and an $s \in S$ such that $sf^n \in I$. If $I \not\subseteq fD$, let $i \in I \setminus fD$. Since fD is a maximal divisorial ideal ([9, Lemma 3.7]), then $(i, f)_v = D$. We have $(sD)_f \subseteq I_f$ and $(sD)_i = sD_i \subseteq D_i = I_i$. Then by Lemma 2.3, $sD = (sD)_v \subseteq I_v$. But by hypothesis I is divisorial, then $sD \subseteq I$. So $sI \subseteq sD \subseteq I$, and hence I is S-principal. If $I \subseteq fD$, set $F = \{m \in \mathbb{N}\ ,\ I \subseteq f^m D\}$, F is nonempty because $1 \in F$. Moreover F is bounded. Indeed, if F is not bounded, then for all $p \in \mathbb{N}$, there exist a $k \geq p + 1$ such that $I \subseteq f^k D$. This implies that $(0) \neq I \subseteq \bigcap_{n\geq 0} f^n D = (0)$, contradiction. So F is bounded, and thus it has a maximum $N \in \mathbb{N}$, $I \subseteq f^N D$ and $I \not\subseteq f^{N+1} D$. Then $f^{-N} I \subseteq D$ and $f^{-N} I \not\subseteq fD$. Set $I^{'} = f^{-N} I$. Since I is divisorial and $I^{'} \subseteq D$, then $I^{'}$ is a divisorial integral ideal of D, and so by the first case applied on $I^{'}$ there exist an $s \in S$ such that $sI^{'} \subseteq sD \subseteq I^{'}$. Thus $sI \subseteq sf^N D \subseteq I$. Hence I is S-principal.

(2) We proceed exactly as in the proof of Theorem 2.5. □

Corollary 3.5. *Let D be an integral domain satisfying $(*)$, $T = \{p^n,\ n \in \mathbb{N}\}$ where p is an Archimedean prime element of D and S another multiplicative subset of D. Then the homomorphism $\varphi : S\text{-}Cl_t(D) \to S\text{-}Cl_t(D_T)$ is an isomorphism.*

Proof. To prove φ is injective, it is sufficient to proceed exactly as in the proof of Theorem 2.6, in which the only difference is by using Lemma 3.4 instead of Theorem 2.5. Moreover, since D satisfies $(*)$, then by [7, Theorem 1], the mapping

$$\begin{array}{cccc} \Psi: & Cl_t(D) & \longrightarrow & Cl_t(D_T) \\ & [I] & \longrightarrow & [I_T] \end{array}$$

is an isomorphism. So by the proof of Theorem 3.1, φ is surjective. □

Proposition 3.6. *Let D be an integral domain, T be a multiplicative subset generated by Archimedeans prime elements of D and S another multiplicative subset of D. If for each multiplicative subset S' of D the localization $D_{S'}$ is an Archimedean domain, then the homomorphism $\varphi : S\text{-}Cl_t(D) \to S\text{-}Cl_t(D_T)$, $[I]^S \mapsto [ID_T]^S$ is injective.*

Proof. We proceed exactly as in the proof of Theorem 2.6. Indeed, we show that for $I \in T(D)$ if I_T is an S-principal ideal of D_T, then I is S-principal. Let $I \in T(D)$ such that I_T is S-principal. Since I is of v-finite type, then $(D : I)_T = (D_T : I_T)$. Since I_T is S-principal, then there exist an $s \in S$ and an $a \in I$ such that $sI_T \subseteq aD_T \subseteq I_T$. Thus $I_T^{-1} \subseteq \frac{1}{a}D_T \subseteq \frac{1}{s}I_T^{-1}$. Set $J = aI^{-1}$. Then J is a divisorial integral ideal of D, $J_T = aI_T^{-1}$ and $sJ_T \subseteq sD_T \subseteq J_T$. So there exists an $h \in T$ such that $sh \in J$. Write $h = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ for some prime elements $p_1, \dots, p_n$ of D such that $p_i \neq p_j$ for all $i \neq j$. Let $f = p_1 \cdots p_n$ and let $m = max\{\alpha_i,\ 1 \leq i \leq n\}$. Then $sf^m \in J$. Thus $J_f \cap S \neq \emptyset$. We proceed then by induction on n:

For $n = 1$, we have $J_{p_1} \cap S = J_f \cap S \neq \emptyset$. Since p_1 is an Archimedean prime element of D, then by Lemma 3.4(1), J is an S-principal ideal of D. Hence by Proposition 2.4, I is S-principal.

Suppose that it remains true until the order n, we show that it holds for $n + 1$:

Let $f = p_1 \cdots p_n p_{n+1}$, $f_1 = p_1 \cdots p_n$ and $R = D_{f_1}$. Then $D_f = R_{p_{n+1}}$. It is easy to show that J_{f_1} is an integral divisorial ideal of R and p_{n+1} is a prime element of R. Moreover, as by the hypothesis that R is Archimedean, then p_{n+1} is an Archimedean element of R. Since $(J_{f_1})_{p_{n+1}} \cap S = J_f \cap S \neq \emptyset$, then by Lemma 3.4(1), J_{f_1} is an S-principal ideal of R. So by the induction hypothesis J is an S-principal ideal of D. Hence by Proposition 2.4, I is an S-principal ideal of D. □

Remark 3.7. There exists an Archimedean domain D such that for each prime ideal P of D the localization D_P is not Archimedean. First, let us recall from [4] that, an element p of D is said to be bounded if p is not Archimedean. Also we define D to be an anti-Archimedean domain if each nonzero element of D is bounded.

Now by [4, Example 2.2], there exists an example of a completely integrally closed (and hence Archimedean) Bezout domain D with no rank-one valuation overrings. Thus while D is not anti-Archimedean, every valuation overring of D is anti-Archimedean. Note that each localization D_P of D (P a prime ideal) is anti-Archimedean.

Since every localization of a Mori domain is a Mori domain (in particular an Archimedean domain), then Proposition 3.6 can be written as follow.

Corollary 3.8. *Let D be a Mori domain, T a multiplicative subset generated by prime elements of D and S another multiplicative subset of D. Then the homomorphism* $\varphi : S\text{-}Cl_t(D) \to S\text{-}Cl_t(D_T)$, $[I]^S \mapsto [ID_T]^S$ *is injective.*

The next Theorem give an S-version of Nagata's Theorem in the case when D is a Mori domain.

Theorem 3.9. *Let D be Mori domain, T a multiplicative subset generated by prime elements of D and S another multiplicative subset of D. Then the homomorphism* $S\text{-}Cl_t(D) \longrightarrow S\text{-}Cl_t(D_T)$ *is an isomorphism.*

Proof. By the previous Corollary, φ is injective.
Since D is a Mori domain, then D satisfies $(*)$. So by [7, Theorem 1], the mapping

$$\begin{array}{cccc} \Psi : & Cl_t(D) & \longrightarrow & Cl_t(D_T) \\ & [I] & \longrightarrow & [I_T] \end{array}$$

is an isomorphism. By the proof of Theorem 3.1, φ is surjective and hence φ is an isomorphism. □

Let D be an integral domain with quotient filed K and S a multiplicative subset of D. The mapping on $\mathcal{F}(D)$ defined by $I \mapsto I_w := \{x \in K \mid xJ \subseteq I$ for some finitely generated ideal J of D such that $J_v = D\}$ is called the w-operation on D. Recall from [12] that, a nonzero ideal I of D is S-w-principal if there exist an $s \in S$ and a principal ideal J of D such that $sI \subseteq J \subseteq I_w$. We also define D to be an S-factorial domain if each nonzero ideal of D is S-w-principal. Our next Theorem is an S-version of a will-known result about factorial domains, that is, if D is a Krull domain and T a multiplicative subset generated by prime elements of D such that D_T is a factorial domain, then D is a factorial domain [9]. To prove this, we need the following Proposition.

Proposition 3.10 ([1, Theorem 4.1])**.** *Let D be a Krull domain and S a multiplicative subset of D. Then* $S\text{-}Cl_t(D) = 0$ *if and only if D is an S-factorial domain.*

Theorem 3.11. *Let D be a Krull domain, T a multiplicative subset generated by prime elements of D and S another multiplicative subset of D. Then D is an S-factorial domain if and only if D_T is an S-factorial domain.*

Proof. ($\Rightarrow$) This implication is always true and need not the Krull hypothesis. Indeed, let I_T be an ideal of D_T with I an ideal of D. Since D is S-factorial, then there exist an $s \in S$ and an $\alpha \in I$ such that $sI \subseteq \alpha D \subseteq I_w$. So $sI_T \subseteq \alpha D_T \subseteq (I_w)_T$. But by [12, lemma 1.2], $(I_w)_T \subseteq (I_T)_w$. Hence I_T is an S-w-principal ideal of D_T.

($\Leftarrow$) Since a Krull domain is a Mori domain, then this implication follows form Theorem 3.9 and Proposition 3.10. □

Recall a couple of definitions from [1]. Let D be an integral domain and S a multiplicative subset of D. We say that a nonzero ideal I of D is S-v-principal if there exist an $s \in S$ and $a \in D$ such that $sI \subseteq aA \subseteq I_v$. We also define D to be an S-GCD-domain if each finitely generated nonzero ideal of D is S-v-principal.

Proposition 3.12 ([1, Theorem 4.2])**.** *Let D be a PvMD. Then S-$Cl_t(D) = 0$ if and only if D is an S-GCD-domain.*

We finish this work with the following Theorem.

Theorem 3.13. *Let D be a PvMD, T a multiplicative subset generated by prime elements of D and S a saturated multiplicative subset of D. Then D is an S-GCD domain if and only if D_T is an S-GCD domain.*

Proof. ($\Rightarrow$) This implication is always true and need not the PvMD hypothesis. Indeed, let J be a finitely generated ideal of D_T. Then we can find a finitely generated ideal I of D such that $J = I_T$. Since D is an S-GCD domain, then there exist an $s \in S$ and an $\alpha \in I$ such that $sI \subseteq \alpha D \subseteq I_v$. As I is a finitely generated ideal of D, then $(I_v)_T \subseteq (I_T)_v$. So $sI_T \subseteq \alpha D_T \subseteq (I_v)_T \subseteq (I_T)_v$. Thus $J = I_T$ is S-v-principal.

($\Leftarrow$) This implication follows from Theorem 3.2 and Proposition 3.12. □

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Ahmed Hamed
Department of Mathematics
Faculty of Sciences
Monastir, Tunisia
Email address: hamed.ahmed@hotmail.fr