

ON UNICITY OF MEROMORPHIC SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we give a uniqueness theorem on meromorphic solutions f of finite order of a class of differential-difference equations such that solutions f are uniquely determined by their poles and two distinct values.

1. Introduction and main results

Let $\mathcal{M}(\mathbb{C})$ be the fields of meromorphic functions on the complex plane \mathbb{C} and let \mathbb{Z}_+ (resp., \mathbb{Z}^+) denote the set of non-negative (resp., positive) integers. Take two integers $m \in \mathbb{Z}_+$, $n \in \mathbb{Z}^+$ and take n multi-indexes

$$\mathbf{j}_k = (j_{k0}, \dots, j_{km}) \in \mathbb{Z}_+^{m+1}, \quad k = 1, \dots, n$$

associated to n elements

$$\mathbf{c}_k = (c_{k0}, \dots, c_{km}) \in \mathbb{C}^{m+1}, \quad k = 1, \dots, n.$$

We define a differential-difference operator $D : \mathcal{M}(\mathbb{C}) \rightarrow \mathcal{M}(\mathbb{C})$ as follows:

$$(1.1) \quad Df = \sum_{k=1}^n a_k f_{c_{k0}}^{j_{k0}} (f'_{c_{k1}})^{j_{k1}} \dots (f_{c_{km}}^{(m)})^{j_{km}},$$

where $a_k \in \mathcal{M}(\mathbb{C}) - \{0\}$ for each $k \in \{1, \dots, n\}$, and where the function f_c associated to $f \in \mathcal{M}(\mathbb{C})$ and a constant c is defined by

$$f_c(z) = f(c + z), \quad z \in \mathbb{C}.$$

Further, take two coprime polynomials over $\mathcal{M}(\mathbb{C})$

$$(1.2) \quad P(w) = \sum_{i=0}^p b_i w^i, \quad Q(w) = \sum_{l=0}^q d_l w^l$$

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with $b_p d_q \neq 0$. We will study admissible meromorphic solutions of the differential-difference equation

$$(1.3) \quad Df = \frac{P(f)}{Q(f)}.$$

A meromorphic solution f of (1.3) is said to be *admissible* if f is non-constant such that the Nevanlinna's characteristic functions of f, a_k, b_i, d_l satisfy

$$(1.4) \quad \sum_{k=1}^n T(r, a_k) + \sum_{i=0}^p T(r, b_i) + \sum_{l=0}^q T(r, d_l) = S(r, f),$$

where $S(r, f)$ denotes any function of r with the following property

$$(1.5) \quad S(r, f) = o(T(r, f))$$

for all r outside of a possible exceptional set with finite logarithmic measure.

If (1.3) is only a differential equation, that is, $\mathbf{c}_1 = \cdots = \mathbf{c}_n = 0$, the general Malmquist's theorem shows that if (1.3) has an admissible meromorphic solution f , then we must have

$$q = 0, \quad p \leq \max_{1 \leq k \leq n} \lambda_k,$$

where

$$(1.6) \quad \lambda_k = \text{Weight}(\mathbf{j}_k) := j_{k0} + 2j_{k1} + \cdots + (m+1)j_{km}.$$

More results related to this topic are referred to Tu [6], Brosch [1], Yang [7].

However, if (1.3) contains really differences, that is, $\mathbf{c}_k \neq 0$ for some k , there are different results. For example, Li [3] notes that (1.3) has admissible meromorphic solutions (or see Remarks below). Some works related to the topics are referred to [2], [7].

Write

$$H[f] = Q(f)Df - P(f), \quad \Lambda = \sum_{k=1}^n \lambda_k.$$

In this paper, we prove the following main theorem:

Theorem 1.1. *Let f be an admissible meromorphic solution of (1.3) and further assume that the order of f is finite. Suppose that $p \leq q = \Lambda$ and take two distinct complex numbers e_1, e_2 with*

$$H[e_1] \neq 0, \quad H[e_2] \neq 0.$$

If $g \in \mathcal{M}(\mathbb{C})$ and f share the values e_1, e_2 and ∞ CM, then $f = g$.

By definition, f and g are said to share a value e CM if $f^{-1}(e) = g^{-1}(e)$ counting multiplicity. For the special case $m = 0, \mathbf{j}_1 = \cdots = \mathbf{j}_n = 1$, Lü, Han and Lü [5] proved Theorem 1.1 by applying main ideas due to [1].

Remark 1.2. The number of shared values in Theorem 1.1 cannot be reduced. For example, define a differential-difference operator $D : \mathcal{M}(\mathbb{C}) \rightarrow \mathcal{M}(\mathbb{C})$ as follows:

$$Df = f'_c + f'_{c'}$$

with $c = \frac{\pi}{4}, c' = -\frac{\pi}{4}$, and take

$$P(f) = 4(f^2 + 1)^2, Q(f) = (f^2 - 1)^2.$$

Obviously, we have

$$H[\pm 1] = -16, p = q = \Lambda = 4.$$

Equation (1.3) has an admissible meromorphic solution $f(z) = \frac{1}{\tan z}$ of order 1. However, the solution f and a different meromorphic function $g(z) = \tan z$ share two values ± 1 CM.

Remark 1.3. The condition $H[e_1] \neq 0, H[e_2] \neq 0$ cannot be dropped. Take in (1.3)

$$Df = f'_c, P(f) = 2 + 2f^2, Q(f) = (f - 1)^2$$

with $c = \frac{\pi}{4}, p = q = \Lambda = 2, H[\pm i] = 0$. Equation (1.3) has an admissible meromorphic solution $f(z) = \tan z$ of order 1 such that $f(z)$ and $g(z) = -\tan z$ share the values $\pm i$ and ∞ CM, but $f \neq g$.

Remark 1.4. The condition $p \leq q$ is sharp in the following meanings. Take in (1.3)

$$Df = f_c f'_{c'}, P(f) = f^2, Q(f) = 1$$

with $c = -1, c' = 1, p = 2, q = 0, \Lambda = 3, H[\pm 1] = -1$. Equation (1.3) has an admissible entire solution $f(z) = e^z$ of order 1 such that $f(z)$ and $g(z) = e^{-z}$ share the values ± 1 and ∞ CM, but $f \neq g$.

Remark 1.5. The condition $q = \Lambda$ is necessary. Take in (1.3)

$$Df = -e^2 f_c + f'_{c'}, P(f) = -e^2, Q(f) = 1$$

with $c = -1, c' = 1, p = 0, q = 0, \Lambda = 3, H[0] = e^2, H[2] = -e^2$. Equation (1.3) has an admissible meromorphic solution $f(z) = e^z + 1$ of order 1 such that $f(z)$ and $g(z) = e^{-z} + 1$ share the values 0, 2 and ∞ CM, but $f \neq g$.

Remark 1.6. The assumption that f is of finite order is necessary. Take in (1.3)

$$Df = f'_c - f'_{c'}, P(f)(z) = 3e^z f(z) - 4e^z, Q(f) = f^4$$

with $e^c = -4, e^{c'} = -3, p = 1, q = \Lambda = 4, H[0](z) = 4e^z, H[e](z) = 4e^z - 3e^{z+1}$. Equation (1.3) has an admissible entire solution $f(z) = e^{e^z}$ of order ∞ , $f(z)$ and $g(z) = e^{2-e^z}$ share the values 0, e and ∞ CM, but $f \neq g$.

2. Preliminary

We assume that the reader is familiar with the standard notations and fundamental results in Nevanlinna theory, Refer to the book [4].

The following lemma is referred to Lemma 2.4 and Lemma 2.5 in [3].

Lemma 2.1. *If f is a non-constant meromorphic function of finite order, then*

$$m\left(r, \frac{f_c^{(k)}}{f}\right) = S(r, f)$$

holds for $c \in \mathbb{C}, k \in \mathbb{Z}_+$.

Lemma 2.2. *Let f be an admissible meromorphic solution of finite order to the equation (1.3). If $b \in \mathcal{M}(\mathbb{C})$ is a small function of f , that is,*

$$T(r, b) = S(r, f),$$

with $H[b] \neq 0$, then

$$(2.1) \quad m\left(r, \frac{1}{f-b}\right) = S(r, f).$$

Proof. Substituting $f = h + b$ into (1.3), we obtain

$$(2.2) \quad A[h] + H[b] = 0,$$

where

$$A[h] = H[h+b] - H[b] = \sum_{1 \leq k_0, k_1, \dots, k_m \leq n} \sum_{\mathbf{i}} c_{\mathbf{i}} h_{c_{k_0 0}}^{i_0} (h'_{c_{k_1 1}})^{i_1} \dots (h_{c_{k_m m}}^{(m)})^{i_m}$$

in which $\mathbf{i} = (i_0, \dots, i_m)$ runs on a finite set of $\mathbb{Z}_+^{m+1} - \{0\}$, and $c_{\mathbf{i}}$ is a combination of $a_k, b_i, d_l, b_{c_{k_0 0}}, \dots, b_{c_{k_m m}}^{(m)}$ satisfying

$$T(r, c_{\mathbf{i}}) = S(r, f).$$

Then, when $|h(z)| \leq 1$ with $|z| = r$, we obtain an estimate

$$\left| \frac{A[h](z)}{h(z)} \right| \leq \sum_{1 \leq k_0, k_1, \dots, k_m \leq n} \sum_{\mathbf{i}} |c_{\mathbf{i}}(z)| \left| \frac{h_{c_{k_0 0}}(z)}{h(z)} \right|^{i_0} \dots \left| \frac{h_{c_{k_m m}}^{(m)}(z)}{h(z)} \right|^{i_m}.$$

By using (2.2) and Lemma 2.1, it follows that

$$\begin{aligned} m\left(r, \frac{1}{f-b}\right) &= m\left(r, \frac{1}{h}\right) \leq m\left(r, \frac{H[b]}{h}\right) + m\left(r, \frac{1}{H[b]}\right) \\ &= m\left(r, \frac{A[h]}{h}\right) + m\left(r, \frac{1}{H[b]}\right) = S(r, f) \end{aligned}$$

since $T(r, h) = T(r, f) + S(r, f)$. When $|h(z)| > 1$ with $|z| = r$, we know $m\left(r, \frac{1}{f-b}\right) = m\left(r, \frac{1}{h}\right) = S(r, f)$ is obvious. hence Lemma 2.2 is proved. \square

Lemma 2.3. *If f is an admissible meromorphic solution of finite order of the equation (1.3) with $p \leq q = \Lambda$, then we have*

$$(2.3) \quad m(r, f) = S(r, f).$$

Proof. Put

$$(2.4) \quad d = \max_{1 \leq l \leq q} \left(1, 2 \left| \frac{d_{q-l}}{d_q} \right|^{\frac{1}{l}} \right).$$

Take $z \in \mathbb{C}$ and write $z = re^{i\theta}$. Set

$$(2.5) \quad E_1 := \{ \theta \in [0, 2\pi) : |f(re^{i\theta})| \leq d(re^{i\theta}) \}, \quad E_2 := [0, 2\pi) \setminus E_1.$$

In the set E_1 , we have the following estimate

$$(2.6) \quad \begin{aligned} |Df| &\leq \sum_{k=1}^n |a_k f^{j_{k0} + j_{k1} + \dots + j_{km}}| \left| \frac{f_{c_{k0}}}{f} \right|^{j_{k0}} \dots \left| \frac{f_{c_{km}}^{(m)}}{f} \right|^{j_{km}} \\ &\leq d^\gamma \sum_{k=1}^n |a_k| \left| \frac{f_{c_{k0}}}{f} \right|^{j_{k0}} \dots \left| \frac{f_{c_{km}}^{(m)}}{f} \right|^{j_{km}}, \end{aligned}$$

where

$$\gamma = \max_{1 \leq k \leq n} \{ j_{k0} + j_{k1} + \dots + j_{km} \}.$$

In the set E_2 , noting that

$$|f| > d \geq 2 \left| \frac{d_{q-l}}{d_q} \right|^{\frac{1}{l}},$$

and hence

$$\left| \frac{d_{q-l}}{d_q f^l} \right| \leq \frac{1}{2^l}$$

for $l = 1, \dots, q$, which means

$$\begin{aligned} |Q(f)| &= |d_q f^q + d_{q-1} f^{q-1} + \dots + d_1 f + d_0| \\ &\geq |d_q f^q| \left(1 - \sum_{l=1}^q \left| \frac{d_{q-l}}{d_q f^l} \right| \right) \geq \frac{|d_q| |f|^q}{2^q}, \end{aligned}$$

we also obtain an estimate

$$(2.7) \quad \begin{aligned} |Df| &= \left| \frac{P(f)}{Q(f)} \right| \leq \frac{2^q}{|d_q| |f|^q} \sum_{i=0}^p |b_i| |f|^i \\ &= \frac{2^q}{|d_q|} \sum_{i=0}^p |b_i| |f|^{i-q} \leq \frac{2^q}{|d_q|} \sum_{i=0}^p |b_i|. \end{aligned}$$

Combing (2.6) and (2.7), we obtain a complete estimate

$$|Df| \leq \frac{2^q}{|d_q|} \sum_{i=0}^p |b_i| + d^\gamma \sum_{k=1}^n |a_k| \left| \frac{f_{c_{k0}}}{f} \right|^{j_{k0}} \dots \left| \frac{f_{c_{km}}^{(m)}}{f} \right|^{j_{km}}.$$

which yields immediately

$$\begin{aligned} m(r, Df) &\leq (\gamma + 1)m\left(r, \frac{1}{d_q}\right) + \sum_{i=0}^p m(r, b_i) + \gamma \sum_{l=0}^q m(r, d_l) \\ &\quad + \sum_{k=1}^n m(r, a_k) + \sum_{k=1}^n \sum_{\nu=0}^m j_{k\nu} m\left(r, \frac{f_{c_{k\nu}}^{(\nu)}}{f}\right) + O(1). \end{aligned}$$

Further, by using Lemma 2.1, it follows that

$$m(r, Df) = S(r, f)$$

since a_k, b_i, d_l are small function of f .

Theorem 2.2 of Chiang and Feng [2] implies

$$N(r, f_{c_{k\nu}}) = N(r, f) + S(r, f)$$

which further yields

$$N(r, f_{c_{k\nu}}^{(\nu)}) \leq (\nu + 1)N(r, f_{c_{k\nu}}) = (\nu + 1)N(r, f) + S(r, f).$$

It follows that

$$\begin{aligned} T(r, Df) &= m(r, Df) + N(r, Df) \leq \sum_{k=1}^n \sum_{\nu=0}^m j_{k\nu} N(r, f_{c_{k\nu}}^{(\nu)}) + S(r, f) \\ &\leq \sum_{k=1}^n \sum_{\nu=0}^m (\nu + 1)j_{k\nu} N(r, f) + S(r, f) = qN(r, f) + S(r, f). \end{aligned}$$

On the other hand, it is well knew that

$$T(r, Df) = T\left(r, \frac{P(f)}{Q(f)}\right) = qT(r, f) + S(r, f).$$

Therefore, we have

$$m(r, f) = \frac{1}{q}\{T(r, Df) - qN(r, f)\} + S(r, f),$$

and hence Lemma 2.3 follows. \square

3. Proof of Theorem 1.1: special cases $m \leq 1$

Now (1.3) has the following form

$$(3.1) \quad Df = \sum_{k=1}^n a_k f_{c_{k0}}^{j_{k0}} (f'_{c_{k1}})^{j_{k1}} = \frac{P(f)}{Q(f)}.$$

Since f is an admissible meromorphic solution of (3.1), it follows that f must be non-constant. By using Theorem 5.25 in [8], we have

$$T(r, g) = \{1 + o(1)\}T(r, f)$$

as $r \rightarrow \infty$, which particularly implies that g and f have the same order. Hence, there exist two polynomials α, β satisfying

$$(3.2) \quad \frac{f - e_1}{g - e_1} = e^\alpha, \quad \frac{f - e_2}{g - e_2} = e^\beta.$$

Assume, to the contrary, that $g \neq f$. Then we obtain easily

$$e^\alpha \neq 1, \quad e^\beta \neq 1, \quad e^\alpha \neq e^\beta$$

and

$$(3.3) \quad f = e_1 + (e_2 - e_1) \frac{e^\beta - 1}{e^\gamma - 1} = e_2 + (e_1 - e_2) \frac{e^\alpha - 1}{e^{-\gamma} - 1},$$

where $\gamma = \beta - \alpha$ is not a constant (see Lemma 2.3). Thus one of α and β at least is not constant. Moreover, by Lemma 2.2 and the first main theorem of Nevanlinna, we have

$$(3.4) \quad T(r, f) = N\left(r, \frac{1}{f - e_j}\right) + S(r, f)$$

for $j = 1, 2$. If one of α and β is constant, it follows that

$$T(r, f) = S(r, f).$$

This is a contradiction. Hence α, β are not constants.

Further, we claim

$$(3.5) \quad d := \text{ord}(f) = \deg \alpha = \deg \beta = \deg \gamma > 0.$$

The first main theorem due to Nevanlinna yields immediately

$$N\left(r, \frac{1}{e^\alpha - 1}\right) \leq T(r, e^\alpha) + O(1),$$

and the second main theorem applied to three values $0, 1, \infty$ implies

$$T(r, e^\alpha) = N\left(r, \frac{1}{e^\alpha - 1}\right) + S(r, e^\alpha).$$

Note that

$$(3.6) \quad T(r, e^\alpha) \leq T(r, f) + T(r, g) + O(1) \leq 2T(r, f) + S(r, f).$$

We obtain

$$(3.7) \quad T(r, e^\alpha) = N\left(r, \frac{1}{e^\alpha - 1}\right) + S(r, f).$$

Similarly, we can obtain

$$(3.8) \quad T(r, e^\beta) = N\left(r, \frac{1}{e^\beta - 1}\right) + S(r, f),$$

and

$$(3.9) \quad T(r, e^\gamma) = N\left(r, \frac{1}{e^\gamma - 1}\right) + S(r, f).$$

It follows from (3.9) that

$$T(r, e^\gamma) = N(r, f) + N\left(r, \frac{1}{\xi}\right) + S(r, f),$$

where ξ is an entire function determined by common zeros of $e^\beta - 1$ and $e^\gamma - 1$. By using Lemma 2.3, we see

$$T(r, e^\gamma) = T(r, f) + N\left(r, \frac{1}{\xi}\right) + S(r, f).$$

Note that (3.8) and (3.4) yield

$$\begin{aligned} T(r, e^\beta) &= N\left(r, \frac{1}{f - e_1}\right) + N\left(r, \frac{1}{\xi}\right) + S(r, f) \\ &= T(r, f) + N\left(r, \frac{1}{\xi}\right) + S(r, f). \end{aligned}$$

Therefore, we have

$$T(r, e^\beta) = T(r, e^\gamma) + S(r, f)$$

which means

$$\deg \beta = \text{ord}(e^\beta) := \limsup_{r \rightarrow \infty} \frac{\log T(r, e^\beta)}{\log r} = \text{ord}(e^\gamma) = \deg \gamma > 0.$$

According to the arguments above, we can prove

$$T(r, e^\alpha) = T(r, e^\gamma) + S(r, f)$$

and hence

$$\deg \alpha = \deg \gamma.$$

Now (3.6) implies

$$\deg \alpha = \text{ord}(e^\alpha) \leq \text{ord}(f).$$

But, we also have

$$T(r, f) \leq T(r, e^\alpha) + 2T(r, e^\beta) + S(r, f) \leq 3T(r, e^\alpha) + S(r, f)$$

which means

$$\text{ord}(f) \leq \text{ord}(e^\alpha) = \deg \alpha.$$

The claim (3.5) is proved.

Substituting the representation (3.3) of f into (3.1), we have

$$(3.10) \quad \sum_{i=0}^p b_i \left[e_1 + (e_2 - e_1) \frac{e^\beta - 1}{e^\gamma - 1} \right]^i = \sum_{l=0}^q d_l \left[e_1 + (e_2 - e_1) \frac{e^\beta - 1}{e^\gamma - 1} \right]^l \\ \sum_{k=1}^n a_k \left[e_1 + (e_2 - e_1) \frac{e^{\beta_{c_{k0}}} - 1}{e^{\gamma_{c_{k0}}} - 1} \right]^{j_{k0}} \left[(e_2 - e_1) \left(\frac{e^{\beta_{c_{k1}}} - 1}{e^{\gamma_{c_{k1}}} - 1} \right)' \right]^{j_{k1}}.$$

Write

$$\beta_{c_{k0}} = \beta + s_{k0}, \quad \gamma_{c_{k0}} = \gamma + t_{k0}, \quad \beta_{c_{k1}} = \beta + s_{k1}, \quad \gamma_{c_{k1}} = \gamma + t_{k1}$$

for $1 \leq k \leq n$, where $s_{k0}, t_{k0}, s_{k1}, t_{k1}$ are polynomials of degrees $\leq d-1$. Then (3.10) becomes the following form

$$(3.11) \quad \sum_{\mu=0}^M \sum_{v=0}^N a_{\mu,v} e^{\mu\beta+v\gamma} - \sum_{\mu=0}^p \sum_{v=0}^{2q} b_{\mu,v} e^{\mu\beta+v\gamma} = 0,$$

where $a_{\mu,v}$ (resp., $b_{\mu,v}$) are combinations of $a_k, d_i, e^{s_{k0}}, e^{t_{k0}}, e^{s_{k1}}, e^{t_{k1}}$ (resp., $b_i, e^{t_{k0}}, e^{t_{k1}}$) with polynomial coefficients (resp., constant coefficients) depending on $\beta', \gamma', s'_{k1}, t'_{k1}$, and where

$$M = q + \max_{1 \leq k \leq n} \{j_{k0} + j_{k1}\}, \quad N = 2q - \min_{1 \leq k \leq n} \{j_{k1}\},$$

or further

$$(3.12) \quad \sum_{\mu=0}^M \sum_{v=0}^{2q} A_{\mu,v} e^{\mu\beta+v\gamma} = 0,$$

where $A_{\mu,v}$ are completely determined by $a_{\mu,v}, b_{\mu,v}$ or 0. Moreover, it is not difficult to show that

$$(3.13) \quad \begin{aligned} A_{0,0} &= H[e_2] \neq 0, \\ A_{0,2q} &= H[e_1] \prod_{k=1}^n e^{j_{k0}t_{k0}+2j_{k1}t_{k1}} \neq 0. \end{aligned}$$

We claim that

$$(3.14) \quad \deg(\mu\beta + v\gamma) = \deg(\mu\beta - v\gamma) = d$$

for $(\mu, v) \in \mathbb{Z}_+^2 - \{(0, 0)\}$, which follows from (3.5) if one of μ and v is zero.

Now we consider the cases $\mu v \neq 0$. First of all, assume, to the contrary, that $\deg(\mu\beta + v\gamma) < d$, so that the entire function $U_1 = e^{\mu\beta+v\gamma}$ is a small function of e^α . We have

$$T(r, U_1 e^{-\mu\alpha}) = T(r, e^{-\mu\alpha}) + S(r, e^\alpha) = \mu T(r, e^\alpha) + S(r, f).$$

On the other hand, we also have

$$\begin{aligned} T(r, U_1 e^{-\mu\alpha}) &= T\left(r, e^{(\mu+v)\gamma}\right) = (\mu+v)T(r, e^\gamma) \\ &= (\mu+v)T(r, e^\alpha) + S(r, f). \end{aligned}$$

So that $v = 0$. This is a contradiction. The first part of (3.14) is confirmed.

Next, assume, to the contrary, that $\deg(\mu\beta - v\gamma) < d$, so that the entire function $U_2 = e^{\mu\beta-v\gamma}$ is a small function of e^α . Thus if $\mu \geq v$, we have

$$T(r, U_2 e^{-\mu\alpha}) = T(r, e^{-\mu\alpha}) + S(r, e^\alpha) = \mu T(r, e^\alpha) + S(r, f).$$

On the other hand, we also have

$$\begin{aligned} T(r, U_2 e^{-\mu\alpha}) &= T\left(r, e^{(\mu-v)\gamma}\right) = (\mu-v)T(r, e^\gamma) \\ &= (\mu-v)T(r, e^\alpha) + S(r, f), \end{aligned}$$

and hence either $\mu = 0$ (if $\mu = v$) or $v = 0$ (if $\mu > v$). This is a contradiction.

If $v > \mu$, we can get $\mu = 0$ in the same way. This is also a contradiction. Therefore, the claim (3.14) is confirmed completely.

By using (3.14), we find that each $A_{\mu,v}$ satisfies the following estimate

$$(3.15) \quad T(r, A_{\mu,v}) = S \left(r, e^{(\mu_1\beta+v_1\gamma)-(\mu_2\beta+v_2\gamma)} \right)$$

for two distinct elements (μ_1, v_1) and (μ_2, v_2) in \mathbb{Z}_+^2 . Thus, by Theorem 1.51 in [8], we have $A_{\mu,v} \equiv 0$. It contradicts to (3.13). Therefore, we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.1: general cases

We can copy completely the procedure of proof in last section up to the claim (3.5). Now a change of proof is to substitute the representation (3.3) of f into the general equation (1.3), so that we obtain

$$(4.1) \quad \sum_{i=0}^p b_i \left[e_1 + (e_2 - e_1) \frac{e^\beta - 1}{e^\gamma - 1} \right]^i = \sum_{l=0}^q d_l \left[e_1 + (e_2 - e_1) \frac{e^\beta - 1}{e^\gamma - 1} \right]^l \\ \sum_{k=1}^n a_k \left[e_1 + (e_2 - e_1) \frac{e^{\beta_{c_{k0}}} - 1}{e^{\gamma_{c_{k0}}} - 1} \right]^{j_{k0}} \prod_{\nu=1}^m \left[(e_2 - e_1) \left(\frac{e^{\beta_{c_{k\nu}}} - 1}{e^{\gamma_{c_{k\nu}}} - 1} \right)^{(\nu)} \right]^{j_{k\nu}}.$$

Write

$$\beta_{c_{k\nu}} = \beta + s_{k\nu}, \quad \gamma_{c_{k\nu}} = \gamma + t_{k\nu}$$

for $1 \leq k \leq n, 0 \leq \nu \leq m$, where $s_{k\nu}, t_{k\nu}$ are polynomials of degrees $\leq d-1$. Then (4.1) becomes the following form

$$(4.2) \quad \sum_{\mu=0}^M \sum_{v=0}^N a_{\mu,v} e^{\mu\beta+v\gamma} - \sum_{\mu=0}^p \sum_{v=0}^{2q} b_{\mu,v} e^{\mu\beta+v\gamma} = 0,$$

where $a_{\mu,v}$ (resp., $b_{\mu,v}$) are combinations of $a_k, d_l, e^{s_{k\nu}}, e^{t_{k\nu}}$ (resp., $b_i, e^{t_{k\nu}}$) with polynomial coefficients (resp., constant coefficients) depending on derivatives $\beta_{c_{k\nu}}$ and $\gamma_{c_{k\nu}}$, and where

$$M = q + \max_{1 \leq k \leq n} \{j_{k0} + j_{k1} + \cdots + j_{km}\},$$

$$N = 2q - \min_{1 \leq k \leq n} \{j_{k1} + 2j_{k2} + \cdots + mj_{km}\},$$

or further

$$(4.3) \quad \sum_{\mu=0}^M \sum_{v=0}^{2q} A_{\mu,v} e^{\mu\beta+v\gamma} = 0,$$

where $A_{\mu,v}$ are completely determined by $a_{\mu,v}, b_{\mu,v}$ or 0. Moreover, it is not difficult to show that

$$(4.4) \quad \begin{aligned} A_{0,0} &= H[e_2] \neq 0, \\ A_{0,2q} &= H[e_1] \prod_{k=1}^n e^{j_{k0}t_{k0}+2j_{k1}t_{k1}+\dots+(m+1)j_{km}t_{km}} \neq 0. \end{aligned}$$

Thus, according to the arguments in last section, we obtain a contradiction, so that the proof of Theorem 1.1 is completed.

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