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CRITICAL VIRTUAL MANIFOLDS AND PERVERSE SHEAVES

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Abstract. In Donaldson-Thomas theory, moduli spaces are locally the critical locus of a holomorphic function defined on a complex manifold. In this paper, we develop a theory of critical virtual manifolds which are the gluing of critical loci of holomorphic functions. We show that a critical virtual manifold X admits a natural semi-perfect obstruction theory and a virtual fundamental class $[X]^{\mathrm{vir}}$ whose degree $DT(X) = \deg[X]^{\mathrm{vir}}$ is the Euler characteristic $\chi_\nu(X)$ weighted by the Behrend function ν. We prove that when the critical virtual manifold is orientable, the local perverse sheaves of vanishing cycles glue to a perverse sheaf P whose hypercohomology has Euler characteristic equal to the Donaldson-Thomas type invariant $DT(X)$. In the companion paper [17], we proved that a moduli space X of simple sheaves on a Calabi-Yau 3-fold Y is a critical virtual manifold whose perverse sheaf categorifies the Donaldson-Thomas invariant of Y and also gives us a mathematical theory of Gopakumar-Vafa invariants.

1. Introduction

The Donaldson-Thomas invariant is a virtual count of stable sheaves on a smooth projective Calabi-Yau 3-fold Y over $\mathbb{C}$ which was defined as the degree of the virtual fundamental class of the moduli space X of stable sheaves on Y (cf. [26]). Using microlocal analysis, Behrend showed that the DT invariant is in fact the Euler number of the moduli space, weighted by a constructible function ν_X, called the Behrend function (cf. [1]). Since the ordinary Euler number is the alternating sum of Betti numbers of cohomology groups, it is reasonable to ask if the Donaldson-Thomas invariant is in fact the Euler number of a cohomology theory on X.

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On the other hand, it is known that the moduli space is locally the critical locus of a holomorphic function, called a local Chern-Simons functional (cf. [15]). Given a holomorphic function f on a complex manifold V, one has the perverse sheaf $\phi_f(\mathbb{Q}[\dim V-1])$ of vanishing cycles supported on the critical locus. So the moduli space is covered by charts each of which comes with the perverse sheaf of vanishing cycles $\phi_f(\mathbb{Q}[\dim V-1])$.

This motivated Joyce and Song to raise the following question ([15, Question 5.7]).

Let X be a moduli space of simple coherent sheaves on Y. Does there exist a natural perverse sheaf P on the underlying analytic space which is locally isomorphic to the sheaf $\phi_f(\mathbb{Q}[\dim V-1])$ of vanishing cycles for (V,f) above?

The purpose of this paper together with its companion paper [17] is to provide an affirmative answer.

Theorem 1.1. *Let X be a moduli space of simple sheaves on a smooth projective Calabi-Yau 3-fold Y. Suppose the reduced scheme $X^{\rm red}$ of X is of finite type and admits a tautological family. Then the collection of perverse sheaves $\phi_f(\mathbb{Q}[\dim V-1])$ geometrically glue to a perverse sheaf P on X. The same holds for polarizable mixed Hodge modules.*

In [17], we proved that a moduli space X of simple sheaves on a smooth projective Calabi-Yau 3-fold Y is a critical virtual manifold, i.e., the complex analytic space X has an open cover $X=\cup X_\alpha$ and each

$$X_\alpha=\mathrm{Crit}(f_\alpha)=\mathrm{zero}(df_\alpha)$$

is the critical locus of a holomorphic function $f_\alpha: V_\alpha\to\mathbb{C}$ on a complex manifold V_α. Furthermore for each pair of indices (α,β), there are open neighborhoods $V_{\alpha\beta}\subset V_\alpha$, $V_{\beta\alpha}\subset V_\beta$ of $X_{\alpha\beta}=X_\alpha\cap X_\beta$ and biholomorphic maps $\varphi_{\alpha\beta}: V_{\alpha\beta}\to V_{\beta\alpha}$ satisfying $f_\beta\circ\varphi_{\alpha\beta}=f_\alpha$ and $\varphi_{\alpha\beta}|_{X_{\alpha\beta}}=\mathrm{id}_{X_{\alpha\beta}}$. We call an analytic space X together with charts

$$X_\alpha=\mathrm{Crit}(f_\alpha:V_\alpha\to\mathbb{C}),\quad\text{and}\quad \varphi_{\alpha\beta}:V_{\alpha\beta}\xrightarrow{\cong}V_{\beta\alpha}$$

a *critical virtual manifold.* See Definition 2.5 for a precise statement. One may say that a critical virtual manifold is the gluing of Landau-Ginzburg models. Usual complex manifolds are special cases of critical virtual manifolds. As mentioned above, by [17], moduli spaces of sheaves on Calabi-Yau 3-folds are all critical virtual manifolds.

In this paper, we investigate several interesting structures on critical virtual manifolds, such as orientability, semi-perfect obstruction theory, virtual fundamental class, Donaldson-Thomas type invariant, weighted Euler characteristic, local perverse sheaves of vanishing cycles, their gluing isomorphisms and mixed Hodge modules.

Since a critical virtual manifold X is locally the critical locus X_α of a holomorphic function f_α on a complex manifold V_α, it comes with a natural symmetric obstruction theory $E_\alpha=[T_{V_\alpha}|_{X_\alpha}\xrightarrow{d(df_\alpha)}\Omega_{V_\alpha}|_{X_\alpha}]\to\mathbb{L}_{X_\alpha}$ on each X_α.

We prove that these local symmetric obstruction theories form a semi-perfect obstruction theory (Proposition 2.30). By [6], we then have a virtual fundamental class $[X]^{\mathrm{vir}} \in A_0(X)$ whose degree gives us the Donaldson-Thomas type invariant

$$DT(X) = \deg[X]^{\mathrm{vir}}$$

when X is compact. By adapting the arguments in [1], we find that $DT(X)$ is indeed the Euler characteristic $\chi_\nu(X)$ of X weighted by the Behrend function ν (Theorem 2.37). Then it is natural to consider the following categorification problem.

Problem 1.2. *Let X be a critical virtual manifold. Does there exist a natural perverse sheaf P on X such that $P|_{X_\alpha}$ is isomorphic to the perverse sheaf $P_\alpha := \phi_{f_\alpha}(\mathbb{Q}[\dim V_\alpha - 1])$ of vanishing cycles for (V_α, f_α)?*

See Definition 3.1 for perverse sheaves of vanishing cycles and Definition 3.7 for perverse sheaves.

The value of the Behrend function ν at a point x in the critical locus $\mathrm{Crit}(f)$ of a holomorphic function f on a complex manifold V equals the Euler characteristic $\chi(\phi_f(\mathbb{Q}[\dim V - 1]))_x$ of the stalk cohomology of the perverse sheaf $\phi_f(\mathbb{Q}[\dim V - 1])$ of vanishing cycles at x (cf. (3.8)). Therefore, the hypercohomology $\mathbb{H}^*_c(X, P)$ of a solution P to Problem 1.2 has Euler characteristic equal to the Donaldson-Thomas type invariant $DT(X)$ (cf. Proposition 3.3), i.e.,

$$\sum_i (-1)^i \dim \mathbb{H}^i_c(X, P) = \chi_\nu(X) = DT(X).$$

Hence a solution to Problem 1.2 categorifies the Donaldson-Thomas type invariant $DT(X)$ into a cohomological invariant $\mathbb{H}^*_c(X, P)$.

A critical virtual manifold X with charts $X = \cup X_\alpha$, $X_\alpha = \mathrm{Crit}(f_\alpha) \subset V_\alpha$ has local orientation bundles $\det(T_{V_\alpha})|_{X^{\mathrm{red}}_\alpha}$ on the reduced space X^{red}_α and their isomorphisms $\det(d\varphi_{\alpha\beta})|_{X^{\mathrm{red}}_{\alpha\beta}}$ on the reduced space $X^{\mathrm{red}}_{\alpha\beta}$ of $X_{\alpha\beta}$ given by the transition maps $\varphi_{\alpha\beta}$ (cf. (2.3)). We say that X is *orientable* if these local orientation bundles glue to a line bundle on X^{red}. We will see that the obstruction to orientability lies in $H^2(X, \mathbb{Z}_2)$ (cf. (2.6)). The main result of this paper is the following (cf. Theorems 3.15 and 3.20).

Theorem 1.3. *Let X be an orientable critical virtual manifold with charts*

$$X_\alpha = \mathrm{Crit}(f_\alpha) \subset V_\alpha \xrightarrow{f_\alpha} \mathbb{C}.$$

Then there are a perverse sheaf P and a polarizable mixed Hodge module $\mathcal{M}$ with $\mathrm{rat}(\mathcal{M}) = P$, which are the geometric gluing of local perverse sheaves P_α and mixed Hodge modules $\mathcal{M}_\alpha$ of vanishing cycles for (V_α, f_α).

It is well known that perverse sheaves glue (cf. Proposition 3.8). We show that the biholomorphic maps $\varphi_{\alpha\beta}$ induce isomorphisms $\sigma_{\alpha\beta} : P_\alpha|_{X_{\alpha\beta}} \to P_\beta|_{X_{\alpha\beta}}$ whenever $X_{\alpha\beta} = X_\alpha \cap X_\beta \neq \emptyset$ (cf. Corollary 3.5). We prove that the 2-cocycle obstruction for gluing the local perverse sheaves P_α to a globally defined perverse sheaf P coincides with the obstruction for the orientability of X (cf. Corollary 3.13). Therefore an orientable critical virtual manifold $X = \cup X_\alpha$ has a perverse sheaf P which is locally the perverse sheaf P_α of vanishing cycles on X_α (cf. Theorem 3.15).

Now Theorem 1.1 is a direct consequence of Theorem 1.3 and the following theorem from [17].

Theorem 1.4 ([17, Theorems 5.5 and 5.10])**.** *A bounded moduli space X of simple sheaves on a smooth projective Calabi-Yau 3-fold Y is a critical virtual manifold. If furthermore the reduced space $X^{\rm red}$ is equipped with a tautological family, then X is an orientable critical virtual manifold.*

As an application of Theorem 1.1, we use the hypercohomology $\mathbb{H}^i_c(X,P)$ of P to deduce the *DT* (*Laurent*) *polynomial*

$$DT_t(X) = \sum_i t^i \dim \mathbb{H}^i_c(X,P)$$

such that $DT_{-1}(X)$ is the ordinary Donaldson-Thomas invariant of a smooth projective Calabi-Yau 3-fold Y when X is a moduli space of stable sheaves on Y, equipped with a tautological family.

Another application is a mathematical theory of Gopakumar-Vafa invariants proposed in [9]. Let X be a moduli space of stable sheaves supported on curves of homology class $\beta \in H_2(Y,\mathbb{Z})$. The Gopakumar-Vafa invariants are integers $n_h(\beta)$ for $h \in \mathbb{Z}_{\ge 0}$ defined by an $sl_2 \times sl_2$ action on *some cohomology* of X such that $n_0(\beta)$ is the Donaldson-Thomas invariant of X and that they give all genus Gromov-Witten invariants $N_g(\beta)$ of Y via a relation in [9] (cf. [17, (6.1)]). Applying Theorem 1.1 to X, we obtain a perverse sheaf P on X which is locally the perverse sheaf of vanishing cycles. By the relative hard Lefschetz theorem for the morphism to the Chow scheme (cf. [22]), we have an action of $sl_2 \times sl_2$ on $\mathbb{H}^*(X,\hat{P})$ where $\hat{P}$ is the gradation of P by the weight filtration of P. This gives us a geometric theory of Gopakumar-Vafa invariants which we conjecture to give all the Gromov-Witten invariants $N_g(\beta)$.

We believe that there will be more applications of Theorem 1.3. For instance, in a subsequent paper, we will show that moduli spaces of sheaves with additional structures on a Calabi-Yau 3-fold are critical virtual manifolds and they will lead us to a wall crossing formula of Gopakumar-Vafa invariants. Also further developments of the theory of critical virtual manifolds are expected and desired, such as deformation theory and algebraic versions for stacks. All these directions will bring us interesting progresses on the Donaldson-Thomas theory and related topics.

Here is the layout of this paper. In Chapter 2, we introduce critical virtual manifolds, and discuss orientability, semi-perfect obstruction theory, Behrend's theorem and the categorification problem. In Chapter 3, we prove the gluing theorem for perverse sheaves and mixed Hodge modules. Chapter 4 is devoted to a proof of Theorem 3.12 which is essential for the gluing of perverse sheaves.

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2. Critical virtual manifolds

In this chapter, we introduce the notion of a critical virtual manifold (Definition 2.5) and the orientability (Definition 2.17). We prove that a critical virtual manifold X has a natural symmetric semi-perfect obstruction theory (Proposition 2.30) in the sense of [6], which gives us the virtual fundamental class $[X]^{\mathrm{vir}} \in A_0(X)$ (Definition 2.34) whose degree

$$DT(X) = \deg[X]^{\mathrm{vir}}$$

is shown to equal the Euler characteristic

$$\chi_\nu(X) = \sum_{n\in\mathbb{Z}} n \cdot \chi_c(\nu^{-1}(n))$$

weighted by the Behrend function ν (Theorem 2.37). Then we pose a categorification problem (Problem 2.41) which will be solved in the subsequent chapter. We end this chapter by compairing the notion of a critical virtual manifold with a d-critical locus from [14].

2.1. Background

Recall that a complex manifold is locally modeled on open subsets in $\mathbb{C}^n$, $n > 0$.

Definition 2.1 ([11, page 14])**.** A *complex manifold* X is a second countable paracompact Hausdorff space together with an open cover $\{X_\alpha\}$ and homeomorphisms $\varphi_\alpha : X_\alpha \to \varphi_\alpha(X_\alpha) \subset \mathbb{C}^n$ onto open sets such that $\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(X_{\alpha\beta}) \subset \mathbb{C}^n$ for all α, β. Here $X_{\alpha\beta} = X_\alpha \cap X_\beta$.

When X is a smooth projective variety (or more generally a compact Kähler manifold), its cohomology $H^*(X, \mathbb{C})$ satisfies nice properties like the Hodge decomposition and the Lefschetz decomposition, collectively known as the Kähler package [11, 18]. These properties are well known to be fundamental tools in complex and algebraic geometry. However when X is singular, these nice properties do not hold unless we use more sophisticated tools.

Our discussion on singular spaces will be based on the notion of analytic spaces.

Definition 2.2. A *local ringed space* is a topological space X together with a sheaf $\mathcal{O}_X$ of rings whose stalks are local rings.

Given holomorphic functions $f_1, \dots, f_m$ on an open $V \subset \mathbb{C}^n$, the common vanishing locus $Y = \mathrm{zero}(f_1, \dots, f_m) \subset V$ equipped with the sheaf $\mathcal{O}_Y = \mathcal{O}_V/(f_1, \dots, f_m)$ of holomorphic functions on Y is a local ringed space.

Definition 2.3. An *analytic space* is a second countable paracompact Hausdorff local ringed space $(X, \mathcal{O}_X)$ together with an open covering $\{X_\alpha\}$ and an isomorphism $\varphi_\alpha : (X_\alpha, \mathcal{O}_X|_{X_\alpha}) \xrightarrow{\cong} (Y_\alpha, \mathcal{O}_{Y_\alpha})$ onto a local ringed space $(Y_\alpha, \mathcal{O}_{Y_\alpha})$ in an open $V_\alpha \subset \mathbb{C}^n$ defined by finitely many holomorphic functions for each α. A morphism of analytic spaces is a morphism of local ringed spaces, i.e., a continuous map $\varphi : X \to Y$ together with a sheaf homomorphism $\varphi^* : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ which induces a local homomorphism of stalks.

Given an analytic space $(X, \mathcal{O}_X)$, its reduced space X^{red} is given by the same topological space and the reduced sheaf $\mathcal{O}_{X^{\mathrm{red}}}$ which is the quotient of $\mathcal{O}_X$ by its radical ideal.

Although the Kähler package may fail for the ordinary cohomology $H^*(X, \mathbb{C})$, if we use the hypercohomology $\mathbb{H}^*(X, P)$ of a perverse sheaf P on X which underlies a polarizable Hodge module, then we still have the nice properties, thanks to the theory of perverse sheaves and mixed Hodge modules due to Kashiwara, Goresky, MacPherson, Beilinson, Berstein, Deligne and Saito, to name a few [3, 10, 22, 23]. So it is reasonable to raise the following question:

How do we get a natural perverse sheaf P on an analytic space X underlying a polarizable Hodge module?

One obvious way is to use the intersection cohomology complex IC_X which extends the constant variation of Hodge structure $\mathbb{Q}_U$ on an open dense smooth subset $U \subset X$. For the Donaldson-Thomas theory which is the focus of this paper, the spaces we deal with are locally the critical locus $\mathrm{Crit}(f)$ of a holomorphic function $f : V \to \mathbb{C}$ on a complex manifold V (cf. [15]). In this case, the perverse sheaf $P_f = \phi_f \mathbb{Q}[\dim V - 1]$ of vanishing cycles is much more relevant than the constant sheaf $\mathbb{Q}$ or the intersection cohomology complex IC_X (cf. [1]).

In this paper, we will introduce the notion of a critical virtual manifold and show that there are natural choices of perverse sheaves underlying polarizable mixed Hodge modules. In [17], we proved that moduli spaces for the Donaldson-Thomas theory are critical virtual manifolds. We thus will obtain a cohomology theory for the Donaldson-Thomas invariant which moreover gives us a mathematical theory of the Gopakumar-Vafa invariant.

2.2. Definition of critical virtual manifolds

A critical virtual manifold is an analytic space locally modeled on the critical locus $\mathrm{Crit}(f)$ of a holomorphic function $f : V \to \mathbb{C}$ on a complex manifold V.

Definition 2.4. (1) An *LG pair* is a complex manifold V together with a holomorphic function $f : V \to \mathbb{C}$ which has only one critical value 0.

(2) Two LG pairs (V_1, f_1) and (V_2, f_2) are called *equivalent* if there exists a biholomorphic map $\varphi : V_1 \to V_2$ satisfying $f_2 \circ \varphi = f_1$.

(3) The *critical locus* of an LG pair (V, f) is the analytic space in V defined by the ideal (df) generated by the partial derivatives of f. We will sometimes denote the critical locus $\mathrm{Crit}(f)$ by X_f.

(4) The reduced analytic space X_f^{red} of the critical locus X_f is the analytic space defined by the radical ideal of (df).

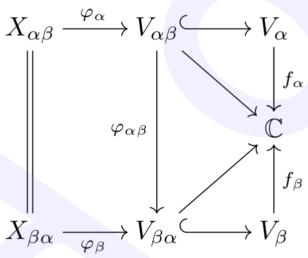
Here LG stands for Landau-Ginzburg. Perhaps a reader familiar with mirror symmetry may think of (V, f) in Definition 2.4(1) as a *Landau-Ginzburg model* with superpotential f. A critical virtual manifold is the gluing of Landau-Ginzburg models.

Definition 2.5. A *critical virtual manifold* is an analytic space $(X, \mathcal{O}_X)$ together with an open covering $X = \cup_\alpha X_\alpha$ and a closed embedding $\varphi_\alpha : X_\alpha \hookrightarrow V_\alpha$ into a complex manifold V_α for each α satisfying the following:

(1) φ_α is an isomorphism onto the critical locus X_{f_α} for an LG pair (V_α, f_α);

(2) for each pair (α, β) of indices and $X_{\alpha\beta} = X_\alpha \cap X_\beta$, we have an open neighborhood $V_{\alpha\beta}$ (resp. $V_{\beta\alpha}$) of $\varphi_\alpha(X_{\alpha\beta})$ in V_α (resp. $\varphi_\beta(X_{\beta\alpha})$ in V_β) and a biholomorphic map $\varphi_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$ that fit into the commutative diagram

(2.1)



i.e., $\varphi_{\alpha\beta} \circ \varphi_\alpha|_{X_{\alpha\beta}} = \varphi_\beta|_{X_{\beta\alpha}}$ and $f_\beta \circ \varphi_{\alpha\beta} = f_\alpha|_{V_{\alpha\beta}}$;

(3) $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$ for each pair (α, β) of indices and $\varphi_{\alpha\alpha} = \mathrm{id}_{V_\alpha}$.

In other words, we have LG pairs (V_α, f_α) whose critical loci are X_α and the restrictions of (V_α, f_α) and (V_β, f_β) to some open neighborhoods of $X_{\alpha\beta}$ are equivalent.

Definition 2.6. A diagram $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$ from Definition 2.5(1) is called a *chart* of the critical virtual manifold X. When $\dim V_\alpha = r$ for each α, we say the critical virtual manifold X has an *atlas of dimension* r.

Remark 2.7. When $X_{\alpha\beta\gamma} = X_\alpha \cap X_\beta \cap X_\gamma \neq \emptyset$, we can find an open neighborhood $V_{\alpha\beta\gamma}$ of $\varphi_\alpha(X_{\alpha\beta\gamma})$ in V_α such that the composition

$$\varphi_{\alpha\beta\gamma} = \varphi_{\gamma\alpha} \circ \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} : V_{\alpha\beta\gamma} \longrightarrow V_\alpha \tag{2.2}$$

is biholomorphic onto its image. For instance, we may let

$$V_{\alpha\beta\gamma} = \varphi_{\beta\alpha}(V_{\beta\alpha} \cap V_{\beta\gamma} \cap \varphi_{\gamma\beta}(V_{\gamma\alpha} \cap V_{\gamma\beta})).$$

By Definition 2.5(2), $\varphi_{\alpha\beta\gamma} \circ \varphi_\alpha|_{X_{\alpha\beta\gamma}} = \varphi_\alpha|_{X_{\alpha\beta\gamma}}$ as morphisms of analytic spaces. Note that we do not require the cocycle condition that $\varphi_{\alpha\beta\gamma}$ should be the identity map of $V_{\alpha\beta\gamma}$. The cocycle condition $\varphi_{\alpha\beta\gamma} = \mathrm{id}$ is guaranteed to hold only at $\varphi_\alpha(X_{\alpha\beta\gamma})$.

Example 2.8. Complex manifolds (Definition 2.1) are critical virtual manifolds by letting $X_\alpha = V_\alpha$, $f_\alpha = 0$ for any open cover $X = \cup_\alpha X_\alpha$.

Nontrivial examples arise from the Landau-Ginzburg theory.

Example 2.9. Let E be a holomorphic vector bundle over a complex manifold W equipped with a holomorphic section $s : W \to E$. Let $\pi : V = E^* \to W$ be the dual vector bundle of E over W, which is a complex manifold. Then s defines a holomorphic function $f : V \to \mathbb{C}$ defined by $f(v) = \langle s(\pi(v)), v\rangle$. Obviously the critical locus $X_f = \mathrm{Crit}(f)$ is a critical virtual manifold.

Suppose the analytic space zero(s) defined by the vanishing of s is smooth, i.e., the Jacobian matrices $(\frac{\partial s_i}{\partial x_j})$ have full rank along $\mathrm{zero}(s) = \mathrm{zero}(s_1, \dots, s_r)$, where $\{x_j\}$ are local coordinates of W and $\{s_i\}_{1\le i\le r}$ are the coordinate functions of the section s after choosing a local trivialization $E|_U \cong U \times \mathbb{C}^r$ over an open $U \subset W$. Then the critical locus $X_f = \mathrm{Crit}(f)$ is precisely the complete intersection $\mathrm{zero}(s) \subset W$. Indeed, in local coordinates,

$$f = \langle s, v\rangle = \sum_{i=1}^r s_i v_i,$$

where $\{v_i\}$ are the vertical coordinates of π. Hence

$$df = \sum_{i,j} v_i \frac{\partial s_i}{\partial x_j} dx_j + \sum_{i=1}^r s_i dv_i = 0$$

amounts to $v_i = 0$ and $s_i = 0$ for all i. Thus $\mathrm{Crit}(f) = \mathrm{zero}(s) \subset W$.

In this way, all smooth complete intersections in complex manifolds come with nontrivial critical virtual manifold structures.

Example 2.10. Let

$$\mathrm{Crit}(yz^2) = \{(y,z)\,|\,yz = z^2 = 0\} \subset \mathbb{C}^2 = \{(y,z)\,|\,y,z \in \mathbb{C}\}$$

and X be the closure of $\mathrm{Crit}(yz^2)$ in the projective plane $\mathbb{P}^2$. In fact, X is the projective line $\mathbb{P}^1 = X^{\mathrm{red}}$ with an embedded point at $(0,0) \in \mathbb{C}^2 \subset \mathbb{P}^2$. We claim that X is a critical virtual manifold. Let $(x:y) \in \mathbb{P}^1$ be the homogeneous coordinates and $((x:y),z)$ denote the coordinates of the line bundle

$$\mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{P}^1, \quad ((x:y),z) \mapsto (x:y).$$

Let $D_1 \subset \mathbb{C}$ denote the unit disk centered at 0 and consider the charts

$$U_0 = \{((1:y),z)\,|\,yz = 0 = z^2, |y| < 1\} = \mathrm{Crit}(yz^2) \subset D_1 \times \mathbb{C} = V_0,$$

$$U_+ = \{((1:y),z)\,|\,y \notin \mathbb{R}_{\geq 0}, z=0\} = \mathrm{Crit}(z^2) \subset (\mathbb{C}-\mathbb{R}_{\geq 0})\times\mathbb{C} = V_+,$$
$$U_- = \{((1:y),z)\,|\,y \notin \mathbb{R}_{\leq 0}, z=0\} = \mathrm{Crit}(z^2) \subset (\mathbb{C}-\mathbb{R}_{\leq 0})\times\mathbb{C} = V_-,$$
$$U_\infty = \{((x:1),z)\,|\,z=0, |x|<1\} \subset \mathrm{Crit}(z^2) \subset D_1\times\mathbb{C} = V_\infty.$$

Then the holomorphic maps $\varphi_{\pm,\infty} = \mathrm{id}$, $\varphi_{+,-} = \mathrm{id}$, and

$$\varphi_{0,\pm} : V_{0,\pm} \longrightarrow V_{\pm,0}, \quad (y,z) \mapsto (y, \sqrt{y}z)$$

give us a critical manifold structure with $f_0 = yz^2$, $f_\pm = z^2$, $f_\infty = z^2$.

In [17], we proved that if Y is a Calabi-Yau 3-fold (a smooth projective variety of dimension 3 whose canonical line bundle K_Y is trivial), then moduli spaces of simple sheaves on Y are all critical virtual manifolds.

Remark 2.11. In [14], Joyce introduced the notion of a d-critical locus and in [4], Brav, Bussi and Joyce proved that a (-1)-shifted symplectic derived scheme is an algebraic d-critical locus. In §2.5, we will prove that critical virtual manifolds are d-critical loci and vice versa. In particular, the analytic space associated to each (-1)-shifted symplectic derived scheme admits a critical virtual manifold structure.

Remark 2.12. Although it was not explicitly stated as a definition, the notion of a critical virtual manifold was the key in our solution to the categorification problem of the Donaldson-Thomas invariants in our previous draft in 2012.

How can we compare two critical virtual manifold structures of different dimensions?

Definition 2.13. Let X be an analytic space with an open cover $\{X_\alpha\}$. Two critical virtual manifold structures

$$\{(X_\alpha = \mathrm{Crit}(f_\alpha) \subset V_\alpha \xrightarrow{f_\alpha} \mathbb{C}, \varphi_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha})\},$$
$$\{(X_\alpha = \mathrm{Crit}(g_\alpha) \subset W_\alpha \xrightarrow{g_\alpha} \mathbb{C}, \psi_{\alpha\beta} : W_{\alpha\beta} \to W_{\beta\alpha})\}$$

with $\dim V_\alpha \leq \dim W_\alpha$ are *compatible* if there exist closed embeddings

$$\imath_\alpha : V_\alpha \longrightarrow W_\alpha, \quad \forall\alpha$$

such that $g_\alpha \circ \imath_\alpha = f_\alpha$ and

$$\imath_\beta \circ \varphi_{\alpha\beta} = \psi_{\alpha\beta} \circ \imath_\alpha, \quad \forall \alpha, \beta.$$

The two charts (V_α, f_α) and (W_α, g_α) are related by the following lemma.

Lemma 2.14. *Let $W \subset V$ be a complex submanifold in a complex manifold. Let $f : V \to \mathbb{C}$ be a holomorphic function and $g = f|_W$. Suppose the critical loci $X_f = \mathrm{Crit}(f)$ and $X_g = \mathrm{Crit}(g)$ are equal. Let x be a closed point of $X_f = X_g$. Then there is a coordinate system $\{z_1, \ldots, z_r\}$ of V centered at x such that W is defined by the vanishing of $z_1, \ldots, z_m$ and*

$$f = q(z_1,\ldots,z_m) + g(z_{m+1},\ldots,z_r), \quad q(z_1,\ldots,z_m) = \sum_{i=1}^m z_i^2, \quad g = f|_W.$$

Proof. We choose coordinates $\{y_1,\dots,y_r\}$ of V centered at x such that W is defined by the vanishing of $y_1,\dots,y_m$. Let I be the ideal generated by $y_1,\dots,y_m$. Since $X_f = X_g$, i.e., $(df) = (dg) + I$, we have

$$\frac{\partial f}{\partial y_i}\Big|_W = \sum_{j=m+1}^{r} a_{ij}\frac{\partial g}{\partial y_j}, \quad i = 1,\dots,m$$

for some functions a_{ij} regular at x. By calculus, we have

$$\begin{aligned} f &= g(y_{m+1},\dots,y_r) + \sum_{i=1}^{m} \frac{\partial f}{\partial y_i}\Big|_W \cdot y_i + I^2 \\ &= g(y_{m+1},\dots,y_r) + \sum_{j=m+1}^{r} \frac{\partial g}{\partial y_j}\left(\sum_{i=1}^{m} a_{ij}y_i\right) + I^2 \\ &= g(z_{m+1},\dots,z_r) + \sum_{i,k=1}^{m} b_{ik}y_iy_k, \end{aligned}$$

where $z_j = y_j + \sum_{i=1}^m a_{ij}y_i$ for $j \ge m+1$ and b_{ik} are some functions holomorphic near x. Since the kernel of the Hessian of f at x is the tangent space of $X_f = X_g \subset W$ at x, the quadratic form $q = \sum_{i,k=1}^m b_{ik}y_iy_k$ is nondegenerate near x. Hence we can diagonalize $q = \sum_{i=1}^m z_i^2$ by changing the coordinates $y_1,\dots,y_m$ to new coordinates $z_1,\dots,z_m$. It follows that $z_1,\dots,z_r$ is the desired coordinate system. □

2.3. Orientability of a critical virtual manifold

For a real (Riemannian) manifold M of dimension n, its orientation bundle is the top exterior power $\det_{\mathbb{R}} T_M = \wedge^n_{\mathbb{R}} T_M$ of the tangent bundle T_M and its associated principal $O(1)$-bundle has $\{\pm 1\}$-valued locally constant transition functions since $O(1) = \{\pm1\} = \mathbb{Z}_2$. So the orientation bundle is determined by a class $\xi \in H^1(M,\mathbb{Z}_2)$ whose vanishing is equivalent to the orientability of M, i.e., the triviality of $\det_{\mathbb{R}} T_M$. There is an analogous story for critical virtual manifolds.

Let X be a critical virtual manifold and $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$ be a chart. Let

$$K_\alpha^\vee = \varphi_\alpha^* \det T_{V_\alpha}|_{X_\alpha^{\rm red}}$$

be the dual of the canonical bundle of V_α restricted to the reduced space $X_\alpha^{\rm red}$ of X_α. These line bundles are related by the isomorphisms $\xi_{\alpha\beta} : K_\alpha^\vee|_{X_{\alpha\beta}^{\rm red}} \to K_\beta^\vee|_{X_{\alpha\beta}^{\rm red}}$ defined by

$$\xi_{\alpha\beta} := (\varphi_\alpha|_{X_{\alpha\beta}^{\rm red}})^* \det(d\varphi_{\alpha\beta}) : \varphi_\alpha^* \det T_{V_\alpha}|_{X_{\alpha\beta}^{\rm red}} \longrightarrow \varphi_\beta^* \det T_{V_\beta}|_{X_{\alpha\beta}^{\rm red}} \tag{2.3}$$

Recall that $\varphi_{\alpha\beta\gamma}$ denotes the composition $\varphi_{\gamma\alpha}\circ\varphi_{\beta\gamma}\circ\varphi_{\alpha\beta} : V_{\alpha\beta\gamma} \longrightarrow V_\alpha$.

Proposition 2.15. *For $X_{\alpha\beta\gamma} = X_\alpha \cap X_\beta \cap X_\gamma \neq \emptyset$,*

$$\xi_{\alpha\beta\gamma} = \xi_{\gamma\alpha}\circ\xi_{\beta\gamma}\circ\xi_{\alpha\beta} = (\varphi_\alpha|_{X_{\alpha\beta\gamma}^{\rm red}})^* \det(d\varphi_{\alpha\beta\gamma}) \tag{2.4}$$

is locally constant, taking values in $\{\pm 1\}$.

Proof. Apply Lemma 2.16, letting $V = V_\alpha$, $U = V_{\alpha\beta\gamma}$, $f = f_\alpha$, $\varphi = \varphi_{\alpha\beta\gamma}$. $\square$

Lemma 2.16. *Let* (V, f) *be an LG pair (Definition 2.4). Let* $U \subset V$ *be open and* $\varphi : U \to V$ *be a biholomorphic map onto its image, such that* $f \circ \varphi = f|_U$ *and* $\varphi|_{X_f \cap U} = \mathrm{id}_{X_f \cap U}$ *where* $X_f = \mathrm{Crit}(f)$. *Then* $\det(d\varphi)|_{X_f^{\mathrm{red}} \cap U}$ *is locally constant, taking values* ± 1.

Proof. Let $H_x = d(df)$ denote the Hessian of f at $x \in X_f \cap U$. Then H_x is a symmetric bilinear form whose kernel is $T_{X_f}|_x$. After diagonalizing H_x, we can choose a subspace W_x, complementary to $T_{X_f}|_x$, such that the Hessian H_x is nondegenerate on W_x. Since $\varphi|_{X_f} = \mathrm{id}_{X_f}$, we can write

$$d\varphi|_x = \begin{pmatrix} \mathrm{id} & A \\ 0 & B \end{pmatrix} : T_V|_x = T_{X_f}|_x \oplus W_x \to T_{X_f}|_x \oplus W_x = T_V|_x, \tag{2.5}$$

where $A : W_x \to T_{X_f}|_x$ and $B : W_x \to W_x$ are homomorphisms of vector spaces. Since $f \circ \varphi = f|_U$, B preserves the nondegenerate symmetric bilinear form H_x and hence B lies in the orthogonal group $O_{H_x}(W_x)$ with respect to H_x. Since $\det(d\varphi|_x) = \det B$ and an orthogonal matrix has determinant ± 1, we find that $\det(d\varphi|_x) \in \{\pm 1\}$ as desired. $\square$

Since $\xi_{\alpha\beta\gamma} = \xi_{\gamma\alpha} \circ \xi_{\beta\gamma} \circ \xi_{\alpha\beta}$ and $\xi_{\alpha\beta} = \xi_{\beta\alpha}^{-1}$ for each pair (α, β),

$$\begin{aligned}
(d\xi)_{\alpha\beta\gamma\delta} &= \xi_{\alpha\beta\gamma}^{-1} \xi_{\alpha\gamma\delta}^{-1} \xi_{\alpha\beta\delta} \xi_{\beta\gamma\delta} \\
&= (\xi_{\beta\alpha}\xi_{\gamma\beta}\xi_{\alpha\gamma})(\xi_{\gamma\alpha}\xi_{\delta\gamma}\xi_{\alpha\delta})(\xi_{\delta\alpha}\xi_{\beta\delta}\xi_{\alpha\beta})\xi_{\beta\gamma\delta} \\
&= (\xi_{\beta\alpha}\xi_{\gamma\beta}\xi_{\delta\gamma}\xi_{\beta\delta}\xi_{\alpha\beta})\xi_{\beta\gamma\delta} \\
&= (\xi_{\alpha\beta}^{-1}\xi_{\beta\gamma\delta}^{-1}\xi_{\alpha\beta})\xi_{\beta\gamma\delta} \\
&= \xi_{\alpha\beta}^{-1} \circ \xi_{\alpha\beta} = 1,
\end{aligned}$$

where the second to the last equality comes from $\xi_{\beta\gamma\delta} = \pm 1$. Therefore, $\{\xi_{\alpha\beta\gamma}\}$ is a Čech 2-cocycle with values in $\{\pm 1\}$ and defines a class

$$\xi := [\xi_{\alpha\beta\gamma}] \in H^2(X, \mathbb{Z}_2) \tag{2.6}$$

whose vanishing amounts to, possibly after a refinement of the open cover $\{X_\alpha\}$, the existence of a 1-cochain $\mu = \{\mu_{\alpha\beta}\}$ taking values in $\{\pm 1\}$ such that for $\bar{\xi}_{\alpha\beta} = \mu_{\alpha\beta}\xi_{\alpha\beta}$, we have

$$\bar{\xi}_{\alpha\beta\gamma} := \bar{\xi}_{\gamma\alpha} \circ \bar{\xi}_{\beta\gamma} \circ \bar{\xi}_{\alpha\beta} = 1 \tag{2.7}$$

whenever $X_{\alpha\beta\gamma} \neq \emptyset$. In particular, the 1-cocycle $\{\bar{\xi}_{\alpha\beta}\}$ with values in $\mathcal{O}^*$ glues the local line bundles $\{K_\alpha^\vee = \varphi_\alpha^* \det T_{V_\alpha}|_{X_\alpha^{\mathrm{red}}}\}$ to a globally defined line bundle $K_X^\vee$ on X^{red}.

Definition 2.17. We say a critical virtual manifold X is *orientable* if the class $\xi \in H^2(X, \mathbb{Z}_2)$ defined in (2.6) is zero. When X is orientable, a line bundle

$K_X^\vee$ on $X^{\rm red}$ obtained by gluing $\{K_\alpha^\vee\}$ by $\{\bar\xi_{\alpha\beta}=\mu_{\alpha\beta}\xi_{\alpha\beta}\}$ above is called an *orientation bundle* of X.

The issue of orientability will be crucial when we construct a natural perverse sheaf and a mixed Hodge module on a critical virtual manifold X by gluing locally defined perverse sheaves and mixed Hodge modules of vanishing cycles on local charts X_α.

Remark 2.18. An orientation bundle of a critical virtual manifold is not unique. One can always twist $\{\bar\xi_{\alpha\beta}\}$ by a $\mathbb{Z}_2$-valued 1-cocycle $\{\lambda_{\alpha\beta}\}$ to obtain a new orientation bundle defined by the gluing isomorphisms $\{\lambda_{\alpha\beta}\bar\xi_{\alpha\beta}\}$. In fact, since $\xi_{\alpha\beta\gamma}\in\{\pm 1\}$, $\{\xi_{\alpha\beta}^2\}$ is a 1-cocycle and glues the locally defined line bundles $\{(K_\alpha^\vee)^2\}$ on $X_\alpha^{\rm red}$ to a globally defined line bundle $(K_X^\vee)^2$ on $X^{\rm red}$. Hence the square of an orientation bundle is canonically defined. So we find that when a critical virtual manifold X is orientable, the set of orientation bundles is an $H^1(X,\mathbb{Z}_2)$-orbit by the exact sequence

$$\cdots\longrightarrow H^1(X,\mathbb{Z}_2)\longrightarrow H^1(X,\mathcal{O}_X^*)\xrightarrow{2} H^1(X,\mathcal{O}_X^*)\longrightarrow\cdots$$

from the short exact sequence

$$1\longrightarrow\{\pm1\}\longrightarrow\mathcal{O}_X^*\xrightarrow{2}\mathcal{O}_X^*\longrightarrow 1.$$

Example 2.19. Every complex manifold is an orientable critical virtual manifold. The critical virtual manifold in Example 2.10 is not orientable because

$$\det(d\varphi_{\infty,+,-})=1\quad\text{and}\quad\det(d\varphi_{0,+,-})=\begin{cases}1 & \mathrm{Im}(y)>0\\ -1 & \mathrm{Im}(y)<0\end{cases}$$

define a nontrivial class in $H^2(\mathbb{P}^1,\mathbb{Z}_2)$.

2.4. Semi-perfect obstruction theory and Behrend's theorem

In this section, we prove that a critical virtual manifold is equipped with a natural semi-perfect obstruction theory (cf. [6]) which gives us a Donaldson-Thomas type invariant of a critical virtual manifold X. We will also prove an analogue of Behrend's theorem for critical virtual manifolds which says that the Donaldson-Thomas type invariant is the weighted Euler number of the analytic space X with the weight function given by the Milnor numbers on the local charts.

2.4.1. *Behrend function.* We first recall the theory of Behrend functions from [1]. We assume that X admits a *global embedding into a complex manifold* $\mathbb{P}$. Let $C_{X/\mathbb{P}}$ be the normal cone of X in $\mathbb{P}$. Then $C_{X/\mathbb{P}}$ defines the *characteristic cycle*

$$\mathfrak{c}_X=\sum_{C'}(-1)^{\dim\pi(C')}\mathrm{mult}(C')\pi(C'),\tag{2.8}$$

where $\pi : C_{X/\mathbb{P}} \to X$ is the obvious projection and the sum is over all irreducible components C' of $C_{X/\mathbb{P}}$. Hence $\pi(C')$ denotes the image of C' by π which is a prime cycle of X. Also, $\mathrm{mult}(C')$ is the length of C' at the generic point of C'.

For a chart $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$, we can use the normal cone C_{X_α/V_α} to define the characteristic cycle $\mathfrak{c}_{X_\alpha}$ by the recipe of (2.8). By [1, Proposition 1.1], we have

$$\mathfrak{c}_X|_{X_\alpha} = \mathfrak{c}_{X_\alpha}$$

because the characteristic cycle is independent of the embedding into a complex manifold.

For a prime cycle W of dimension p on X, the Nash blow-up $\mu : \tilde{W} \to W$ gives us a constructible function

$$Eu(W) = \int_{\mu^{-1}(p)} c(\tilde{T}) \cap s(\mu^{-1}(p), \tilde{W}), \tag{2.9}$$

where $\tilde{T}$ is the dual of the universal quotient bundle on $\tilde{W}$ in the Grassmannian of rank p quotients of Ω_W (or $\Omega_{\mathbb{P}}$). Here $c(\tilde{T})$ is the total Chern class of $\tilde{T}$ and $s(\mu^{-1}(p), \tilde{W})$ is the Segre class of the normal cone to $\mu^{-1}(p)$ in $\tilde{W}$. Now the Behrend function is defined as the $\mathbb{Z}$-valued constructible function ([1, Definition 1.4])

$$\nu_X = Eu(\mathfrak{c}_X). \tag{2.10}$$

When X is a critical virtual manifold so that X is locally the critical locus $X_\alpha \cong \mathrm{Crit}(f_\alpha)$ of a holomorphic function $f_\alpha : V_\alpha \to \mathbb{C}$, by [21, Corollary 2.4 (iii)] or [1, (4)],

$$\nu_X(x) = (-1)^{\dim V_\alpha}(1 - \chi(MF_x)), \tag{2.11}$$

where $\chi(MF_x)$ is the Euler characteristic of the Milnor fiber MF_x of f_α at x (cf. (3.6)).

The constructible function ν_X defines the weighted Euler characteristic

$$\chi_\nu(X) = \sum_{n \in \mathbb{Z}} n \cdot \chi_c(\nu_X^{-1}(n)), \tag{2.12}$$

where $\chi_c(\cdot) = \sum (-1)^j \dim H_c^j(\cdot)$ is the Euler characteristic with compact support.

The cotangent bundle $\Omega_{\mathbb{P}}$ of a complex manifold $\mathbb{P}$ admits a canonical symplectic structure: for any system of local coordinates $x_1, \ldots, x_n$ and the coordinates $p_1, \ldots, p_n$ for the basis $dx_1, \ldots, dx_n$ for the fibers of $\pi : \Omega_{\mathbb{P}} \to \mathbb{P}$, $\sum_{i=1}^n dp_i \wedge dx_i$ is a symplectic form which is independent of the choice of the coordinates $\{x_1, \ldots, x_n\}$.

We say a closed analytic subset $W \subset \Omega_{\mathbb{P}}$ is *conic Lagrangian* if the Euler vector field $\theta = \sum p_i \frac{\partial}{\partial p_i}$ is tangent to W, $\dim W = \dim \mathbb{P}$ and the symplectic form vanishes identically on the smooth locus of W. A conic Lagrangian subset in the cotangent bundle $\Omega_{\mathbb{P}}$ is completely determined by its intersection with the zero section or the characteristic cycle.

Lemma 2.20 ([1, Lemma 4.2]). *Let $W \subset \Omega_{\mathbb{P}}$ be a closed irreducible analytic subset. Let $\bar{W} = \pi(W) \subset \mathbb{P}$ be its image and let $\ell(\bar{W}) \subset \Omega_{\mathbb{P}}$ denote the closure of the conormal bundle of any smooth dense open subset of $\bar{W}$. If W is conic Lagrangian, then $\ell(\bar{W}) = W$.*

Here the conormal bundle of a submanifold $\imath : S \hookrightarrow \mathbb{P}$ means the kernel bundle $\ker(\Omega_{\mathbb{P}}|_S \xrightarrow{\imath^*} \Omega_S)$ of the pullback homomorphism.

Note that conic Lagrangian is a local property. The conormal bundle construction $\ell(\bar{W})$ for a prime cycle $\bar{W}$ extends to a homomorphism

$$\ell : Z_*(\mathbb{P}) \longrightarrow Z_{\dim \mathbb{P}}(\Omega_{\mathbb{P}}). \tag{2.13}$$

Proposition 2.21 ([1, Propositions 1.12 and 4.6]). *If we apply the homomorphism ℓ to the characteristic cycle $\mathfrak{c}_X$ in* (2.8) *together with the Gysin homomorphism*

$$0^!_{\Omega_{\mathbb{P}}} : Z_{\dim \mathbb{P}}(\Omega_{\mathbb{P}}) \longrightarrow A_0(\mathbb{P})$$

and the degree map $\deg : A_0(\mathbb{P}) \xrightarrow{p_*} A_0(pt) = \mathbb{Z}$ *where* $p : \mathbb{P} \to \operatorname{Spec} \mathbb{C}$ *is the constant morphism, then we get the Euler characteristic $\chi_\nu(X)$ weighted by the Behrend function, i.e.,*

$$\chi_\nu(X) = \deg 0^!_{\Omega_{\mathbb{P}}}[\ell(\mathfrak{c}_X)]. \tag{2.14}$$

Remark 2.22. By [1, Proposition 1.1], the Behrend function ν_X depends only on the analytic space structure $(X, \mathcal{O}_X)$. Therefore, the weighted Euler characteristic $\chi_\nu(X)$ is an invariant of the analytic space $(X, \mathcal{O}_X)$.

In the subsequent subsection, we will see that $\ell(\mathfrak{c}_X)$ is actually the obstruction cone in $\Omega_{\mathbb{P}}$ for a natural semi-perfect obstruction theory when X is a critical virtual manifold, so that

$$0^!_{\Omega_{\mathbb{P}}}[\ell(\mathfrak{c}_X)] = [X]^{\mathrm{vir}}. \tag{2.15}$$

Therefore, the weighted Euler characteristic $\chi_\nu(X)$ is the virtual invariant of X which satisfies expected properties such as deformation invariance.

2.4.2. *Semi-perfect obstruction theories on critical virtual manifolds.* In this subsection, we show that a critical virtual manifold admits a natural semi-perfect obstruction theory and a virtual fundamental class whose degree gives us a Donaldson-Thomas type invariant.

Recall the following definition from [2].

Definition 2.23. A *perfect obstruction theory* on an analytic space X refers to a morphism $\phi : E \to \mathbb{L}_X$ in the derived category $D(\mathcal{O}_X)$ such that

(1) E is locally isomorphic to a 2-term complex $[E^{-1} \to E^0]$ of locally free sheaves;
(2) $H^{-1}(\phi)$ is surjective and $H^0(\phi)$ is an isomorphism.

Here $\mathbb{L}_X = \mathbb{L}_X^{\geq -1}$ denotes the cotangent complex of X in [13], truncated at ≥ -1. The perfect obstruction theory $\phi : E \to \mathbb{L}_X$ is called *symmetric* if there is an isomorphism $\theta : E \to E^\vee[1]$ satisfying $\theta^\vee[1] = \theta$.

By [1, Remark 3.7], the obstruction sheaf $Ob_X = H^1(E^\vee)$ of a symmetric obstruction theory is canonically isomorphic to the cotangent sheaf Ω_X.

Definition 2.24 ([6])**.** Let X be an analytic space. A *semi-perfect obstruction theory* of X consists of an open covering $\{X_\alpha\}$ and perfect obstruction theories

$$\phi_\alpha : E_\alpha \longrightarrow \mathbb{L}_{X_\alpha}$$

for each α such that

(1) for each pair α, β of indices, there is an isomorphism

$$\psi_{\alpha\beta} : H^1(E_\alpha^\vee)|_{X_{\alpha\beta}} \to H^1(E_\beta^\vee)|_{X_{\alpha\beta}} \tag{2.16}$$

satisfying $\psi_{\alpha\alpha} = \mathrm{id}$, $\psi_{\beta\alpha} = \psi_{\alpha\beta}^{-1}$ and $\psi_{\beta\gamma} \circ \psi_{\alpha\beta} = \psi_{\alpha\gamma}$ for each triple α, β, γ;

(2) for each pair α, β, the perfect obstruction theories $E_\alpha|_{X_{\alpha\beta}}$ and $E_\beta|_{X_{\alpha\beta}}$ on $X_{\alpha\beta}$ give the same obstruction assignment via $\psi_{\alpha\beta}$.

By (1) above, we obtain a sheaf Ob_X which is the gluing of $\{Ob_{X_\alpha} = H^1(E_\alpha^\vee)\}$ via $\psi_{\alpha\beta}$ and is called the *obstruction sheaf* of X with respect to the semi-perfect obstruction theory.

The second condition requires further explanation.

Definition 2.25. Let $x \in X$ be a point in an analytic space. An *infinitesimal lifting problem* of X at x consists of

(1) an extension $0 \to I \to B \to \bar{B} \to 0$ of Artin local rings by an ideal I with $I \cdot m_B = 0$;

(2) a morphism $\bar{g} : \mathrm{Spec}\bar{B} \to X$ sending the unique closed point to x.

Let $\Delta = \mathrm{Spec}B$ and $\bar{\Delta} = \mathrm{Spec}\bar{B}$. By [2, §4], for any infinitesimal lifting problem, there is a canonical obstruction

$$\omega(\bar{g}, B, \bar{B}) := (\bar{g}^*\mathbb{L}_X \xrightarrow{\bar{g}} \mathbb{L}_{\bar{\Delta}} \to \mathbb{L}_{\bar{\Delta}/\Delta} \xrightarrow{\tau^{\geq -1}} I[1]) \in Ext^1(\bar{g}^*\mathbb{L}_X, I) \tag{2.17}$$

whose vanishing is necessary and sufficient for the existence of a lifting

$$g : \Delta = \mathrm{Spec}B \longrightarrow X$$

such that $g|_{\bar{\Delta}} = \bar{g}$. Here the first morphism $\bar{g}^*\mathbb{L}_X \xrightarrow{\bar{g}} \mathbb{L}_{\bar{\Delta}}$ is the pullback by $\bar{g}$; the second $\mathbb{L}_{\bar{\Delta}} \to \mathbb{L}_{\bar{\Delta}/\Delta}$ is the natural morphism from the embedding $\bar{\Delta} \hookrightarrow \Delta$; the third morphism $\tau^{\geq -1}$ is the truncation to terms of degree ≥ -1.

If $\phi : E \to \mathbb{L}_X$ is a perfect obstruction theory on X, its composition with $\omega(\bar{g}, B, \bar{B}) : \bar{g}^*\mathbb{L}_X \to I[1]$ gives us

$$ob_X(\phi, \bar{g}, B, \bar{B}) : \bar{g}^*E \longrightarrow \bar{g}^*\mathbb{L}_X \longrightarrow I[1] \tag{2.18}$$

which is an element of $Ext^1(\bar{g}^*E, I) = I \otimes_{\mathbb{C}} H^1(E^\vee)|_x$.

Definition 2.26. Let $\phi : E \to \mathbb{L}_X$ and $\phi' : E' \to \mathbb{L}_X$ be two perfect obstruction theories and $\psi : H^1(E^\vee) \to H^1(E'^\vee)$ be an isomorphism. We say the two perfect obstruction theories ϕ and ϕ' give the *same obstruction assignment* via ψ if

$$\bar{g}^*(\psi)(ob_X(\phi, \bar{g}, B, \bar{B})) = ob_X(\phi', \bar{g}, B, \bar{B}) \in I \otimes_{\mathbb{C}} H^1(E'^\vee)|_x.$$

This explains the second condition of Definition 2.24.

Definition 2.27. A semi-perfect obstruction theory $(\phi_\alpha : E_\alpha \to \mathbb{L}_{X_\alpha})$ is called *symmetric* if ϕ_α are all symmetric (cf. Definition 2.23) and the gluing isomorphisms $\psi_{\alpha\beta}$ for the obstruction sheaf Ob_X are the identity maps of $\Omega_{X_{\alpha\beta}}$ via the canonical isomorphism $Ob_{X_\alpha} \cong \Omega_{X_\alpha}$.

Let us suppose X is now a critical virtual manifold equipped with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$. Since $X_\alpha = \mathrm{zero}(df_\alpha)$ in V_α and df_α is a section of the cotangent bundle Ω_{V_α}, we have a perfect obstruction theory (cf. [1, §3])

$$\phi_\alpha : E_\alpha = [T_{V_\alpha}|_{X_\alpha} \xrightarrow{d(df_\alpha)} \Omega_{V_\alpha}|_{X_\alpha}] \longrightarrow \mathbb{L}_{X_\alpha}, \tag{2.19}$$

where $T_{V_\alpha} = \Omega_{V_\alpha}^\vee$ is the tangent bundle of V_α. This is symmetric because the Hessian $d(df_\alpha)$ is symmetric. Moreover, $H^1(E_\alpha^\vee) = \Omega_{X_\alpha}$ from the exact sequence

$$J_\alpha/J_\alpha^2 \xrightarrow{d} \Omega_{V_\alpha}|_{X_\alpha} \longrightarrow \Omega_{X_\alpha} \longrightarrow 0,$$

where $J_\alpha = (df_\alpha)$ is the ideal generated by the partial derivatives of f_α.

The obstruction assignment for an infinitesimal lifting problem at $x \in X$

$$(0 \to I \to B \to \bar{B} \to 0, \bar{g} : \mathrm{Spec}\bar{B} \to X_\alpha) \tag{2.20}$$

can be described as follows. We extend $\bar{g} : \bar{\Delta} = \mathrm{Spec}\bar{B} \to X_\alpha \hookrightarrow V_\alpha$ to a morphism $g' : \Delta = \mathrm{Spec}B \to V_\alpha$. Since X_α is the vanishing locus of the section df_α of the cotangent bundle $\Omega_{V_\alpha} \to V_\alpha$, g' factors through X_α if and only if $df_\alpha \circ g' : \Delta \to g'^*\Omega_{V_\alpha}$ is zero. As $\bar{g} = g'|_{\bar{\Delta}}$ factors through X_α, $df_\alpha \circ g' \in I \otimes \Omega_{V_\alpha}|_x$.

Let $\rho : I \otimes_{\mathbb{C}} \Omega_{V_\alpha}|_x \to I \otimes_{\mathbb{C}} \Omega_{X_\alpha}|_x$ be the tautological projection.

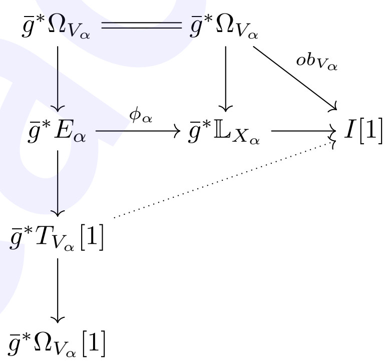
Lemma 2.28. $ob_X(\phi_\alpha, \bar{g}, B, \bar{B}) = \rho(df_\alpha \circ g') \in I \otimes_{\mathbb{C}} \Omega_X|_x$.

Proof. By (2.17), (2.18) and (2.19), $ob_X(\phi_\alpha, \bar{g}, B, \bar{B})$ is the composition

$$\bar{g}^*E_\alpha \longrightarrow \bar{g}^*\mathbb{L}_{X_\alpha} \longrightarrow \mathbb{L}_{\bar{\Delta}/\Delta} \longrightarrow I[1]$$

which fits into a commutative diagram:

(2.21)



The first column is the distinguished triangle

$$T_{V_\alpha}|_{X_\alpha} \longrightarrow \Omega_{V_\alpha}|_{X_\alpha} \longrightarrow [T_{V_\alpha} \to \Omega_{V_\alpha}]|_{X_\alpha} \longrightarrow T_{V_\alpha}|_{X_\alpha}[1].$$

We have an exact sequence

$$\operatorname{Hom}(\bar g^*\Omega_{V_\alpha}, I) = I \otimes_{\mathbb{C}} T_{V_\alpha}|_x \xrightarrow{d(df_\alpha)} \operatorname{Hom}(\bar g^* T_{V_\alpha}, I) = I \otimes_{\mathbb{C}} \Omega_{V_\alpha}|_x \tag{2.22}$$
$$\longrightarrow Ext^1(\bar g^* E_\alpha, I) \longrightarrow Ext^1(\bar g^*\Omega_{V_\alpha}, I).$$

Since V_α is smooth, the morphism

$$\bar\Delta = \operatorname{Spec}\bar B \longrightarrow X_\alpha \hookrightarrow V_\alpha$$

extends to a morphism $g' : \Delta \to V_\alpha$ and hence the homomorphism ob_{V_α} above is zero. Thus $ob_X(\phi_\alpha, \bar g, B, \bar B)$ lives in the cokernel $I \otimes_{\mathbb{C}} \Omega_X|_x$ of the first homomorphism in (2.22). The distinguished triangles

$$\Omega_{V_\alpha}|_{X_\alpha} \longrightarrow E_\alpha \longrightarrow T_{V_\alpha}|_{X_\alpha}[1],$$
$$\mathbb{L}_{V_\alpha} \longrightarrow \mathbb{L}_{X_\alpha} \longrightarrow \mathbb{L}_{X_\alpha/V_\alpha},$$
$$\mathbb{L}_\Delta|_{\bar\Delta} \longrightarrow \mathbb{L}_{\bar\Delta} \longrightarrow \mathbb{L}_{\bar\Delta/\Delta}$$

fit into the diagram:

$$\begin{array}{ccccccc}
\bar g^*\Omega_{V_\alpha}|_{X_\alpha} & \longrightarrow & \bar g^* E_\alpha & \longrightarrow & \bar g^* T_{V_\alpha}|_{X_\alpha}[1] & \overset{df_\alpha\circ g'}{\searrow} & \\
\Vert & & \downarrow{\scriptstyle \phi_\alpha} & & \downarrow & & \\
\bar g^*\mathbb{L}_{V_\alpha}|_{X_\alpha} & \longrightarrow & \bar g^*\mathbb{L}_{X_\alpha} & \longrightarrow & \bar g^*\mathbb{L}_{X_\alpha/V_\alpha} & \xrightarrow{\tau^{\geq -1}} & \bar g^* J_\alpha/J_\alpha^2[1] \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle \bar g} & & \downarrow & & \downarrow \\
\mathbb{L}_\Delta|_{\bar\Delta} & \longrightarrow & \mathbb{L}_{\bar\Delta} & \longrightarrow & \mathbb{L}_{\bar\Delta/\Delta} & \xrightarrow{\tau^{\geq -1}} & I[1]
\end{array} \tag{2.23}$$

From (2.22) and (2.23), we find that $ob_X(\phi_\alpha, \bar g, B, \bar B)$ equals $\rho(df_\alpha \circ g')$. □

Since $E_\alpha = [T_{V_\alpha}|_{X_\alpha} \xrightarrow{d(df_\alpha)} \Omega_{V_\alpha}|_{X_\alpha}]$ and $X_\alpha = \operatorname{zero}(df_\alpha) \subset V_\alpha$, $H^1(E_\alpha^\vee) = \Omega_{X_\alpha}$ and the transition maps $\varphi_{\alpha\beta} : V_{\alpha\beta} \xrightarrow{\cong} V_{\beta\alpha}$ send f_α to f_β and $\Omega_{X_{\alpha\beta}}$ to $\Omega_{X_{\beta\alpha}}$ respectively. Hence we have isomorphisms

$$\psi_{\alpha\beta} : H^1(E_\alpha^\vee)|_{X_{\alpha\beta}} = \Omega_{X_{\alpha\beta}} \longrightarrow H^1(E_\beta^\vee)|_{X_{\beta\alpha}} = \Omega_{X_{\beta\alpha}}$$

induced from $\varphi_{\alpha\beta}$. Certainly these $\{\Omega_{X_\alpha}, \psi_{\alpha\beta}\}$ glue to the cotangent sheaf Ω_X. Moreover these local perfect obstruction theories $\{\phi_\alpha\}$ give the same obstruction assignment.

Lemma 2.29. *Let $f : V \to \mathbb{C}$ and $g : W \to \mathbb{C}$ be two equivalent LG pairs, i.e., f and g have only one critical value 0 and we have a biholomorphic $\Phi : V \to W$ satisfying $g \circ \Phi = f$. Let $X_f = \operatorname{Crit}(f)$ and $X_g = \operatorname{Crit}(g)$ be the critical loci of f and g respectively. Then under the induced isomorphism $\hat\Phi = \Phi|_{X_f} : X_f \xrightarrow{\cong} X_g$, the perfect obstruction theories $E = [T_V|_{X_f} \to \Omega_V|_{X_f}]$ and*

$E' = [T_W|_{X_g} \to \Omega_W|_{X_g}]$ *give the same obstruction assignment via the canonical isomorphism*

$$H^1(E^\vee) = \Omega_{X_f} \cong \hat{\Phi}^*\Omega_{X_g} = \hat{\Phi}^* H^1(E'^\vee).$$

Proof. Because (V, f) and (W, g) are equivalent under Φ, Φ induces an isomorphism of the critical loci $\hat{\Phi} : X_f \cong X_g$. Hence, we have the induced $\hat{\Phi}^*\Omega_{X_g} \cong \Omega_{X_f}$ making the following square commutative

$$\begin{array}{ccc} \Omega_V|_{X_f} & \xrightarrow{\cong} & \hat{\Phi}^*(\Omega_W|_{X_g}) \\ \Big\downarrow \rho & & \Big\downarrow \varrho \\ \Omega_{X_f} & \xrightarrow{\cong} & \hat{\Phi}^*\Omega_{X_g} \end{array} \tag{2.24}$$

where the vertical arrows are natural quotient homomorphisms.

We now compare the obstruction assignments. Let $\bar{B} = B/I$ and $\bar{\lambda} : \operatorname{Spec} \bar{B} \to X_f$ supported at $x \in X_f$, as in (2.20). We let $\bar{\mu} = \hat{\Phi} \circ \bar{\lambda} : \operatorname{Spec} \bar{B} \to X_g$. We need to show that

$$ob_{X_f}(\phi, \bar{\lambda}, B, \bar{B}) = ob_{X_g}(\phi', \bar{\mu}, B, \bar{B}) \in I \otimes_{\mathbb{C}} \Omega_{X_f}|_x = I \otimes_{\mathbb{C}} \Omega_{X_g}|_{\hat{\Phi}(x)}. \tag{2.25}$$

We extend $\bar{\lambda}$ to $\lambda' : \operatorname{Spec} B \to V$, and let $\mu' = \Phi \circ \lambda' : \operatorname{Spec} B \to W$. Using $df \circ \bar{\lambda} = 0$ and $I \cdot \mathbf{m}_B = 0$, $df \circ \lambda' \in I \otimes_{\mathbb{C}} \Omega_V|_x$. By a similar reason, $dg \circ \mu' \in I \otimes_{\mathbb{C}} \Omega_W|_{\hat{\Phi}(x)}$. From $\Phi^*(dg) = df$ and $\mu' = \Phi \circ \lambda'$, we conclude

$$df \circ \lambda' = \Phi^*(dg \circ \mu') \in I \otimes_{\mathbb{C}} \Omega_V|_x \cong I \otimes_{\mathbb{C}} \Omega_W|_{\hat{\Phi}(x)}.$$

From (2.24), we obtain

$$\rho(df \circ \lambda') = \varrho(dg \circ \mu') \in I \otimes_{\mathbb{C}} \Omega_{X_f}|_x \cong I \otimes_{\mathbb{C}} \Omega_{X_g}|_{\hat{\Phi}(x)}.$$

Thus $ob_{X_f}(\phi, \bar{\lambda}, B, \bar{B}) = ob_{X_g}(\phi', \bar{\mu}, B, \bar{B})$, which proves the lemma. □

So we proved the following.

Proposition 2.30. *A critical virtual manifold X with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$ admits a symmetric semi-perfect obstruction theory $\{E_\alpha = [T_{V_\alpha}|_{X_\alpha} \to \Omega_{V_\alpha}|_{X_\alpha}]\}$ whose obstruction sheaf Ob_X is the cotangent sheaf Ω_X of X.*

Remark 2.31. In [6, Definition-Theorem 4.7], moduli spaces of simple gluable objects with fixed determinant in the derived category $D^b(Y)$ of coherent sheaves on a Calabi-Yau 3-fold Y are proven to have semi-perfect obstruction theories. Here an object $E \in D^b(Y)$ is simple if $\operatorname{Hom}_{D^b(Y)}(E, E) = \mathbb{C}$ and gluable if $Ext^{<0}(E, E) = 0$. It is expected that in fact these moduli spaces are critical virtual manifolds. In [17], we proved that this is indeed true for moduli spaces of simple sheaves.

Remark 2.32. The isomorphism $\varphi_{\alpha\beta}$ in Definition 2.5 induces the isomorphism

$$\begin{CD} E_\alpha|_{X_{\alpha\beta}} @= [T_{V_\alpha}|_{X_{\alpha\beta}} @>{d(df_\alpha)}>> \Omega_{V_\alpha}|_{X_{\alpha\beta}}] \\ @VVV @VVV @VVV \\ E_\beta|_{X_{\alpha\beta}} @= [T_{V_\beta}|_{X_{\alpha\beta}} @>{d(df_\beta)}>> \Omega_{V_\beta}|_{X_{\alpha\beta}}] \end{CD} \tag{2.26}$$

whose vertical maps are isomorphisms induced from $\varphi_{\alpha\beta}$ since $f_\beta \circ \varphi_{\alpha\beta} = f_\alpha$. However, because $\varphi_{\alpha\beta\gamma}$ in (2.2) need *not* be the identity morphism, there is no obvious reason for the local perfect obstruction theories $\{E_\alpha\}$ to glue to a global perfect obstruction theory.

2.4.3. *Virtual cycle and Donaldson-Thomas type invariant.* In [1], Behrend proved that when a scheme admits a symmetric perfect obstruction theory, the Donaldson-Thomas type invariant, which is the degree of the virtual cycle, is the weighted Euler characteristic (cf. §2.4.1). We extend this result to the case of symmetric semi-perfect obstruction theory and apply it to critical virtual manifolds.

Let X be an analytic space equipped with a symmetric semi-perfect obstruction theory $\{\phi_\alpha : E_\alpha \to \mathbb{L}_{X_\alpha}\}$. By [6, (3.5)], we then have the obstruction cone cycle $C_X \in Z_*(Ob_X)$ where Ob_X denotes the obstruction sheaf of X defined in Definition 2.24. By applying the Gysin map $s^!$ for the sheaf stack Ob_X when X is proper (cf. [6, Proposition 3.4]), we obtain the virtual cycle of X defined by

$$[X]^{\mathrm{vir}} := s^![C_X] \in A_0(X). \tag{2.27}$$

In case X admits a global embedding $\imath : X \hookrightarrow \mathbb{P}$ into a complex manifold $\mathbb{P}$, we can give an alternative description of $[X]^{\mathrm{vir}}$ by using the following proposition which is an immediate consequence of [1, Proposition 2.2].

Proposition 2.33. *Let X be an analytic space equipped with a symmetric semi-perfect obstruction theory $\{\phi_\alpha : E_\alpha \to \mathbb{L}_{X_\alpha}\}$. There exists a unique closed subcone $C_{\mathbb{P}} \subset \Omega_{\mathbb{P}}$ such that for each local resolution $F \to E_\alpha^\vee[1]|_U$ for open $U \subset X_\alpha$ by a locally free sheaf F on U and every lift ϕ*

$$\begin{array}{ccc} & & F \\ & \overset{\phi}{\nearrow} & \downarrow \\ \Omega_{\mathbb{P}}|_U & \xrightarrow{\imath^*} & \Omega_U \end{array}$$

we have $C_{\mathbb{P}}|_U = \phi^{-1}(C_F)$ where C_F is the fiber product of the intrinsic normal cone

$$\mathfrak{C}_{X_\alpha} \hookrightarrow h^1/h^0(E_\alpha^\vee)$$

and the quotient morphism $F \to h^1/h^0(E_\alpha^\vee)|_U$.

Indeed, by [1, Proposition 2.2], there are unique closed subcones $C_{\mathbb{P}}|_{X_\alpha} \subset \Omega_{\mathbb{P}}|_{X_\alpha}$ satisfying the above lifting property. By the uniqueness over the intersections $X_{\alpha\beta}$, these subcones uniquely glue to a cone $C_{\mathbb{P}} \subset \Omega_{\mathbb{P}}$. See [2] for the theory of intrinsic normal cones.

Note that

$$C_{\mathbb{P}} \in Z_{\dim \mathbb{P}}(\Omega_{\mathbb{P}})$$

since $\mathfrak{C}_{X_\alpha}$ is 0-dimensional.

Definition 2.34. Suppose X is a compact analytic space equipped with a symmetric semi-perfect obstruction theory and with an embedding into a complex manifold $\mathbb{P}$. The *virtual fundamental class* of X is defined to be

$$[X]^{\mathrm{vir}} = 0^!_{\Omega_{\mathbb{P}}}[C_{\mathbb{P}}] \in A_0(X) \tag{2.28}$$

using the cone in Proposition 2.33. We define the Donaldson-Thomas type invariant of X to be

$$DT(X) = \deg[X]^{\mathrm{vir}}, \tag{2.29}$$

where $\deg : A_0(X) \to A_0(\mathrm{pt}) = \mathbb{Z}$ is the pushforward by the constant map $X \to \mathrm{pt} = \operatorname{Spec} \mathbb{C}$.

Lemma 2.35. $[X]^{\mathrm{vir}}$ *is independent of the embedding* $X \hookrightarrow \mathbb{P}$ *into a complex manifold.*

Proof. Given two embeddings $\imath, \imath'$ of X into $\mathbb{P}$ and $\mathbb{P}'$, we have an embedding

$$(\imath, \imath') : X \hookrightarrow \mathbb{P} \times \mathbb{P}'$$

and exact sequences

$$0 \longrightarrow p_2^*\Omega_{\mathbb{P}'}|_X \longrightarrow C_{\mathbb{P}\times\mathbb{P}'} \longrightarrow C_{\mathbb{P}} \longrightarrow 0,$$
$$0 \longrightarrow p_1^*\Omega_{\mathbb{P}}|_X \longrightarrow C_{\mathbb{P}\times\mathbb{P}'} \longrightarrow C'_{\mathbb{P}} \longrightarrow 0,$$

where p_1, p_2 are the projections from $\mathbb{P} \times \mathbb{P}'$ to $\mathbb{P}$ and $\mathbb{P}'$ respectively. Hence

$$0^!_{\Omega_{\mathbb{P}}}[C_{\mathbb{P}}] = 0^!_{\Omega_{\mathbb{P}\times\mathbb{P}'}}[C_{\mathbb{P}\times\mathbb{P}'}] = 0^!_{\Omega_{\mathbb{P}'}}[C_{\mathbb{P}'}]$$

as desired. □

By the construction of $s^!$ in [6, Proposition 3.4], it is obvious that the virtual fundamental class in Definition 2.34 coincides with (2.27).

We next show that the Donaldson-Thomas type invariant (2.29) is the Euler characteristic (2.12) weighted by the Behrend function ν_X.

Lemma 2.36. *Let X be a compact analytic space equipped with a symmetric semi-perfect obstruction theory. Fix an embedding $\imath : X \hookrightarrow \mathbb{P}$ of X into a complex manifold $\mathbb{P}$. Then the cone $C_{\mathbb{P}}$ from Proposition 2.33 is conic Lagrangian, and $C_{\mathbb{P}} = \ell(\mathfrak{c}_X)$ where $\mathfrak{c}_X$ is the characteristic cycle of X in* (2.8).

Proof. Being conic Lagrangian is a local property. Hence $C_{\mathbb{P}}$ is conic Lagrangian by [1, Theorem 4.14]. For $C_{\mathbb{P}} = \ell(\mathfrak{c}_X)$, use Lemma 2.20 and the fact that $C_{\mathbb{P}}|_{X_\alpha}$ is the pullback of intrinsic normal cone $\mathfrak{C}_{X_\alpha}$ for each α. □

Combining Lemma 2.36 and Proposition 2.21, we obtain the following generalization of [1, Theorem 4.18] to the setting of semi-perfect obstruction theory.

Theorem 2.37. *Let X be a compact analytic space equipped with a symmetric semi-perfect obstruction theory and a closed immersion into a complex manifold $\mathbb{P}$. Then*

$$DT(X) = \chi_\nu(X), \tag{2.30}$$

where $DT(X) = \deg[X]^{\mathrm{vir}}$ and $\chi_\nu(X) = \sum n \cdot \chi_c(\nu^{-1}(n))$ is the Euler characteristic of X weighted by the Behrend function defined by (2.10).

Corollary 2.38. *The Donaldson-Thomas type invariant $DT(X) = \deg[X]^{\mathrm{vir}}$ depends only on the analytic space $(X, \mathcal{O}_X)$, i.e., $DT(X)$ is independent of the symmetric semi-perfect obstruction theory or the embedding into a complex manifold $\mathbb{P}$.*

Proof. By Remark 2.22, the weighted Euler characteristic $\chi_\nu(X)$ depends only on the analytic space $(X, \mathcal{O}_X)$. The corollary follows from Theorem 2.37. □

Let X be a critical virtual manifold with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$. Let

$$\{E_\alpha = [T_{V_\alpha}|_{X_\alpha} \xrightarrow{d(df_\alpha)} \Omega_{V_\alpha}|_{X_\alpha}]\}$$

be the symmetric semi-perfect obstruction theory from Proposition 2.30 so that the cotangent sheaf Ω_X is the gluing of local obstruction sheaves $\{H^1(E_\alpha^\vee)\}$. By Corollary 2.38, we therefore obtain the following.

Corollary 2.39. *Let X be a compact critical virtual manifold equipped with an embedding into a complex manifold $\mathbb{P}$. Then $DT(X) = \chi_\nu(X)$ and the invariant depends only on the analytic space underlying the critical virtual manifold.*

Remark 2.40. In [6], the theory of semi-perfect obstruction was developed in the more general setting of a separated proper Deligne-Mumford stack X over a smooth Artin stack $\mathcal{M}$ of pure dimension. The Donaldson-Thomas type invariant $DT(X) = \deg[X]^{\mathrm{vir}}$ is deformation invariant (cf. [6, Proposition 3.8]): If

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow & & \downarrow \\ \mathcal{M}' & \xrightarrow{v} & \mathcal{M} \end{array}$$

is a fiber product and v is a regular embedding of smooth Artin stacks of pure dimensions,

$$v^![X]^{\mathrm{vir}} = [X']^{\mathrm{vir}}.$$

2.4.4. *A categorification problem.* Categorification roughly means finding a deeper structure underlying a known invariant. The categorification problem for critical virtual manifolds is the following.

Problem 2.41. *Find a cohomology theory $\mathcal{H}^*(X)$ for a compact critical virtual manifold X whose Euler characteristic*

$$\sum_m (-1)^m \dim \mathcal{H}^m(X)$$

is equal to the Donaldson-Thomas type invariant

$$DT(X) = \deg[X]^{\mathrm{vir}} = \chi_\nu(X).$$

Originally this problem was proposed by Joyce and Song in [15, Question 5.7] for the case where X is a moduli space of simple coherent sheaves on a Calabi-Yau 3-fold Y.

In subsequent chapters, we will construct such a cohomology theory as the hypercohomology of a perverse sheaf on X which is locally the perverse sheaf P_α of vanishing cycles for $f_\alpha : V_\alpha \to \mathbb{C}$ whenever X is an *orientable* critical virtual manifold.

2.5. Critical virtual manifolds and *d*-critical loci

In [14], Joyce introduced the notion of a d-critical locus as a classical model for a (-1)-shifted symplectic derived scheme. In this section, we prove that critical virtual manifolds are d-critical loci and vice versa.

According to [14, Definition 2.3], a d-critical locus is a pair (X, s) of an analytic space X and a section s of a sheaf $\mathcal{S}_X^0$ satisfying the condition that for any $x \in X$ there exists an open neighborhood U of x which is isomorphic to the critical locus $\mathrm{Crit}(f)$ of a holomorphic function f on a complex manifold V such that f defines the section $s|_U$. Roughly speaking, $\mathcal{S}_X^0$ sheafifies the choices of charts (V, f) and their compatibility (cf. Definition 2.13) while allowing the dimensions of the charts to vary. The definition of the sheaf $\mathcal{S}_X^0$ is rather involved (cf. [14, Theorem 2.1]). We will not reproduce the precise definition here but instead we will use the following.

Proposition 2.42. *An analytic space X is a d-critical locus if and only if there is an open cover $X = \cup_\alpha X_\alpha$ and LG pairs $\{(V_\alpha, f_\alpha)\}$ such that*

1. *for each α, $\mathrm{Crit}(f_\alpha) \cong X_\alpha$ as analytic spaces;*
2. *for each pair (α, β) of indices, there exist open neighborhoods $V_{\alpha\beta}$ (resp. $V_{\beta\alpha}$) of $X_{\alpha\beta} = X_\alpha \cap X_\beta$ in V_α (resp. V_β) and closed embeddings*

$$V_{\alpha\beta} \xhookrightarrow{\imath_{\alpha\beta}} W_{\alpha\beta} \xhookleftarrow{\imath_{\beta\alpha}} V_{\beta\alpha} \tag{2.31}$$

into a complex manifold $W_{\alpha\beta}$ satisfying

$$g_{\alpha\beta} \circ \imath_{\alpha\beta} = f_\alpha|_{V_{\alpha\beta}}, \quad g_{\alpha\beta} \circ \imath_{\beta\alpha} = f_\beta|_{V_{\beta\alpha}} \tag{2.32}$$

for some holomorphic function $g_{\alpha\beta}$ on $W_{\alpha\beta}$ whose critical locus $\mathrm{Crit}(g_{\alpha\beta})$ is $X_{\alpha\beta}$.

Proof. If we have an open cover $X = \cup_\alpha X_\alpha$ and LG pairs $\{(V_\alpha, f_\alpha)\}$ satisfying the stated conditions, then we certainly have a d-critical locus structure on X because the local sections s_α defined by f_α glue to a globally defined section s by [14, Theorem 2.1(ii)].

Conversely, suppose we have a d-critical locus (X, s) and at every point $x \in X$ we can find a neighborhood U of x in X which is the critical locus of an LG pair (V, f) such that f defines the section $s|_U$. Hence there is an open cover $X = \cup_{\lambda \in \Lambda} X_\lambda$ and LG pairs (V_λ, f_λ) such that $\mathrm{Crit}(f_\lambda) \cong X_\lambda$. Since X is paracompact, we may assume the covering is locally finite. For each $x \in X$,

$$\Lambda_x = \{\lambda \in \Lambda \,|\, x \in X_\lambda\}$$

is a finite set. Since a second countable paracompact space is a metric space, we can choose a metric d on X. By [14, Theorem 2.20], we can pick $\varepsilon_x > 0$ such that

$$B(x, 3\varepsilon_x) \subset \cap_{\lambda \in \Lambda_x} X_\lambda$$

and that for any $\lambda_1, \lambda_2 \in \Lambda_x$, there is an LG pair $(W_{\lambda_1\lambda_2}, g_{\lambda_1\lambda_2})$ satisfying

$$\mathrm{Crit}(g_{\lambda_1\lambda_2}) = B(x, 3\varepsilon_x)$$

together with closed embeddings

$$V^\circ_{\lambda_1} \xhookrightarrow{\imath_1} W_{\lambda_1\lambda_2} \xhookleftarrow{\imath_2} V^\circ_{\lambda_2}$$

of open neighborhoods $V^\circ_{\lambda_i}$ of $B(x, 3\varepsilon_x)$ in V_{λ_i} for $i = 1, 2$, satisfying

$$g_{\lambda_1\lambda_2} \circ \imath_1 = f_{\lambda_1}|_{V^\circ_{\lambda_1}}, \quad g_{\lambda_1\lambda_2} \circ \imath_2 = f_{\lambda_2}|_{V^\circ_{\lambda_2}}.$$

Fix a map $\lambda : X \to \Lambda$ such that $\lambda(x) \in \Lambda_x$ for all x. Then we can find an open neighborhood V_x of

$$U_x = B(x, \varepsilon_x)$$

in $V_{\lambda(x)}$ such that the critical locus of $f_x = f_{\lambda(x)}|_{V_x}$ is U_x. We claim that

$$X = \cup_{x \in X} U_x, \quad (V_x, f_x)$$

satisfy the conditions in Proposition 2.42. We have to check a diagram similar to (2.31) for $\{V_x\}$. Indeed, if $U_x \cap U_{x'} \neq \emptyset$ with $\varepsilon_x \geq \varepsilon_{x'}$, then

$$U_x \cup U_{x'} \subset B(x, 3\varepsilon_x)$$

and thus $\Lambda_x \subset \Lambda_{x'}$. In particular, both $\lambda(x)$ and $\lambda(x')$ lie in $\Lambda_{x'}$. Now by assumption, we have closed embeddings

$$V^\circ_{\lambda(x)} \xhookrightarrow{\imath} W_{\lambda(x)\lambda(x')} \xhookleftarrow{\imath'} V^\circ_{\lambda(x')}$$

of open neighborhoods $V^\circ_{\lambda(x)}$, $V^\circ_{\lambda(x')}$ of $U_{x'}$ in $V_{\lambda(x)}$, $V_{\lambda(x')}$ respectively, for an LG pair $(W_{\lambda(x)\lambda(x')}, g)$ with $\mathrm{Crit}(g) = U_{x'}$ and

$$g \circ \imath = f_x|_{V^\circ_{\lambda(x)}} \quad \text{and} \quad g \circ \imath' = f_{x'}|_{V^\circ_{\lambda(x')}}.$$

The diagram certainly restricts to such a diagram for the open subset $U_x \cap U_{x'}$. This proves the proposition. □

Definition 2.43. A d-critical locus (X, s) is called *finite dimensional* if there are an open cover $X = \cup_\alpha X_\alpha$ and LG pairs $\{(V_\alpha, f_\alpha)\}$ such that $\{\dim V_\alpha, \dim W_{\alpha\beta}\}$ is bounded, in the notation of Proposition 2.42.

Proposition 2.44. *A critical virtual manifold is a finite dimensional d-critical locus. Conversely a finite dimensional d-critical locus is a critical virtual manifold.*

Proof. The first statement is obvious from Proposition 2.42 because if

$$X = \cup_\alpha X_\alpha, \quad X_\alpha = \mathrm{Crit}(f_\alpha) \subset V_\alpha \xrightarrow{f_\alpha} \mathbb{C}$$

is a critical virtual manifold structure, then the biholomorphic map $\varphi_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$ gives a diagram

$$V_{\alpha\beta} \xhookrightarrow{\imath_{\alpha\beta}} W_{\alpha\beta} \xhookleftarrow{\imath_{\beta\alpha}} V_{\beta\alpha}$$

with $W_{\alpha\beta} = V_{\alpha\beta}$, $\imath_{\alpha\beta} = \mathrm{id}$ and $\imath_{\beta\alpha} = \varphi_{\beta\alpha}$.

Conversely, if X is a finite dimensional d-critical locus, then we have an open cover $X = \cup X_\alpha$, charts

$$\{(X_\alpha = \mathrm{Crit}(f_\alpha) \subset V_\alpha \xrightarrow{f_\alpha} \mathbb{C})\}$$

and diagrams (2.31) satisfying (2.32). Moreover, we can find an integer r such that

$$r \geq \dim W_{\alpha\beta}, \quad r \geq \dim V_\alpha, \quad \forall \alpha, \beta.$$

By Lemma 2.14, for any $x \in X$, we have a commutative diagram

$$\begin{array}{ccccc} V^\circ_{\alpha\beta} \times \mathbb{C}^{r-\dim V_\alpha} & \xrightarrow{\cong} & W^\circ_{\alpha\beta} \times \mathbb{C}^{r-\dim W_{\alpha\beta}} & \xleftarrow{\cong} & V^\circ_{\beta\alpha} \times \mathbb{C}^{r-\dim V_\beta} \\ & {\scriptstyle \tilde f_\alpha} \searrow & \downarrow {\scriptstyle \tilde g_{\alpha\beta}} & \swarrow {\scriptstyle \tilde f_\beta} & \\ & & \mathbb{C} & & \end{array}$$

for open neighborhoods $V^\circ_{\alpha\beta}$, $V^\circ_{\beta\alpha}$ and $W^\circ_{\alpha\beta}$ of x in $V_{\alpha\beta}$, $V_{\beta\alpha}$ and $W_{\alpha\beta}$ respectively. Here $\tilde f_\alpha = f_\alpha + \sum_{i=1}^{r-\dim V_\alpha} z_i^2$, $\tilde f_\beta = f_\beta + \sum_{i=1}^{r-\dim V_\beta} z_i^2$, $\tilde g_{\alpha\beta} = g_{\alpha\beta} + \sum_{i=1}^{r-\dim W_{\alpha\beta}} z_i^2$. So we have new charts

$$(\tilde V_\alpha = V_\alpha \times \mathbb{C}^{r-\dim V_\alpha}, \tilde f_\alpha = f_\alpha + \sum_{i=1}^{r-\dim V_\alpha} z_i^2)$$

for X and for any $x \in X_\alpha \cap X_\beta$, we have a biholomorphic map

$$\tilde\varphi_{\alpha\beta} : \tilde V^\circ_\alpha \longrightarrow \tilde V^\circ_\beta$$

of open neighborhoods of x such that $\tilde f_\beta \circ \tilde\varphi_{\alpha\beta} = \tilde f_\alpha$. Then the theorem follows from the same argument as in the proof of Proposition 2.42 or more explicitly from [17, Proposition 3.18]. □

The notion of finite dimensionality (Definition 2.43) is equivalent to the following condition:

$$\{\dim T_x X \mid x \in X\} \text{ is bounded.} \tag{2.33}$$

Indeed, at each $x \in X$, we can find a chart $f_x : V_x \to \mathbb{C}$ with $\dim V_x = \dim T_x X$ whose critical locus $\mathrm{Crit}(f_x)$ is open in X. When $z \in \mathrm{Crit}(f_x) \cap \mathrm{Crit}(f_y) \neq \emptyset$ and $\dim T_x X \geq \dim T_y X$, then we can find a critical chart $g : W_{xy} \to \mathbb{C}$ with $\dim W_{xy} = \dim T_x X$ whose critical locus $\mathrm{Crit}(g)$ is an open neighborhood of z. Hence (2.33) implies Definition 2.43 and vice versa.

3. Perverse sheaves on critical virtual manifolds

In this chapter, we provide a solution to the categorification problem (Problem 2.41). We prove that when X is an orientable critical virtual manifold with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$, there is a perverse sheaf $P \in Perv(\mathbb{Q}_X)$ which underlies a polarizable mixed Hodge module $\mathcal{M} \in MHM(X)^p$ such that the hypercohomology $\mathbb{H}^*_c(X, P)$ has Euler characteristic equal to the Euler characteristic $\chi_\nu(X)$ of X weighted by the Behrend function (cf. Theorems 3.15 and 3.20). Our perverse sheaf P on X is the gluing of the perverse sheaves P_α of vanishing cycles for f_α and so is the mixed Hodge module $\mathcal{M}$. The obstruction for gluing $\{P_\alpha\}$ is shown to coincide with that for the orientability of X. Moreover, we construct a perverse sheaf $\hat{P}$ with $\chi_c(X, \hat{P}) = \chi_\nu(X)$, which underlies a direct sum of polarizable Hodge modules (cf. Corollary 3.21). The hypercohomology $\mathbb{H}^*_c(X, \hat{P})$ will be of fundamental use for a mathematical theory of the Gopakumar-Vafa invariant in [17].

3.1. Perverse sheaves of vanishing cycles

In this section, we recall the perverse sheaf of vanishing cycles for the constant sheaf on a complex manifold V and a holomorphic function f on V. We show that if X is a critical virtual manifold with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$, there are sheaf complexes $P_\alpha = \varphi_\alpha^* P_{f_\alpha} \in D^b_c(\mathbb{Q}_{X_\alpha})$ and isomorphisms $\sigma_{\alpha\beta} : P_\alpha|_{X_{\alpha\beta}} \to P_\beta|_{X_{\alpha\beta}}$ (cf. Proposition 3.4 and Corollary 3.5).

Recall that an LG pair (V, f) refers to a complex manifold V together with a holomorphic function f on V which has only one critical value 0. Let

$$V_{>0} = \{x \in V \mid \mathrm{Re}\, f(x) > 0\} \quad \text{and} \quad V_{\leq 0} = \{x \in V \mid \mathrm{Re}\, f(x) \leq 0\}.$$

For any sheaf $\mathcal{F}$ of $\mathbb{Q}$-vector spaces on V, let $\Gamma_{V_{\leq 0}}\mathcal{F}$ denote the sheaf

$$U \mapsto \ker[\Gamma(U, \mathcal{F}) \to \Gamma(U \cap V_{>0}, \mathcal{F})], \quad \forall U \text{ open}.$$

Let $R\Gamma_{V_{\leq 0}} : D^b_c(\mathbb{Q}_V) \to D^b_c(\mathbb{Q}_V)$ be the right derived functor of the left exact functor $\Gamma_{V_{\leq 0}}$ where $D^b_c(\mathbb{Q}_V)$ denotes the derived category of bounded (cohomologically) constructible complexes of sheaves of $\mathbb{Q}$-vector spaces on V.

Definition 3.1. The *perverse sheaf of vanishing cycles* for an LG pair (V, f) is defined as

$$P_f = R\Gamma_{V_{\le 0}} \mathbb{Q}_V[\dim V]|_{f^{-1}(0)} \in D^b_c(\mathbb{Q}_{f^{-1}(0)}). \tag{3.1}$$

Here $[\dim V]$ denotes the translation functor, shifting a complex to the left by $\dim V$.

Example 3.2. Let $q = \sum_{i=1}^r y_i^2 : V = \mathbb{C}^r \to \mathbb{C}$. The set $V_{>0} \subset V = \mathbb{C}^r$ is a disk bundle over $\mathbb{R}^r - \{0\}$ which is homotopic to S^{r-1}. From the distinguished triangle

$$R\Gamma_{V_{\le 0}}\mathbb{Q} \longrightarrow \mathbb{Q} \longrightarrow R\imath_*\imath^*\mathbb{Q} \xrightarrow{+1} \tag{3.2}$$

where $\imath : V_{>0} \hookrightarrow V$ is the inclusion, we find that $P_q[1-r] = R\Gamma_{V_{\le 0}}\mathbb{Q}[1]$ is a sheaf complex supported at the origin 0 satisfying $P_q[1-r]|_0 \cong \mathbb{Q}[-r+1]$, i.e., $P_q \cong \mathbb{Q}_0$.

By [16, Exercise VIII.13] or [8, Proposition 4.2.9], (3.1) is equivalent to the following definition: Let $\widetilde{\mathbb{C}}^\times \to \mathbb{C}^\times$ denote the universal cover of $\mathbb{C}^\times = \mathbb{C} - \{0\}$ and let $\widetilde{V}^\times$ be the fiber product of $V^\times = V - f^{-1}(0) \xrightarrow{f} \mathbb{C}^\times$ with the universal cover $\widetilde{\mathbb{C}}^\times \to \mathbb{C}^\times$ so that we have the diagram:

$$\begin{array}{ccccccc} & & & & \widetilde{V}^\times & \longrightarrow & \widetilde{\mathbb{C}}^\times \\ & & & \swarrow^{\bar\pi} & \downarrow{\scriptstyle \pi} & & \downarrow \\ f^{-1}(0) & \xhookrightarrow{\imath} & V & \xhookleftarrow{\jmath} & V^\times & \longrightarrow & \mathbb{C}^\times \end{array} \tag{3.3}$$

Then $P_f[1-\dim V]$ is isomorphic to the cone of the morphism

$$\mathbb{Q}_{f^{-1}(0)} = \imath^*\mathbb{Q}_V \longrightarrow \imath^*R\jmath_*R\pi_*\pi^*\jmath^*\mathbb{Q}_V = \imath^*R\jmath_*R\pi_*\mathbb{Q}_{\widetilde{V}^\times} = \imath^*R\bar\pi_*\mathbb{Q}_{\widetilde{V}^\times} \tag{3.4}$$

in the triangulated category $D^b_c(\mathbb{Q}_{f^{-1}(0)})$, given by the adjunctions $\mathrm{id} \to R\pi_*\pi^*$ and $\mathrm{id} \to R\jmath_*\jmath^*$.

By [8, Example 4.2.6], the stalk cohomology of P_f at $x \in f^{-1}(0)$ is the reduced cohomology

$$\mathcal{H}^k(P_f)_x = \tilde{H}^{k+\dim V-1}(MF_x, \mathbb{Q}) \tag{3.5}$$

of the Milnor fiber

$$MF_x = f^{-1}(\delta) \cap B_\epsilon(x) \quad \text{for } 0 < \delta << \epsilon << 1. \tag{3.6}$$

Since f is submersive away from $X_f = \mathrm{Crit}(f)$, the Milnor fibers MF_x for $x \in f^{-1}(0) - X_f$ are contractible and P_f is trivial on $f^{-1}(0) - X_f$. Hence we have

$$P_f \in D^b_c(\mathbb{Q}_{X_f}).$$

Moreover, by (3.5), we have

$$\chi(P_f)_x = \sum_m (-1)^m \dim \mathcal{H}^m(P_f)_x = (-1)^{\dim V}(1 - \chi(MF_x)). \tag{3.7}$$

By (2.11), we find that the right side of (3.7) equals the value $\nu_{X_f}(x)$ of the Behrend function for the analytic space $X_f = \mathrm{Crit}(f)$ defined by the partial derivatives of f. Therefore we have

$$\chi(P_f)_x = \nu_{X_f}(x). \tag{3.8}$$

Given any bounded constructible complex of sheaves P on a topological space X and open $U \subset X$, we have a distinguished triangle

$$R\jmath_!\jmath^{-1}P \longrightarrow P \longrightarrow R\imath_*\imath^{-1}P \longrightarrow$$

where $\imath : X - U \to X$ and $\jmath : U \to X$ are inclusion maps. Taking the compact support hypercohomology, we obtain an exact sequence

$$\cdots \longrightarrow \mathbb{H}^k_c(U, P) \longrightarrow \mathbb{H}^k_c(X, P) \longrightarrow \mathbb{H}^k_c(X - U, P) \longrightarrow \mathbb{H}^{k+1}_c(U, P) \longrightarrow \cdots.$$

So we have the additive property of the Euler characteristic:

$$\chi_c(X, P) = \chi_c(U, P) + \chi_c(X - U, P), \tag{3.9}$$

where $\chi_c(-, P) = \sum_m (-1)^m \dim \mathbb{H}^m_c(-, P)$.

By using the usual spectral sequence from sheaf cohomology to hypercohomology

$$E_2^{p,q} = H^p_c(X, \mathcal{H}^q(P)) \Rightarrow \mathbb{H}^{p+q}_c(X, P)$$

together with (3.8) and (3.9), we find that the Euler characteristic of the hypercohomology $\mathbb{H}^*_c(X_f, P_f)$ equals the Euler characteristic

$$\chi_\nu(X_f)$$

weighted by the Behrend function $\nu = \nu_{X_f}$, since P_f is (cohomologically) constructible. By the same argument, we obtain the following solution to Problem 2.41.

Proposition 3.3. *Let X be a critical virtual manifold with charts*

$$(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$$

and let $P_\alpha = \varphi_\alpha^* P_{f_\alpha}$ denote the pullback of the perverse sheaf P_{f_α} of vanishing cycles for $f_\alpha : V_\alpha \to \mathbb{C}$ via $X_\alpha \cong \varphi_\alpha(X_\alpha) = \mathrm{Crit}(f_\alpha) = X_{f_\alpha}$. Suppose there are an object $P \in D^b_c(\mathbb{Q}_X)$ and isomorphisms $P|_{X_\alpha} \cong P_\alpha$ in $D^b_c(\mathbb{Q}_{X_\alpha})$ for all α. Then the Euler characteristic of the compact support hypercohomology $\mathbb{H}^*_c(X, P)$ equals the Euler characteristic $\chi_\nu(X)$ weighted by the Behrend function (2.12).

In other words, our categorification problem (Problem 2.41) will be solved if we can glue $\{P_\alpha\}$ to a globally defined sheaf complex P. In the subsequent sections we will construct canonical gluing isomorphisms of $\{P_\alpha\}$, called of geometric origin, and show that these isomorphisms enable us to glue $\{P_\alpha\}$ to a perverse sheaf P on X when the critical virtual manifold X is orientable. Moreover P is unique up to twisting by a $\mathbb{Z}_2$-local system in $H^1(X, \mathbb{Z}_2)$.

For gluing $\{P_\alpha\}$, we will use the following.

Proposition 3.4 ((cf. [8, Proposition 4.2.11] or [16, Exercise VIII.15]))**.** *Let (V,f) and (W,g) be two LG pairs (Definition 2.4). Let $\Phi : V \to W$ be a homeomorphism satisfying $g \circ \Phi = f$. Then*

$$R\hat{\Phi}_*(P_f) \cong P_g, \tag{3.10}$$

where $\hat{\Phi} : f^{-1}(0) \to g^{-1}(0)$ is the restriction of Φ. Since $R\hat{\Phi}_*$ and $\hat{\Phi}^{-1} = \hat{\Phi}^*$ are adjoints, we also have an isomorphism

$$P_f \cong \hat{\Phi}^* P_g. \tag{3.11}$$

Proof. Using the notation of (3.3), consider the diagram

$$\begin{array}{ccccccc} f^{-1}(0) & \xhookrightarrow{\imath_V} & V & \xleftarrow{\bar{\pi}_V} & \widetilde{V}^\times & \longrightarrow & \widetilde{\mathbb{C}}^\times \\ \downarrow{\scriptstyle\hat{\Phi}} & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\tilde{\Phi}} & & \| \\ g^{-1}(0) & \xhookrightarrow{\imath_W} & W & \xleftarrow{\bar{\pi}_W} & \widetilde{W}^\times & \longrightarrow & \widetilde{\mathbb{C}}^\times \end{array}$$

where the vertical maps except the last are the homeomorphisms induced from Φ. Then we have

$$\begin{aligned} R\hat{\Phi}_*[\imath_V^* R(\bar{\pi}_V)_* \mathbb{Q}_{\widetilde{V}^\times}] &= \imath_W^* R\Phi_* R(\bar{\pi}_V)_* \mathbb{Q}_{\widetilde{V}^\times} \\ &= \imath_W^* R(\bar{\pi}_W)_* (R\tilde{\Phi}_* \mathbb{Q}_{\widetilde{V}^\times}) \\ &= \imath_W^* R(\bar{\pi}_W)_* \mathbb{Q}_{\widetilde{W}^\times}. \end{aligned}$$

Since $R\hat{\Phi}_*$ is an exact functor of triangulated categories and $\hat{\Phi}$ is a homeomorphism, we have a commutative diagram of distinguished triangles

$$\begin{array}{ccccccc} R\hat{\Phi}_*\mathbb{Q}_{f^{-1}(0)} & \longrightarrow & R\hat{\Phi}_*[\imath_V^* R(\bar{\pi}_V)_* \mathbb{Q}_{\widetilde{V}^\times}] & \longrightarrow & R\hat{\Phi}_* P_f[1-\dim V] & \longrightarrow & \\ \| & & \| & & \vdots & & \\ \mathbb{Q}_{g^{-1}(0)} & \longrightarrow & \imath_W^* R(\bar{\pi}_W)_* \mathbb{Q}_{\widetilde{W}^\times} & \longrightarrow & P_g[1-\dim W] & \longrightarrow & \end{array}$$

which gives us the isomorphism (3.10) because $\dim V = \dim W$ by Φ. □

We can give an alternative proof by using (3.1). Indeed, by (3.1), P_f fits into the distinguished triangle

$$P_f[-n] \longrightarrow \mathbb{Q}_V|_{f^{-1}(0)} \longrightarrow Rv_*\mathbb{Q}_{V_{>0}}|_{f^{-1}(0)} \longrightarrow$$

where $v : V_{>0} = \{x \in V \,|\, \mathrm{Re}\, f(x) > 0\} \to V$ is the inclusion and $n = \dim V$. Applying $R\hat{\Phi}_*$, we get a distinguished triangle

$$R\Phi_* P_f[-n] \longrightarrow \mathbb{Q}_W|_{g^{-1}(0)} \longrightarrow Rw_*\mathbb{Q}_{W_{>0}}|_{g^{-1}(0)} \longrightarrow$$

where $w : W_{>0} \to W$ is the inclusion, because Φ sends $V_{>0}$ to $W_{>0}$. By (3.1), P_g fits into the distinguished triangle:

$$P_g[-n] \longrightarrow \mathbb{Q}_W|_{g^{-1}(0)} \longrightarrow Rw_*\mathbb{Q}_{W_{>0}}|_{g^{-1}(0)} \longrightarrow$$

Therefore, $P_g \cong R\hat{\Phi}_* P_f$.

Corollary 3.5. *Let X be a critical virtual manifold equipped with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$ and let $P_\alpha = \varphi_\alpha^* P_{f_\alpha}$. Then we have an isomorphism*

$$\sigma_{\alpha\beta} : P_\alpha|_{X_{\alpha\beta}} \longrightarrow P_\beta|_{X_{\alpha\beta}} \tag{3.12}$$

in $D^b_c(\mathbb{Q}_{X_{\alpha\beta}})$.

Proof. To simplify the notation, we drop the restriction to $X_{\alpha\beta}$ below. We use the notation of Definition 2.5. We have a biholomorphic map $\varphi_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$ which gives an isomorphism $P_{f_\alpha} \xrightarrow{\cong} \varphi_{\alpha\beta}^* P_{f_\beta}$ by (3.11). Applying φ_α^*, we get an isomorphism

$$P_\alpha = \varphi_\alpha^* P_{f_\alpha} \xrightarrow{\cong} \varphi_\alpha^* \varphi_{\alpha\beta}^* P_{f_\beta} = \varphi_\beta^* P_{f_\beta} = P_\beta$$

because $\varphi_\beta = \varphi_{\alpha\beta} \circ \varphi_\alpha$. □

The perverse sheaves of vanishing cycles satisfy the following self-duality.

Proposition 3.6 ([8, Proposition 4.2.10])**.** *Let (V, f) be an LG pair. Then there exists an isomorphism*

$$\mathbf{D} P_f \cong P_f,$$

where $\mathbf{D} P_f = R\mathcal{H}om_{X_f}(P_f, \omega_{X_f})$ is the Verdier dual of P_f with respect to $\omega_{X_f} = p^! \mathbb{Q}_{\mathrm{pt}}$ via the constant map $p : X_f \to \mathrm{pt}$.

3.2. Gluing of perverse sheaves

In this section, we recall the category of perverse sheaves $Perv(\mathbb{Q}_X)$ and their gluing properties. We then state a theorem (Theorem 3.12) which implies that the 2-cocycle obstruction for gluing the local perverse sheaves $P_\alpha = \varphi_\alpha^* P_{f_\alpha} \in Perv(\mathbb{Q}_{X_\alpha})$ from §3.1 coincides with the 2-cocycle obstruction for the orientability of the critical virtual manifold X. We thus obtain a global perverse sheaf $P \in Perv(\mathbb{Q}_X)$ for an *orientable* critical virtual manifold X with charts $X = \cup X_\alpha$ which is the gluing of the local perverse sheaves $\{P_\alpha\}$ (cf. Theorem 3.15).

The full subcategory of perverse sheaves in $D^b_c(\mathbb{Q}_X)$ on an analytic space X is an abelian subcategory of the derived category $D^b_c(\mathbb{Q}_X)$ of bounded constructible complexes of sheaves of $\mathbb{Q}$-vector spaces on X, whose objects are defined as follows.

Definition 3.7. An object $P \in D^b_c(\mathbb{Q}_X)$ is called a *perverse sheaf* (with respect to the middle perversity) if

(1) $\dim\{x \in X \mid H^i(\imath_x^* P) = \mathbb{H}^i(B_\varepsilon(x); P) \neq 0\} \leq -i$ for all i;
(2) $\dim\{x \in X \mid H^i(\imath_x^! P) = \mathbb{H}^i(B_\varepsilon(x), B_\varepsilon(x) - \{x\}; P) \neq 0\} \leq i$ for all i,

where $\imath_x : \{x\} \hookrightarrow X$ is the inclusion and $B_\varepsilon(x)$ is the open ball of radius ε centered at x for ε small enough.

Perverse sheaves form an abelian category $Perv(\mathbb{Q}_X)$ which is the core of a t-structure ([3, §2]). An example of perverse sheaf is the perverse sheaf P_f of vanishing cycles defined in Definition 3.1 (cf. [8, Theorem 5.2.21]).

Although sheaf complexes do not have the gluing property in general, it is known that perverse sheaves and their isomorphisms glue.

Proposition 3.8 ([3, Paragraph 2]). *Let X be a reduced complex analytic space and let $\{X_\alpha\}$ be an open covering of X.*

(1) *Suppose that for each α we have $P_\alpha \in Perv(\mathbb{Q}_{X_\alpha})$ and for each pair α, β we have isomorphisms*

$$\sigma_{\alpha\beta} : P_\alpha|_{X_{\alpha\beta}} \xrightarrow{\cong} P_\beta|_{X_{\alpha\beta}}$$

satisfying $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}^{-1}$, $\sigma_{\alpha\alpha} = \mathrm{id}_{V_\alpha}$ and the cocycle condition $\sigma_{\gamma\alpha} \circ \sigma_{\beta\gamma} \circ \sigma_{\alpha\beta} = 1$. Then $\{P_\alpha\}$ glue to define a perverse sheaf P on X equipped with isomorphisms

$$\sigma_\alpha : P|_{X_\alpha} \xrightarrow{\cong} P_\alpha$$

such that $\sigma_{\alpha\beta}$ equals the composition

$$\sigma_\beta \circ \sigma_\alpha^{-1} : P_\alpha|_{X_{\alpha\beta}} \xleftarrow{\cong} P|_{X_{\alpha\beta}} \xrightarrow{\cong} P_\beta|_{X_{\alpha\beta}}.$$

(2) *Suppose $P, Q \in Perv(\mathbb{Q}_X)$ and $\sigma_\alpha : P|_{X_\alpha} \xrightarrow{\cong} Q|_{X_\alpha}$ are isomorphisms such that $\sigma_\alpha|_{X_{\alpha\beta}} = \sigma_\beta|_{X_{\alpha\beta}}$. Then there exists an isomorphism $\sigma : P \to Q$ such that $\sigma|_{X_\alpha} = \sigma_\alpha$ for all α.*

By the Riemann-Hilbert correspondence established by Kashiwara and Mebkhout, perverse sheaves correspond to regular holonomic D-modules. As sheaves of D-modules glue, so do perverse sheaves.

Our gluing isomorphism $\sigma_{\alpha\beta}$ in Corollary 3.5 is not an arbitrary isomorphism but arose from a biholomorphic map. Recall that an LG pair (V, f) is a holomorphic function f on a complex manifold V that has only one critical value 0. For an LG pair (V, f), X_f denotes the critical locus $\mathrm{Crit}(f)$ defined by the partial derivatives of f.

Definition 3.9 (Geometric origin). Let (V_1, f_1) and (V_2, f_2) be two LG pairs, and let $\zeta : X_{f_1} \to X_{f_2}$ be an isomorphism of analytic spaces. An isomorphism $\sigma : P_{f_1} \xrightarrow{\cong} \zeta^* P_{f_2}$ of perverse sheaves is said to be *of geometric origin* if there exists an open neighborhood $X_{f_1} \subset U_1 \subset V_1$, a holomorphic map $\varphi : U_1 \to V_2$ biholomorphic onto its image such that

$$\varphi|_{X_{f_1}} = \zeta, \quad f_2 \circ \varphi = f_1|_{U_1} \tag{3.13}$$

and σ is the isomorphism defined in Proposition 3.4 induced from φ.

Definition 3.10. Let X be a critical virtual manifold equipped with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$. A *geometric gluing* of the perverse sheaves $P_\alpha = \varphi_\alpha^* P_{f_\alpha}$ of vanishing cycles on X_α is a perverse sheaf P on X together with isomorphisms $\sigma_\alpha : P|_{X_\alpha} \to P_\alpha$ such that

$$\sigma_\beta \circ \sigma_\alpha^{-1} : P_\alpha|_{X_{\alpha\beta}} \longrightarrow P_\beta|_{X_{\alpha\beta}}$$

are of geometric origin.

The following is immediate from Definition 3.10 and Proposition 3.8.

Corollary 3.11. *Let X be a critical virtual manifold equipped with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$. There exists a geometric gluing of $\{P_\alpha\}$ if possibly after a refinement of the covering $X = \cup_\alpha X_\alpha$ of X, there exists a 1-cochain $\mu = \{\mu_{\alpha\beta}\}$ taking values in $\mathbb{Z}_2 = \{\pm 1\}$ such that for $\bar\sigma_{\alpha\beta} = \mu_{\alpha\beta}\sigma_{\alpha\beta}$, and for $X_{\alpha\beta\gamma} \neq \emptyset$, we have*

$$\bar\sigma_{\alpha\beta\gamma} := \bar\sigma_{\gamma\alpha} \circ \bar\sigma_{\beta\gamma} \circ \bar\sigma_{\alpha\beta} = 1$$

where $X_{\alpha\beta\gamma} = X_\alpha \cap X_\beta \cap X_\gamma$.

The gluing condition in this corollary demands, in particular, that the 2-cocycle

$$\sigma_{\alpha\beta\gamma} := \sigma_{\gamma\alpha} \circ \sigma_{\beta\gamma} \circ \sigma_{\alpha\beta} : P_\alpha|_{X_{\alpha\beta\gamma}} \longrightarrow P_\alpha|_{X_{\alpha\beta\gamma}} \tag{3.14}$$

be locally constant with values $\mu_{\alpha\beta\gamma} = \mu_{\gamma\alpha}\mu_{\beta\gamma}\mu_{\alpha\beta} \in \mathbb{Z}_2 = \{\pm 1\}$. This is indeed true by the following.

Theorem 3.12. *Let (V, f) be an LG pair and $X_f = \mathrm{Crit}(f)$ (cf. Definition 2.4). Let U be an open subset such that $X_f \subset U \subset V$ and let $\Phi : U \to V$ be a holomorphic map, biholomorphic onto its image such that $f \circ \Phi = f|_U$ and $\Phi|_{X_f} = \mathrm{id}_{X_f}$. Then the isomorphism σ from Proposition 3.4 is equal to*

$$\det(d\Phi|_{X_f}) \cdot \mathrm{id} : P_f \longrightarrow P_f$$

where $\det(d\Phi|_{X_f})$ is locally constant with values in $\{\pm 1\}$.

Our proof of this theorem is rather lengthy and independent of the rest of this chapter. So we postpone the proof of Theorem 3.12 to Chapter 4. See [5, Corollary 3.2] for a different proof.

The following are immediate consequences of Theorem 3.12.

Corollary 3.13. *Let X be a critical virtual manifold equipped with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$. Let $\{\xi_{\alpha\beta\gamma}\}$ be the $\{\pm 1\}$-valued 2-cocycle in (2.4) for the gluing of the anticanonical line bundles $\{K_\alpha^\vee = \varphi_\alpha^* \det T_{V_\alpha}|_{X_\alpha^{\mathrm{red}}}\}$. Let $\{\sigma_{\alpha\beta\gamma}\}$ be the 2-cocycle in (3.14). Then $\sigma_{\alpha\beta\gamma} = \xi_{\alpha\beta\gamma}$ whenever $X_{\alpha\beta\gamma} \neq \emptyset$.*

Proof. Simply let Φ be the composition $\varphi_{\alpha\beta\gamma}$ in (2.2) and use the definitions of $\xi_{\alpha\beta\gamma}$ and $\sigma_{\alpha\beta\gamma}$, together with Theorem 3.12. □

Corollary 3.14. *Two geometric gluings P and $\tilde P$ of perverse sheaves $\{P_\alpha\}$ can differ only by a $\mathbb{Z}_2$-local system, i.e., there exists a $\mathbb{Z}_2$-local system $\rho \in H^1(X, \mathbb{Z}_2)$ such that*

$$\tilde P \cong P \otimes \rho.$$

Proof. We have isomorphisms $\tilde\sigma_\alpha : \tilde P|_{X_\alpha} \to P_\alpha$ and $\tilde\sigma_{\alpha\beta} = \tilde\sigma_\beta \circ \tilde\sigma_\alpha^{-1}$. Then $\sigma_{\alpha\beta}$ and $\tilde\sigma_{\alpha\beta}$ are two isomorphisms from $P_\alpha|_{X_{\alpha\beta}}$ to $P_\beta|_{X_{\alpha\beta}}$ of geometric origin, arising from biholomorphic $\varphi_{\alpha\beta}$ and $\tilde\varphi_{\alpha\beta}$ up to sign. The composition

$$\tilde\sigma_{\alpha\beta}^{-1} \circ \sigma_{\alpha\beta} : P_\alpha|_{X_{\alpha\beta}} \to P_\alpha|_{X_{\alpha\beta}}$$

is also an isomorphism of geometric origin, i.e., it is the pullback (cf. Proposition 3.4) by the biholomorphic map $\tilde\varphi_{\beta\alpha} \circ \varphi_{\alpha\beta}$ from an open neighborhood of $X_{\alpha\beta}$

in V_α to itself preserving f_α and $X_{\alpha\beta}$ up to sign. By Theorem 3.12, $\rho_{\alpha\beta} = \tilde{\sigma}_{\alpha\beta}^{-1} \circ \sigma_{\alpha\beta} = \pm 1$ which coincides with the determinant of the tangent map $d(\tilde{\varphi}_{\beta\alpha} \circ \varphi_{\alpha\beta})|_{X_{\alpha\beta}}$. Since both $\{\sigma_{\alpha\beta}\}$ and $\{\tilde{\sigma}_{\alpha\beta}\}$ are cocycles, $\{\rho_{\alpha\beta} \in \mathbb{Z}_2\}$ is also a cocycle and defines a $\mathbb{Z}_2$-local system $\rho \in H^1(X, \mathbb{Z}_2)$. □

If X is an orientable critical virtual manifold, then possibly after a refinement of the covering $\{X_\alpha\}$ of X, we can find a 1-cochain $\{\mu_{\alpha\beta}\}$ with values in $\{\pm 1\}$ such that $\bar{\sigma}_{\alpha\beta\gamma} = \bar{\xi}_{\alpha\beta\gamma} = 1$ by using the notation of Definition 2.17 and Corollary 3.11. Therefore we obtain a geometric gluing of $\{P_\alpha\}$. We summarize the above discussion as follows.

Theorem 3.15. *Let X be an* orientable *critical virtual manifold with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$. Then there exists a geometric gluing P of the local perverse sheaves $P_\alpha = \varphi_\alpha^* P_{f_\alpha}$ which is unique up to twisting by a $\mathbb{Z}_2$-local system.*

We thus obtain a solution to the categorification problem (Problem 2.41).

Corollary 3.16. *Let X be an orientable critical virtual manifold and P be the perverse sheaf in Theorem 3.15. Then the Euler characteristic $\chi_c(X, P)$ of the hypercohomology $\mathbb{H}_c^*(X, P)$ of P is equal to the Euler characteristic $\chi_\nu(X)$ of X weighted by the Behrend function. When X is compact, this in turn equals the Donaldson-Thomas type invariant $DT(X) = \deg[X]^{\mathrm{vir}}$.*

Remark 3.17. The geometric gluing condition is highly nontrivial. For instance, when $f_\alpha = 0$ for all α so that $P_\alpha = \mathbb{Q}|_{X_\alpha}$, the gluing isomorphisms $\mathbb{Q}|_{X_{\alpha\beta}} \to \mathbb{Q}|_{X_{\alpha\beta}}$ can only be either 1 or -1. In particular, a geometric gluing P can only be the trivial bundle $\mathbb{Q}_X$ twisted by a $\mathbb{Z}_2$ local system when X is smooth.

In [17], we proved that a moduli space X of simple sheaves on a Calabi-Yau 3-fold Y admits a structure of critical virtual manifold which is orientable when there is a tautological family. Hence the Donaldson-Thomas invariant of Y along X can be categorified by the hypercohomology $\mathbb{H}_c^*(X, P)$ of the perverse sheaves in Theorem 3.15. As an application, this cohomological Donaldson-Thomas invariant will provide us with a mathematical theory of the Gopakumar-Vafa invariant [9].

3.3. Gluing of mixed Hodge modules

We proved that there is a perverse sheaf P on an orientable critical virtual manifold X whose hypercohomology $\mathbb{H}_c^*(X, P)$ has Euler characteristic equal to the Euler characteristic $\chi_\nu(X)$ of X weighted by the Behrend function, which in turn coincides with the Donaldson-Thomas type invariant $DT(X)$ when X is compact. In this section, we show that there is a Hodge theory on $\mathbb{H}_c^*(X, P)$ so that the decomposition theorem and the hard Lefschetz theorem hold. A Hodge theory for a perverse sheaf means a mixed Hodge module defined and studied by Morihiko Saito [22, 23]. (See [25] for a quick survey.) We prove that there is a mixed Hodge module $\mathcal{M}$ on X whose underlying perverse sheaf is the perverse sheaf P constructed in the previous section (cf. Theorem 3.20).

3.3.1. *Hodge modules.* Let X be an analytic space embedded into a complex manifold $\mathbb{P}$. The category of mixed Hodge modules is independent of $\mathbb{P}$ thanks to Kashiwara's equivalence [12, Theorem 1.6.1]. Considering vector fields on $\mathbb{P}$ as $\mathbb{C}_{\mathbb{P}}$-derivations of holomorphic functions in $\mathcal{O}_{\mathbb{P}}$, the sheaf $D_{\mathbb{P}}$ of differential operators is defined as the subsheaf of $End_{\mathbb{C}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}})$ generated by the sheaf $\mathcal{O}_{\mathbb{P}}$ and the tangent bundle $T_{\mathbb{P}}$. There is a natural filtration of $D_{\mathbb{P}}$ defined inductively by $F_0D_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}$ and

$$F_lD_{\mathbb{P}} = \{P \in End_{\mathbb{C}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}) \,|\, [P, f] \in F_{l-1}D_{\mathbb{P}},\ \forall f \in \mathcal{O}_{\mathbb{P}}\}.$$

It is straightforward to see that

$$\mathrm{gr}D_{\mathbb{P}} = \oplus_l F_lD_{\mathbb{P}}/F_{l-1}D_{\mathbb{P}} \cong \mathrm{Sym}T_{\mathbb{P}}.$$

A *Hodge module* $\mathcal{M}$ on $X \subset \mathbb{P}$ consists of

(1) a regular holonomic $D_{\mathbb{P}}$-module M whose support lies in X;
(2) a perverse sheaf P on X;
(3) an isomorphism $DR(M) \cong \mathbb{C} \otimes_{\mathbb{Q}} P$;
(4) a good filtration of M by $\mathcal{O}_{\mathbb{P}}$-coherent subsheaves $\{M_i\}$ such that $M_i \cdot D_j \subset M_{i+j}$ and $\mathrm{gr}M = \oplus M_i/M_{i-1}$ is coherent over $\mathrm{gr}D_{\mathbb{P}} \cong \mathrm{Sym}T_{\mathbb{P}}$

which satisfy suitable local conditions. See [12, 22, 25] for precise statements of these local conditions. For our purpose of gluing (mixed) Hodge modules, these local conditions are always satisfied from the start. We refer to [12] for the definitions of regular holonomic $D_{\mathbb{P}}$-modules, de Rham functor DR etc. The isomorphism (3) is said to give a rational structure on M. The category $HM(X)$ of Hodge modules is a full subcategory of the category $HW(X)$ of tuples $(M, P, M_\bullet)$ as above without local conditions.

When V is a complex manifold of dimension n and $\omega_V = \wedge^n T_V^*$, the constant variation of Hodge structures gives a Hodge module which consists of $P = \mathbb{Q}_V[n]$, $M = \omega_V$, and $M_i = M$ for $i \geq -n$ and 0 for $i < -n$. We denote this Hodge module by $\mathbb{Q}_V^H[n]$ and call it the constant Hodge module.

A polarization of a Hodge module $\mathcal{M}$ of weight w refers to an isomorphism $\mathcal{M}(w) \cong \mathbf{D}\mathcal{M}$ in $HM(X)$ where $\mathbf{D}$ denotes the Verdier dual. A Hodge module $\mathcal{M} \in HM(X)$ is called *polarizable* if it admits a polarization. We let $HM(X)^p \subset HM(X)$ denote the full subcategory of polarizable Hodge modules. For example, the constant Hodge module $\mathbb{Q}_V^H[n]$ is a polarizable Hodge module because $R\mathcal{H}om(\mathbb{Q}[n], \mathbb{Q}[2n]) \cong \mathbb{Q}[n]$ and this extends to the canonical isomorphism $\mathbb{Q}_V^H[n] \cong \mathbf{D}\mathbb{Q}_V^H[n]$ of Hodge modules. Polarizable Hodge modules satisfy the following useful properties.

Theorem 3.18 ([22, Theorem 5.3.1])**.** *Let $f : X \to Y$ be a projective morphism of analytic spaces (admitting an embedding into complex manifolds). Let ℓ be the first Chern class of a relatively ample line bundle on X. Then for a polarizable Hodge module $\mathcal{M}$ on X,*

(1) ${}^pR^if_*\mathcal{M} \in HM(Y)^p$ *for all* i;
(2) $\ell^i : {}^pR^{-i}f_*\mathcal{M} \longrightarrow {}^pR^if_*\mathcal{M}$ *is an isomorphism of Hodge modules.*

The hard Lefschetz property (2) above gives us an isomorphism

$$\ell^i : {}^pR^{-i}f_*P \to {}^pR^if_*P,$$

where P is the underlying perverse sheaf of $\mathcal{M} = (M, P, M_\bullet)$. Together with Deligne's argument from [7] (cf. [11, p. 466]) for degeneration of spectral sequences, this isomorphism gives us the decomposition theorem.

Corollary 3.19. *Under the hypothesis of Theorem 3.18, we have a (non-canonical) isomorphism*

$$Rf_*P \cong \oplus_i {}^pR^if_*P[-i]$$

and each ${}^pR^if_*P[-i]$ *is a perverse sheaf underlying a polarizable Hodge module.*

3.3.2. *Mixed Hodge modules.* As in §3.3.1, X is an analytic space embeddable into a complex manifold $\mathbb{P}$.

A *polarizable mixed Hodge module* on X consists of

(1) a regular holonomic $D_{\mathbb{P}}$-module M whose support lies in X;
(2) a perverse sheaf P on X;
(3) an isomorphism $DR(M) \cong \mathbb{C} \otimes_{\mathbb{Q}} P$;
(4) a good filtration of M by $\mathcal{O}_{\mathbb{P}}$-coherent subsheaves $M_\bullet = \{M_i\}$ such that $M_i \cdot D_j \subset M_{i+j}$ and $\mathrm{gr}M = \oplus M_i/M_{i-1}$ is coherent over $\mathrm{gr}D_{\mathbb{P}} \cong \mathrm{Sym}T_{\mathbb{P}}$;
(5) a finite increasing filtration $W_\bullet$ of $\mathcal{M} = (M, P, M_\bullet)$ with $\mathrm{gr}_i^W\mathcal{M} \in HM(X)^p$ for all i

which satisfy suitable local conditions (cf. [23]). As our purpose is gluing mixed Hodge modules, these local conditions are always satisfied from the start (before gluing). The category $MHM(X)^p$ of polarizable mixed Hodge modules is the full subcategory of the category $MHW(X)$ of tuples $(M, P, M_\bullet, W_\bullet)$ without local conditions.

By the definition, we have a forgetful functor

$$\mathrm{rat} : MHM(X)^p \longrightarrow Perv(\mathbb{Q}_X), \quad (M, P, M_\bullet, W_\bullet) \mapsto P \tag{3.15}$$

which is an exact and faithful functor via the Riemann-Hilbert correspondence.

The constant Hodge module $\mathbb{Q}_V^H[n]$ on a complex manifold V of dimension n is a polarizable mixed Hodge module (cf. [23, Theorem 3.8, (4.5.5)]). If $f : V \to \mathbb{C}$ is a holomorphic function on a complex manifold V, then there is a polarizable mixed Hodge module $\mathcal{M}_f := \phi_f\mathbb{Q}_V^H[n]$ supported on $f^{-1}(0)$ such that

$$\mathrm{rat}(\mathcal{M}_f) = P_f$$

is the perverse sheaf of vanishing cycles defined in Definition 3.1 where ϕ_f denotes the vanishing cycle functor. This is actually part of the local conditions required for mixed Hodge modules.

By [23, Theorem 0.1], when there is a morphism Φ of analytic spaces, we have natural functors Φ_*, $\Phi_!$, Φ^*, $\Phi^!$, ψ_g, $\phi_{g,1}$, $\mathbf{D}$, $\boxtimes$, $\otimes$, and $\mathcal{H}om$ between the derived categories of mixed Hodge modules. By [23, Theorem 2.14], when

$\Phi : V \to W$ is a biholomorphic map of complex manifolds and $g : W \to \mathbb{C}$ is a holomorphic function with $f = g \circ \Phi$, we have a *canonical* isomorphism

$$\sigma^H : \mathcal{M}_f = \phi_f \mathbb{Q}_V^H[n] \xrightarrow{\cong} \hat{\Phi}^* \phi_g \mathbb{Q}_W^H[n] = \hat{\Phi}^* \mathcal{M}_g \tag{3.16}$$

which induces (3.11) when rat is applied. Here $\hat{\Phi} : f^{-1}(0) \to g^{-1}(0)$ is the isomorphism of analytic spaces induced from Φ. By [23, Proposition 2.6], if $f : V \to \mathbb{C}$ is a holomorphic function, we have a *canonical* isomorphism

$$\phi_f \mathbf{D}\mathbb{Q}_V^H[n] \cong \mathbf{D}\phi_f \mathbb{Q}_V^H[n] = \mathbf{D}\mathcal{M}_f, \tag{3.17}$$

where $n = \dim V$. The canonical isomorphism $\mathbb{Q}_V^H[n] \cong \mathbf{D}\mathbb{Q}_V^H[n]$ sending 1 to 1 gives us a *canonical* isomorphism

$$\mathcal{M}_f = \phi_f \mathbb{Q}_V^H[n] \cong \phi_f \mathbf{D}\mathbb{Q}_V^H[n]. \tag{3.18}$$

Composing (3.18) with (3.17), we obtain a *canonical* isomorphism

$$\mathcal{M}_f \longrightarrow \mathbf{D}\mathcal{M}_f, \tag{3.19}$$

i.e., $\mathcal{M}_f$ has a *canonical* polarization. Since (3.19) and (3.16) are canonical, the isomorphism (3.16) is an isomorphism of polarizable mixed Hodge modules.

By [23, §2] and [24, §1.6], if $X = \cup_\alpha X_\alpha$ is an open cover, the category $MHM(X)^p$ of polarizable mixed Hodge modules is equivalent to the category of the collections $\mathcal{M}_\alpha \in MHM(X_\alpha)^p$ together with isomorphisms

$$\sigma^H_{\alpha\beta} : \mathcal{M}_\alpha|_{X_{\alpha\beta}} \to \mathcal{M}_\beta|_{X_{\alpha\beta}} \tag{3.20}$$

in $MHM(X_{\alpha\beta})^p$, satisfying

$$\sigma^H_{\alpha\beta\gamma} = \sigma^H_{\gamma\alpha} \circ \sigma^H_{\beta\gamma} \circ \sigma^H_{\alpha\beta} = 1$$

whenever $X_{\alpha\beta\gamma} \neq \emptyset$. Likewise, isomorphisms can be glued. In other words, Proposition 3.8 holds for polarizable mixed Hodge modules.

Combining all in this subsection, we obtain the following theorem. We use the notation of Definition 2.5.

Theorem 3.20. *Let X be an orientable critical virtual manifold with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$. Let $\mathcal{M}_\alpha = \varphi_\alpha^* \mathcal{M}_{f_\alpha} \in MHM(X_\alpha)^p$ such that* $\mathrm{rat}(\mathcal{M}_\alpha) = P_\alpha$*. Let $\sigma^H_{\alpha\beta} : \mathcal{M}_\alpha|_{X_{\alpha\beta}} \to \mathcal{M}_\beta|_{X_{\alpha\beta}}$ be the isomorphism induced from* (3.16) *by the biholomorphic map $\varphi_{\alpha\beta}$ as in Corollary 3.5. There is a polarizable mixed Hodge module $\mathcal{M} \in MHM(X)^p$ together with isomorphisms $\sigma^H_\alpha : \mathcal{M}|_{X_\alpha} \xrightarrow{\cong} \mathcal{M}_\alpha$ such that $\sigma^H_{\alpha\beta}$ equals the composition*

$$\sigma^H_\beta \circ {\sigma^H_\alpha}^{-1} : \mathcal{M}_\alpha|_{X_{\alpha\beta}} \xleftarrow{\cong} \mathcal{M}|_{X_{\alpha\beta}} \xrightarrow{\cong} \mathcal{M}_\beta|_{X_{\alpha\beta}}.$$

Moreover, $\mathrm{rat}(M)$ *is the perverse sheaf P in Theorem 3.15.*

Proof. Since (3.16) is an isomorphism of polarizable mixed Hodge modules, it gives an isomorphism $\sigma_{\alpha\beta}$ of polarizable mixed Hodge modules for each pair (α, β), as in the proof of Corollary 3.5. Since the functor rat is faithful and $\mathrm{rat}(\sigma^H_{\alpha\beta\gamma}) = \sigma_{\alpha\beta\gamma}$ where $\sigma_{\alpha\beta\gamma}$ is from (3.14), the gluing condition $\sigma_{\alpha\beta\gamma} = 1$ for the perverse sheaves $\{P_\alpha\}$ implies the gluing condition $\sigma^H_{\alpha\beta\gamma} = 1$ for the mixed Hodge modules $\{\mathcal{M}_\alpha\}$. As the perverse sheaves $\{P_\alpha\}$ glue, so do the polarizable mixed Hodge modules $\{\mathcal{M}_\alpha\}$. □

Let P be the perverse sheaf which underlies a polarizable mixed Hodge module. Let

$$\hat{P} = \mathrm{gr}^W P$$

denote the gradation with respect to the weight filtration. Then $\hat{P}$ is a direct sum of polarizable Hodge modules. If $\psi : X \to Y$ is a projective morphism of analytic spaces, then the hard Lefschetz theorem (Theorem 3.18) and the decomposition theorem (Corollary 3.19) hold for $\hat{P}$. On the other hand, by the long exact sequence for hypercohomology groups and Corollary 2.39, we have

$$\chi_c(X, \hat{P}) = \chi_c(X, P) = \chi_\nu(X) = DT(X).$$

We therefore have another solution to Problem 2.41.

Corollary 3.21. *Let X be an orientable critical virtual manifold. Then there is a perverse sheaf $\hat{P}$ underlying a direct sum of polarizable Hodge modules such that the Euler characteristic $\chi_c(X, \hat{P})$ of the hypercohomology $\mathbb{H}^*_c(X, \hat{P})$ equals the Euler characteristic $\chi_\nu(X)$ of X weighted by the Behrend function.*

We will use $\hat{P}$ for a mathematical theory of the Gopakumar-Vafa invariant.

4. Rigidity of perverse sheaves

In this chapter, we provide a proof of Theorem 3.12. The main points are the following:

1. Perverse sheaves and their isomorphisms are rigid under continuous deformations.
2. We can use vector fields to produce isotopies of equivalences of LG pairs which generate continuous deformations of isomorphisms of perverse sheaves.
3. We can find isotopies from a self-equivalence Φ of an LG pair to linear transformations.

Our proof reveals the dependence of the perverse sheaf of vanishing cycles on the obstruction theory of the critical locus.

4.1. Sebastiani-Thom isomorphism and rigidity

We begin with a few properties of perverse sheaves that we will use.

Proposition 4.1 ([20, §2], Sebastiani-Thom isomorphism)**.** *Let $g : V \to \mathbb{C}$ and $h : W \to \mathbb{C}$ be holomorphic functions on complex manifolds. Let $f = g + h : V \times W \to \mathbb{C}$ be defined by $f(z, y) = g(z) + h(y)$. Then we have an isomorphism of perverse sheaves*

$$P_f \cong pr_1^{-1} P_g \otimes pr_2^{-1} P_h,$$

where $pr_1 : V \times W \to V$ and $pr_2 : V \times W \to W$ denote the projections.

Corollary 4.2. *When h is the quadratic function $\sum_{i=1}^r y_i^2$ on $\mathbb{C}^r$ in Example 3.2, $P_h = \mathbb{Q}_0$ and hence $P_f \cong P_g$ as perverse sheaves defined on the critical locus $X_g = X_f$.*

We recall the following "elementary construction" of perverse sheaves by MacPherson and Vilonen.

Theorem 4.3 ([19, Theorem 4.5])**.** *Let X be an analytic space. Let $S \subset X$ be a closed stratum of complex codimension c. The category $Perv(\mathbb{Q}_X)$ is equivalent to the category of objects $(B, C) \in Perv(\mathbb{Q}_{X-S}) \times Sh_{\mathbb{Q}}(S)$ together with a commutative triangle*

$$\begin{array}{ccc} R^{-c-1}\pi_*\kappa_*\kappa^* B & \longrightarrow & R^{-c}\pi_*\gamma_!\gamma^* B \\ & {\scriptstyle m}\searrow \quad \nearrow{\scriptstyle n} & \\ & C & \end{array} \tag{4.1}$$

such that $\ker(n)$ *and* $\operatorname{coker}(m)$ *are local systems on S, where $\kappa : K \hookrightarrow L$ and $\gamma : L - K \hookrightarrow L$ are inclusions of the perverse link bundle K and its complement $L - K$ in the link bundle $\pi : L \to S$. The equivalence of categories is explicitly given by sending $P \in Perv(\mathbb{Q}_X)$ to $B = P|_{X-S}$ together with the natural morphisms*

$$\begin{array}{ccc} R^{-c-1}\pi_*\kappa_*\kappa^* B & \longrightarrow & R^{-c}\pi_*\gamma_!\gamma^* B \\ & {\scriptstyle m}\searrow \quad \nearrow{\scriptstyle n} & \\ & R^{-c}\pi_*\varphi_!\varphi^* P & \end{array}$$

where $\varphi : D - K \hookrightarrow D$ is the inclusion into the normal slice bundle.

See [19, §4] for precise definitions of links K, L and D. Morally the above theorem says that an extension of a perverse sheaf on $X - S$ to X is obtained by adding a sheaf on S.

An application of Theorem 4.3 is the following *rigidity property of perverse sheaves.*

Lemma 4.4. *Let $P \in Perv(\mathbb{Q}_U)$ be a perverse sheaf on an analytic space U. Let $\pi : T \to U$ be a continuous map from a topological space T with connected fibers and let T' be a subspace of T such that $\pi|_{T'}$ is surjective. Suppose that an isomorphism $\mu : \pi^{-1}P \xrightarrow{\cong} \pi^{-1}P$ satisfies $\mu|_{T'} = \mathrm{id}_{(\pi^{-1}P)|_{T'}}$. Then $\mu = \mathrm{id}_{\pi^{-1}P}$.*

Proof. We first prove the simple case. If we let C be a locally constant sheaf over $\mathbb{Q}$ of finite rank supported on a subset $Z \subset U$ and $\bar{\mu} : \pi^{-1}C \to \pi^{-1}C$ be a homomorphism such that $\bar{\mu}|_{T' \cap \pi^{-1}(Z)} = \mathrm{id}$, then $\bar{\mu}$ is the identity morphism. Indeed, since the issue is local, we may assume that Z is connected and that $C \cong \mathbb{Q}^r$ so that $\bar{\mu} : \mathbb{Q}^r \to \mathbb{Q}^r$ is given by a continuous map $\pi^{-1}(Z) \to GL(r, \mathbb{Q})$. By fiber connectedness, this obviously is a constant map which is 1 along $T' \cap \pi^{-1}(Z)$. We thus proved the lemma in the sheaf case.

For the general case, we use Theorem 4.3. By replacing U by the support of P if necessary, we may assume that the support of P is U. We stratify U and let $U^{(i)}$ be the union of strata of codimension $\le i$. Since P is a perverse sheaf, $P|_{U^{(0)}}[-\dim U]$ is isomorphic to a locally constant sheaf and hence $\mu|_{U^{(0)}}$ is the identity map. For $U^{(1)} = U^{(0)} \cup S$, using the notation of Theorem 4.3, $C = R^{-1}\pi_*\varphi_!\varphi^* P$ is a locally constant sheaf and μ induces a homomorphism $\pi^{-1}C \to \pi^{-1}C$ which is identity on $T' \cap \pi^{-1}(S)$. Therefore μ induces the identity morphism of the pullback of (4.1) by π to itself and hence μ is the identity on $U^{(1)}$. Continuing in this fashion, we obtain Lemma 4.4. □

Typically we will consider the case where $\pi : T = U\times I \to U$ is the projection and I is the interval $[0, 1]$ or a disc in $\mathbb{C}$. If there is a continuous family μ of isomorphisms $P \to P$ parameterized by I which is id over $U \times \{0\}$, all the isomorphisms are the identity. More precisely, let $\Phi : V \times I \to V$ be a continuous family of homeomorphisms of a complex manifold V, i.e., Φ is continuous and $\Phi_t := \Phi|_{V\times\{t\}}$ are homeomorphisms for all $t \in I$. Let $f : V \to \mathbb{C}$ be a holomorphic function, satisfying $f(\Phi(x, t)) = f(x)$, i.e., $f \circ \Phi_t = f$ for all t. As always, we let $X_f = \mathrm{Crit}(f)$ denote the critical locus of f. Suppose $\Phi_t|_{X_f}$ is the identity map for all t. Then the homeomorphism Φ_t gives us the isomorphism

$$\sigma_t : P_f \longrightarrow P_f \tag{4.2}$$

by Proposition 3.4.

Let $P_f = R\Gamma_{\{x\in V\,|\,\mathrm{Re} f(x)\le 0\}}\mathbb{Q}_V[\dim V]|_{f^{-1}(0)}$ be the perverse sheaf of vanishing cycles from Definition 3.1. Since $\pi^{-1} = \pi^*$ is an exact functor (on $\mathbb{Q}$-sheaves), we have $\pi^{-1}P_f = R\Gamma_{\{x\in V\,|\,\mathrm{Re} f(x)\le 0\}\times I}\mathbb{Q}_{V\times I}[\dim V]|_{f^{-1}(0)\times I}$. By the proof of Proposition 3.4, we then have an isomorphism

$$\mu : \pi^{-1}P_f \longrightarrow \pi^{-1}P_f \tag{4.3}$$

whose restriction to $X_f \times \{t\}$ is the isomorphism $\sigma_t : P_f \to P_f$ in (4.2). In this situation, Lemma 4.4 gives the following.

Proposition 4.5. *Let (V, f) be an LG pair (Definition 2.4). Let Φ_t, P_f, σ_t be as above. Suppose Φ_0 is the identity map of V and $\Phi_t|_{X_f} : X_f \to X_f$ is the identity map for all $t \in I$. Then $\sigma_t = \mathrm{id}_{P_f}$ for all $t \in I$.*

Proof. The proposition is immediate from Lemma 4.4 by letting $U = X_f$, $T = U \times I \xrightarrow{\pi} U$ and μ given by (4.3), because $\mu|_{U\times\{0\}} = \mathrm{id}$ by $\Phi_0 = \mathrm{id}_V$. □

4.2. Vector fields and isotopies

As the issue of Theorem 3.12 is local, we may restrict our concern to an open submanifold $V \subset \mathbb{C}^n$. In this section, we use vector fields to generate isotopies. Together with the rigidity (Proposition 4.5), this will give us a 2-cocycle property of perverse sheaves of vanishing cycles (Lemma 4.7).

Lemma 4.6. *Let $V \subset \mathbb{C}^n$ be an open submanifold and $f : V \to \mathbb{C}$ be an LG pair (Definition 2.4). Let (df) be the ideal generated by the partial derivatives of f and let $X_f = \mathrm{Crit}(f)$ denote the analytic subspace defined by the ideal (df). Then there is an open neighborhood V' of X_f in V such that*

$$f|_{V'} \in (df)|_{V'} \quad \text{and} \quad \frac{f}{\|df\|} \to 0 \textit{ as } df \to 0.$$

Proof. Let $\pi : \tilde{V} \to V$ be a resolution of X_f so that $\pi^{-1}(X_f)$ is a normal crossing divisor. Near every $\tilde{x} \in \pi^{-1}(X_f)$, the pullback $\pi^*(df)$ of the ideal (df) is a principal ideal sheaf generated by some monomial $\varphi = z_1^{k_1} \cdots z_r^{k_r}$ with $k_i > 0$ for a system $z_1, \ldots, z_n$ of local coordinates of $\tilde{V}$ centered at $\tilde{x}$. Let $x = \pi(\tilde{x})$, and let $w_1, \ldots, w_n$ be the coordinate functions of $\mathbb{C}^n$. Then φ divides $\pi^* \frac{\partial f}{\partial w_i}$ for all i.

We claim that $\pi^* f$ is divisible by $z_1^{k_1+1} \cdots z_r^{k_r+1}$. We first write $\pi^* f = c_{m_1} z_1^{m_1} + c_{m_1+1} z_1^{m_1+1} + \cdots$, where c_k are holomorphic functions of $\{z_2, \ldots, z_n\}$. Because near $\tilde{x}$ and away from $\pi^{-1}(X_f)$, π is biholomorphic,

$$\frac{\partial(\pi^* f)}{\partial z_1} = \sum_i \pi^* \left(\frac{\partial f}{\partial w_i} \right) \cdot \frac{\partial(\pi^* w_i)}{\partial z_1} \tag{4.4}$$

holds away from $\pi^{-1}(X_f)$. As all terms in this identity are holomorphic, it holds in a neighborhood of $\tilde{x}$. Because the right hand side is divisible by φ, so is the left hand side. Since

$$\frac{\partial(\pi^* f)}{\partial z_1} = m_1 c_{m_1} z_1^{m_1-1} + (m_1+1) c_{m_1+1} z_1^{m_1} + \cdots ,$$

we must have $m_1 - 1 \geq k_1$. Therefore $z_1^{k_1+1}$ divides $\pi^* f$. Likewise $z_i^{k_i+1}$ divides $\pi^* f$ for each i. Therefore by (4.4),

$$\pi^*(f) \subset \pi^*(df) \sqrt{\pi^*(df)} \subset \pi^*(df). \tag{4.5}$$

Since $\pi^* : \mathcal{O}_V \to \pi_* \mathcal{O}_{\tilde{V}}$ is injective, we have $f \in (df)$.

Finally, using (4.5), near $\tilde{x}$ we write $\pi^* f = \sum b_i \pi^* \big(\frac{\partial f}{\partial w_i} \big)$ for some holomorphic functions b_i such that b_i vanishes along $\pi^{-1}(X_f)$. Since $\pi : \tilde{V} \to V$ is proper, we have $\frac{f}{\|df\|} \to 0$ as $df \to 0$. This proves the lemma. ☐

Lemma 4.7. *Let $V \subset \mathbb{C}^n$ be open. Let (V, f_0) and (V, f_1) be two LG pairs. Let $f_t = (1-t) f_0 + t f_1$ for $0 \leq t \leq 1$. Suppose the critical locus $X_{f_t} = \mathrm{Crit}(f_t)$ is independent of t as an analytic subspace of V. Then there is an isomorphism*

$$\tau_{01} : P_{f_0} \xrightarrow{\cong} P_{f_1}.$$

If we have a third LG pair (V, f_2) such that the critical locus of $(1-t-s) f_0 + t f_1 + s f_2$ is independent of $t, s \in [0,1]$. Then

$$\tau_{12} \circ \tau_{01} = \tau_{02} : P_{f_0} \longrightarrow P_{f_2}.$$

Proof. We use the standard hermitian inner product on $\mathbb{C}^n$. Let ∇f_t be the gradient vector field of f_t defined by $df_t(v) = \nabla f_t \cdot v$ for tangent vectors v. By Lemma 4.6, we have $\frac{f}{\|df\|} \to 0$ as $df \to 0$ in a neighborhood of X_f. We define a time dependent vector field

$$\xi_t = \frac{f_0 - f_1}{\|\nabla f_t\|^2}\, \overline{\nabla f_t}.$$

We claim that this is a well defined vector field on V. It suffices to show that

$$\|\xi_t\| = \frac{|f_0 - f_1|}{\|\nabla f_t\|} = \frac{|f_0 - f_1|}{\|df_t\|}$$

approaches zero as the point approaches $X_{f_t} = X_{f_0} = X_{f_1}$. Since $(df_t) \supset (df_0) = (df_1)$ by assumption, we can express the partial derivatives of f_0 and f_1 as linear combinations of the partial derivatives of f_t. Hence,

$$\|df_0\| \le C\|df_t\| \quad \text{and} \quad \|df_1\| \le C\|df_t\|$$

for some $C > 0$. Thus by Lemma 4.6,

$$\|\xi_t\| = \frac{|f_0 - f_1|}{\|df_t\|} \le C^{-1}\left(\frac{|f_0|}{\|df_0\|} + \frac{|f_1|}{\|df_1\|}\right) \to 0 \text{ as } df_0, df_1 \to 0.$$

So we proved the claim.

Let x_t for $t \in [0,1]$ be an integral curve of the vector field ξ_t, so that

$$\frac{dx_t}{dt} = \dot{x}_t = \xi_t(x_t).$$

Then $f_t(x_t)$ is constant because $\frac{d}{dt} f_t(x_t)$ is equal to

$$df_t(\dot{x}_t) + f_1 - f_0 = \nabla f_t \cdot \dot{x}_t + f_1 - f_0 = \nabla f_t \cdot \frac{f_0 - f_1}{\|\nabla f_t\|^2}\, \overline{\nabla f_t} + f_1 - f_0 = 0.$$

Therefore the flow of the vector field ξ_t from $t = 0$ to $t = 1$ gives a homeomorphism $\Phi_{01} : U \to U'$ of neighborhoods of X_{f_0} such that $f_1(\Phi_{01}(x)) = f_0(x)$ for $x \in U$. If $x \in X_{f_0} = X_{f_1}$, $f_0(x) = f_1(x) = 0$ and hence $\xi_t(x) = 0$ for all t. So $\Phi_{01}|_{X_{f_0}}$ is the identity map of X_{f_0}. By Proposition 3.4, we obtain an isomorphism $\tau_{01} : P_{f_0} \cong P_{f_1}$.

Suppose that we have three holomorphic functions f_0, f_1, f_2 on V as stated in Lemma 4.7. The composition $\tau_{12} \circ \tau_{01}$ is obtained from the composition of the diffeomorphism $\Phi_{01} : U \to U'$ with $f_1(\Phi_{01}(x)) = f_0(x)$ and the diffeomorphism $\Phi_{12} : U' \to U''$ with $f_2(\Phi_{12}(x)) = f_1(x)$. By replacing f_1 by $(1-s)f_1 + sf_2$ for $0 \le s \le 1$, we obtain an isotopy from $\Phi_{12} \circ \Phi_{01}$ to Φ_{02}, so that $\Phi_{02}^{-1} \circ \Phi_{12} \circ \Phi_{01}$ is isotopic to the identity map. By Proposition 4.5, we have $\tau_{02}^{-1} \circ \tau_{12} \circ \tau_{01} = \text{id}$. □

The following is a consequence of Lemmas 4.6 and 4.7.

Lemma 4.8. *Let $V \subset \mathbb{C}^n$ be open and let (V, g) be an LG pair (Definition 2.4). Let u be a nowhere vanishing holomorphic function on V and let $f = ug$. Then there is an open neighborhood $X_g \subset U \subset V$ so that $X_f \cap U = X_g$. Suppose further $u|_{X_f \cap U} = 1$. Then $g = uf$ induces a canonical isomorphism $P_f|_U \cong P_g$.*

Proof. By Lemma 4.6, $g \in (dg)$ on an open neighborhood of X_g. Since $f = ug$ and $df = u\,dg + g\,du$, the ideal (df) is contained in (dg), thus $X_g \subset X_f$. Let $\mathfrak{A} = X_f \cap (f = 0)$. By Lemma 4.6, $\mathfrak{A} \subset X_f$ is both open and closed. We let $U \subset V$ be an open neighborhood of $\mathfrak{A} \subset V$ so that $U \cap X_f = \mathfrak{A}$. Then using $g = u^{-1}f$, the same argument shows that (dg) is contained in $(df|_U)$. Hence $X_f \cap U = X_g$.

Suppose $u|_{X_f} = 1$. We let $u_t = (1-t) + tu : V \to \mathbb{C}$. Then u_t are invertible in a neighborhood of X_f for any t. Let $f_t = u_t g$; then $f_0 = g$ and $f_1 = f$. By the same argument, and using that $[0,1]$ is compact, we can find an open neighborhood $X_g \subset U' \subset V$ so that $X_{f_t} \cap U' = X_{f_0}$ for all $t \in [0,1]$. Applying Lemma 4.7, we obtain an induced isomorphism $P_f|_U = P_{f_1}|_{U'} \cong P_{f_0}|_{U'} = P_g$, thus proving the lemma. □

4.3. Obstruction theory and isotopy

In order to use Lemma 4.7 for a proof of Theorem 3.12, we have to make sure that the critical loci X_{f_t} is independent of t. In this section, we will use the obstruction assignments (Definition 2.26) to give a criterion for the constancy of the critical loci.

Recall that an LG pair (V, f) gives us a symmetric obstruction theory

$$E_V = [T_V|_{X_f} \xrightarrow{d(df)} \Omega_V|_{X_f}] \longrightarrow \mathbb{L}_{X_f}$$

and there is an obstruction class $ob_{X_f}(\phi, \bar{g}, B, \bar{B}) \in I \otimes_{\mathbb{C}} \Omega_{X_f}|_x$ for an infinitesimal lifting problem (Definition 2.8). We proved that two equivalent LG pairs give the same obstruction assignment (Lemma 2.29). A natural case where non-equivalent LG pairs give the same obstruction assignment is the following.

Lemma 4.9. *Let $V \subset \mathbb{C}^n$ be open and let (V, f) be an LG pair with a critical point $x \in X_f = \mathrm{Crit}(f)$. Then there exist a system of coordinates $y_1, \ldots, y_n$ centered at x in a neighborhood U of x in V, a holomorphic function h on U in the form $h = h(y_{r+1}, \ldots, y_n)$, and an invertible function u on U with $u|_{X_f \cap U} = 1$ such that*

$$f|_U = u \cdot (y_1^2 + \cdots + y_r^2 + h) \quad \text{and} \quad T_x X_f = T_x U',$$

where $U' = (y_1 = \cdots = y_r = 0) \cap U$. If we let $h' = h|_{U'}$, then $x \in X_{h'} = X_f \cap U$. Moreover, we have a canonical isomorphism $P_f|_U \cong P_{h'}$.

Proof. If $T_x X_f = T_x V$, then there is nothing to prove. So we assume $T_x X_f \neq T_x V$. Choose a local coordinate system $w_1, \ldots, w_n$ of V centered at x, and let

$$H = \left(\frac{\partial^2 f}{\partial w_i \partial w_j}\Big|_x \right)$$

be the Hessian matrix of f at x. Since X_f is the vanishing locus of $\frac{\partial f}{\partial w_i}$, we have $T_x X_f = \ker H \neq T_x V$. Since H is symmetric, there exists an element $v \in T_x V$ such that $v^T H v \neq 0$. After a coordinate change, we may assume that $v = (1,0,0,\dots,0)$, i.e., $\frac{\partial^2 f}{\partial w_1^2}|_x \neq 0$. By the Weierstrass preparation lemma [11, p. 7], we can write

$$f = u_1 \cdot (w_1^2 + w_1 p_1 + q_1)$$

for some invertible holomorphic function u_1 and holomorphic functions p_1, q_1 in $w_2, \dots, w_n$. By completing the square, we can write

$$f = u_1 \cdot (\bar{w}_1^2 + \tilde{q}_1(w_2, \dots, w_n))$$

with $\bar{w}_1 = w_1 + p_1/2$. We can repeat the same argument with $\tilde{q}_1$ in place of f to obtain $f = u_1 \cdot (\bar{w}_1^2 + u_2(\bar{w}_2^2 + \tilde{q}_2(w_3, \dots, w_n)))$ for some invertible u_2. We continue this way until we reach

$$f = u_1 \bar{w}_1^2 + u_1 u_2 \bar{w}_2^2 + \cdots + \left(\prod_{i=1}^r u_i\right) \bar{w}_r^2 + \left(\prod_{i=1}^r u_i\right) \tilde{h}(w_{r+1}, \dots, w_n)$$

for invertible functions $u_1, \dots, u_r$. Let $\tilde{u} = \prod_{i=1}^r u_i$, $\tilde{y}_j = (\prod_{i=j+1}^r u_i)^{-\frac{1}{2}} \bar{w}_j$ for $j = 1, \dots, r$ and $y_{r+k} = w_{r+k}$ for $k \geq 1$. Then

$$f = \tilde{u} \cdot (\tilde{h}(y_{r+1}, \dots, y_n) + \tilde{y}_1^2 + \cdots + \tilde{y}_r^2)$$

in a neighborhood U of x, expressed in terms of the coordinate variables $(\tilde{y}, y) = (\tilde{y}_1, \dots, \tilde{y}_r, y_{r+1}, \dots, y_n)$. Let $u = \tilde{u}(\tilde{y}, y)/\tilde{u}(0, y)$, $y_j = \sqrt{\tilde{u}(0,y)}\tilde{y}_j$ for $j \leq r$ and $h = \tilde{u}(0,y)\tilde{h}$ so that we have

$$f = u \cdot (y_1^2 + \cdots + y_r^2 + h(y_{r+1}, \dots, y_n)), \quad u(0,z) = 1.$$

We let

$$g(y_1, \dots, y_n) = y_1^2 + \cdots + y_r^2 + h(y_{r+1}, \dots, y_n).$$

By Lemma 4.8, possibly after shrinking $x \in U$, we have $X_f \cap U = X_g \cap U$, and canonical isomorphism $P_f|_U \cong P_g|_U$. Applying the Sebastiani-Thom isomorphism, we get $P_g|_U \cong P_{h'}|_{U'}$, and thus an isomorphism $P_g|_U \cong P_{h'}|_{U'}$. This proves the lemma. □

Lemma 4.10. *Let the situation be as in Lemma 4.9. Then the symmetric obstruction theories defined by the LG pairs (V, f) and (V', h') give the same obstruction assignment of $X_f \cap U = X_{h'}$ at x.*

Proof. The proof is parallel to that of Lemma 2.29, using that $u(x) = 1$. We will omit the details here. □

Proposition 4.11. *Let $V \subset \mathbb{C}^n$ be open. Let (V, f_0) and (V, f_1) be two LG pairs such that $X_{f_0} = X_{f_1}$ with $x \in X_{f_0}$. Let $f_t = (1-t)f_0 + tf_1$. Suppose that the symmetric obstruction theories defined by (V, f_0) and (V, f_1) give the same obstruction assignment at x, and that $f_1 - f_0 \in \mathfrak{m}_x^3$ where $\mathfrak{m}_x$ is the ideal of holomorphic functions on V vanishing at x. Then there is an open neighborhood $x \in U \subset V$ so that $X_{f_t} \cap U$ is independent of $t \in [0,1]$.*

Proof. We consider $F:\mathbb{C}\times V\to\mathbb{C}$ defined by $F(t,z)=(1-t)f_0(z)+tf_1(z)$, and define the relative critical locus $X_{F/\mathbb{C}}$ to be defined by the ideal $(d_V F)$, where d_V is the differential along the V directions. Because $X_{f_0}=X_{f_1}$,

$$\mathbb{C}\times X_{f_0}\subset X_{F/\mathbb{C}}\subset\mathbb{C}\times V \tag{4.6}$$

are closed analytic subspaces. We prove that there is an open subset $x\in U\subset V$ and $[0,1]\subset W\subset\mathbb{C}$ so that

$$W\times X_{f_0}=X_{F/\mathbb{C}}\cap(W\times U). \tag{4.7}$$

Let $\pi:\mathbb{C}\times V\to V$ be the projection. Because $\pi^*(df_0)\subset(d_V F)$ is a coherent subsheaf of $\mathcal{O}_V$-modules, the quotient $(d_V F)/\pi^*(df_0)$ is a coherent sheaf of $\mathcal{O}_V$-modules. Thus

$$\Sigma=\{z\in\mathbb{C}\times V\mid (d_V F)/\pi^*(df_0)\big|_z\ne 0\}$$

is a closed analytic subset of V. So if $\Sigma\cap(\mathbb{C}\times x)=\emptyset$, we can find open neighborhoods $x\in U\subset V$ and $[0,1]\subset W\subset\mathbb{C}$ such that (4.7) holds.

Let $t\in\mathbb{C}$ and let $\hat X_{f_t}$ be the formal completion of X_{f_t} at x. Since $t\times_{\mathbb{C}}X_{F/\mathbb{C}}=X_{f_t}$, $(t,x)\notin\Sigma$ if and only if $\hat X_{f_t}=\hat X_{f_0}$. We now prove this identity. Because $\hat X_{f_0}\subset\hat X_{f_t}$, we have a surjective homomorphism $\mathcal{O}_{\hat X_{f_t}}\twoheadrightarrow\mathcal{O}_{\hat X_{f_0}}$. We will show $\mathcal{O}_{\hat X_{f_t}}=\mathcal{O}_{\hat X_{f_0}}$ by showing that for any $k\ge 1$,

$$\mathcal{O}_{X_{f_t}}\big/\big(\mathcal{O}_{X_{f_t}}\cap\mathfrak{m}_x^k\big)=\mathcal{O}_{X_{f_0}}\big/\big(\mathcal{O}_{X_{f_0}}\cap\mathfrak{m}_x^k\big). \tag{4.8}$$

The identity for $k=2$ follows from $f_1-f_0\in\mathfrak{m}_x^3$, thus $T_xX_{f_t}=T_xX_{f_0}$. Suppose $\mathcal{O}_{\hat X_{f_t}}\ne\mathcal{O}_{\hat X_{f_0}}$. Then there is a $k_0\ge 2$ so that (4.8) is true for $k=k_0$ but not for $k=k_0+1$.

We let $B=\mathcal{O}_{X_{f_t}}/(\mathcal{O}_{X_{f_t}}\cap\mathfrak{m}_x^{k_0+1})$, and let $I\subset B$ be the kernel of the restriction homomorphism

$$B\to\bar B=\mathcal{O}_{X_{f_t}}/(\mathcal{O}_{X_{f_t}}\cap\mathfrak{m}_x^{k_0}).$$

We let $\bar g:\operatorname{Spec}\bar B\to X_{f_t}$ be the tautological morphism. Because the identity (4.8) holds for k_0, the composite $\operatorname{Spec}\bar B\to X_{f_t}\to V$ factors through $\bar g':\operatorname{Spec}\bar B\to X_{f_0}$.

By the definition of $(I,B,\bar g)$, $\bar g$ extends to $\operatorname{Spec}B\to X_{f_t}$. Thus the obstruction class $ob_{X_{f_t}}(\bar g,B,\bar B)$ is zero. We claim that the obstruction class $ob_{X_{f_0}}(\bar g',B,\bar B)$ to extending $\bar g'$ to $\operatorname{Spec}B\to X_{f_0}$ is trivial too. Indeed, Using the identity $\hat X_{f_1}=\hat X_{f_0}\subset\hat V$, we can view $\bar g'$ as a morphism from $\operatorname{Spec}\bar B$ to X_{f_1}. Because $f_0-f_1\in\mathfrak{m}_x^3$, we have

$$\Omega_{X_{f_0}}|_x=\Omega_{X_{f_1}}|_x=\Omega_{X_{f_t}}|_x, \tag{4.9}$$

as quotient spaces of $\Omega_V|_x$. Adding that $f_t=(1-t)f_0+tf_1$, we obtain

$$ob_{X_{f_t}}(\bar g,B,\bar B)=(1-t)\cdot ob_{X_{f_0}}(\bar g',B,\bar B)+t\cdot ob_{X_{f_1}}(\bar g',B,\bar B). \tag{4.10}$$

Here we can equate and add because they are elements in $I\otimes_{\mathbb{C}}\Omega_{X_{f_0}}|_x$.

Because X_{f_0} and X_{f_1} have identical obstruction assignments at x, we have

$$ob_{X_{f_0}}(\bar{g}', B, \bar{B}) = ob_{X_{f_1}}(\bar{g}', B, \bar{B}).$$

Using (4.10) and $ob_{X_{f_t}}(\bar{g}, B, \bar{B}) = 0$, we get $ob_{X_{f_0}}(\bar{g}', B, \bar{B}) = 0$. Hence $\bar{\varphi}'$ also extends to $\varphi' : \operatorname{Spec} B \to X_{f_0}$. Composing with the ring homomorphism induced by $\hat{X}_{f_0} \subset \hat{X}_{f_t}$, we obtain a composite ring homomorphism

(4.11)
$$\mathcal{O}_{X_{f_t}}/\big(\mathcal{O}_{X_{f_t}} \cap \mathfrak{m}_x^{k_0+1}\big) \longrightarrow \mathcal{O}_{X_{f_0}}/\big(\mathcal{O}_{X_{f_0}} \cap \mathfrak{m}_x^{k_0+1}\big) \xrightarrow{\varphi'^*} \mathcal{O}_{X_{f_t}}/\big(\mathcal{O}_{X_{f_t}} \cap \mathfrak{m}_x^{k_0+1}\big)$$

whose restriction to $\mathcal{O}_{X_{f_t}}/(\mathcal{O}_{X_{f_t}} \cap \mathfrak{m}_x^2)$ is the identity map. Thus the composite (4.11) is an isomorphism. Because $\hat{X}_{f_0} \subset \hat{X}_{f_t}$, we conclude that (4.8) holds for $k = k_0 + 1$, a contradiction. This proves $\hat{X}_{f_t} = \hat{X}_{f_0}$, and the proposition. □

Remark 4.12. The proposition may fail when $f_0 - f_1 \notin \mathfrak{m}_x^3$. For example, let $V = \mathbb{C}$, $f_0 = z^2$ and $f_1 = -z^2$. Then $X_{f_0} = X_{f_1}$ while $X_{f_{1/2}} \neq X_{f_0}$.

Corollary 4.13. *Let the situation be as in Proposition 4.11. Then there is an open neighborhood $x \in U \subset V$ such that the family of holomorphic functions f_t induces an isomorphism*

$$\tau_{01} : P_{f_0}|_{X_{f_0} \cap U} \cong P_{f_1}|_{X_{f_1} \cap U}.$$

Proof. Applying Lemma 4.7 to the family f_t gives the corollary. □

4.4. Proof of Theorem 3.12

In this section, we complete our proof of Theorem 3.12.

Lemma 4.14. *Let $\Psi : \mathbb{C}^r \to \mathbb{C}^r$ be the linear isomorphism defined by the diagonal matrix* $\operatorname{diag}(\pm 1, 1, \ldots, 1)$. *Let $q(z_1, \ldots, z_r) = z_1^2 + \cdots + z_r^2$. Then the pullback isomorphism $P_q \longrightarrow P_q$ induced from Ψ by Proposition 3.4 is* $\pm\mathrm{id}$. *Moreover if $\Phi : V \times \mathbb{C}^r \to V \times \mathbb{C}^r$ is $\mathrm{id} \times \Psi$ and $f(z, y) = g(y) + q(z)$ for $y \in V$ and $z \in \mathbb{C}^r$, then the pullback isomorphism induced from Φ by Proposition 3.4 is $\pm\mathrm{id} : P_f \to P_f$.*

Proof. The Milnor fiber at 0 is homotopic to the sphere S^{r-1} whose reduced cohomology is $\tilde{H}^{r-1}(S^{r-1}) = \mathbb{Q}$. Obviously, Ψ acts $\tilde{H}^{r-1}(S^{r-1})$ on as ± 1. For the second statement, use Proposition 4.1. □

Theorem 4.15 (Theorem 3.12)**.** *Let (V, f) be an LG pair. Let U be an open neighborhood of X_f in V and $\Phi : U \to V$ be a holomorphic map, biholomorphic onto its image, such that $f \circ \Phi = f|_U$ and $\Phi|_{X_f} = \mathrm{id}_{X_f}$. Then the isomorphism $\sigma : P_f \to P_f$ induced from Φ by Proposition 3.4 is* $\det(d\Phi|_{X_f}) \cdot \mathrm{id}$.

Note that by Lemma 2.16, $\det(d\Phi|_{X_f})$ is locally constant with values in $\{\pm 1\}$.

Proof. Since this is a local problem, for any $x \in X_f$, by shrinking V, we can assume that V embeds into $\mathbb{C}^n$ as an open subset, so that and $x \in V$

corresponds to $0 \in \mathbb{C}^n$. Applying Lemma 4.9, we can assume that for the standard complex coordinate variables $(y_1, \ldots, y_n)$ of $\mathbb{C}^n$,

$$f = u \cdot \big(y_1^2 + \cdots + y_r^2 + h(y_{r+1}, \ldots, y_n)\big), \tag{4.12}$$

where $u : V \to \mathbb{C}$ is a holomorphic function such that $T_0 X_f$ is the linear space $V_0 := \{y_1 = \cdots = y_r = 0\} \cap V$ and $u|_{V_0} = 1$.

Using $V \subset \mathbb{C}^n$ and the canonical isomorphism $T_x\mathbb{C}^n \cong \mathbb{C}^n$, we form the linear transformation

$$\Psi := d\Phi|_0 : \mathbb{C}^n \longrightarrow \mathbb{C}^n.$$

Because $\Phi|_{X_f} = \mathrm{id}_{X_f}$, and because $T_x X_f$ is the linear subspace $\{y_1 = \cdots = y_r = 0\}$, we see that $\Psi(V_0) \subset V_0$ and $\Psi|_{V_0} = \mathrm{id}_{V_0}$. Because of (4.12), possibly after shrinking $x \in V_0$, we have $X_f \subset V_0$. Thus $\Psi|_{X_f} = \mathrm{id}_{X_f}$.

For $s \in [0, 1]$, we define

$$\varphi_s = (1 - s) \cdot \Phi + s \cdot \Psi : U \longrightarrow \mathbb{C}^n.$$

Since $d\varphi_s|_0 = \Psi$ are invertible, by shrinking U if necessary, φ_s map into $V \subset \mathbb{C}^n$, and are biholomorphic onto their images. Note that $\varphi_0 = \Phi|_U$ and $\varphi_1 = \Psi|_U$ is linear. Because both $\Phi|_{X_f} = \Psi|_{X_f} = \mathrm{id}_{X_f}$, we have $\varphi_s|_{X_f \cap U} = \mathrm{id}_{X_f \cap U}$.

We now construct various isomorphisms. We let $g = u^{-1} f$, and let f_t, $t \in [0, 1]$, be $u_t \cdot g|_U$, as in the proof of Lemma 4.8. Thus f_t interpolates between $f|_U$ and $g|_U$, with $f_0 = f|_U$ and $f_1 = g|_U$. We then form the composite isomorphism

$$\eta_{s,t} : P_{f_t}|_{X_{f_t} \cap U} \xrightarrow{\cong} P_{f_t \circ \varphi_s}|_{X_{f_t \circ \varphi_s}} \xrightarrow{\cong} P_{f_t}|_{X_{f_t} \cap U}$$

where the first isomorphism is obtained by applying Proposition 3.4 to the map φ_s from (U, f_t) to $(U, f_t \circ \varphi_s)$, and the second isomorphism is obtained by applying Corollary 4.13 to $f_t \circ \varphi_s - f_t \in \mathfrak{m}_x^3$. Applying Corollary 4.13 to f_t, possibly after shrinking $x \in U$, we have a family of isomorphisms τ_t as shown:

$$\begin{array}{ccc} P_{f_t}|_{X_{f_t} \cap U} & \xrightarrow{\tau_t} & P_{f_0}|_{X_{f_0} \cap U} \\ \eta_{s,t} \big\downarrow & & \eta_{0,0} \big\downarrow \\ P_{f_t}|_{X_{f_t} \cap U} & \xrightarrow{\tau_t} & P_{f_0}|_{X_{f_0} \cap U}. \end{array}$$

Since for $(s, t) = (0, 0)$, the square is commutative, and since τ_t and $\eta_{s,t}$ are continuous families of isomorphisms, applying Lemma 2.16, the above square is commutative at $(s, t) = (1, 1)$.

Since $\Phi|_{V_0} = \mathrm{id}_{V_0}$, Ψ is of the form

$$\begin{pmatrix} A & 0 \\ B & I \end{pmatrix}$$

with $A \in O(r)$. Since $O(r)$ has only two connected components, we can find a continuous path A_t in $O(r)$ from A to the diagonal $r \times r$ matrix $\mathrm{diag}(e, 1 \cdots, 1)$ where $e = \det A = \det d\Phi|_0$. Let $B_t = (1 - t)B$. So that we have a path Ψ_t from Ψ above to the $n \times n$ diagonal matrix $D := \mathrm{diag}(e, 1, \ldots, 1)$. Then

$g \circ \Psi_t - g \in \mathfrak{m}_x^3$ for all t because $h \in \mathfrak{m}_x^3$. Applying the same argument as above, we obtain a commutative diagram of perverse sheaves

$$\begin{array}{ccccc} P_g & \xrightarrow{\tau'} & P_g & \xrightarrow{\tau_1} & P_f \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle \eta_{1,1}} & & \downarrow{\scriptstyle \eta_{0,0}} \\ P_g & \xrightarrow{\tau'} & P_g & \xrightarrow{\tau_1} & P_f \end{array}$$

all restricted to U. By Lemma 4.14, the left vertical is $e \cdot \mathrm{id}$. The right vertical is the isomorphism σ in Theorem 4.15. Since the horizontal maps are the same, we find that $\sigma = e \cdot \mathrm{id}$. □

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