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NUCLEARITY PROPERTIES AND C^*-ENVELOPES OF OPERATOR SYSTEM INDUCTIVE LIMITS

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Abstract. We investigate the relationship between C^*-envelopes and inductive limit of operator systems. Various operator system nuclearity properties of inductive limit for a sequence of operator systems are also discussed.

1. Introduction

In last few years, the development of the theory of operator systems has seen a fair amount of attention. All the important notions from the theory of C^*-algebras including exactness, nuclearity, weak expectation property and lifting properties have been explicitly defined in the category of operator systems. Associated to every representation ϕ of an operator system $\mathcal{S}$ into C^*-algebra of bounded operator $B(H)$, for some Hilbert Space H, one can always consider a C^*-cover generated by $\phi(\mathcal{S})$ that is, the C^*-algebra $C^*(\phi(\mathcal{S})) \subset B(H)$. The minimal C^*-cover among all such representations is known as the C^*-envelope of $\mathcal{S}$. It is thus quite natural to ask which C^*-algebraic properties of the C^*-envelopes are carried over to the generating operator systems in terms of their definitions in the operator system category, and to what extent. Some attempts done in this direction can be found in [5, 12].

It is well known (see [2]) that for the category of C^*-algebras, inductive limit preserves many intrinsic properties, viz., exactness, nuclearity, simplicity etc. The analysis of inductive limit of ascending sequences of finite dimensional C^*-algebras, known as approximately finite dimensional (AF) C^*-algebras, has played an important role in theory of operator algebras. Existence of inductive limits in the category of operator systems has been shown in [10]. But unlike in the category of C^*-algebras, there are several notion of nuclearity in the operator system category (see [7–9]). It is thus natural to check if these nuclearity properties are preserved under operator system inductive limit. This

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paper deals with C^*-envelopes of operator system inductive limit and various nuclearity properties of operator system inductive limit.

After recalling all the prerequisite definitions and results including that of inductive limit for the categories of C^*-algebras and operator systems in Section 2 of preliminaries, the commutativity of inductive limit and C^*-envelopes has been studied in Section 3. We also do this for particular cases of direct sequences. In the last section, various nuclearity properties of inductive limit of operator systems are discussed.

2. Preliminaries

2.1. Operator systems

The concept of operator systems and their tensor products is the familiar one now and most of the details can be seen in [7–9].

Recall that a concrete operator system is a unital self-adjoint subspace of $B(H)$ for some Hilbert space H. A *C^*-cover* ([6, §2]) of an operator system $\mathcal{S}$ is a pair (A, i) consisting of a unital C^*-algebra A and a complete order embedding $i : \mathcal{S} \to A$ such that $i(A)$ generates the C^*-algebra A. The *C^*-envelope* as defined by Hamana [6], of an operator system $\mathcal{S}$ is a C^*-cover generated by $\mathcal{S}$ in its injective envelope $I(\mathcal{S})$ and is denoted by $C_e^*(\mathcal{S})$. The C^*-envelope $C_e^*(\mathcal{S})$ enjoys the following universal "minimality" property ([6, Corollary 4.2]):

Identifying $\mathcal{S}$ with its image in $C_e^*(\mathcal{S})$, for any C^*-cover (A, i) of $\mathcal{S}$, there is a unique surjective unital $*$-homomorphism $\pi : A \to C_e^*(\mathcal{S})$ such that $\pi(i(s)) = s$ for every s in $\mathcal{S}$.

One more fundamental C^*-cover, the maximal one, is associated to an operator system $\mathcal{S}$, namely, the universal C^*-algebra $C_u^*(\mathcal{S})$ introduced by Kirchberg and Wassermann ([11, §3]). $C_u^*(\mathcal{S})$ satisfies the following universal "maximality" property:

Every unital completely positive map $\phi : \mathcal{S} \to A$, where A is a unital C^*-algebra extends uniquely to a unital $*$-homomorphism $\pi : C_u^*(\mathcal{S}) \to A$.

Proposition 2.1. *For operator systems $\mathcal{S}$ and $\mathcal{T}$, let $\theta : \mathcal{S} \to \mathcal{T}$ be a unital complete order isomorphism. Then*

- (i) [11, Proposition 9] *θ can be uniquely extended to a unital $*$-monomorphism $\overline{\theta} : C_u^*(\mathcal{S}) \to C_u^*(\mathcal{T})$, that is, $\overline{\theta} \circ i_{\mathcal{S}} = i_{\mathcal{T}} \circ \theta$, where $i_{\mathcal{S}} : \mathcal{S} \to C_u^*(\mathcal{S})$ and $i_{\mathcal{T}} : \mathcal{T} \to C_u^*(\mathcal{T})$ denote the natural inclusion maps.*
- (ii) [1, Theorem 2.2.5] *In case θ is surjective, then θ can be uniquely extended to a unital $*$-monomorphism $\overline{\theta} : C_e^*(\mathcal{S}) \to C_e^*(\mathcal{T})$, that is, $\overline{\theta} \circ i_{\mathcal{S}} = i_{\mathcal{T}} \circ \theta$, where $i_{\mathcal{S}} : \mathcal{S} \to C_e^*(\mathcal{S})$ and $i_{\mathcal{T}} : \mathcal{T} \to C_e^*(\mathcal{T})$ denote the natural inclusion maps.*

The collection of all operator system tensor products is a complete lattice with respect to the order: $\tau_2 \leq \tau_1$ if and only if for any two operator system $\mathcal{S}$ and $\mathcal{T}$, $M_n(\mathcal{S} \otimes_{\tau_1} \mathcal{T})^+ \subseteq M_n(\mathcal{S} \otimes_{\tau_2} \mathcal{T})^+$ for every $n \in \mathbb{N}$.

We now give the formal definitions of the operator system tensor products that we shall use frequently. For any two operator systems $(\mathcal{S}, \{M_n(\mathcal{S})^+\}_{n=1}^\infty, e_{\mathcal{S}})$ and $(\mathcal{T}, \{M_n(\mathcal{T})^+\}_{n=1}^\infty, e_{\mathcal{T}})$,

- *The minimal tensor product*, denoted by $\mathcal{S} \otimes_{\min} \mathcal{T}$, is the operator system $(\mathcal{S} \otimes \mathcal{T}, \{\mathcal{C}_n^{min}\}_{n=1}^\infty, e_{\mathcal{S}} \otimes e_{\mathcal{T}})$, where

$$\mathcal{C}_n^{\min} = \big\{(u_{i,j}) \in M_n(\mathcal{S} \otimes \mathcal{T}) \;\big|\; \big((\phi \otimes \psi)(u_{i,j})\big)_{i,j} \in M_{nkm}^+, \\ \text{for } \phi \in S_k(\mathcal{S}), \psi \in S_m(\mathcal{T}) \text{ for all } k, m \in \mathbb{N}\big\}.$$

  For any other operator system tensor product τ, $\min \leq \tau$, that is, $\mathcal{C}_n^{\min}$ is the largest cone structure on $\mathcal{S} \otimes \mathcal{T}$.
- The *maximal operator system tensor product*,

$$\mathcal{S} \otimes_{\max} \mathcal{T} = (\mathcal{S} \otimes \mathcal{T}, \{\mathcal{C}_n^{\max}\}_{n=1}^\infty, e_{\mathcal{S}} \otimes e_{\mathcal{T}}),$$

  is the largest of all operator system tensor product, where $\mathcal{C}_n^{\max} = \mathcal{D}_n^{\mathrm{Arch}}$ (Archimedeanization of $\mathcal{D}_n$) and

$$\mathcal{D}_n = \{A(P \otimes Q)A^* : P \in M_r(\mathcal{S})^+, Q \in M_s(\mathcal{T})^+, A \in M_{n,rs}, r, s \in \mathbb{N}\}.$$

- *The maximal commuting operator system tensor product*, denoted by $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$ is defined by the inclusion

$$\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T} \subseteq_{c.o.i.} C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}).$$

- Making use of one more association of operator systems with the unique minimal injective C^*-algebras $I(\mathcal{S})$, three more tensor products were defined in [8]:

$$\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T} \subseteq_{c.o.i.} I(\mathcal{S}) \otimes_{\max} \mathcal{T},$$
$$\mathcal{S} \otimes_{\mathrm{er}} \mathcal{T} \subseteq_{c.o.i.} \mathcal{S} \otimes_{\max} I(\mathcal{T}),$$
$$\mathcal{S} \otimes_{\mathrm{e}} \mathcal{T} \subseteq_{c.o.i.} I(\mathcal{S}) \otimes_{\max} I(\mathcal{T}).$$

These tensor products admit the following partial order:

$$\min \leq \mathrm{e} \leq \mathrm{el}, \mathrm{er} \leq \mathrm{c} \leq \max.$$

In [4], a natural operator system tensor product “ess” arising from the enveloping C^*-algebras, viz., $\mathcal{S} \otimes_{\mathrm{ess}} \mathcal{T} \subseteq \mathrm{C}_{\mathrm{e}}^*(\mathcal{S}) \otimes_{\max} \mathrm{C}_{\mathrm{e}}^*(\mathcal{T})$, was also defined. It is known from [4, §8] that ess $\leq$ c. See also [5, Proposition 4.4] for comparison of ess with other operator system tensor products. Given two operator system tensor products α and β, an operator system $\mathcal{S}$ is said to be (α, β)-*nuclear* if the identity map between $\mathcal{S} \otimes_\alpha \mathcal{T}$ and $\mathcal{S} \otimes_\beta \mathcal{T}$ is a complete order isomorphism for every operator system $\mathcal{T}$, i.e.,

$$\mathcal{S} \otimes_\alpha \mathcal{T} = \mathcal{S} \otimes_\beta \mathcal{T}.$$

Also, an operator system $\mathcal{S}$ is said to be C^*-*nuclear*, if

$$\mathcal{S} \otimes_{\min} A = \mathcal{S} \otimes_{\max} A$$

for all unital C^*-algebras A.

Given an operator system $\mathcal{S}$, a subspace $J \subseteq \mathcal{S}$ is said to be a *kernel* ([9, Definition 3.2]) if there exist an operator system $\mathcal{T}$ and a unital completely positive map $\phi : \mathcal{S} \to \mathcal{T}$ such that $J = \ker \phi$. For such a kernel $J \subset \mathcal{S}$, Kavruk et al. have shown that the quotient space $\mathcal{S}/J$ forms an operator system ([9, Proposition 3.4]) with respect to the natural involution, whose positive cones are given by

$$\begin{aligned}\mathcal{C}_n &= \mathcal{C}_n(\mathcal{S}/J)\\ &= \{(s_{ij} + J)_{i,j} \in M_n(\mathcal{S}/J) : \ (s_{ij}) + \varepsilon(1 + J)_n \in \mathcal{D}_n \ \text{ for every } \varepsilon > 0\},\end{aligned}$$

where

$$\mathcal{D}_n = \{(s_{ij} + J)_{i,j} \in M_n(\mathcal{S}/J) : \ (s_{ij} + y_{ij})_{i,j} \in M_n(\mathcal{S})^+ \ \text{ for some } y_{ij} \in J\}$$

and the Archimedean unit is the coset $1 + J$.

An operator system $\mathcal{S}$ is said to be *exact* ([9]) if for every unital C^*-algebra A and a closed ideal I in A the induced map

$$(\mathcal{S}\hat{\otimes}_{\min} A)/(\mathcal{S}\bar{\otimes} I) \to \mathcal{S}\hat{\otimes}_{\min}(A/I)$$

is a complete order isomorphism, where $\mathcal{S}\bar{\otimes} I$ denotes the closure of $\mathcal{S} \otimes I$ in the completion $\mathcal{S}\hat{\otimes}_{\min} A$ of the minimal tensor product $\mathcal{S} \otimes_{\min} A$. Recall that an operator subsystem $\mathcal{S}$ of a unital C^*-algebra A is said *to contain enough unitaries of* A if the unitaries in $\mathcal{S}$ generate A as a C^*-algebra ([9, §9]).

Following characterizations established in [9, §5], [7, §4], [5] and [12] are used quite often:

Theorem 2.2. (i) *An operator system with exact C^*-envelope is exact. Converse is true if the operator system contains enough unitaries in its C^*-envelope.*

(ii) *An operator system is* (min, ess)*-nuclear if its C^*-envelope is nuclear. Moreover, a unital C^*-algebra is* (min, ess)*-nuclear as an operator system if and only if it is nuclear as a C^*-algebra.*

(iii) *Let $\mathcal{S} \subset A$ contains enough unitaries of the unital C^*-algebra A. Then $\mathcal{S}$ is* (min, ess)*-nuclear if and only if A is a nuclear C^*-algebra.*

(iv) *Suppose $\mathcal{S} \subset A$ contains enough unitaries. Then, upto a $*$-isomorphism fixing $\mathcal{S}$, we have $A = C_e^*(\mathcal{S})$.*

2.2. Inductive limits

For convenience, we recall some fundamental definitions and facts about inductive limits in the general context of a category [17].

An inductive sequence in a category C is a sequence $\{X_n\}_{n=1}^{\infty}$ of objects in C and a sequence $\phi_n : X_n \to X_{n+1}$ of morphisms in C,

$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3 \xrightarrow{\phi_3} \cdots .$$

For $m > n$, the composed morphisms

$$\phi_{m,n} = \phi_{m-1} \circ \phi_{m-2} \circ \cdots \circ \phi_n : X_n \to X_m$$

which, together with the morphisms ϕ_n, are called the *connecting morphisms*. Conventionally for categories with zero object, $\phi_{n,n} = Id_{X_n}$ and $\phi_{m,n} = 0$ when $m < n$.

An *inductive limit* of the inductive sequence $\{X_n, \phi_n\}_{n=1}^{\infty}$ in a category C is a system $\{X, \mu_n\}_{n=1}^{\infty}$, where X is an object in C and $\mu_n : X_n \to X$ is a morphism in C for each $n \in \mathbb{N}$ satisfying the following conditions:

(i) The diagram

$$\begin{tikzcd} X_n \arrow[rr, "\phi_n"] \arrow[dr, "\mu_n"'] & & X_{n+1} \arrow[dl, "\mu_{n+1}"] \\ & X & \end{tikzcd}$$

commutes for each n.

(ii) If $(Y, \{\lambda_n\}_{n=1}^{\infty})$ is a system, where Y is an object in C, $\lambda_n : X_n \to Y$ is a morphism in C for each $n \in \mathbb{N}$, and $\lambda_n = \lambda_{n+1} \circ \phi_n$ for all $n \in \mathbb{N}$, then there exists a unique $\lambda : X \to Y$ such that the diagram

$$\begin{tikzcd} & X_n \arrow[dl, "\mu_n"'] \arrow[dr, "\lambda_n"] & \\ X \arrow[rr, "\lambda"] & & Y \end{tikzcd}$$

commutes for each n.

The inductive limit of the sequence is denoted by $\varinjlim(X_n, \phi_n)$, or more briefly by $\varinjlim X_n$.

Inductive limits do not exist in all categories (see [17, Exercise 6.4]). We deal with the inductive limits only in the categories of C^*-algebras and operator systems.

2.2.1. *Inductive limits in the category of C^*-algebras.* [17, Proposition 6.2.4]

Every inductive system $\{A_n, \mu_n\}_{n=1}^{\infty}$ with $\phi_n : A_n \to A_{n+1}$, $n = 1, 2, 3, \ldots$ the connecting $*$-homomorphisms, has an inductive limit A satisfying

(i) $A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$;

(ii) $\|\mu_n(a)\| = \lim_{m\to\infty} \|\phi_{m,n}(a)\|$ for all $n \in \mathbb{N}$ and $a \in A$.

(iii) If $(B, \{\lambda_n\}_{n=1}^{\infty})$ is another inductive system with $\lambda : A \to B$ as in (ii) above in the definition of inductive limit, then λ is injective if and only if $\mathrm{Ker}(\lambda_n) \subseteq \mathrm{Ker}(\mu_n)$ for all $n \in \mathbb{N}$ and is surjective if and only if $B = \overline{\bigcup_{n=1}^{\infty} \lambda_n(B_n)}$.

Following hold for the inductive limit of C^*-algebras:

Theorem 2.3. (i) [2, II.8.2.5] *The class of simple C^*-algebras is closed under inductive limits.*

(ii) *The class of nuclear C^*-algebras is closed under inductive limits.*

(iii) [2, IV.3.4.4] *An inductive limit (with injective connecting maps) of exact C^*-algebras is exact.*

(iv) [2, II.8.3.24] *Any countable inductive limit of AF algebras is an AF algebra.*
(v) [2, II.9.6.5] *The maximal C^*-tensor product commutes with arbitrary inductive limits.*
(vi) [2, II.9.6.5] *The minimal C^*-tensor product commutes with inductive limits with injective connecting maps.*

2.2.2. *Inductive limits in the category of operator systems.* The existence of inductive limit for the category of (complete) operator systems was shown in [10, §2] with connecting morphism as the unital complete positive maps:

Let $(\mathcal{S}, \{\lambda_n\}_{n=1}^{\infty})$ denote the inductive limit of operator systems $\{\mathcal{S}_n\}$ with $u_n : \mathcal{S}_n \to \mathcal{S}_{n+1}$ the unital completely contractive maps (and hence unital completely positive maps), then the completely positive $\lambda_n : \mathcal{S}_n \to \mathcal{S}$, for $n = 1, 2, \ldots$ are such that $\lambda_n = \lambda_{n+1} \circ u_n$. Moreover $\mathcal{S}$ has the universal property that if $\mathcal{T}$ is an operator system and $\phi_n : \mathcal{S}_n \to \mathcal{T}$ are unital completely positive maps such that $\phi_n = \phi_{n+1} \circ u_n$ for each n, then there is a unital completely positive map $\phi : \mathcal{S} \to \mathcal{T}$ such that $\phi_n = \phi \circ \lambda_n$. Thus Hahn-Banach separation theorem, Riesz decomposition property for hermitian functionals on operator systems and uniqueness of ϕ, implies that $\mathcal{S} = \overline{\bigcup_{n=1}^{\infty} \lambda_n(\mathcal{S}_n)}$ ([10, p. 40]).

The next proposition about the inductive limit of operator systems and their universal C^*-covers is well known, and was also used in the proof of [11, Proposition 16]. For the sake of completeness, we outline the proof.

Proposition 2.4. *$C_u^*(\varinjlim \mathcal{S}_n) = \varinjlim C_u^*(\mathcal{S}_n)$ for an inductive sequence $\{\mathcal{S}_n\}_{n=1}^{\infty}$ with $u_n : \mathcal{S}_n \to \mathcal{S}_{n+1}$ $\forall n \in \mathbb{N}$ as the connecting completely positive maps.*

Proof. Let $(\mathcal{S}, \{\lambda_n\}_{n=1}^{\infty})$ denote the inductive limit of the inductive sequence $\{\mathcal{S}_n\}_{n=1}^{\infty}$. Using Proposition 2.1(i), there exists a unique $*$-homomorphism $\widetilde{u_n} : C_u^*(\mathcal{S}_n) \to C_u^*(\mathcal{S}_{n+1})$ such that $\widetilde{u_n} \circ i_n = i_{n+1} \circ u_n$, where $i_n : \mathcal{S}_n \to C_u^*(\mathcal{S}_n)$ denote the natural complete order inclusion for all n. Now, the universal property of inductive limits implies that there exist a unital complete order embedding i such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{S}_1 \arrow[r, "u_1"] \arrow[d, "i_1"] & \mathcal{S}_2 \arrow[r, "u_2"] \arrow[d, "i_2"] & \mathcal{S}_3 \arrow[r, "u_3"] \arrow[d, "i_3"] & \cdots \arrow[r] & \mathcal{S} \arrow[d, "i"] \\
C_u^*(\mathcal{S}_1) \arrow[r, "\widetilde{u_1}"] & C_u^*(\mathcal{S}_3) \arrow[r, "\widetilde{u_2}"] & C_u^*(\mathcal{S}_3) \arrow[r, "\widetilde{u_3}"] & \cdots \arrow[r] & \varinjlim(C_u^*(\mathcal{S}_n))
\end{tikzcd}$$

Clearly, $C^*(i(S)) = \varinjlim C_u^*(\mathcal{S}_n)$. Let $\theta : \mathcal{S} \to B$ be any other complete order embedding. Using the complete order isomorphisms $\theta_n = \theta \circ \lambda_n : \mathcal{S}_n \to B$, there exist surjective $*$-homomorphisms $\widetilde{\theta_n} : C_u^*(\mathcal{S}_n) \to C^*(\theta_n(\mathcal{S}_n)) \subset B$ for all $n \in \mathbb{N}$. Again $\{C^*(\theta_n(\mathcal{S}_n))\}_{n=1}^{\infty}$ is an inductive sequence with inclusion maps as connecting maps, and $\varinjlim C^*(\theta_n(\mathcal{S}_n)) = C^*(\theta(\mathcal{S})) \subset B$. Thus there exists a surjective $*$-homomorphism $\pi : \varinjlim C_u^*(\mathcal{S}_n) \to \varinjlim C^*(\theta_n(\mathcal{S}_n)) = C^*(\theta(\mathcal{S}))$, and hence $C_u^*(\mathcal{S}) = \varinjlim C_u^*(\mathcal{S}_n)$. □

Our aim is to explore (complete) operator system inductive limits with complete order embeddings as connecting maps through the C^*-covers. We thank the referee for bringing to our attention the more recent article [13], where one can find some of the similar results for the non-complete operator system inductive limit.

3. Operator system inductive limits and C^*-envelopes

3.1. C^*-envelopes of operator systems pair

There is no general method to determine the C^*-envelopes of even the lower dimensional operator systems. Unlike universal C^*-covers, the unital complete order embedding $\mathcal{S} \subset \mathcal{T}$ may not, in general, be extended to a $*$-homomorphism $C_e^*(\mathcal{S}) \subset C_e^*(\mathcal{T})$. It is rather strange but amid the list of operator systems whose C^*-envelopes are known, majority of operator system pairs are such that the respective pair of C^*-envelopes behave as nicely as a pair of universal C^*-covers does, that is, for $\mathcal{S} \subset \mathcal{T}$ the C^*-algebra generated by $\mathcal{S}$ in $C_e^*(\mathcal{T})$ coincides with the C^*-envelope of $\mathcal{S}$, thus giving $C_e^*(\mathcal{S}) \subset C_e^*(\mathcal{T})$:

Example 3.1.1 (Operator systems associated to discrete groups)**.** Let G be a countable discrete group, $\mathfrak{u}$ denote a generating set of G and $\mathcal{S}(\mathfrak{u})$ the operator system associated to $\mathfrak{u}$ by Farenick et al. in [4], i.e., $\mathcal{S}(\mathfrak{u}) := \text{span}\{1, u, u^* : u \in \mathfrak{u}\} \subset C^*(G)$, where $C^*(G)$ denotes the full group C^*-algebra of the group G ([15, Chapter 8]). It was shown in [4] that if $\mathfrak{u}$ is a generating set of the free group $\mathbb{F}_n$, then $\mathcal{S}(\mathfrak{u})$ is independent of the generating set $\mathfrak{u}$ and is simply denoted by $\mathcal{S}_n$. Also, $C_e^*(\mathcal{S}(\mathfrak{u})) = C^*(G)$ ([4, Proposition 2.2]) and $C_e^*(\mathcal{S}_r(\mathfrak{u})) = C_r^*(G)$, the reduced group C^*-algebra ([5, Proposition 2.9]. Recall from [3, Propositions 2.5.8-2.5.9]: If H is a subgroup of a discrete group G, then there is a canonical inclusion

$$C^*(H) \subset C^*(G).$$

Also,

$$C_r^*(H) \subset C_r^*(G)$$

canonically.

Using the fact that the C^*-envelope of an operator system associated to the group is the group C^*-algebra itself, the preceding statement can be translated in terms of operator systems:

In case $\mathfrak{u}$ and $\mathfrak{v}$ are generating sets of H, the subgroup of G and G respectively, then the complete order inclusion $\mathcal{S}(\mathfrak{u}) \subset \mathcal{S}(\mathfrak{v})$ can be extended canonically to their C^*-envelopes: $C_e^*(\mathcal{S}(\mathfrak{u})) \subset C_e^*(\mathcal{S}(\mathfrak{v}))$.

Example 3.1.2 (Graph Operator Systems)**.** Given a finite graph G with n-vertices, Kavruk et al. in [8] associated an operator system $\mathcal{S}_G$ as the finite dimensional operator subsystem of $M_n(\mathbb{C})$ given by

$$\mathcal{S}_G = \operatorname{span}\{\{E_{i,j} : (i,j) \in G\} \cup \{E_{i,i} : 1 \le i \le n\}\} \subseteq M_n(\mathbb{C}),$$

where $\{E_{i,j}\}$ is the standard system of matrix units in $M_n(\mathbb{C})$ and (i,j) denotes (an unordered) edge in G. Ortiz and Paulsen proved in [14, Theorem 3.2] that $C_e^*(\mathcal{S}_G) = C^*(\mathcal{S}_G) \subseteq M_n(\mathbb{C})$ for any graph G. Thus, we have:

For graphs $G_1 \subset G_2$, the complete order inclusion $\mathcal{S}_{G_1} \subset \mathcal{S}_{G_2}$ extends canonically to a $*$-isomorphism

$$C_e^*(\mathcal{S}_{G_1}) \subset C_e^*(\mathcal{S}_{G_2}) \subset M_n.$$

Example 3.1.3 (Unital C^*-algebras)**.** Since the C^*-envelope of an operator system which is completely order isomorphic to a unital C^*-algebra coincides with itself ([5, Proposition 2.2(iii)]), we naturally have:

If $\mathcal{S}$ and $\mathcal{T}$ are both unitally completely order isomorphic to unital C^*-algebras, then the inclusion $\mathcal{S} \subset \mathcal{T}$ extends canonically to $C_e^*(\mathcal{S}) \subset C_e^*(\mathcal{T})$.

Example 3.1.4 (Universal Operator Systems)**.** The universal operator systems ([7, §1.1]) are those for which the universal C^*-cover and the C^*-envelope coincide. Since for $\mathcal{S} \subset \mathcal{T}$, $C_u^*(\mathcal{S}) \subset C_u^*(\mathcal{T})$, trivially:

For a pair of universal operator systems $\mathcal{S} \subset \mathcal{T}$, $C_e^*(\mathcal{S}) \subset C_e^*(\mathcal{T})$.

Example 3.1.5 (Operator systems for Non-Commuting n-cubes)**.** Farenick et al. (in [4]) introduced an $(n+1)$-dimensional operator system $NC(n)$ as follows:

Let $\mathcal{G} = \{h_1, \dots, h_n\}$, let $\mathcal{R} = \{h_j^* = h_j, \|h_j\| \le 1, 1 \le j \le n\}$ be the relation in the set $\mathcal{G}$, and let $C^*(\mathcal{G}|\mathcal{R})$ denote the universal unital C^*-algebra generated by $\mathcal{G}$ subject to the relation $\mathcal{R}$. The operator system

$$NC(n) := \operatorname{span}\{1, h_1, \dots, h_n\} \subset C^*(\mathcal{G}|\mathcal{R})$$

is called *the operator system of the non-commuting n-cube.*

They showed that upto a $*$-isomorphism, $C_e^*(NC(n)) = C^*(*_n\mathbb{Z}_2)$ ([4, Corollary 5.6]), so that $*_n\mathbb{Z}_2 \subset *_{n+1}\mathbb{Z}_2$, and hence $C^*(*_n\mathbb{Z}_2) \subset C^*(*_{n+1}\mathbb{Z}_2)$ for all n (as in Example 3.1.2), so that:

For $n \in \mathbb{N}$, the complete order inclusion $NC(n) \subset NC(n+1)$ extends to complete order inclusion $C_e^*(NC(n)) \subset C_e^*(NC(n+1))$.

Example 3.1.6 (Operator system with simple C^*-envelope)**.** In case $\mathcal{S} \subset \mathcal{T}$ is such that $C_e^*(\mathcal{S})$ is simple, trivially every homomorphism from $C_e^*(\mathcal{S})$ is injective, thus $C_e^*(\mathcal{S}) \subset C_e^*(\mathcal{T})$.

Example 3.1.7 (Cuntz Operator System)**.** From [18], for the generators $s_1, s_2, \dots, s_n$ $(n \ge 2)$ of the Cuntz algebra $\mathcal{O}_n$ and identity I, the Cuntz operator system $\mathcal{S}_n$ denotes the operator system generated by $s_1, s_2, \dots, s_n$, that is,

$$\mathcal{S}_n = \operatorname{span}\{I, s_1, s_2, \dots, s_n, s_1^*, s_2^*, \dots, s_n^*\} \subset \mathcal{O}_n.$$

Similarly, for the generators $s_1, s_2, \dots$ of $\mathcal{O}_\infty$,

$$\mathcal{S}_\infty = \operatorname{span}\{I, s_1, s_2, \dots, s_1^*, s_2^*, \dots\} \subset \mathcal{O}_\infty.$$

Also, by [18], for each $n < m$, $\mathcal{S}_n \subset \mathcal{S}_m$ and $C_e^*(\mathcal{S}_n) = \mathcal{O}_n$ (see also [12, Proposition 2.8]). Clearly, $\mathcal{O}_n \subset \mathcal{O}_m$.

3.2. C^*-envelope of operator system inductive limit

Under a restricted setting, we have an analogy of Proposition 2.4:

Theorem 3.1. *Let $\{\mathcal{S}_n\}_{n=1}^{\infty}$ be an increasing collection of operator systems such that the complete order embedding $\mathcal{S}_n \subset \mathcal{S}_{n+1}$ extends to a $*$-homomorphism from $C_e^*(\mathcal{S}_n)$ into $C_e^*(\mathcal{S}_{n+1})$, then we have*

$$C_e^*(\varinjlim \mathcal{S}_n) = \varinjlim C_e^*(\mathcal{S}_n). \tag{3.1}$$

And, moreover if each $\mathcal{S}_n$ is separable, exact and contains enough unitaries of $C_e^*(\mathcal{S}_n)$, then $\varinjlim \mathcal{S}_n$ embeds into $\mathcal{O}_2$.

Proof. Consider the complete order embeddings $u_n : \mathcal{S}_n \to \mathcal{S}_{n+1} \subset B(H)$ and let $i_n : \mathcal{S}_n \hookrightarrow C_e^*(\mathcal{S}_n) \subset B(K_n)$ be the complete order embedding generating $C_e^*(\mathcal{S}_n)$, K_n being a Hilbert space. Then if $\widetilde{u_n} : C_e^*(\mathcal{S}_n) \to C_e^*(\mathcal{S}_{n+1})$ is the extended $*$-homomorphism, we have $\widetilde{u_n} \circ i_n(s) = i_{n+1} \circ u_n(s)$ for all $s \in \mathcal{S}_n$.

We can thus consider

$$A_n = C^*(i_{n+1} \circ u_n(\mathcal{S}_n)) \subset C_e^*(\mathcal{S}_{n+1}) \subset B(K_{n+1}).$$

By minimality of C^*-envelopes there exist a unique $*$-epimorphism

$$\rho_n : A_n \to C_e^*(\mathcal{S}_n)$$

with

$$\rho_n \circ i_{n+1} \circ u_n(s) = i_n(s) \ \ \forall \ \ s \in \mathcal{S}_n.$$

Then

$$\widetilde{u_n} \circ \rho_n : A_n \to C_e^*(\mathcal{S}_{n+1})$$

is a map with

$$\widetilde{u_n} \circ \rho_n \circ i_{n+1} \circ u_n(s) = \widetilde{u_n} \circ i_n(s) \ \ \forall \ s \in \mathcal{S}_n.$$

Therefore,

$$\widetilde{u_n} \circ \rho_n \circ i_{n+1} \circ u_n(s) = i_{n+1} \circ u_n(s) \ \ \forall \ \ s \in \mathcal{S}_n,$$

thus $\widetilde{u_n} \circ \rho_n$ is identity on $i_{n+1} \circ u_n(\mathcal{S}_n)$ and hence on A_n. Since ρ_n is surjective, we have the injectivity of $\widetilde{u_n}$.

Let $(\mathcal{S}, \{\lambda_n\}_{n=1}^{\infty})$ denote the inductive limit of the increasing sequence $\{\mathcal{S}_n\}_{n=1}^{\infty}$. Now, the universal property of inductive limits implies that there exist a unital complete order embedding i such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{S}_1 \arrow[r, "u_1"] \arrow[d, "i_1"] & \mathcal{S}_2 \arrow[r, "u_2"] \arrow[d, "i_2"] & \mathcal{S}_3 \arrow[r, "u_3"] \arrow[d, "i_3"] & \cdots \arrow[r] & \mathcal{S} \arrow[d, "i"] \\
C_e^*(\mathcal{S}_1) \arrow[r, "\widetilde{u_1}"] & C_e^*(\mathcal{S}_3) \arrow[r, "\widetilde{u_2}"] & C_e^*(\mathcal{S}_3) \arrow[r, "\widetilde{u_3}"] & \cdots \arrow[r] & \varinjlim(C_e^*(\mathcal{S}_n))
\end{tikzcd}$$

Clearly, $C^*(i(\mathcal{S})) = \varinjlim(C^*_e(\mathcal{S}_n))$.
Claim: $C^*_e(\mathcal{S}) = C^*(i(\mathcal{S})) = \varinjlim(C^*_e(\mathcal{S}_n))$, that is, if $\phi : \mathcal{S} \to B$ is any other unital complete order embedding, where B is any unital C^*-algebra, then there exists a surjective $*$-homomorphism $\pi : C^*(\phi(\mathcal{S})) \to C^*(i(\mathcal{S})) = \varinjlim(C^*_e(\mathcal{S}_n))$ such that $\pi(\phi(s)) = i(s)$ for all $s \in \mathcal{S}$.

For each n, $\phi_n = \phi \circ \lambda_n : \mathcal{S}_n \to B$ is a complete order isomorphism. Now, by minimality of C^*-envelopes, there exists a surjective $*$-homomorphism $\widetilde{\phi_n} : C^*(\phi_n(\mathcal{S}_n)) \to C^*_e(\mathcal{S}_n)$ for $n \in \mathbb{N}$ such that $\widetilde{\phi_n}(\phi_n(s_n)) = s_n$ for all $s_n \in \mathcal{S}_n$. Again $\{C^*(\phi_n(\mathcal{S}_n))\}_{n=1}^{\infty}$ is an inductive sequence with inclusion maps as the connecting morphisms, and $\varinjlim C^*(\phi_n(\mathcal{S}_n)) = C^*(\phi(\mathcal{S})) \subset B$. So using the universal property of inductive limits there exists a surjective $*$-homomorphism $\pi : \varinjlim C^*(\phi_n(\mathcal{S}_n)) \to \varinjlim C^*_e(\mathcal{S}_n)$ with $\pi(\phi(s)) = i(s)$ for all $s \in \mathcal{S}$. Hence $C^*_e(\mathcal{S}) = C^*(i(\mathcal{S})) = \varinjlim(C^*_e(\mathcal{S}_n))$. In case each $\mathcal{S}_i$ contains enough unitaries of $C^*_e(\mathcal{S}_i)$ by Theorem 2.2(i) exactness of $\mathcal{S}_i$ implies exactness of $C^*_e(\mathcal{S}_i)$, and since inductive limit of exact C^*-algebras is exact (Theorem 2.3), we have $\varinjlim C^*_e(\mathcal{S}_i)$ is exact.

Thus, using the fact that a separable operator system embeds into $\mathcal{O}_2$ if and only if it's C^*-envelope is exact ([12, Theorem 3.1]), we have $\varinjlim \mathcal{S}_i$ embeds into $\mathcal{O}_2$. □

Recall that if for an operator system $\mathcal{S}$ the universal (maximal) $C^*_u(\mathcal{S})$ and enveloping (minimal) $C^*_e(\mathcal{S})$ C^*-covers coincide, the operator system is known as *universal operator system* [11]. Since $\mathcal{S}_n \subseteq \mathcal{S}_{n+1}$ extends to their universal C^*-covers, $C^*_u(\mathcal{S}_n) \subseteq C^*_u(\mathcal{S}_{n+1})$, preceding proof implies the following.

Corollary 3.2. *For a collection $\{\mathcal{S}_n\}$ of exact universal increasing operator systems with each $\mathcal{S}_n$ containing enough unitaries of $C^*_e(\mathcal{S}_n)$, we have that $\varinjlim C^*_e(\mathcal{S}_n)$ is exact.*

Corollary 3.3. *For an increasing collection of operator systems $\{\mathcal{S}_n\}$, with each $C^*_e(\mathcal{S}_n)$ an approximately finite dimensional von Neumann algebra, we have that $C^*_e(\varinjlim \mathcal{S}_n)$ is an AF-algebra.*

Proof. Given such a sequence, using injectivity of approximately finite dimensional von Neumann algebra one can extend the complete order embedding to $*$-homomorphims on the C^*-envelopes, so that from Theorem 3.1 we have $C^*_e(\varinjlim \mathcal{S}_i) = \varinjlim C^*_e(\mathcal{S}_i)$, with each $C^*_e(\mathcal{S}_i)$ an AF-algebra. Thus, $C^*_e(\varinjlim \mathcal{S}_i)$ being inductive limit of AF-algebra is again an AF-algebra (Theorem 2.3(iv)). □

3.2.1. *Infinite tensor product of operator systems.* Infinite tensor product of a collection $\{A_i : i \in \Omega\}$ of C^*-algebras, $\otimes_{C^*\text{-}\min} A_i$; $i \in \Omega$ has been defined as the inductive limit of the collection $B_{\mathcal{F}}$, where $B_{\mathcal{F}} = A_{i_1} \otimes_{C^*\text{-}\min} \cdots \otimes_{C^*\text{-}\min} A_{i_n}$, for $\mathcal{F} = \{i_1, \ldots, i_n\} \subseteq \Omega$ ([2, II.9.8]). Using the similar technique, Pisier in [15, Page 390] discussed the infinite Haagerup tensor product.

Extending the same to "min" operator system tensor product, infinite tensor product of a set $\{\mathcal{S}_i : i \in \Omega\}$ of operator systems, $\otimes_{i\in\Omega}\mathcal{S}_i$, can be defined in terms of inductive limit. In fact, for $\mathcal{F} = \{i_1, \ldots, i_n\} \subseteq \Omega$, set

$$\mathcal{T}_{\mathcal{F}} = \mathcal{S}_{i_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{S}_{i_n}.$$

Then if $\mathcal{F} \subset \mathcal{G}$, $\mathcal{T}_{\mathcal{G}} \cong \mathcal{T}_{\mathcal{F}} \otimes_{\min} \mathcal{T}_{\mathcal{G}\setminus\mathcal{F}}$, so that there is a natural inclusion of $\mathcal{T}_{\mathcal{F}}$ into $\mathcal{T}_{\mathcal{G}}$ by $t \mapsto t \otimes 1_{\mathcal{G}\setminus\mathcal{F}}$. This way, the collection $\mathcal{T}_{\mathcal{G}}$ forms an inductive system and $\otimes_{\min}\mathcal{S}_i; i \in \Omega$ is defined to be the inductive limit.

Corollary 3.4. *For an ascending sequence of operator systems $\{\mathcal{S}_i\}$ such that for each i, $\mathcal{S}_i \subseteq C_e^*(\mathcal{S}_i)$ contains enough unitaries in $C_e^*(\mathcal{S}_i)$ or each $C_e^*(\mathcal{S}_i)$ is simple. Then we have*

$$C_e^*(\otimes_{\min}\mathcal{S}_i) = \otimes_{C^*\text{-}\min} C_e^*(\mathcal{S}_i).$$

Proof. For the finite case, $\mathcal{S}_{i_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{S}_{i_n}$ has enough unitaries of $C_e^*(\mathcal{S}_{i_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{S}_{i_n})$ and therefore by Theorem 2.2(iv) upto $*$-isomorphism that fixes $\mathcal{S}_{i_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{S}_{i_n}$;

$$C_e^*(\mathcal{S}_{i_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{S}_{i_n}) = C_e^*(\mathcal{S}_{i_1}) \otimes_{C^*\text{-}\min} \cdots \otimes_{C^*\text{-}\min} C_e^*(\mathcal{S}_{i_n}).$$

Thus, using Theorem 3.1, for the infinite tensor product we have:

$$\begin{aligned} C_e^*(\otimes_{\min}\mathcal{S}_i) &= C_e^*(\varinjlim \mathcal{S}_{i_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{S}_{i_n}) \\ &= \varinjlim C_e^*(\mathcal{S}_{i_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{S}_{i_n}) \\ &= \varinjlim C_e^*(\mathcal{S}_{i_1}) \otimes_{C^*\text{-}\min} \cdots \otimes_{C^*\text{-}\min} C_e^*(\mathcal{S}_{i_n}) \\ &= \otimes_{C^*\text{-}\min} C_e^*(\mathcal{S}_i). \end{aligned}$$

□

Following is an immediate generalization of Corollary 3.5 of [12] to infinite tensor product using the C^*-isomorphism $\bigotimes_{C^*\text{-}\min} \mathcal{O}_2 \cong \mathcal{O}_2$ ([16]). We give a short proof for the sake of completeness:

Corollary 3.5. *Let $\{\mathcal{T}_i\}$; $i \in \mathbb{N}$ be a collection of operator systems with separable C^*-envelopes. If, for each $i = 1, 2, \ldots$, $C_e^*(\mathcal{T}_i)$ is exact, then the operator system $\otimes_{\min}\mathcal{T}_i$ embeds into $\mathcal{O}_2$. Converse holds, if either $C_e^*(\mathcal{T}_i)$ is simple for all $i = 1, 2, \ldots$ or each $\mathcal{T}_i$, $i = 1, 2, \ldots$, contains enough unitaries of $C_e^*(\mathcal{T}_i)$, respectively.*

Proof. Let $C_e^*(\mathcal{T}_i)$ be exact for $i \in \mathbb{N}$. Since min is injective and operator system min tensor product of C^*-algebras embeds complete order isomorphically into their C^*-min tensor product [8, Theorem 4.6, Corollary 4.10], using $\bigotimes_{C^*\text{-}\min} \mathcal{O}_2 \cong \mathcal{O}_2$ ([16]), we have

$$\bigotimes_{\min} \mathcal{T}_i \hookrightarrow \bigotimes_{C^*\text{-}\min} \mathcal{O}_2 \cong \mathcal{O}_2.$$

Conversely, let there be an embedding of $\bigotimes_{\min} \mathcal{T}_i$ into $\mathcal{O}_2$. In case, $C_e^*(\mathcal{T}_i)$ is simple for $i \in \mathbb{N}$ or each $\mathcal{T}_i$, $i \in \mathbb{N}$, contains enough unitaries of $C_e^*(\mathcal{T}_i)$, respectively, then using Corollary 3.4,

$$C_e^*(\otimes_{\min}\mathcal{T}_i) = \otimes_{C^*\text{-}\min} C_e^*(\mathcal{T}_i),$$

which is separable (being the minimal C^*-tensor product of separable C^*-algebra). Since an operator system is exact if and only if its separable C^*-envelope embeds into $\mathcal{O}_2$ [12, Theorem 3.1]), we have $\otimes_{C^*\text{-}\min} C_e^*(\mathcal{T}_i)$ is exact and hence, for each i, the C^*-subalgebras $C_e^*(\mathcal{T}_i)$ is exact ([2]). $\square$

Proposition 3.6. *For an increasing collection $\{\mathfrak{u}_i\}$, where each $\mathfrak{u}_i$ denotes a generating set of the group G_i (Example 3.1.2), $C_e^*(\varinjlim \mathcal{S}(\mathfrak{u}_i)) = \varinjlim C_e^*(\mathcal{S}(\mathfrak{u}_i))$ (resp. $C_e^*(\varinjlim \mathcal{S}_r(\mathfrak{u}_i)) = \varinjlim C_e^*(\mathcal{S}_r(\mathfrak{u}_i))$.*

In particular, for the operator system $\mathcal{S}_n \subset C^*(\mathbb{F}_n)$ associated to free group $\mathbb{F}_n$ and $\mathcal{S}_\infty \subset C^*(\mathbb{F}_\infty)$, we have $C_e^*(\mathcal{S}_\infty) = C^*(\mathbb{F}_\infty)$.

Proof. It follows directly by using the fact that whenever $G_i \subseteq G_{i+1}$, $C^*(G_i) \subseteq C^*(G_{i+1})$ and $C_r^*(G_i) \subseteq C_r^*(G_{i+1})$ in Theorem 3.1.

Also, as the sequence of $\mathcal{S}_n \subset C^*(\mathbb{F}_n)$ of operator subsystems of full group C^*-algebra generated by free group with n-generators increases to the operator subsystem $\mathcal{S}_\infty \subset C^*(\mathbb{F}_\infty)$, therefore we have $C_e^*(\mathcal{S}_\infty) = C^*(\mathbb{F}_\infty)$. $\square$

Theorem 3.1 gives an easy proof of the following known result from [18]:

Proposition 3.7. *For the Cuntz operator system of infinite degree, $C_e^*(\mathcal{S}_\infty) = \mathcal{O}_\infty$.*

Proof. By considering the inductive sequence $\{\mathcal{S}_n\}_{n=1}^\infty$ of Cuntz operator system, since each C^*-envelopes of a pair of increasing Cuntz operator system is increasing, by Theorem 3.1, $C_e^*(\mathcal{S}_\infty) = C_e^*(\varinjlim \mathcal{S}_n) = \varinjlim C_e^*(\mathcal{S}_n) = \varinjlim \mathcal{O}_n = \mathcal{O}_\infty$. $\square$

Proposition 3.8. *For an increasing sequence of graphs as in Example 3.1.2, we have $C_e^*(\varinjlim \mathcal{S}_{G_i}) = C^*(\varinjlim \mathcal{S}_{G_i})$, the C^*-algebra generated inside an AF C^*-algebra B.*

Proof. Since $C_e^*(\mathcal{S}_{G_i}) = C^*(\mathcal{S}_{G_i}) \subset M_{n_i}$ ([14, Thereom 3.2]) and G_i's are increasing, we have $\{M_{n_i}\}$ are also ascending. So that by Theorem 3.1, we have $C_e^*(\varinjlim \mathcal{S}_{G_i}) = \varinjlim C_e^*(\mathcal{S}_{G_i}) = \varinjlim C^*(\mathcal{S}_{G_i}) \subset \varinjlim M_{n_i}$. Using $B = \varinjlim M_{n_i}$ which is an AF-algebra, we are done. $\square$

Remark 3.9. Note that B being nuclear is exact C^*-algebra and hence $C_e^*(\varinjlim \mathcal{S}_{G_i})$ embeds into $\mathcal{O}_2$.

4. Nuclearity properties of inductive limits in operator system category

In this section, we study various notions of nuclearity of operator systems. The approach to study nuclearity of inductive limit is through C^*-covers.

Using injectivity of minimal operator system tensor product ([8, Theorem 4.6] and the fact that the maximal commuting operator system tensor product c is induced by the max tensor product of universal C^*-covers ([8, Theorem 6.4]), we have the next proposition:

Proposition 4.1. *For an inductive sequence $\{\mathcal{S}_n\}_{n=1}^{\infty}$ with $u_n : \mathcal{S}_n \to \mathcal{S}_{n+1}$ $\forall n \in \mathbb{N}$ as the connecting complete order embeddings and any operator system $\mathcal{T}$, the maximal commuting operator system tensor product* c *commutes with the inductive limit, that is,*

$$\varinjlim(\mathcal{S}_n \otimes_c \mathcal{T}) = (\varinjlim \mathcal{S}_n) \otimes_c \mathcal{T};$$

provided $\{\mathcal{S}_n \otimes_{\mathrm{c}} \mathcal{T}\}_{n=1}^{\infty}$ is an inductive system with the complete order embeddings $\{u_n \otimes Id_{\mathcal{T}}\}_{n=1}^{\infty}$ as the connecting maps. For the inductive system $\{\mathcal{S}_n \otimes_{\min} \mathcal{T}\}_{n=1}^{\infty}$ with the complete order embeddings $\{u_n \otimes Id_{\mathcal{T}}\}_{n=1}^{\infty}$ as the connecting maps, the minimal operator system tensor product min *also commutes with the inductive limit,*

$$\varinjlim(\mathcal{S}_n \otimes_{\min} \mathcal{T}) = (\varinjlim \mathcal{S}_n) \otimes_{\min} \mathcal{T}.$$

Proof. First we give the proof for the tensor product c. The maximal C^*-tensor product commutes with arbitrary inductive limits using Theorem 2.3(v), so that

$$\varinjlim(C_u^*(\mathcal{S}_n) \otimes_{C^*\text{-}\max} C_u^*(\mathcal{T})) = (\varinjlim C_u^*(\mathcal{S}_n)) \otimes_{C^*\text{-}\max} C_u^*(\mathcal{T}) \tag{4.1}$$

for the inductive sequence $\{\mathcal{S}_n\}_{n=1}^{\infty}$ and operator system $\mathcal{T}$. Also, by [8, Theorem 6.4], the operator system tensor product c is naturally induced by the maximal operator system tensor product of universal C^*-covers, that is for each n, we have

$$\mathcal{S}_n \otimes_c \mathcal{T} \subset C_u^*(\mathcal{S}_n) \otimes_{\max} C_u^*(\mathcal{T}).$$

Therefore, we have the following commutative diagram,

$$\begin{tikzcd}
\mathcal{S}_1 \otimes_{\mathrm{c}} \mathcal{T} \arrow[r, "u_1 \otimes Id_{\mathcal{T}}"] \arrow[d, hook] & \mathcal{S}_2 \otimes_{\mathrm{c}} \mathcal{T} \arrow[r, "u_2 \otimes Id_{\mathcal{T}}"] \arrow[d, hook] & \cdots \arrow[r] & \varinjlim(\mathcal{S}_n \otimes_{\mathrm{c}} \mathcal{T}) \arrow[d, hook] \\
C_u^*(\mathcal{S}_1) \otimes_{\max} C_u^*(\mathcal{T}) \arrow[r, "\widetilde{u_1} \otimes Id_{C_u^*(\mathcal{T})}"] & C_u^*(\mathcal{S}_2) \otimes_{\max} C_u^*(\mathcal{T}) \arrow[r, "\widetilde{u_2} \otimes Id_{C_u^*(\mathcal{T})}"] & \cdots \arrow[r] & \varinjlim(C_u^*(\mathcal{S}_n) \otimes_{\max} C_u^*(\mathcal{T}))
\end{tikzcd}$$

which further implies,

$$\begin{aligned}
\varinjlim(\mathcal{S}_n \otimes_{\mathrm{c}} \mathcal{T}) &\subset \varinjlim(C_u^*(\mathcal{S}_n) \otimes_{\max} C_u^*(\mathcal{T})) \\
&\subset \varinjlim(C_u^*(\mathcal{S}_n) \otimes_{C^*\text{-}\max} C_u^*(\mathcal{T})) \text{ ([8, Theorem 5.12])} \\
&= (\varinjlim C_u^*(\mathcal{S}_n)) \otimes_{C^*\text{-}\max} C_u^*(\mathcal{T}) \text{ (Equation (4.1))}
\end{aligned}$$

$$= C_u^*(\varinjlim \mathcal{S}_n) \otimes_{C^*\text{-}\max} C_u^*(\mathcal{T}) \text{ (Proposition 2.4).}$$

Hence, using $(\varinjlim \mathcal{S}_n) \otimes_{\mathrm{c}} \mathcal{T} \subset C_u^*(\varinjlim \mathcal{S}_n) \otimes_{\max} C_u^*(\mathcal{T})$, the complete order inclusion map above, $\varinjlim(\mathcal{S}_n \otimes_{\mathrm{c}} \mathcal{T}) \to C_u^*(\varinjlim \mathcal{S}_n) \otimes_{C^*\text{-}\max} C_u^*(\mathcal{T})$ induces the complete order isomorphism $\varinjlim(\mathcal{S}_n \otimes_{\mathrm{c}} \mathcal{T}) = (\varinjlim \mathcal{S}_n) \otimes_{\mathrm{c}} \mathcal{T}$.
Similarly, since min is injective,

$$\varinjlim(\mathcal{S}_n \otimes_{\min} \mathcal{T}) \subset \varinjlim(C_u^*(\mathcal{S}_n) \otimes_{\min} C_u^*(\mathcal{T})) \subset \varinjlim(C_u^*(\mathcal{S}_n) \otimes_{C^*\text{-}\min} C_u^*(\mathcal{T})).$$

But from Theorem 2.3(vi),

$$\begin{aligned}\varinjlim(C_u^*(\mathcal{S}_n) \otimes_{C^*\text{-}\min} C_u^*(\mathcal{T})) &= (\varinjlim C_u^*(\mathcal{S}_n)) \otimes_{C^*\text{-}\min} C_u^*(\mathcal{T}) \\ &= C_u^*(\varinjlim \mathcal{S}_n) \otimes_{C^*\text{-}\min} C_u^*(\mathcal{T}).\end{aligned}$$

Again, using the injectivity of min, we have $\varinjlim(\mathcal{S}_n \otimes_{\min} \mathcal{T})$ is complete order isomorphic to $(\varinjlim \mathcal{S}_n) \otimes_{\min} \mathcal{T}$. □

In case of non-complete operator systems, results similar to the preceding proposition have been provided in [13, Lemma 4.38 and Theorem 4.39].

It is known that the inductive limit of nuclear C^*-algebras (in C^*-algebra category) and the inductive limit of nuclear operator spaces (in operator space category) is exact. Since by [9, Proposition 5.5] an operator system is $(\min, \max)$-nuclear (as an operator system) if and only if it is 1-nuclear as an operator space, we have the inductive limit of $(\min, \max)$-nuclear operator systems is $(\min, \max)$-nuclear:

Proposition 4.2. *In case $\{\mathcal{S}_n\}_{n=1}^{\infty}$ is an inductive system of $(\min, \max)$-nuclear operator system, then $\varinjlim \mathcal{S}_n$ is also $(\min, \max)$-nuclear.*

For finite dimensional operator system $(\min, \mathrm{c})$-nuclearity is the highest nuclearity that can be attained by an operator system which is not a C^*-algebra ([7, Proposition 4.12]). Next result proves that it is preserved under inductive limit:

Theorem 4.3. *In case $\{\mathcal{S}_n\}_{n=1}^{\infty}$ is an inductive system of $(\min, \mathrm{c})$-nuclear operator system with $u_n : \mathcal{S}_n \to \mathcal{S}_{n+1}$ $\forall n \in \mathbb{N}$ as the connecting complete order embedding, $\varinjlim \mathcal{S}_n$ is also $(\min, \mathrm{c})$-nuclear.*

Proof. Note that if each operator system $\mathcal{S}_n$ in the inductive system is $(\min, \mathrm{c})$-nuclear, then using the injectivity of min, the connecting maps $\{u_n \otimes Id_{\mathcal{T}}\}_{n=1}^{\infty}$ are complete order embeddings, so that $\{\mathcal{S}_n \otimes_{\mathrm{c}} \mathcal{T}\}_{n=1}^{\infty}$ forms an inductive system for every operator system $\mathcal{T}$. Thus result follows directly by Proposition 4.1 as: $(\varinjlim \mathcal{S}_n) \otimes_{\min} \mathcal{T} = \varinjlim(\mathcal{S}_n \otimes_{\min} \mathcal{T}) = \varinjlim(\mathcal{S}_n \otimes_{\mathrm{c}} \mathcal{T}) = (\varinjlim \mathcal{S}_n) \otimes_{\mathrm{c}} \mathcal{T}$. □

It is known that the inductive limit of exact C^*-algebras (in C^*-algebra category) and the inductive limit of exact operator spaces (in operator space category) is exact. And, since by [9, Proposition 5.5] an operator system is exact as an operator system if and only if it is exact as an operator space, we have the inductive limit of exact operator systems is exact. Further we know by [7] that $(\min, \mathrm{el})$-nuclearity in the operator system category is equivalent to exactness thus, we have:

Theorem 4.4. *If $\mathcal{S}_n$ is an inductive system of $(\min, \mathrm{el})$-nuclear operator system, $\varinjlim \mathcal{S}_n$ is also $(\min, \mathrm{el})$-nuclear.*

In an attempt to study nuclearity of operator systems through their C^*-envelopes, it was proved in [5, Proposition 4.2] that an operator system is $(\min, \mathrm{ess})$-nuclear if its C^*-envelope is nuclear. Moreover, if an operator system contains enough unitaries of its C^*-envelope, then its $(\min, \mathrm{ess})$-nuclearity is equivalent to the nuclearity of its C^*-envelope. We make use of this result to prove $(\min, \mathrm{ess})$-nuclearity of an inductive system of $(\min, \mathrm{ess})$-nuclear operator systems.

Theorem 4.5. *Let $\{\mathcal{S}_n\}_{n=1}^{\infty}$ be an increasing collection of operator systems such that, for each n, $\mathcal{S}_n \subseteq C_e^*(\mathcal{S}_n)$ contain enough unitaries in $C_e^*(\mathcal{S}_n)$. Moreover, assume that for each pair $\mathcal{S}_n \subseteq \mathcal{S}_{n+1}$, $C_e^*(\mathcal{S}_n) \subseteq C_e^*(\mathcal{S}_{n+1})$. If each $\mathcal{S}_n$ is $(\min, \mathrm{ess})$-nuclear, then $\varinjlim \mathcal{S}_n$ is $(\min, \mathrm{ess})$-nuclear.*

Proof. From Theorem 3.1, $\varinjlim C_e^*(\mathcal{S}_n) = C_e^*(\varinjlim \mathcal{S}_n)$. Now using [5, Proposition 4.2], $(\min, \mathrm{ess})$-nuclearity of $\mathcal{S}_n$ implies C^*-nuclearity of $C_e^*(\mathcal{S}_n)$, and since inductive limit of nuclear C^*-algebras is nuclear (Theorem 2.3(ii)), we have $\varinjlim C_e^*(\mathcal{S}_n)$ and hence $C_e^*(\varinjlim \mathcal{S}_n)$ is nuclear. Therefore, $\varinjlim \mathcal{S}_n$ is $(\min, \mathrm{ess})$-nuclear ([5, Proposition 4.2]). □

It is known from Theorem 7.1 and Theorem 7.6 of [9] (see also [7, Theorem 4.7]) that an operator system $\mathcal{S}$ is $(\mathrm{el}, \mathrm{c})$-nuclear if and only if it has DCEP which is true, if and only if $\mathcal{S} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{S} \otimes_{\max} C^*(\mathbb{F}_\infty)$. This characterization is useful to study $(\mathrm{el}, \mathrm{c})$-nuclearity of inductive limit of operator systems:

Theorem 4.6. *Let $\{\mathcal{S}_n\}_{n=1}^{\infty}$ be an inductive sequence of $(\mathrm{el}, \mathrm{c})$-nuclear operator systems (equivalently, with DCEP), then $\varinjlim \mathcal{S}_n$ is also $(\mathrm{el}, \mathrm{c})$-nuclear (equivalently, has DCEP).*

Proof. Since for each n, $\mathcal{S}_n$ is $(\mathrm{el}, \mathrm{c})$-nuclear, we have

$$\mathcal{S}_n \otimes_{\max} C^*(\mathbb{F}_\infty) = \mathcal{S}_n \otimes_{\min} C^*(\mathbb{F}_\infty).$$

Also, el is left injective ([8, Theorem 7.5]), therefore for each n, $u_n \otimes Id_{\mathcal{T}}$ is a complete order embedding, so that $\{\mathcal{S}_n \otimes_{\mathrm{c}} \mathcal{T}\}_{n=1}^{\infty}$ forms an inductive system with $\{u_n \otimes Id_{\mathcal{T}}\}_{n=1}^{\infty}$ as the connecting maps. Using this and the fact that maximal commuting tensor product c coincides with the operator system max tensor product if one of the tensorial factor is a C^*-algebra ([8, Theorem 6.7]), we have

$$\begin{aligned}
(\varinjlim \mathcal{S}_n) \otimes_{\min} C^*(\mathbb{F}_\infty) &= \varinjlim (\mathcal{S}_n \otimes_{\min} C^*(\mathbb{F}_\infty)) \text{ (Proposition 4.1)} \\
&= \varinjlim (\mathcal{S}_n \otimes_{\max} C^*(\mathbb{F}_\infty)) \\
&= \varinjlim (\mathcal{S}_n \otimes_{\mathrm{c}} C^*(\mathbb{F}_\infty)) \\
&= (\varinjlim \mathcal{S}_n) \otimes_{\mathrm{c}} C^*(\mathbb{F}_\infty) \text{ (Proposition 4.1)} \\
&= (\varinjlim \mathcal{S}_n) \otimes_{\max} C^*(\mathbb{F}_\infty)
\end{aligned}$$

Hence, $\varinjlim \mathcal{S}_n$ has DCEP provided each $\mathcal{S}_n$ has DCEP. □

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