

## EQUIVALENT DEFINITIONS OF RESCALED EXPANSIVENESS

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ABSTRACT. Recently, a new version of expansiveness which is closely attached to some certain weak version of hyperbolicity was given for  $C^1$  vector fields as following: a  $C^1$  vector field  $X$  will be called *rescaling expansive* on a compact invariant set  $\Lambda$  of  $X$  if for any  $\epsilon > 0$  there is  $\delta > 0$  such that, for any  $x, y \in \Lambda$  and any time reparametrization  $\theta : \mathbb{R} \rightarrow \mathbb{R}$ , if  $d(\varphi_t(x), \varphi_{\theta(t)}(y)) \leq \delta \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ , then  $\varphi_{\theta(t)}(y) \in \varphi_{(-\epsilon, \epsilon)}(\varphi_t(x))$  for all  $t \in \mathbb{R}$ . In this paper, some equivalent definitions for rescaled expansiveness are given.

### 1. Introduction

Expansiveness is a strong symbol of chaotic dynamics that has been studied extensively. For the discrete differentiable dynamical system, the expansiveness is well investigated and now we know that it is closely related to the hyperbolicity. But for flows generated by  $C^1$  vector fields, the things are more complicated than the situation of discrete differential dynamical systems.

Firstly, let us recall the expansiveness proposed by Bowen-Walters ([3]). Let  $\varphi_t$  be a continuous flow defined on a compact metric space  $M$ .  $\varphi_t$  is called *expansive* if for any  $\epsilon > 0$ , there is  $\delta > 0$  such that if  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta$  for all  $t \in \mathbb{R}$  hold for a pair of points  $x, y \in M$  and a continuous map  $s : \mathbb{R} \rightarrow \mathbb{R}$ , then  $\varphi_{s(0)}(y) = \varphi_t(x)$  where  $|t| < \epsilon$ . It is proved in [3] that if  $\varphi_t$  is expansive, then every fixed point of  $\varphi_t$  is an isolated point. So one can assume  $\varphi_t$  has no fixed point once the expansiveness is assumed. In [3] there are several equivalent definitions of this kind of expansiveness for fixed point-free flows that are given.

Beside the hyperbolic basic sets, there are some important examples with chaotic phenomena such as the geometric Lorenz attractors [5, 9]. Note that singularities (or fixed points) are involved in the geometric Lorenz attractor,

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which was persistently accumulated by periodic orbits. It is obviously that geometric Lorenz attractor does not admit the expansiveness proposed by Bowen and Walters. To understand the expansiveness for Lorenz attractor, there is a following definition introduced by Komuro [7]: A flow  $\varphi_t$  is  $K^*$ -*expansive* on a compact invariant set  $\Lambda$  of  $\varphi_t$  if for any  $\epsilon > 0$  there is  $\delta > 0$  such that, for any  $x$  and  $y$  in  $\Lambda$  and any surjective increasing continuous functions  $\theta : \mathbb{R} \rightarrow \mathbb{R}$ , if  $d(\varphi_t(x), \varphi_{\theta(t)}(y)) \leq \delta$  for all  $t \in \mathbb{R}$ , then  $\varphi_{\theta(t_0)}(y) \in \varphi_{(-\epsilon, \epsilon)}(\varphi_{t_0}(x))$  for some  $t_0 \in \mathbb{R}$ . Komuro [7] proved that the geometrical Lorenz attractor [5] is  $K^*$ -expansive. Now we know that the geometric Lorenz attractor can be characterized by the singular hyperbolicity proposed by Morales, Pacifico and Pujals [11]. It is naturally to ask whether the  $K^*$  expansiveness is satisfied for singular hyperbolic sets. Araujo-Pacifico-Pujals-Viana [1] proved that every singular hyperbolic attractor in a 3-dimensional manifold is  $K^*$ -expansive. For the relation between Bowen-Walters' expansiveness and  $K^*$ -expansiveness, one can see a discussion in [12].

Recently, Bonatti-da Luz [2] give a more general weak hyperbolicity which is called multisingular hyperbolicity. Multisingular hyperbolicity extends the notion of singular hyperbolicity and it well characterizes the classical star flows proposed by Liao and Mañé [8, 10] in the study of structural stabilities. Towards this new version of hyperbolicity for  $C^1$  vector fields, Wen-Wen proposed a new version of expansiveness for  $C^1$ -vector fields which is called *rescaled expansiveness*. They proved that every multisingular hyperbolicity set is rescaling expansive and the converse holds generically [13]. Let us recall the definition of rescaled expansiveness precisely. Let  $M$  be a compact Riemannian manifold without boundary and  $X$  be a  $C^1$  vector field on  $M$  that generates a flow  $\varphi_t$  on  $M$ .

**Definition 1.1.** A flow  $\varphi_t$  generated by a  $C^1$  vector field  $X$  is rescaling expansive on a compact invariant set  $\Lambda$  if for any  $\epsilon > 0$  there is  $\delta > 0$  such that, for any  $x, y \in \Lambda$  and any increasing continuous functions  $\theta : \mathbb{R} \rightarrow \mathbb{R}$ , if  $d(\varphi_t(x), \varphi_{\theta(t)}(y)) \leq \delta \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ , then  $\varphi_{\theta(t)}(y) \in \varphi_{[-\epsilon, \epsilon]}(\varphi_t(x))$  for all  $t \in \mathbb{R}$ .

This new definition of expansiveness was hinted by Liao's work on standard system ([8]) and some recent progress of Gan-Yang ([4]). Here we give a explanation that why the size of neighborhoods of a regular point will be rescaled by the flow speed. It is obvious that every regular point  $x$  has a neighborhood  $B_\delta(x)$  in  $M$  that is free of singularities. Here the radius  $\delta > 0$  depends on  $x$  and shrinks to 0 when  $x$  approaches a singularity. Nevertheless it can be proved that, for  $C^1$  vector field  $X$ , there is a uniform constant  $\delta_0 > 0$  such that, for every regular point  $x$ ,  $B_{\delta_0 \|X(x)\|}(x)$  is free of singularities. In other words, there is a "relative size"  $\delta_0$  of singularity-free neighborhoods of  $x$  for all regular points  $x$ . This uniformity is not a consequence of the compactness of the set of regular points (which is in fact noncompact) but the Lipschitz property of the vector field  $f$ . Likewise, there is also a "uniform relative"

feature for the continuity of maps and functions. For instance, let us take a look on line bundles  $\langle X(x) \rangle$  on the set of regular points  $x$ , where  $\langle X(x) \rangle$  denotes the span of the vector  $X(x)$ . The angle function is continuous, i.e., for every regular point  $x$  and every  $\epsilon > 0$ , there is  $\delta > 0$  such that for every regular point  $y$ , if  $d(x, y) < \delta$ , then  $\angle(\langle X(x) \rangle, \langle X(y) \rangle) < \epsilon$ . Clearly, the angle function is not uniformly continuous and the line bundle  $\langle X(x) \rangle$  is distorted badly near singularities. Nevertheless it can be proved that, for  $C^1$  vector fields, for every  $\epsilon > 0$  there is a uniform constant  $\delta_0 > 0$  such that, for every pair of regular points  $x$  and  $y$ , if  $d(x, y) < \delta_0 \|X(x)\|$ , then  $\angle(\langle X(x) \rangle, \langle X(y) \rangle) < \epsilon$ . In other words, the angle function is “relatively” uniformly continuous for  $C^1$  vector fields. These two observations for  $C^1$ -vector fields hint us to consider the rescaled expansiveness.

In [3], there are four equivalent definitions of expansiveness for nonsingular flows that are given (see Theorem 1 of [3]). Similarly, in this article, we give several equivalent definitions for rescaled expansiveness as following which is almost parallel to Theorem 1 of [3].

**Theorem 1.1.** *Let  $M$  be a compact Riemannian manifold without boundary and  $X$  be a  $C^1$  vector field on  $M$ . Let  $\varphi_t$  be the flow generated by  $X$  and  $\Lambda$  be a compact invariant set of  $\varphi_t$ . The following are mutually equivalent for  $X$ .*

- (i)  $X$  is rescaling expansive on  $\Lambda$ .
- (ii) For any  $\epsilon > 0$  there is  $\delta > 0$  such that, for any  $x, y \in \Lambda$  and a continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$ , if  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ , then  $\varphi_{s(t)}(y) \in \varphi_{[-\epsilon, \epsilon]}(\varphi_t(x))$  for all  $t \in \mathbb{R}$ .
- (iii) For any  $\epsilon > 0$  there is  $\alpha > 0$  such that, for any  $x, y \in \Lambda$  and an increasing homeomorphism  $h : \mathbb{R} \rightarrow \mathbb{R}$ , if

$$d(\varphi_t(x), \varphi_{h(t)}(y)) \leq \alpha \|X(\varphi_t(x))\|$$

for all  $t \in \mathbb{R}$ , then  $\varphi_{h(0)}(y) \in \varphi_{[-\epsilon, \epsilon]}(x)$ .

- (iv) For any  $\eta > 0$  there is  $\delta > 0$  such that, for any  $x, y \in \Lambda$  and any continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$ , if  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ , then  $y$  is on the same orbit as  $x$  and the orbit from  $\varphi_t(x)$  to  $\varphi_{s(t)}(y)$  lies inside  $B_{\eta \|X(\varphi_t(x))\|}(\varphi_t(x))$  for all  $t \in \mathbb{R}$ .
- (v) For any  $\epsilon > 0$  there is  $\alpha > 0$  with the following property: if  $\{t_i\}_{-\infty}^{+\infty}$  and  $\{u_i\}_{-\infty}^{+\infty}$  are doubly infinite sequences of real numbers with  $0 < t_i - t_{i-1} \leq \alpha$ ,  $|u_i - u_{i-1}| \leq \alpha$ ,  $t_i \rightarrow \infty$  and  $t_{-i} \rightarrow -\infty$  as  $i \rightarrow +\infty$ , and if  $x, y \in \Lambda$  satisfy  $d(\varphi_{t_i}(x), \varphi_{u_i}(y)) \leq \alpha \|X(\varphi_{t_i}(x))\|$  for all  $i \in \mathbb{Z}$ , then  $\varphi_{u_i}(y) \in \varphi_{[-\epsilon, \epsilon]}(\varphi_{t_i}(x))$  for all  $i \in \mathbb{Z}$ .

In Section 3 we give an application of Theorem 1.1, which says that with an additional hypothesis, rescaled expansiveness is an invariant quantity for  $C^1$ -orbit equivalent vector fields. This theorem and its application show that for  $C^1$  vector fields, after a rescaling by the flow speed, one can treat something for the singular flow similar to the nonsingular case.

## 2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. It is easy to see that (ii) implies (i) and (i) implies (iii), hence once we prove (iii) implies (ii), then items (i)-(iii) are equivalent. Now we proceed to prove (iii) $\Rightarrow$ (ii). Firstly we collect some known results for  $C^1$  vector fields.

As usual, denote

$$T_x M(r) = \{v \in T_x M : \|v\| \leq r\},$$

$$B_r(x) = \exp_x(T_x M(r)).$$

By the compactness of  $M$  and the  $C^1$  smoothness of  $X$ , there are constants  $L > 0$  and  $a > 0$  such that for any  $x \in M$  the vector fields

$$\bar{X} = (\exp_x^{-1})_*(X|_{B_a(x)})$$

in  $T_x M(a)$  are locally Lipschitz vector fields with a Lipschitz constant  $L$ . We call  $L$  a *local Lipschitz constant* of  $X$ . We may assume

$$m(D_p \exp_x) > 2/3, \quad \|D_p \exp_x\| < 3/2$$

for any  $p \in T_x M(a)$ .

We call  $x \in M$  a *singularity* of  $X$  if  $X(x) = 0$ . Denote  $\text{Sing}(X)$  the set of singularities of  $X$ . We call  $x \in M$  a *regular point* if  $x \in M \setminus \text{Sing}(X)$ .

**Lemma 2.1.** *There is  $r_0 > 0$  such that for any regular point  $x \in M$ , if  $d(y, x) \leq r_0 \|X(x)\|$ , then*

$$\frac{1}{2} \|X(x)\| \leq \|X(y)\| \leq 2 \|X(x)\|.$$

*Proof.* Let  $\bar{X} = (\exp_x^{-1})_*(X|_{B_a(x)})$  and  $L$  be the local Lipschitz constant as before. By the compactness of  $M$ , there is  $r_0$  such that  $r_0 \|X(x)\| \leq a$  for all  $x \in M$  and  $r_0 \leq \frac{1}{4L}$ . For any  $y \in B_{r_0 \|X(x)\|}(x)$ ,

$$\|\bar{X}(\exp_x^{-1}(y)) - X(x)\| = \|\bar{X}(\exp_x^{-1}(y)) - \bar{X}(0)\| \leq L \|\exp_x^{-1}(y)\| \leq \frac{1}{4} \|X(x)\|.$$

Then

$$\frac{3}{4} \|X(x)\| \leq \|\bar{X}(\exp_x^{-1}(y))\| \leq \frac{5}{4} \|X(x)\|.$$

Thus

$$\frac{1}{2} \|X(x)\| \leq \|X(y)\| = \|D \exp_x(\bar{X}(\exp_x^{-1}(y)))\| \leq 2 \|X(x)\|.$$

This ends the proof of lemma.  $\square$

For a regular point  $x \in M$  of  $X$ , denote the *normal space* of  $X(x)$  to be

$$N_x = N_x(X) = \{v \in T_x M : v \perp X(x)\}.$$

Given a constant  $r > 0$ , we can take a box

$$U_x(r \|X(x)\|) = \{v + tX(x) \in T_x M : v \in N_x, \|v\| \leq r \|X(x)\|, |t| \leq r\}$$

in  $T_x M$ . Define a  $C^1$  map

$$F_x : U_x(r\|X(x)\|) \rightarrow M$$

to be

$$F_x(v + tX(x)) = \varphi_t(\exp_x(v)).$$

This map  $F_x$  is called a *flowbox* of  $X$  at  $x$ . In [13], the following relative uniform version of flowbox theorem is proved.

**Proposition 2.2** ([13]). *For any  $C^1$  vector field  $X$  on  $M$ , there is  $0 < r_1 \leq \frac{1}{10L}$  such that for any regular point  $x$  of  $X$ ,  $F_x : U_x(r_1\|X(x)\|) \rightarrow M$  is an embedding whose image contains no singularities of  $X$ , and  $m(D_p F_x) > 1/3$  and  $\|D_p F_x\| < 3$  for every  $p \in U_x(r_1\|X(x)\|)$ .*

For any regular point  $x \in M$  and  $t \in \mathbb{R}$  with  $|t| \leq r_1$ , it is easy to see

$$(1) \quad d(x, \varphi_t(x)) = d(F_x(0), F_x(t\|X(x)\|)) < 3|t| \cdot \|X(x)\|$$

by the fact that  $\|D_p F_x\| < 3$  for all  $p \in U_x(r_1\|X(x)\|)$ .

By the fact that  $m(D_p F_x) > 1/3$  for all  $p \in U_x(r_1\|X(x)\|)$ , we can also get that for any regular point  $x \in M$  and any  $t \in \mathbb{R}$  with  $|t| \leq r_1$ , one has

$$(2) \quad d(x, \varphi_t(x)) = d(F_x(0), F_x(t\|X(x)\|)) > \frac{1}{3}|t| \cdot \|X(x)\|.$$

**Lemma 2.3.** *For any  $0 < T \leq \min\{r_0/3, r_1/20\}$  there is  $\delta_T = T/18$  such that for any  $x \in M \setminus \text{Sing}(X), y \in M$  and any continuous functions  $s : \mathbb{R} \rightarrow \mathbb{R}$ , if  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta_T \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ , then  $|s(t+T) - s(t)| < r_1$  for all  $t \in \mathbb{R}$ .*

*Proof.* Let  $x \in M \setminus \text{Sing}(X), y \in M$  and continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$  be given with property  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta_T \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ . It is easy to see that

$$\begin{aligned} & d(\varphi_{s(t)}(y), \varphi_{s(t+T)}(y)) \\ & \leq d(\varphi_t(x), \varphi_{t+T}(x)) + d(\varphi_t(x), \varphi_{s(t)}(y)) + d(\varphi_{s(t+T)}(y), \varphi_{t+T}(x)) \\ & < 3T\|X(\varphi_t(x))\| + \delta_T\|X(\varphi_t(x))\| + \delta_T\|X(\varphi_{t+T}(x))\| \end{aligned}$$

for any given  $t \in \mathbb{R}$ . Here  $d(\varphi_t(x), \varphi_{t+T}(x)) < 3T\|X(\varphi_t(x))\|$  is from (1). By the choice of  $T < r_0/3$ , we have  $d(\varphi_t(x), \varphi_{t+T}(x)) < r_0\|X(\varphi_t(x))\|$ , then  $\|X(\varphi_{t+T}(x))\| \leq 2\|X(\varphi_t(x))\|$  by Lemma 2.1. Hence

$$d(\varphi_{s(t)}(y), \varphi_{s(t+T)}(y)) \leq 3(T + \delta_T)\|X(\varphi_t(x))\|.$$

By the fact  $\delta_T \leq r_0$  and  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta_T\|X(\varphi_t(x))\|$ , we have

$$\|X(\varphi_t(x))\| \leq 2\|X(\varphi_{s(t)}(y))\|.$$

Hence

$$d(\varphi_{s(t)}(y), \varphi_{s(t+T)}(y)) \leq 6(T + \delta_T)\|X(\varphi_{s(t)}(y))\|.$$

From (2), we have

$$d(\varphi_{s(t)}(y), \varphi_{s(t+T)}(y)) > \frac{1}{3}|s(t+T) - s(t)| \cdot \|X(\varphi_{s(t)}(y))\|.$$

Thus we have

$$|s(t+T) - s(t)| < 18(T + \delta_T) = 19T < r_1.$$

This ends the proof of lemma.  $\square$

**Lemma 2.4.** *Let  $0 < T \leq \min\{r_0/3, r_1/20\}$  and  $\delta_T = T/18$  be given. Then for any  $x \in M \setminus \text{Sing}(X)$  and  $y \in M$  and a continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$ , if  $d(\varphi_t(x), \varphi_{s(t)}(y)) < \delta_T \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ , then  $s(t+T) - s(t) \geq 0$  for all  $t \in \mathbb{R}$ .*

*Proof.* We prove the lemma by a contradiction. Assume there is  $t \in \mathbb{R}$  such that  $s(t+T) - s(t) < 0$ . Let us consider the flow box at  $\varphi_t(x)$ . Note that  $|s(t+T) - s(t)| < r_1$ , by the same reason of (2), we have

$$d(\varphi_{t+T}(x), \varphi_{t+s(t+T)-s(t)}(x)) > \frac{1}{3}|T - (s(t+T) - s(t))| \cdot \|X(\varphi_t(x))\|.$$

Let  $L$  be the local Lipschitz constant of  $X$  given as before. Then by the classical continuous dependence on initial conditions theorem of ODE, we have

$$d(\varphi_{t+s(t+T)-s(t)}(x), \varphi_{s(t+T)}(y)) < e^{L(s(t)-s(t+T))} d(\varphi_t(x), \varphi_{s(t)}(y)).$$

From Lemma 2.3, we have  $s(t) - s(t+T) < r_1 \leq \frac{1}{10L}$ . Hence

$$d(\varphi_{t+s(t+T)-s(t)}(x), \varphi_{s(t+T)}(y)) < e^{Lr_1} \delta_T \|X(\varphi_t(x))\| < 2\delta_T \|X(\varphi_t(x))\|.$$

Then we have

$$\begin{aligned} & d(\varphi_{t+T}(x), \varphi_{s(t+T)}(y)) \\ & \geq d(\varphi_{t+T}(x), \varphi_{t+s(t+T)-s(t)}(x)) - d(\varphi_{t+s(t+T)-s(t)}(x), \varphi_{s(t+T)}(y)) \\ & > \left(\frac{1}{3}|T - (s(t+T) - s(t))| - 2\delta_T\right) \cdot \|X(\varphi_t(x))\| \\ & > \left(\frac{1}{3}T - 2\delta_T\right) \cdot \|X(\varphi_t(x))\|. \end{aligned}$$

Note that  $\|X(\varphi_t(x))\| \geq \frac{1}{2}\|X(\varphi_{t+T}(x))\|$  by the choice of  $T$ . We have a contradiction

$$d(\varphi_{t+T}(x), \varphi_{s(t+T)}(y)) > \frac{1}{2}\left(\frac{1}{3}T - 2\delta_T\right) \cdot \|X(\varphi_{t+T}(x))\| > \delta_T \|X(\varphi_{t+T}(x))\|.$$

This ends the proof of lemma.  $\square$

**Lemma 2.5.** *For any  $0 < T \leq \min\{r_0/3, r_1/20\}$  there is  $\delta_T = T/18$ , such that for any  $x \in M \setminus \text{Sing}(X)$ ,  $y \in M$  and any continuous functions  $s : \mathbb{R} \rightarrow \mathbb{R}$ , if  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta_T \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ , then  $s(t+T) - s(t) \geq T/36 > 0$  for all  $t \in \mathbb{R}$ .*

*Proof.* Let  $0 < T \leq \min\{r_0/3, r_1/20\}$  and  $\delta_T = T/18$  be given. Assume there is  $x \in M \setminus \text{Sing}(X), y \in M$  and a continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$  such that  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta_T \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ . Then for any  $t \in \mathbb{R}$

$$\begin{aligned} & d(\varphi_{s(t)}(y), \varphi_{s(t+T)}(y)) \\ & \geq d(\varphi_t(x), \varphi_{t+T}(x)) - d(\varphi_t(x), \varphi_{s(t)}(y)) - d(\varphi_{s(t+T)}(y), \varphi_{t+T}(x)) \\ & > \frac{1}{3}T \cdot \|X(\varphi_t(x))\| - \delta_T \|X(\varphi_t(x))\| - \delta_T \|X(\varphi_{t+T}(x))\| \\ & > \frac{1}{3}T \cdot \|X(\varphi_t(x))\| - \delta_T \|X(\varphi_t(x))\| - 2\delta_T \|X(\varphi_t(x))\|. \end{aligned}$$

Here we used the fact  $d(\varphi_t(x), \varphi_{t+T}(x)) > \frac{1}{3}T \cdot \|X(\varphi_t(x))\|$  from (2) and  $\|X(\varphi_{t+T}(x))\| \leq 2\|X(\varphi_t(x))\|$  from Lemma 2.1. Note that  $d(\varphi_{s(t)}(y), \varphi_t(x)) \leq \delta_T \|X(\varphi_t(x))\| < r_0 \|X(\varphi_t(x))\|$ , we have  $\|X(\varphi_{s(t)}(y))\| \geq \frac{1}{2}\|X(\varphi_t(x))\|$ . Hence

$$d(\varphi_{s(t)}(y), \varphi_{s(t+T)}(y)) > \left(\frac{1}{3}T - 3\delta_T\right) \cdot \|X(\varphi_t(x))\| \geq \frac{1}{2} \left(\frac{1}{3}T - 3\delta_T\right) \cdot \|X(\varphi_{s(t)}(y))\|.$$

Note that  $s(t+T) - s(t) \geq 0$ . From (1), we can get

$$s(t+T) - s(t) \geq \frac{1}{3} \cdot \frac{1}{2} \cdot \left(\frac{1}{3}T - 3\delta_T\right) = \frac{T}{36}.$$

This proves the lemma.  $\square$

*Remark 2.6.* From this lemma we can see that the condition  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta_T \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$  forces the continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$  to be a surjective on  $\mathbb{R}$  since  $s(nT) \rightarrow +\infty$  and  $s(-nT) \rightarrow -\infty$  as  $n \rightarrow +\infty$ .

Now we proceed to prove (iii) implies (ii). Suppose (iii) is satisfied on a compact invariant set  $\Lambda$ . Let  $\epsilon > 0$  be given and  $\alpha > 0$  be the constant in (iii) corresponding to  $\epsilon$ . Choose  $0 < T \leq \min\{r_0/3, r_1/20, \alpha/6\}$  and  $\delta = \min\{\alpha/4, \delta_T\}$  where  $\delta_T$  is the constant in Lemma 2.5 corresponding to  $T$ . We will prove that  $\delta$  satisfies the request in (ii).

Let  $x, y \in \Lambda$  and a continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$  with  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ . If  $x \in \text{Sing}(X)$ , then it is easy to see that  $y = x$ . It is trivial that  $\varphi_{s(t)}(y) \in \varphi_{[-\epsilon, \epsilon]}(\varphi_t(x))$  for all  $t \in \mathbb{R}$ . Now we assume  $x \in \Lambda \setminus \text{Sing}(X)$ . By the choice of  $\delta \leq \delta_T$ , we have  $s((n+1)T) - s(nT) \geq T/36 > 0$  for all  $n \in \mathbb{Z}$ . Define  $h : \mathbb{R} \rightarrow \mathbb{R}$  by  $h(nT) = s(nT)$ ,  $n \in \mathbb{Z}$  and by linearity on each interval  $[nT, (n+1)T]$ . It is easy to see that  $h$  is an increasing homeomorphism of  $\mathbb{R}$ . For any  $t \in [nT, (n+1)T]$ , by the intermediate value theorem of continuous function, there exists  $t' \in [nT, (n+1)T]$  with  $h(t) = s(t')$ . Then we have

$$\begin{aligned} d(\varphi_t(x), \varphi_{h(t)}(y)) &= d(\varphi_t(x), \varphi_{s(t')}(y)) \\ &\leq d(\varphi_t(x), \varphi_{t'}(x)) + d(\varphi_{t'}(x), \varphi_{s(t')}(y)) \\ &< 3|t - t'| \cdot \|X(\varphi_t(x))\| + \delta \|X(\varphi_{t'}(x))\| \\ &< 3T \cdot \|X(\varphi_t(x))\| + 2\delta \|X(\varphi_t(x))\| \end{aligned}$$

$$< \alpha \|X(\varphi_t(x))\|.$$

By (iii), we have

$$\varphi_{s(0)}(y) = \varphi_{h(0)}(y) \in \varphi_{[-\epsilon, \epsilon]}(x).$$

Now we have proved that for any  $x, y \in \Lambda$  and continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$ , if  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ , then  $\varphi_{s(0)}(y) \in \varphi_{[-\epsilon, \epsilon]}(x)$ . We proceed to prove that  $\varphi_{s(\tau)}(y) \in \varphi_{[-\epsilon, \epsilon]}(\varphi_\tau(x))$  for all  $\tau \in \mathbb{R}$ . For any given  $\tau \in \mathbb{R}$ , let  $x' = \varphi_\tau(x)$  and  $y' = \varphi_{s(\tau)}(y)$  and  $s'(t) = s(t + \tau) - s(\tau)$ . Then we have

$$d(\varphi_t(x'), \varphi_{s'(t)}(y')) = d(\varphi_{t+\tau}(x), \varphi_{s(t+\tau)}(y)) \leq \delta \|X(\varphi_{t+\tau}(x))\| = \delta \|X(\varphi_t(x'))\|$$

for all  $t \in \mathbb{R}$ . Hence we have

$$\varphi_{s(\tau)}(y) = y' = \varphi_{s'(0)}(y') \in \varphi_{[-\epsilon, \epsilon]}(x') = \varphi_{[-\epsilon, \epsilon]}(\varphi_\tau(x)).$$

This proves  $\delta$  satisfies the request of (ii) and (iii) implies (ii).

Now we proceed to prove (ii) implies (v). Suppose (ii) is satisfied on a compact invariant set  $\Lambda$ . Let  $\epsilon > 0$  be given and  $\delta > 0$  be the constant in (ii) corresponding to  $\epsilon$ . Choose  $\alpha \leq \min\{r_1, r_0/5, \delta/11\}$ . We will prove that  $\alpha$  satisfies the request in (v).

Let  $(t_i)_{-\infty}^{+\infty}$ ,  $(u_i)_{-\infty}^{+\infty}$  and  $x, y \in \Lambda$  satisfy the hypotheses of (v) corresponding to the constant  $\alpha$ . If  $x$  is a singularity, then it is trivially to see that  $y = x$ , then  $\varphi_{u_i}(y) \in \varphi_{[-\epsilon, \epsilon]}(\varphi_{t_i}(x))$  automatically. Now we assume  $x \in \Lambda \setminus \text{Sing}(X)$ . Define  $s : \mathbb{R} \rightarrow \mathbb{R}$  by  $s(t_i) = u_i$  and by extending linearly on each interval  $[t_i, t_{i+1}]$ . Then for any  $t \in [t_i, t_{i+1}]$ ,

$$d(\varphi_{t_i}(x), \varphi_t(x)) < 3|t_i - t| \cdot \|X(\varphi_t(x))\| \leq 3\alpha \|X(\varphi_t(x))\| < r_0 \|X(\varphi_t(x))\|.$$

Hence  $\|X(\varphi_{t_i}(x))\| \leq 2\|X(\varphi_t(x))\|$ . Then we have

$$\begin{aligned} d(\varphi_{u_i}(y), \varphi_t(x)) &\leq d(\varphi_{u_i}(y), \varphi_{t_i}(x)) + d(\varphi_{t_i}(x), \varphi_t(x)) \\ &\leq \alpha \|X(\varphi_{t_i}(x))\| + 3\alpha \|X(\varphi_t(x))\| \\ &\leq 5\alpha \|X(\varphi_t(x))\| \leq r_0 \|X(\varphi_t(x))\|. \end{aligned}$$

Hence  $\|X(\varphi_{t_i}(x))\| \leq 2\|X(\varphi_t(x))\|$ . Then we have

$$\begin{aligned} &d(\varphi_t(x), \varphi_{s(t)}(y)) \\ &\leq d(\varphi_t(x), \varphi_{t_i}(x)) + d(\varphi_{t_i}(x), \varphi_{s(t_i)}(y)) + d(\varphi_{s(t_i)}(y), \varphi_{s(t)}(y)) \\ &\leq 3\alpha \cdot \|X(\varphi_t(x))\| + \alpha \|X(\varphi_{t_i}(x))\| + 3|s(t_i) - s(t)| \cdot \|X(\varphi_{s(t_i)}(y))\| \\ &\leq 11\alpha \cdot \|X(\varphi_t(x))\| \leq \delta \|X(\varphi_t(x))\|. \end{aligned}$$

By the assumption of (ii) is satisfied, we have  $\varphi_{s(t)}(y) \in \varphi_{[-\epsilon, \epsilon]}(\varphi_t(x))$  for all  $t \in \mathbb{R}$ . Then  $\varphi_{u_i}(y) \in \varphi_{[-\epsilon, \epsilon]}(\varphi_{t_i}(x))$  for all  $i \in \mathbb{Z}$ . This ends the proof of (ii) implies (v).

Now we show (v) implies (iii). Suppose (v) is satisfied on a compact invariant set  $\Lambda$ . Let  $\epsilon > 0$  be given and  $\alpha > 0$  be the constant in (v) corresponding to  $\epsilon$ . Let  $x, y \in \Lambda$  and a continuous function  $h : \mathbb{R} \rightarrow \mathbb{R}$  with  $d(\varphi_t(x), \varphi_{h(t)}(y)) < \alpha \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ . If  $x \in \text{Sing}(X)$ , then it is easy to see that  $y = x$ .



It is trivial that  $\varphi_{h(0)}(y) \in \varphi_{[-\epsilon, \epsilon]}(x)$ . Now let us assume  $x \in \Lambda \setminus \text{Sing}(X)$ . Suppose  $d(\varphi_t(x), \varphi_{h(t)}(y)) \leq \alpha \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$  and some increasing homomorphism  $h$  of  $\mathbb{R}$ . Let  $t_0 = 0$ . We will give  $t_i, u_i$  inductively. For any  $i \in \mathbb{N}$ , if  $t_i$  is given, then we take  $t_{i+1} > t_i$  such that

$$\max\{t_{i+1} - t_i, h(t_{i+1}) - h(t_i)\} = \alpha.$$

Similarly, if  $t_{-i}$  is given, then take  $t_{-i-1}$  such that

$$\max\{t_{-i} - t_{-i-1}, h(t_{-i}) - h(t_{-i-1})\} = \alpha.$$

Let  $u_i = h(t_i)$ . From the construction, it is obvious that  $0 < t_{i+1} - t_i \leq \alpha$ ,  $0 < u_{i+1} - u_i \leq \alpha$  for all  $i \in \mathbb{Z}$ . For any  $i > 0$ , it is easy to see that  $t_{i+1} + h(t_{i+1}) - h(0) \geq i\alpha$ . Hence either  $t_i \rightarrow +\infty$  or  $h(t_i) \rightarrow +\infty$  as  $i \rightarrow +\infty$ . By the assumption that  $h$  is a homeomorphism on  $\mathbb{R}$ , we have  $t_i \rightarrow +\infty$  in both cases. Similarly, we have  $t_{-i} \rightarrow -\infty$  as  $i \rightarrow +\infty$ . By the assumption that  $d(\varphi_t(x), \varphi_{h(t)}(y)) \leq \alpha \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ , we have  $d(\varphi_{t_i}(x), \varphi_{u_i}(y)) \leq \alpha \|X(\varphi_{t_i}(x))\|$  for all  $i \in \mathbb{Z}$ . Hence we have

$$\varphi_{h(0)}(y) = \varphi_{h(t_0)}(y) = \varphi_{u_0}(y) \in \varphi_{[-\epsilon, \epsilon]}(\varphi_{t_0}(x)) = \varphi_{[-\epsilon, \epsilon]}(x).$$

This proves the fact that (v) implies (iii).

We have shown that (i), (ii), (iii), (v) are equivalent, so it remains to show (ii) and (iv) are equivalent.

To show (ii) implies (iv). Suppose (ii) is satisfied on a compact invariant set  $\Lambda$ . Let  $\eta > 0$  be given. Then we can take  $\epsilon = \min\{\eta/3, r_1\}$  and  $\delta > 0$  be the constant in (ii) corresponding to  $\epsilon$ . We will prove that  $\delta$  satisfy the request of (iv). Let  $x, y \in \Lambda$  and a continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$  be given such that  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ . Then by the choice of  $\delta$ , we have  $\varphi_{s(t)}(y) \in \varphi_{[-\epsilon, \epsilon]}(\varphi_t(x))$ . It is obvious that  $y$  is in the orbit of  $x$  and since  $\varphi_{s(t)}(y) = \varphi_{t+t'}(x)$  for some  $|t'| \leq \epsilon$ , then from (1) we have that the orbit from  $\varphi_t(x)$  to  $\varphi_{s(t)}(y)$  lies inside  $B_{\eta \|X(\varphi_t(x))\|}(\varphi_t(x))$ .

Conversely, suppose (iv) is given and we prove (ii). Let  $\epsilon > 0$  be given. Take  $\eta = \min\{\epsilon/3, r_1\}$  and  $\delta > 0$  be the constant in (iv) corresponding to  $\eta$ . Let  $x, y \in \Lambda$  and a continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$  be given such that  $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \delta \|X(\varphi_t(x))\|$  for all  $t \in \mathbb{R}$ . Then by the choice of  $\delta$ , we have  $y$  is in the orbit of  $x$  and the orbit from  $\varphi_t(x)$  to  $\varphi_{s(t)}(y)$  lies in  $B_{\eta \|X(\varphi_t(x))\|}(\varphi_t(x))$ . That is, there is  $\tau \in \mathbb{R}$  such that  $\varphi_{s(t)}(y) = \varphi_\tau(x)$  and  $\varphi_{[t, \tau]}(x) \subset B_{\eta \|X(\varphi_t(x))\|}(\varphi_t(x))$ . We will to show that  $|t - \tau| \leq \epsilon$ . Assume the contrary,  $|t - \tau| > \epsilon$ . Without loss of generality, we assume  $\tau - t > \epsilon$ . Then  $\varphi_{t+\epsilon}(x) \in \varphi_{[t, \tau]}(x)$ , but we have

$$d(\varphi_{t+\epsilon}(x), \varphi_t(x)) > \frac{1}{3} \epsilon \|X(\varphi_t(x))\| \geq \eta \|X(\varphi_t(x))\|.$$

This contradicts with  $\varphi_{[t, \tau]}(x) \subset B_{\eta \|X(\varphi_t(x))\|}(\varphi_t(x))$ . This proves that (iv) implies (v) and ends the proof of theorem.

### 3. An application of Theorem 1.1

In this section we will give an application of Theorem 1.1. We will show that under a certain kind of orbit equivalence, the rescaled expansiveness is an invariant quantity. Recall that a  $C^1$  flow  $\phi_t$  on  $M$  is a *time change* of  $\varphi_t$  on  $M$  if the orbits  $\{\phi_t(x) : t \in \mathbb{R}\}$  and  $\{\varphi_t(x) : t \in \mathbb{R}\}$  coincide and the orientations given by the change of  $t$  in the positive direction are the same for all  $x \in M$ . Let  $Y$  be a  $C^1$  vector field on a manifold  $N$  and  $\phi_t$  is the  $C^1$  flow generated by  $Y$ , we say that  $Y$  (or  $\phi_t$ ) is  *$C^1$ -orbit equivalent* to  $X$  (or  $\varphi_t$ ) if there is a  $C^1$  diffeomorphism  $\lambda : M \rightarrow N$  such that  $\lambda^{-1} \circ \phi_t \circ \lambda$  is a time change of  $\varphi_t$ . One see [6] for a precise introduction on orbit equivalence of flows.

**Lemma 3.1.** *Let  $X$  and  $Y$  be two  $C^1$  vector fields on  $M$  and  $N$  respectively. If  $X$  and  $Y$  are  $C^1$ -orbit equivalent with a  $C^1$  conjugacy map  $\lambda : M \rightarrow N$  and all of their singularities are hyperbolic, then there exist two constants  $0 < c_1 < c_2$  such that*

$$c_1 \|X(x)\| \leq \|Y(\lambda(x))\| \leq c_2 \|X(x)\|$$

for all  $x \in M$ .

*Proof.* Since all singularity of  $X$  are hyperbolic we know that  $X$  has only finitely many singularities. Since  $X$  and  $Y$  are  $C^1$ -orbit equivalent we know that  $\sigma$  is a singularity of  $X$  if and only if  $\lambda(\sigma)$  is a singularity of  $Y$ . At every point  $x \in M \setminus \text{Sing}(X)$ , we know that  $\|Y(\lambda(x))\|/\|X(x)\|$  is strictly positive and varies continuously on  $x \in M \setminus \text{Sing}(X)$ . To prove the lemma we just need to prove that for every singularity  $\sigma$  of  $X$  there exist a neighborhood  $U \subset M$  of  $\sigma$  and constants  $0 < c_1(\sigma) < c_2(\sigma)$  such that for any  $x \in U$ , one has

$$c_1(\sigma) \|X(x)\| \leq \|Y(\lambda(x))\| \leq c_2(\sigma) \|X(x)\|$$

for any  $x \in U$ .

By the fact that  $\lambda$  is a  $C^1$  diffeomorphism from a compact manifold  $M$  to  $N$ , we can take  $K > 0$  such that  $\max\{\|D_x \lambda\|, \|D_{\lambda(x)} \lambda^{-1}\|\} \leq K$  for all  $x \in M$ . Fix a singularity  $\sigma$  of  $X$ . By the fact that  $\sigma$  is hyperbolic we know that  $DX(\sigma)$  has no 0 eigenvalue, hence there exists a neighborhood  $U$  with a constants  $C$  such that for any  $x \in U$ , one has  $C^{-1}d(x, \sigma) \leq \|X(x)\| \leq Cd(x, \sigma)$ . Similarly, from the fact that 0 is not an eigenvalue of  $DY(\lambda(\sigma))$ , we can choose a neighborhood  $V \subset N$  of  $\lambda(\sigma)$  with a constant  $C'$  such that for any  $y \in V$ , one has  $C'^{-1}d(y, \lambda(\sigma)) \leq \|Y(y)\| \leq C'd(y, \lambda(\sigma))$ . Without loss of generality, we assume that  $U$  is choosing small enough such that  $\lambda(U) \subset V$ . Then for any  $x \in U$ , it is not hard to see that

$$K^{-1}C'^{-1}C^{-1} \leq \frac{\|Y(\lambda(x))\|}{\|X(x)\|} \leq KC'C.$$

Then

$$K^{-1}C'^{-1}C^{-1}\|X(x)\| \leq \|Y(\lambda(x))\| \leq KC'C\|X(x)\|.$$

This ends the proof of the lemma.  $\square$

**Proposition 3.2.** *Suppose vector field  $X$  on manifold  $M$  and vector field  $Y$  on manifold  $N$  are  $C^1$ -orbit equivalent and all of their singularities are hyperbolic. If  $X$  is rescaling expansive on  $M$ , then  $Y$  is rescaling expansive on  $N$ .*

*Proof.* We will use item (iv) in Theorem 1.1 to prove the proposition. Let  $\eta > 0$  be given. Let  $K > 0$  be the constant with property  $\max\{\|D_x \lambda\|, \|D_{\lambda(x)} \lambda^{-1}\|\} \leq K$  for all  $x \in M$  and  $c_2 > c_1 > 0$  be the constants given in the above lemma. Then we can take  $\eta' = K^{-1}c_1\eta$  such that

$$\lambda(B_{\eta'\|X(x)\|}(x)) \subset B_{\eta(\|Y(\lambda(x))\|)}(x).$$

Let  $\delta'$  be the number in item (iv) of Theorem 1.1 corresponds to  $\eta'$  and  $\delta = K^{-1}c_2^{-1}\delta'$ .

Denote by  $\phi_t$  the flow generated by  $Y$ . Since  $Y$  is  $C^1$ -orbit equivalent to  $X$ , letting  $\tilde{\phi}_t = \lambda^{-1} \circ \phi_t \circ \lambda$ , then there is a function  $\tau(x, t) : M \times \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\tilde{\phi}_t(x) = \varphi_{\tau(x,t)}(x).$$

We know that  $\tau(x, t)$  is a increasing diffeomorphism from  $\mathbb{R}$  to  $\mathbb{R}$  for every fixed regular point  $x \in M \setminus \text{Sing}(X)$ . Write  $\tau(x, t)$  to be  $\tau_x(t)$  for convenience. Assume there are two points  $y_1, y_2 \in N$  with a continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$d(\phi_t(y_1), \phi_{s(t)}(y_2)) \leq \delta \|Y(\phi_t(y_1))\|$$

for all  $t \in \mathbb{R}$ . Then we have

$$d(\lambda^{-1}(\phi_t(y_1)), \lambda^{-1}(\phi_{s(t)}(y_2))) \leq K\delta \|Y(\phi_t(y_1))\|$$

for all  $t \in \mathbb{R}$ . Let  $x_1 = \lambda^{-1}(y_1), x_2 = \lambda^{-1}(y_2)$ . Then  $\lambda^{-1}(\phi_t(y_1)) = \varphi_{\tau_{x_1}(t)}(x_1)$  and  $\lambda^{-1}(\phi_{s(t)}(y_2)) = \varphi_{\tau_{x_2}(s(t))}(x_2)$ . Hence we have

$$d(\varphi_{\tau_{x_1}(t)}(x_1), \varphi_{\tau_{x_2}(s(t))}(x_2)) \leq K\delta \|Y(\phi_t(y_1))\| \leq \delta' \|X(\varphi_{\tau_{x_1}(t)}(x_1))\|$$

for all  $t \in \mathbb{R}$  or

$$d(\varphi_t(x_1), \varphi_{\tau_{x_2}(s(\tau_{x_1}^{-1}(t)))}(x_2)) \leq \delta' \|X(\varphi_t(x_1))\|$$

for all  $t \in \mathbb{R}$ . By the choice of  $\delta'$  we know that  $x_2$  is in the orbit of  $x_1$  and the orbit segment from  $\varphi_t(x_1)$  to  $\varphi_{\tau_{x_2}(s(\tau_{x_1}^{-1}(t)))}(x_2)$  lies inside  $B_{\eta'\|X(\varphi_t(x_1))\|}(\varphi_t(x_1))$  for every  $t \in \mathbb{R}$ . Then we know that  $y_2$  is contained in the orbit of  $y_1$  with respect to  $\phi_t$  and the orbit segment from  $\phi_t(y_1) = \lambda(\varphi_{\tau_{x_1}(t)}(x_1))$  to  $\phi_{s(t)}(y_2) = \lambda(\varphi_{\tau_{x_2}(s(t))}(x_2))$  is contained in  $\lambda(B_{\eta'\|X(\varphi_{\tau_{x_1}(t)}(x_1))\|}(\varphi_{\tau_{x_1}(t)}(x_1)))$  and then in  $B_{\eta\|Y(\phi_t(y_1))\|}(\phi_t(y_1))$ . Now the item (iv) is verified for  $Y$ , hence  $Y$  is also rescaling expansive on  $N$ . This ends the proof of the proposition.  $\square$

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