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OPENNESS OF ANOSOV FAMILIES

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Abstract. Anosov families were introduced by A. Fisher and P. Arnoux motivated by generalizing the notion of Anosov diffeomorphism defined on a compact Riemannian manifold. Roughly, an Anosov family is a two-sided sequence of diffeomorphisms (or non-stationary dynamical system) with similar behavior to an Anosov diffeomorphisms. We show that the set consisting of Anosov families is an open subset of the set consisting of two-sided sequences of diffeomorphisms, which is equipped with the strong topology (or Whitney topology).

1. Introduction

Anosov families were introduced by P. Arnoux and A. Fisher in [1], motivated by generalizing the notion of Anosov diffeomorphisms. Roughly, an Anosov family is a two-sided sequence of diffeomorphisms $\boldsymbol{f} = (f_i)_{i\in\mathbb{Z}}$ defined on a two-sided sequence of compact Riemannian manifolds $(M_i)_{i\in\mathbb{Z}}$, which has a similar behavior to an Anosov diffeomorphisms, that is, each tangent bundle TM_i has a splitting into two subbundles, called stable and unstable subbundles, where the elements in the stable subbundle are contracted by $D(f_{i+n-1}\circ\cdots\circ f_i)$ and the elements in the unstable subbundle are contracted by $D(f_{i-n}^{-1}\circ\cdots\circ f_{i-1}^{-1})$, for $n \geq 1$. The study of sequences of maps is know in the literature with several different names: non-stationary dynamical systems, non-autonomous dynamical systems, sequences of mappings, among other names (see [1], [2], [3], [7]).

Other approaches dealing sequences of diffeomorphisms with hyperbolic behaviour can be found in [2], [3], [10], among other works. One difference between the notion considered in this paper and the considered in the works above mentioned is that the f_i's of the Anosov families do not necessarily are Anosov diffeomorphisms (see [1], Example 3). Furthermore, the M_i's, although they are diffeomorphic, they are not necessarily isometric, thus, the hyperbolicity could be induced by the Riemannian metrics (see [1], [6] for more detail).

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Let $\mathbf{M}$ be the disjoint union of the M_i's, for $i \in \mathbb{Z}$, and $\mathcal{F}(\mathbf{M})$ the set consisting of the families of C^1-diffeomorphisms on $\mathbf{M}$ endowed with the *strong topology* (see Definition 2.3). We denote by $\mathcal{A}(\mathbf{M})$ the subset of $\mathcal{F}(\mathbf{M})$ consisting of Anosov families. We will prove that $\mathcal{A}(\mathbf{M})$ is an open subset of $\mathcal{F}(\mathbf{M})$: for any $(f_i)_{i\in\mathbb{Z}} \in \mathcal{A}(\mathbf{M})$, there exists a sequence of positive numbers $(\varepsilon_i)_{i\in\mathbb{Z}}$ such that if $(g_i)_{i\in\mathbb{Z}} \in \mathcal{F}(\mathbf{M})$, with $d^1(f_i, g_i) < \varepsilon_i$ for all $i \in \mathbb{Z}$, where d^1 is the C^1-metric on $\mathrm{Diff}^1(M_i, M_{i+1})$, then $(g_i)_{i\in\mathbb{Z}} \in \mathcal{A}(\mathbf{M})$. An important implication of this result is the great variety of non-trivial examples that it provides (in [1] and [8] can be found non-trivial examples of Anosov families, thus with the openness of $\mathcal{A}(\mathbf{M})$ we have that, in a certain way, these examples are not isolated), since we only ask that the family be Anosov and we do not ask for any additional condition. For that reason, the sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ could no be taken constant (it can converge to zero as $i \to \pm\infty$). In [8] we proved that, if the second derivative of the family is bounded and the angles between the stable and unstable subspaces induced by the family are bounded away from zero, then $(\varepsilon_i)_{i\in\mathbb{Z}}$ can be taken constant.

Young in [12] proved that families consisting of C^{1+1} random small perturbations of an Anosov diffeomorphism of class C^2 are Anosov families (see Remark 2.7). Our result is a generalization of this fact, since, as we said, Anosov families are not necessarily sequences of Anosov diffeomorphisms. This fact will be fundamental to prove the structural stability of certain elements in $\mathcal{A}(\mathbf{M})$, considering the uniform conjugacies to be given in Definition 2.2 (see [8]), which generalizes the structural stability of random small perturbations of hyperbolic diffeomorphisms, shown by P. D. Liu in [5].

In the next section we define the class of objects to be studied in this work. We define the composition law for a two-sided sequence of diffeomorphisms, the strong topology and a type of conjugations which work for the class of families of diffeomorphisms. Furthermore, we introduce the notion of Anosov family and we present some examples of such families. In Section 3 we will see several properties that satisfy the Anosov families. It is important to keep fixed the Riemannian metric on each M_i, since the notion of Anosov family depends on the Riemannian metric (see [1], Example 4). Other examples and properties of Anosov families can be found in [1], [6] and [8]. In Section 4 we will prove that each family close to an Anosov family satisfies the property of the invariant cones (see Lemma 4.7). This fact will be fundamental for showing the openness of Anosov families, which will be proved in Theorem 5.4.

2. Anosov families: definition, examples and uniform conjugacy

Given a two-sided sequence of Riemannian manifolds M_i with Riemannian metric $\langle \cdot, \cdot \rangle_i$ for $i \in \mathbb{Z}$, consider the *disjoint union*

$$\mathbf{M} = \coprod_{i\in\mathbb{Z}} M_i = \bigcup_{i\in\mathbb{Z}} M_i \times i.$$

The set $\mathbf{M}$ will be called *total space* and the M_i will be called *components*. We give the total space $\mathbf{M}$ the Riemannian metric $\langle\cdot,\cdot\rangle$ induced by $\langle\cdot,\cdot\rangle_i$ setting

$$\langle\cdot,\cdot\rangle|_{M_i} = \langle\cdot,\cdot\rangle_i \quad \text{for } i\in\mathbb{Z}, \tag{2.1}$$

and we will use the notation $(\mathbf{M},\langle\cdot,\cdot\rangle)$ for point out that we are considering the Riemannian metric given in (2.1). We denote by $\|\cdot\|_i$ the induced norm by $\langle\cdot,\cdot\rangle_i$ on TM_i and we will take $\|\cdot\|$ defined on $\mathbf{M}$ as $\|\cdot\||_{M_i} = \|\cdot\|_i$ for $i\in\mathbb{Z}$. If $d_i(\cdot,\cdot)$ is the metric on M_i induced by $\langle\cdot,\cdot\rangle_i$, the total space is equipped with the metric

$$d(x,y) = \begin{cases} \min\{1, d_i(x,y)\} & \text{if } x,y\in M_i \\ 1 & \text{if } x\in M_i, y\in M_j \text{ and } i\neq j. \end{cases} \tag{2.2}$$

Definition 2.1. A *non-stationary dynamical system* (or *n.s.d.s.*) $(\mathbf{M},\langle\cdot,\cdot\rangle,\boldsymbol{f})$ is a map $\boldsymbol{f}:\mathbf{M}\to\mathbf{M}$ such that, for each $i\in\mathbb{Z}$, $\boldsymbol{f}|_{M_i} = f_i : M_i\to M_{i+1}$ is a C^1-diffeomorphism. Sometimes we use the notation $\boldsymbol{f} = (f_i)_{i\in\mathbb{Z}}$. The n-th composition is defined to be

$$\boldsymbol{f}_i^n := \begin{cases} f_{i+n-1}\circ\cdots\circ f_i : M_i\to M_{i+n} & \text{if } n>0 \\ f_{i-n}^{-1}\circ\cdots\circ f_{i-1}^{-1} : M_i\to M_{i-n} & \text{if } n<0 \\ I_i : M_i\to M_i & \text{if } n=0, \end{cases}$$

where $I_i : M_i\to M_i$ is the identity on M_i (see Figure 1).

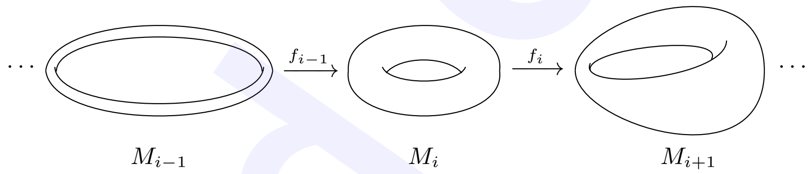

FIGURE 1. A non-stationary dynamical system on a sequence of 2-torus endowed with different Riemannian metrics.

One type of conjugacy that works for the class of non-stationary dynamical systems is the *uniform conjugacy*:

Definition 2.2. A *uniform conjugacy* between two n.s.d.s. $\boldsymbol{f} = (f_i)_{i\in\mathbb{Z}}$ and $\boldsymbol{g} = (g_i)_{i\in\mathbb{Z}}$ on $\mathbf{M}$ is a map $\boldsymbol{h}:\mathbf{M}\to\mathbf{M}$, such that $\boldsymbol{h}|_{M_i} = h_i : M_i\to M_i$ is a homeomorphism, $(h_i : M_i\to M_i)_{i\in\mathbb{Z}}$ and $(h_i^{-1} : M_i\to M_i)_{i\in\mathbb{Z}}$ are equicontinuous families and $\boldsymbol{h}$ is a *topological conjucacy* between the systems, i.e., $h_{i+1}\circ f_i = g_i\circ h_i : M_i\to M_{i+1}$ for every $i\in\mathbb{Z}$. This fact means that the following diagram commutes:

$$\begin{array}{ccccccc} M_{-1} & \xrightarrow{f_{-1}} & M_0 & \xrightarrow{f_0} & M_1 & \xrightarrow{f_1} & M_2 \\ \cdots \big\downarrow h_{-1} & & \big\downarrow h_0 & & \big\downarrow h_1 & & \big\downarrow h_2 \cdots \\ M_{-1} & \xrightarrow{g_{-1}} & M_0 & \xrightarrow{g_0} & M_1 & \xrightarrow{g_1} & M_2 \end{array}$$

In that case, we will say the families are *uniformly conjugate*.

The reason for considering uniform conjugacy instead of the topological conjugacy is that every n.s.d.s. is topologically conjugate to the n.s.d.s. whose maps are all the identity (see [1], Proposition 2.1). Uniform conjugacies are also considered to characterize random dynamical systems (see [5]). In [7] we showed that the topological entropy for non-autonomous dynamical systems is a continuous map. The invariance of that entropy by uniform conjugacies is a fundamental tool to classify non-stationary dynamical systems by uniform conjugacies.

Consider $\mathcal{F}(\mathbf{M}) = \{\boldsymbol{f} = (f_i)_{i\in\mathbb{Z}} : f_i : M_i \to M_{i+1} \text{ is a } C^1\text{-diffeomorphism}\}$. We endow $\mathcal{F}(\mathbf{M})$ with the *strong topology*:

Definition 2.3. Let $\varepsilon = (\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of positive numbers and $\boldsymbol{f} \in \mathcal{F}(\mathbf{M})$. The set

$$B(\boldsymbol{f}, \varepsilon) = \{\boldsymbol{g} \in \mathcal{F}(\mathbf{M}) : d_{\mathbf{D}_i}(f_i, g_i) < \varepsilon_i \text{ for all } i\}$$

is called a *strong basic neighborhood* of $\boldsymbol{f}$, where $d_{\mathbf{D}_i}(\cdot,\cdot)$ is the C^1-metric on $\mathbf{D}_i = \mathrm{Diff}^1(M_i, M_{i+1})$, the set consisting of C^1-diffeomorphisms on M_i to M_{i+1}. The *strong topology* (or *Whitney topology*) is generated by the strong basic neighborhoods of each $\boldsymbol{f} \in \mathcal{F}(\mathbf{M})$.

Definition 2.4. A subset $\mathcal{A}$ of $\mathcal{F}(\mathbf{M})$ is open if for each $\boldsymbol{f} \in \mathcal{A}$ there exists $\varepsilon = (\varepsilon_i)_{i\in\mathbb{Z}}$ such that $B(\boldsymbol{f}, \varepsilon) \subseteq \mathcal{A}$. An element $\boldsymbol{f} \in \mathcal{F}(\mathbf{M})$ is called *structurally stable* if there exists $\varepsilon = (\varepsilon_i)_{i\in\mathbb{Z}}$ such that any $\boldsymbol{g} \in B(\boldsymbol{f}, \varepsilon)$ is uniformly conjugate to $\boldsymbol{f}$. If each element in $\mathcal{A}$ is structurally stable, we say that $\mathcal{A}$ is *structurally stable*.

Definition 2.5. A n.s.d.s $\boldsymbol{f}$ on $\mathbf{M}$ is called an *Anosov family* if:

i. the tangent bundle $T\mathbf{M}$ has a continuous splitting $E^s \oplus E^u$ which is $D\boldsymbol{f}$-*invariant*, i.e., for each $p \in \mathbf{M}$, $T_p\mathbf{M} = E^s_p \oplus E^u_p$ with $D_p\boldsymbol{f}(E^s_p) = E^s_{\boldsymbol{f}(p)}$ and $D_p\boldsymbol{f}(E^u_p) = E^u_{\boldsymbol{f}(p)}$, where $T_p\mathbf{M}$ is the tangent space at p;
ii. there exist constants $\lambda \in (0,1)$ and $c > 0$ such that for each $i \in \mathbb{Z}$, $n \geq 1$, and $p \in M_i$, we have:

$$\|D_p(\boldsymbol{f}^n_i)(v)\| \leq c\lambda^n \|v\| \text{ if } v \in E^s_p \quad \text{and} \quad \|D_p(\boldsymbol{f}^{-n}_i)(v)\| \leq c\lambda^n \|v\| \text{ if } v \in E^u_p.$$

The subspaces E^s_p and E^u_p are called stable and unstable subspaces, respectively.

The set consisting of Anosov family on $(\mathbf{M}, \langle\cdot,\cdot\rangle)$ will be denoted by $\mathcal{A}(\mathbf{M})$. If we can take $c = 1$ we say the family is *strictly Anosov*.

A clear example of an Anosov family is the *constant family associated* to an Anosov diffeomorphism (see [1], Definition 2.2). Is well-known the notion of Anosov diffeomorphism does not depend on the Riemannian metric on the manifold (see [9]). However, Example 4 in [1] shows that suitably changing the metric on each M_i the notion of Anosov family could not be satisfied.

Example 2.6. Let F be a hyperbolic linear cocycle defined by $A : X \to SL(\mathbb{Z}, d)$ over a homeomorphism $\phi : X \to X$ on a compact metric space X (see [11]). For each $x \in X$, the family $(A(f^n(x)))_{n\in\mathbb{Z}}$ defined on $M_i = \mathbb{R}^d/\mathbb{Z}^d$, the torus d-dimensional equipped with the Riemannian metric inherited from $\mathbb{R}^d$, determines an Anosov family.

Remark 2.7. Let $\phi : M \to M$ be an Anosov diffeomorphism of class C^2 on a compact Riemannian manifold M and $\beta > 0$ such that $L(D\phi) < \beta$, where $L(D\phi)$ is a Lipchitz constant of the derivative application $x \mapsto D_x\phi$. For $\alpha > 0$, take

$$\Omega_{\alpha,\beta}(\phi) = \{\psi \in C^1(M) : d(\phi, \psi) \leq \alpha \text{ and } L(D\psi) \leq \beta\},$$

where $d(\cdot, \cdot)$ is the C^1-metric on $\mathrm{Diff}^1(M)$. If α is small enough, any sequence $(\psi_i)_{i\in\mathbb{Z}}$ in $\Omega_{\alpha,\beta}(\phi)$ defines an Anosov family in $\mathbf{M} = \coprod_{i\in\mathbb{Z}} M$ (see [12], Proposition 2.2). Consequently, the set consisting of the constant families associated to Anosov diffeomorphisms of class C^2 is open in $\mathcal{F}(\mathbf{M})$.

Using the above fact we have:

Example 2.8. Given $\alpha \in \mathbb{R}$, consider $\phi_\alpha : \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$\phi_\alpha(x, y) = (2x + y - (1 + \alpha)\sin x \bmod 2\pi, x + y - (1 + \alpha)\sin x \bmod 2\pi).$$

For all $\alpha \in [-1, 0)$, ϕ_α is an Anosov diffeomorphism (see [4]). We have that given $\alpha^\star \in [-1, 0)$ there exists $\varepsilon > 0$ such that, if $(\alpha_i)_{i\in\mathbb{Z}}$ is a sequence in $[-1, 0)$ with $|\alpha_i - \alpha^\star| < \varepsilon$, then $(f_i)_{i\in\mathbb{Z}}$ is an Anosov family, where $f_i = \phi_{\alpha_i}$ for $i \in \mathbb{Z}$.

The existence of Anosov diffeomorphisms $\phi : M \to M$ imposes strong restrictions on the manifold M. All known examples of Anosov diffeomorphisms are defined on *infranilmanifolds* (see [4], [9], [11]). The circle $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ does not admit any Anosov diffeomorphism. In [6] we show that $\mathbf{S}^1$ does not admit Anosov families in the following sense: let $\mathbf{M} = \bigcup_{i\in\mathbb{Z}} M_i$ where $M_i = \mathbb{S}^1 \times \{i\}$ equipped with the Riemannian metric inherited from $\mathbb{R}^2$ for each i. Thus, there is not any Anosov family on $\mathbf{M}$. As mentioned above, the Anosov families are not necessarily formed by Anosov diffeomorphisms. Then, a natural question that arises from the notion of Anosov families is: which compact Riemannian manifolds admit Anosov families?

3. Some properties of the Anosov families

We now show some properties that the Anosov families satisfy and that will be used in the rest of the work. In this section, if we do not say otherwise,

$(\mathbf{M}, \langle\cdot,\cdot\rangle, \boldsymbol{f})$ will represent an Anosov family with constants $\lambda \in (0,1)$ and $c \geq 1$. Sometimes we will omit the index i of f_i if it is clear that we are considering the i-th diffeomorphism of $\boldsymbol{f}$.

In [1], Proposition 2.12, is shown for an Anosov family the splitting $T_p\mathbf{M} = E^s_p \oplus E^u_p$ is unique. Actually, we have:

Lemma 3.1. *For each $p \in M_i$ we have*

i. $E^s_p = \{v \in T_pM_i : \|D_p(\boldsymbol{f}^n)(v)\|$ *is bounded, for* $n \geq 1\}$.
ii. $E^u_p = \{v \in T_pM_i : \|D_p(\boldsymbol{f}^{-n})(v)\|$ *is bounded, for* $n \geq 1\}$.

Proof. We will prove i. Set $B^s_p = \{v \in T_pM_i : \sup_{n\geq 1} \|D_p(\boldsymbol{f}^n)(v)\| < +\infty\}$. It is clear that $E^s_p \subseteq B^s_p$. Suppose there exists $v \in T_pM_i$ such that $v \notin E^s_p$. Thus $v = v_s + v_u$ for some $v_s \in E^s_p$ and $v_u \in E^u_p$ with $v_u \neq 0$. Therefore, we have $\|D_p(\boldsymbol{f}^n)(v)\| \geq c^{-1}\lambda^{-n}\|v_u\| - c\lambda^n\|v_s\|$, where $\|D_p(\boldsymbol{f}^n)(v)\| \to +\infty$, that is, $v \notin B^s_p$. Thus $B^s_p \subseteq E^s_p$. □

Definition 3.2. For $p \in \mathbf{M}$ and $\alpha > 0$, set

$$\begin{aligned} K^s_{\alpha,\boldsymbol{f},p} &= \{(v_s, v_u) \in E^s_p \oplus E^u_p : \|v_u\| < \alpha\|v_s\|\} \cup \{(0,0)\} \\ &:= \textit{stable } \alpha\textit{-cone of } \boldsymbol{f} \textit{ atp}, \\ K^u_{\alpha,\boldsymbol{f},p} &= \{(v_s, v_u) \in E^s_p \oplus E^u_p : \|v_s\| < \alpha\|v_u\|\} \cup \{(0,0)\} \\ &:= \textit{unstable } \alpha\textit{-cone of } \boldsymbol{f} \text{ at} p. \end{aligned}$$

(see Figure 2).

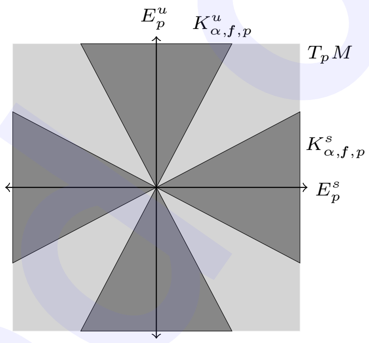

FIGURE 2. Stable and unstable α-cones at p.

Taking a suitable α, the following lemma shows that the cones are invariant by the derivative of the family and, in addition, the derivative of the family restricted $K^u_{\alpha,\boldsymbol{f},p}$ is an expansion and restricted to $K^s_{\alpha,\boldsymbol{f},p}$ is a contraction:

Lemma 3.3. *Suppose that $\boldsymbol{f}$ is a strictly Anosov family. Fix $\alpha \in (0, \frac{1-\lambda}{1+\lambda})$ and take $\lambda' = \lambda\frac{1+\alpha}{1-\alpha} < 1$. Thus:*

i. $D_p\boldsymbol{f}(K^u_{\alpha,\boldsymbol{f},p}) \subseteq K^u_{\alpha,\boldsymbol{f},\boldsymbol{f}(p)}$. *Furthermore,* $\|D_p\boldsymbol{f}(v)\| \geq \frac{1}{\lambda'}\|v\|$ *for* $v \in K^u_{\alpha,\boldsymbol{f},p}$.

ii. $D_{\boldsymbol{f}(p)}\boldsymbol{f}^{-1}(K^s_{\alpha,\boldsymbol{f},\boldsymbol{f}(p)}) \subseteq K^s_{\alpha,\boldsymbol{f},p}$. *Furthermore,* $\|D_{\boldsymbol{f}(p)}\boldsymbol{f}^{-1}(v)\| \geq \frac{1}{\lambda'}\|v\|$ *for* $v \in K^s_{\alpha,\boldsymbol{f},\boldsymbol{f}(p)}$.

Proof. For $(v_s, v_u) \in K^u_{\alpha,\boldsymbol{f},p}$ we have

$$\|D_p\boldsymbol{f}(v_s)\| \leq \lambda\|v_s\| \leq \lambda\alpha\|v_u\| \leq \lambda^2\alpha\|D_p\boldsymbol{f}(v_u)\| \leq \alpha\|D_p\boldsymbol{f}(v_u)\|.$$

Therefore $D_p\boldsymbol{f}(K^u_{\alpha,\boldsymbol{f},p}) \subseteq K^u_{\alpha,\boldsymbol{f},\boldsymbol{f}(p)}$. On the other hand, we have

$$\|D_p\boldsymbol{f}(v_s, v_u)\| \geq \|D_p\boldsymbol{f}(v_u)\| - \|D_p\boldsymbol{f}(v_s)\| \geq \frac{1-\alpha}{\lambda(1+\alpha)}\|(v_s, v_u)\|,$$

and this fact proves i. The part ii. can be proved analogously. □

Next proposition proves the continuity of the splitting $E^s \oplus E^u$ can be obtained from both the condition ii. in Definition 2.5 and the $D\boldsymbol{f}$-invariance of the splitting. We adapt the ideas of the proof of Proposition 2.2.9 in [4] (which is done for diffeomorphisms defined on compact Riemannian manifolds) to show the following result.

Proposition 3.4. *Let* $\boldsymbol{f} \in \mathcal{F}(\boldsymbol{M})$. *Suppose that* $T\boldsymbol{M}$ *has a splitting* $E^s \oplus E^u$ *which is* $D\boldsymbol{f}$*-invariant and satisfies the property* ii. *from Definition 2.5. Thus,* E^s_p *and* E^u_p *depend continuously on* p.

Proof. First we prove that the dimensions of E^u and E^s are locally constants. Let $p \in \mathbf{M}$ and $k = \dim E^s_p$. Suppose by contradiction that there exists a sequence $(p_m)_{m\in\mathbb{N}} \subseteq \mathbf{M}$ converging to p such that $\dim E^s_{p_m} \geq k+1$ for all m. Take a sequence of orthonormal vectors

$$v_1(p_m), \ldots, v_k(p_m), v_{k+1}(p_m) \quad \text{in } E^s_{p_m}, \text{ for each } m.$$

Choosing a suitable subsequence, we can suppose that

$$v_1(p_m) \to v_1 \in T_p\mathbf{M}, \ldots, v_{k+1}(p_m) \to v_{k+1} \in T_p\mathbf{M} \quad \text{as } m \to \infty.$$

Therefore, by continuity of the Riemannian metric, it follows from condition ii. in Definition 2.5 that, for all $n \geq 1$, we have

$$\|D_p(\boldsymbol{f}^n_i)(v_s)\| \leq c\lambda^n\|v_s\| \quad \text{for each } s = 1, \ldots, k+1. \tag{3.1}$$

By Lemma 3.1 we obtain $v_1, \ldots, v_{k+1} \in E^s_p$. Since $v_1(p_m), \ldots, v_k(p_m)$, and $v_{k+1}(p_m)$ are orthonormal for all $m \geq 1$, we have that $v_1, \ldots, v_{k+1}$ are orthonormal, which contradicts that $\dim E^s_p = k$. Similarly we can prove that there is not any sequence $(p_m)_{m\in\mathbb{N}}$ converging to p with $\dim E^s_{p_m} < k$ for all m. Therefore, the dimension of E^s_p is locally constant.

Analogously we obtain that the dimension of E^u_p is locally constant.

Now, let $(p_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbf{M}$ such that $p_m \to p \in \mathbf{M}$ as $m \to \infty$. Without loss of generality, we can suppose that $(p_m)_{m\in\mathbb{N}} \subseteq M_i$ and $p \in M_i$ for some $i \in \mathbb{Z}$. This fact follows from the definition of the metric on $\mathbf{M}$ given in (2.2). Furthermore, we can assume that $\dim E^s_{p_m} = \dim E^s_p = k$ for every $m \geq 1$. Let $\{v_1(p_m), \dots, v_k(p_m)\}$ be an orthonormal basis of $E^s_{p_m}$, for each $m \geq 1$, such that $v_1(p_m) \to v_1 \in T_pM_i, \dots, v_k(p_m) \to v_k \in T_pM_i$ as $m \to \infty$. By the continuity of the Riemannian metric we have that $v_1, \dots, v_k$ are orthonormal and

$$\|D_p(\boldsymbol{f}^n_i)(v_s)\| \leq c\lambda^n \|v_s\| \quad \text{for each } s = 1, \dots, k,$$

that is, $v_1, \dots, v_k$ belong to E^s_p. This fact proves that E^s_p depends continuously on p. Analogously we can prove that E^u_p depends continuously on p. □

The notion of Anosov diffeomorphism does not depend of the Riemannian metric on the manifold (see [9]). In contrast, the notion of Anosov family depends on the Riemannian metric taken on each M_i (see [1, Example 4]). However, the next proposition proves that the notion of Anosov family does not depend on the Riemannian metric chosen uniformly equivalent on $\mathbf{M}$.[1]

Proposition 3.5. *Let $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^\star$ be Riemannian metrics uniformly equivalent on $\boldsymbol{M}$. We have that $(\boldsymbol{M}, \langle \cdot, \cdot \rangle, \boldsymbol{f})$ is an Anosov family if, and only if, $(\boldsymbol{M}, \langle \cdot, \cdot \rangle^\star, \boldsymbol{f})$ is an Anosov family.*

Proof. Let $\|\cdot\|$ and $\|\cdot\|^\star$ be the norms induced by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^\star$, respectively. Since $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^\star$ are uniformly equivalent on $\mathbf{M}$, there exist $k > 0$ and $K > 0$ such that $k\|v\|^\star \leq \|v\| \leq K\|v\|^\star$ for all $v \in T\mathbf{M}$. Suppose that $(\mathbf{M}, \langle \cdot, \cdot \rangle, \boldsymbol{f})$ is an Anosov family with constant $\lambda \in (0, 1)$ and $c \geq 1$. Thus, for $v \in T_p\mathbf{M}, n \geq 1$,

$$\|D_p(\boldsymbol{f}^n_i)(v)\|^\star \leq (1/k)\|D_p(\boldsymbol{f}^n_i)(v)\| \leq (c/k)\lambda^n\|v\| \leq (Kc/k)\lambda^n\|v\|^\star.$$

Analogously we have $\|D_p(\boldsymbol{f}^{-n}_i)(v)\|^\star \leq (Kc/k)\lambda^n\|v\|^\star$, for $v \in T_p\mathbf{M}, n \geq 1$. Therefore, $(\mathbf{M}, \langle \cdot, \cdot \rangle^\star, \boldsymbol{f})$ is an Anosov family with constant λ and $\tilde{c} = Kc/k$.

Similarly we can prove if $(\mathbf{M}, \langle \cdot, \cdot \rangle^\star, \boldsymbol{f})$ is an Anosov family then $(\mathbf{M}, \langle \cdot, \cdot \rangle, \boldsymbol{f})$ is an Anosov family. □

In Proposition 3.7 we will show there exists a Riemannian metric $\langle \cdot, \cdot \rangle^\star$, equivalent to $\langle \cdot, \cdot \rangle$ on each M_i ($\langle \cdot, \cdot \rangle^\star$ is not necessarily uniformly equivalent to $\langle \cdot, \cdot \rangle$ on the total space $\mathbf{M}$), with which, $(\mathbf{M}, \langle \cdot, \cdot \rangle^\star, \boldsymbol{f})$ is a strictly Anosov family. That is a version for families of a well-known Lemma of Mather for Anosov diffeomorphisms (see [9]). In order to prove this fact, we introduce the following notion: Fix $i \in \mathbb{Z}$. Since for each $p \in M_i$, the subspaces E^s_p and E^u_p are transversal, that is, $E^s_p \oplus E^u_p = T_pM_i$, then, by the compactness of M_i and the continuity of both the Riemannian metric and the subspaces E^s_p and E^u_p, we obtain that there exists $\mu_i \in (0, 1)$ such that, if v_s and v_u are unit vectors in E^s_p and E^u_p, respectively, then

$$\cos(\widehat{v_s v_u}) \in [\mu_i - 1, 1 - \mu_i], \tag{3.2}$$

[1]Two Riemannian metrics $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_*$ defined on a manifold M are *uniformly equivalent* if there exist positive numbers k and K such that $k\langle v, v\rangle \leq \langle v, v\rangle_* \leq K\langle v, v\rangle$ for any $v \in TM$.

where $\widehat{v_s v_u}$ is the angle between v_s and v_u. In the case of Anosov diffeomorphisms defined on compact manifolds those angles are uniformly bounded away from 0. In [6] we gave an example where the angles between the unstable and stable subspaces along the orbit of a point of M_0 converge to zero.

Definition 3.6. We say that an Anosov family satisfies the *property of the angles* (or *s. p. a.*) if there exists $\mu \in (0,1)$ such that, for all $p \in \mathbf{M}$, if $v_s \in E^s_p$ and $v_u \in E^u_p$, then $\cos(\widehat{v_s v_u}) \in [\mu - 1, 1 - \mu]$, that is, μ does not depend on i.

Proposition 3.7. *There exists a C^∞ Riemannian metric $\langle \cdot,\cdot\rangle^\star$ on $\boldsymbol{M}$, which is uniformly equivalent to $\langle \cdot,\cdot\rangle$ on each M_i, such that $(\boldsymbol{M}, \langle \cdot,\cdot\rangle^\star, \boldsymbol{f})$ is a strictly Anosov family. Furthermore, $(\boldsymbol{M}, \langle \cdot,\cdot\rangle^\star, \boldsymbol{f})$ satisfies the property of the angles.*

Proof. Let $\varepsilon \in (0, 1-\lambda)$. For $p \in \mathbf{M}$, if $(v_s, v_u) \in E^s_p \oplus E^u_p$, take

$$\|(v_s, v_u)\|_1 = \sqrt{\|v_s\|_1{}^2 + \|v_u\|_1{}^2}, \tag{3.3}$$

where

$$\|v_s\|_1 = \sum_{n=0}^{\infty} (\lambda+\varepsilon)^{-n} \|D_p(\boldsymbol{f}^{\,n}) v_s\| \text{ and } \|v_u\|_1 = \sum_{n=0}^{\infty} (\lambda+\varepsilon)^{-n} \|D_p(\boldsymbol{f}^{\,-n}) v_u\|.$$

Note that if $v_s \in E^s_p$ we have

$$\|v_s\|_1 = \sum_{n=0}^{\infty} (\lambda+\varepsilon)^{-n} \|D_p(\boldsymbol{f}^{n}) v_s\| \leq \sum_{n=0}^{\infty} (\lambda+\varepsilon)^{-n} c\lambda^n \|v_s\| = \frac{\lambda+\varepsilon}{\varepsilon} c\|v_s\|. \tag{3.4}$$

Analogously, $\|v_u\|_1 \leq \frac{\lambda+\varepsilon}{\varepsilon} c\|v_u\|$ for $v_u \in E^u_p$. Consequently the series $\|v_s\|_1$ and $\|v_u\|_1$ converge uniformly. That is, $\|\cdot\|_1$ is well defined.

We prove that $\|\cdot\|_1$ is uniformly equivalent to $\|\cdot\|$ on each M_i. It is clear that $\|v_s\| \leq \|v_s\|_1$ and $\|v_u\| \leq \|v_u\|_1$. Thus,

$$\begin{aligned}\|(v_s, v_u)\| &\leq \|v_s\| + \|v_u\| \leq 2(\|v_s\|^2 + \|v_u\|^2)^{1/2} \\ &\leq 2(\|v_s\|_1^2 + \|v_u\|_1^2)^{1/2} = 2\|(v_s, v_u)\|_1.\end{aligned}$$

This fact implies

$$\|v\| \leq 2\|v\|_1 \quad \text{for all } v \in T\mathbf{M}. \tag{3.5}$$

Fix $p \in M_i$. Let θ_p be the angle between two vectors $v_s \in E^s_p$ and $v_u \in E^u_p$, for $p \in M_i$. Take μ_i as in (3.2). Since $(1-\mu_i)(\|v_s\|^2 + \|v_u\|^2) \geq 2(1-\mu_i)\|v_s\|\|v_u\|$, we have

$$\|v_s\|^2 + \|v_u\|^2 + 2(\mu_i - 1)\|v_s\|\|v_u\| \geq \mu_i(\|v_s\|^2 + \|v_u\|^2).$$

Therefore

$$\begin{aligned}\|(v_s, v_u)\|^2 &= \|v_s\|^2 + \|v_u\|^2 - 2\cos\theta_p \|v_s\|\|v_u\| \\ &\geq \|v_s\|^2 + \|v_u\|^2 + 2(\mu_i - 1)\|v_s\|\|v_u\| \\ &\geq \mu_i(\|v_s\|^2 + \|v_u\|^2).\end{aligned}$$

Consequently,

$$\begin{aligned}\|(v_s,v_u)\|_1^2 = \|v_s\|_1^2+\|v_u\|_1^2 &\leq (\frac{\lambda+\varepsilon}{\varepsilon}c)^2(\|v_s\|^2+\|v_u\|^2)\\ &\leq \frac{1}{\mu_i}(\frac{\lambda+\varepsilon}{\varepsilon}c)^2\|(v_s,v_u)\|^2.\end{aligned}$$

Thus,

$$\|v\|_1 \leq \frac{1}{\mu_i}(\frac{\lambda+\varepsilon}{\varepsilon}c)^2\|v\| \quad \text{for all } v\in TM_i. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\frac{1}{2}\|v\| \leq \|v\|_1 \leq \frac{1}{\mu_i}(\frac{\lambda+\varepsilon}{\varepsilon}c)^2\|v\| \quad \text{for all } v\in TM_i. \tag{3.7}$$

Hence, the norm $\|\cdot\|_1$ is uniformly equivalent to the norm $\|\cdot\|$ on each M_i.

We have also that

$$\begin{aligned}&\|D_p\boldsymbol{f}v_s\|_1 \leq (\lambda+\varepsilon)\|v_s\|_1 \text{ if } v_s\in E^s_p \quad \text{and}\\ &\|D_p(\boldsymbol{f}^{\,-1})v_u\|_1 \leq (\lambda+\varepsilon)\|v_u\|_1 \text{ if } v_u\in E^u_p.\end{aligned}$$

Note that the norm $\|\cdot\|_1$ comes from an inner product $\langle\cdot,\cdot\rangle_1$, which defines a continuous Riemannian metric on $\mathbf{M}$. Consequently, for each i, we can choose a C^∞-Riemannian metric $\langle\cdot,\cdot\rangle_i^\star$ such that $|\langle v,v\rangle_i^\star - \langle v,v\rangle_1| < \varepsilon$ for each $v\in TM_i$. We take $\langle\cdot,\cdot\rangle^\star$ on $\mathbf{M}$, defined on each M_i as $\langle\cdot,\cdot\rangle^\star|_{M_i} = \langle\cdot,\cdot\rangle_i^\star$. Hence $(\mathbf{M},\langle\cdot,\cdot\rangle^\star,\boldsymbol{f})$ is a strictly Anosov family with constant $\lambda'=\lambda+\varepsilon$, which s.p.a.. □

By (3.7) we have that $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle^\star$ are uniformly equivalent on each M_i. However, this fact does not imply that they are uniformly equivalent on $\mathbf{M}$, because μ_i could converge to 0 as $i\to\pm\infty$ (notice that $\mathbf{M}$ is not compact). If the angles between the stable and unstable subspaces converge to zero along an orbit, then μ_i converges to zero. In that case the two metrics are not uniformly equivalent on the total space. On the other hand:

Corollary 3.8. *If $(\boldsymbol{M},\langle\cdot,\cdot\rangle,\boldsymbol{f})$ s.p.a., then there exists a C^∞-Riemannian metric $\langle\cdot,\cdot\rangle^\star$, uniformly equivalent to $\langle\cdot,\cdot\rangle$ on $\boldsymbol{M}$, such that $(\boldsymbol{M},\langle\cdot,\cdot\rangle^\star,\boldsymbol{f})$ is a strictly Anosov family that s.p.a..*

Proof. Since $\boldsymbol{f}$ satisfies the property of the angles, we can take a μ as in Definition 3.6. From (3.7) we have for all $v\in T\mathbf{M}$,

$$\frac{1}{2}\|v\| \leq \|v\|_1 \leq \frac{1}{\mu}(\frac{\lambda+\varepsilon}{\varepsilon}c)^2\|v\|,$$

where $\|\cdot\|_1$ is the metric defined in (3.3). Thus, $\|\cdot\|$ and $\|\cdot\|_1$ are uniformly equivalent on the total space. The corollary follows from the proof of Proposition 3.7. □

A Riemannian metric is *adapted to an hyperbolic set* of a diffeomorphism if, in this metric, the expansion (contraction) of the unstable (stable) subspaces is seen after only one iteration. The metric obtained in Proposition 3.7 is adapted to $\mathbf{M}$ for the family $\boldsymbol{f}$. This metric is not always uniformly equivalent to $\langle\cdot,\cdot\rangle$, because there exist Anosov families which do not s.p.a..

4. Invariant cones

In order to prove the openness of $\mathcal{A}(\mathbf{M})$, we use the method of the invariant cones (see [4]). We will prove that there exists a strong basic neighborhood $B(\boldsymbol{f},(\varepsilon_i)_{i\in\mathbb{Z}})$ of $\boldsymbol{f}$ such that each family in $B(\boldsymbol{f},(\varepsilon_i)_{i\in\mathbb{Z}})$ satisfies Lemma 3.3.

We will use the exponential application to work on a Euclidian ambient space. For each $i\in\mathbb{Z}$, there exists $\delta_i>0$ such that, if $p\in M_i$, then the exponencial application at p, $\exp_p : B_p(0,\delta_i)\to B(p,\delta_i)$, is a diffeomorphism, and $\|v\| = d(\exp_p(v),p)$, for all $v\in B_p(0,\delta_i)$, where $B_p(0,\delta_i)$ is the ball in T_pM_i with radius δ_i and center $0\in T_pM_i$ and $B(p,\delta_i)$ is the ball in M_i with radius δ_i and center p, i.e., δ_i is the *injectivity radius* of the exponential application at each $p\in M_i$. The injectivity radius could decrease as $|i|$ increases, since the M_i's are different. We need a radius small enough such that the inequality in (4.2) be valid. This inequality depends also on the behavior of each f_i.

By simplicity, in this section we will suppose that $\boldsymbol{f}\in\mathcal{F}(\mathbf{M})$ is an Anosov family that satisfies the property of the angles.

Remark 4.1. We can choose $\beta_i>0$, with $\beta_i<\min\{\delta_{i-1},\delta_i,\delta_{i+1}\}/2$, such that, if $p\in M_i$, $f(B(p,2\beta_i))\subseteq B(f(p),\delta_{i+1}/2)$ and $f^{-1}(B(f(p),2\beta_{i+1}))\subseteq B(p,\delta_i/2)$. Thus, if $\boldsymbol{g}=(g_i)_{i\in\mathbb{Z}}\in\mathcal{F}(\mathbf{M})$ with $d_{\mathbf{D}_i}(f_i,g_i)<\beta_i$ for all i, we have

$$g(B(p,\beta_i))\subseteq B(f(p),\delta_{i+1}) \quad\text{and}\quad g^{-1}(B(f(p),\beta_{i+1}))\subseteq B(p,\delta_i). \tag{4.1}$$

Consider a linear isomorphism $\tau_p : T_p\mathbf{M}\to\mathbb{R}^d$, depending continuously on p, which maps an orthonormal basis of E^s_p to an orthonormal basis of $\mathbb{R}^k$ and maps an orthonormal basis of E^u_p to an orthonormal basis of $\mathbb{R}^{d-k}$, where d is the dimension of each M_i and k the dimension of E^s_p. Since $\boldsymbol{f}$ satisfies the property of the angles, the norm $\|\cdot\|_1$ defined in (3.3) is uniformly equivalent to the norm $\|\cdot\|$ (Corollary 3.8). Hence, without loss of generality, we can suppose that $\|\cdot\| = \|\cdot\|_1$, because a family of diffeomorphisms in any strong basic neighborhood of $\boldsymbol{f}$ is Anosov with $\|\cdot\|$ if and only if is Anosov with $\|\cdot\|_1$ (see Proposition 3.5). Therefore, we can suppose that $\boldsymbol{f}$ is strictly Anosov. Note that $\|\tau_p(v)\|=\|v\|$ for all $v\in T_p\mathbf{M}$.

For $g\in\mathbf{D}_i$, with $d_{\mathbf{D}_i}(f_i,g)<\beta_i$, we set

$$\tilde{g}_p=\tau_{f(p)}\circ\exp^{-1}_{f(p)}\circ g_i\circ\exp_p\circ\tau_p^{-1} : B_p(0,\beta_i)\to B_{f(p)}(0,\delta_{i+1})$$

$$\text{and}\quad \tilde{g}_p^{-1}=\tau_p\circ\exp_p^{-1}\circ g_i^{-1}\circ\exp_{f(p)}\circ\tau^{-1}_{f(p)} : B_{f(p)}(0,\beta_{i+1})\to B_p(0,\delta_i),$$

which are well-defined as a consequence of (4.1).

Definition 4.2. Let $B^k(0,\beta_i) \subseteq \mathbb{R}^k$ and $B^{d-k}(0,\beta_i) \subseteq \mathbb{R}^{d-k}$ be the open balls with center at 0 and radius β_i. For $x \in \mathbb{R}^d$, we denote by $(x)_1$ and $(x)_2$ the orthogonal projections of x on E^s and E^u, respectively. If $(v,w) \in B^k(0,\beta_i) \times B^{d-k}(0,\beta_i)$, then

$$\tilde{f}_p(v,w) = ((\tilde{f}_p)_1(v,w), (\tilde{f}_p)_2(v,w)) = (\tilde{a}_p(v,w) + \tilde{F}_p(v), \tilde{b}_p(v,w) + \tilde{F}_p(w)),$$

where $\tilde{a}_p(v,w) = (\tilde{f}_p)_1(v,w) - \tilde{F}_p(v)$, $\tilde{b}_p(v,w) = (\tilde{f}_p)_2(v,w) - \tilde{F}_p(w)$, and $\tilde{F}_p = D_0(\tilde{f}_p)$. Analogously we have that, for each $(v,w) \in B^k(0,\beta_{i+1}) \times B^{d-k}(0,\beta_{i+1})$,

$$\tilde{f}_p^{-1}(v,w) = (\tilde{c}_p(v,w) + \tilde{G}_p(v), \tilde{d}_p(v,w) + \tilde{G}_p(w)),$$

with $\tilde{c}_p(v,w) = (\tilde{f}_p^{-1})_1(v,w) - \tilde{G}_p(v)$; $\tilde{d}_p(v,w) = (\tilde{f}_p^{-1})_2(v,w) - \tilde{G}_p(w)$; $\tilde{G}_p = D_0(\tilde{f}_p^{-1})$.

Consider

$$\begin{aligned} &\sigma_{1,p} = \sup\{\|D_{(v,w)}(\tilde{a}_p, \tilde{b}_p)\| : (v,w) \in B^k(0,\beta_i) \times B^{d-k}(0,\beta_i)\} \\ \text{and} \quad &\sigma_{2,p} = \sup\{\|D_{(v,w)}(\tilde{c}_p, \tilde{d}_p)\| : (v,w) \in B^k(0,\beta_{i+1}) \times B^{d-k}(0,\beta_{i+1})\}. \end{aligned}$$

Note that $\sigma_{1,p}$ and $\sigma_{2,p}$ depend on β_i. Take $\sigma_p = \max\{\sigma_{1,p}, \sigma_{2,p}\}$.

Lemma 4.3. *Fix $\alpha \in (0, \frac{1-\lambda}{1+\lambda})$. For each $i \in \mathbb{Z}$ there exists β_i such that*

$$\sigma_i := \max_{p \in M_i} \sigma_p \leq \min\left\{ \frac{(\lambda^{-1} - \lambda)\alpha}{2(1+\alpha)^2}, \frac{\lambda^{-1}(1-\alpha) - (1+\alpha)\alpha}{2(1+\alpha)} \right\}. \tag{4.2}$$

Proof. Note that $D_0(\tilde{f}_p) = \tau_{f(p)} D_p f \tau_p^{-1}$. Hence, if $(v,w) \in \mathbb{R}^k \oplus \mathbb{R}^{d-k}$, we have

$$\begin{aligned} (\tilde{F}_p v, \tilde{F}_p w) &= (\tau_{f(p)} D_p f \tau_p^{-1}(v), \tau_{f(p)} D_p f \tau_p^{-1}(w)) = \tau_{f(p)} D_p f \tau_p^{-1}(v,w) \\ &= D_0(\tilde{f}_p)(v,w) = (D_0(\tilde{f}_p)_1(v,w), D_0(\tilde{f}_p)_2(v,w)). \end{aligned}$$

Consequently, $D_0(\tilde{a}_p) = 0$ and $D_0(\tilde{b}_p) = 0$. Analogously, we can prove that $D_0(\tilde{c}_p) = 0$ and $D_0(\tilde{d}_p) = 0$. Thus, since f is of class C^1 and M_i is compact, it follows that for each i we can choose β_i small enough such that (4.2) is valid. □

We chose $\alpha \in (0, \frac{1-\lambda}{1+\lambda})$ for the minimum in (4.2) be positive. Set

$$\begin{aligned} K^s_\alpha &= \{(v,w) \in \mathbb{R}^k \oplus \mathbb{R}^{d-k} : \|w\| < \alpha \|v\|\}; \\ K^u_\alpha &= \{(v,w) \in \mathbb{R}^k \oplus \mathbb{R}^{d-k} : \|v\| < \alpha \|w\|\}. \end{aligned}$$

Lemma 4.4. *Let $\alpha \in (0, \frac{1-\lambda}{1+\lambda})$ and β_i be as in Lemma (4.3). Thus, there exists a $\varepsilon_i > 0$ such that, if $g \in \boldsymbol{D}_i$ with $d_{\boldsymbol{D}_i}(f_i, g) < \varepsilon_i$, for all $p \in M_i$ we have:*

i. $D_{(v,w)}\tilde{g}_p(\overline{K^u_\alpha}) \subseteq K^u_\alpha$ *for all* $(v,w) \in B^k(0,\beta_i) \times B^{d-k}(0,\beta_i)$, *and*
ii. $D_{(v,w)}\tilde{g}_p^{-1}(\overline{K^s_\alpha}) \subseteq K^s_\alpha$ *for all* $(v,w) \in B^k(0,\beta_{i+1}) \times B^{d-k}(0,\beta_{i+1})$.

Proof. We will prove i. Take $\varepsilon_i < \min\{\beta_i, \beta_{i+1}, \sigma_i\}$. Fix $(v, w) \in B^k(0, \beta_i) \times B^{d-k}(0, \beta_i)$. If $(x, y) \in \overline{K^u_\alpha} \setminus \{(0, 0)\}$, then

$$\begin{aligned}
&\|(D_{(v,w)}\tilde{g}_p(x, y))_1\| \\
&\leq \|(D_{(v,w)}\tilde{g}_p(x, y))_1 - (D_{(v,w)}\tilde{f}_p(x, y))_1\| + \|(D_{(v,w)}\tilde{f}_p(x, y))_1\| \\
&\leq \sigma_i(\alpha\|y\| + \|y\|) + \sigma_i\|(x, y)\| + \lambda\|x\| \leq ((\alpha + 1)2\sigma_i + \lambda\alpha)\|y\|.
\end{aligned}$$

Analogously, we have $\|(D_{(v,w)}\tilde{g}_p(x, y))_2\| \geq (\lambda^{-1} - 2\sigma_i(\alpha + 1))\|y\|$. Since $\sigma_i < \frac{\alpha(\lambda^{-1}-\lambda)}{2(1+\alpha)^2}$, then $\frac{(\alpha+1)2\sigma_i+\lambda\alpha}{\lambda^{-1}-2\sigma_i(\alpha+1)} < \alpha$, and hence,

$$\|(D_{(v,w)}\tilde{g}_p(x, y))_1\| < \alpha\|(D_{(v,w)}\tilde{g}_p(x, y))_2\|.$$

Therefore, $D_{(v,w)}\tilde{g}_p(x, y) \in K^u_\alpha$. Consequently, $D_{(v,w)}\tilde{g}_p(\overline{K^u_\alpha}) \subseteq K^u_\alpha$. □

Lemma 4.5. *If $\varepsilon_i < \min\{\beta_i, \beta_{i+1}, \sigma_i\}$, there exists $\eta < 1$ such that, if $g \in \boldsymbol{D}_i$ is such that $d_{\boldsymbol{D}_i}(f_i, g) < \varepsilon_i$, then, for $p \in M_i$,*

i. $\|D_{(v,w)}\tilde{g}_p(x, y)\| \geq \eta^{-1}\|(x, y)\|$ *if* $(x, y) \in \overline{K^u_\alpha}$;
ii. $\|D_{(v,w)}\tilde{g}_p^{-1}(x, y)\| \geq \eta^{-1}\|(x, y)\|$ *if* $(x, y) \in \overline{K^s_\alpha}$.

Proof. We will prove i. Let $g \in \mathbf{D}_i$ be such that $d_{\mathbf{D}_i}(f_i, g) < \varepsilon_i$. Fix $p \in M_i$ and take $(x, y) \in \overline{K^u_\alpha}$. By Lemma 4.4 we have $\|(D_{(v,w)}\tilde{f}_p(x, y))_1\| \leq \alpha\|(D_{(v,w)}\tilde{f}_p(x, y))_2\|$ for $(v, w) \in B^k(0, \beta_i) \times B^{d-k}(0, \beta_i)$. Thus,

$$\begin{aligned}
\|D_{(v,w)}\tilde{g}_p(x, y)\| &\geq \|D_{(v,w)}\tilde{f}_p(x, y)\| - \|D_{(v,w)}\tilde{f}_p(x, y) - D_{(v,w)}\tilde{g}_p(x, y)\| \\
&\geq \|(D_{(v,w)}\tilde{f}_p(x, y))_2\| - \|(D_{(v,w)}\tilde{f}_p(x, y))_1\| - \varepsilon_i\|(x, y)\| \\
&\geq (1 - \alpha)(\|\tilde{F}_p(y)\| - \|D_{(v,w)}\tilde{b}_p(x, y)\|) - \sigma_i\|(x, y)\| \\
&\geq (1 - \alpha)(\frac{\lambda^{-1}}{1 + \alpha}\|(x, y)\| - \sigma_i\|(x, y)\|) - \sigma_i\|(x, y)\|.
\end{aligned}$$

Consequently, $\|D_{(v,w)}\tilde{g}_p(x, y)\| \geq \frac{1}{\eta}\|(x, y)\|$, where $\frac{1}{\eta} := (1-\alpha)(\frac{\lambda^{-1}}{1+\alpha} - \sigma_i) - \sigma_i$. Since $\sigma_i < \frac{(1-\alpha)\lambda^{-1}-(1+\alpha)}{2(1+\alpha)}$, $\eta < 1$. □

Fix $\boldsymbol{g} = (g_i)_{i\in\mathbb{Z}} \in B(\boldsymbol{f}, (\varepsilon_i)_{i\in\mathbb{Z}})$. For each $i \in \mathbb{Z}$, let $m_i \in \mathbb{N}$ be such that $M_i = \cup_{j=1}^{m_j} B(p_{j,i}, \beta_i)$, where $p_{j,i} \in M_i$ for $j = 1, \ldots, m_i$. Take the set of charts

$$\phi_{j,i} : B^k(0, \beta_i) \times B^{d-k}(0, \beta_i) \to B(p_{j,i}, \beta_i) \text{ where } \phi_{j,i} = \exp_{p_{j,i}} \circ \tau^{-1}_{p_{j,i}}.$$

It follows from Lemmas 4.4 and 4.5 that:

Lemma 4.6. *For all $i \in \mathbb{Z}$ and $j = 1, \ldots, m_i$:*

i. $M_i = \bigcup_{j=1}^{m_i} \phi_{j,i}(B^k(0, \beta_i) \times B^{d-k}(0, \beta_i))$,
ii. $\phi^{-1}_{j,i+1}\boldsymbol{g}\phi_{j,i}(B^k(0, \beta_i) \times B^{d-k}(0, \beta_i)) \subseteq B^k(0, \delta_{i+1}) \times B^{d-k}(0, \delta_{i+1})$.
iii. $\phi^{-1}_{j,i}\boldsymbol{g}^{-1}\phi_{j,i+1}(B^k(0, \beta_{i+1}) \times B^{d-k}(0, \beta_{i+1})) \subseteq B^k(0, \delta_i) \times B^{d-k}(0, \delta_i)$.
iv. *For all $v \in B^k(0, \beta_i) \times B^{d-k}(0, \beta_i)$, if $x \in \overline{K^u_\alpha}$, we have*

$$D_v(\phi^{-1}_{j,i+1}\boldsymbol{g}\phi_{j,i})(\overline{K^u_\alpha}) \subseteq K^u_\alpha \quad \textit{and} \quad \|D_v(\phi^{-1}_{j,i+1}\boldsymbol{g}\phi_{j,i})(x)\| \geq \eta^{-1}\|x\|.$$

v. *For all* $v \in B^k(0,\beta_{i+1}) \times B^{d-k}(0,\beta_{i+1})$*, if* $x \in \overline{K^s_\alpha}$*, we have*

$$D_v(\phi_{j,i}^{-1}\boldsymbol{g}^{-1}\phi_{j,i+1})(\overline{K^s_\alpha}) \subseteq K^s_\alpha \quad \textit{and} \quad \|D_v(\phi_{j,i}^{-1}\boldsymbol{g}^{-1}\phi_{j,i+1})(x)\| \geq \eta^{-1}\|x\|.$$

Hence, since $D_0\exp_p = Id_{T_pM}$, $\tilde{g}_p = \tau_{f(p)} \circ \exp^{-1}_{f(p)} \circ g_i \circ \exp_p \circ \tau_p^{-1}$ and τ_p is an isometry, by choosing β_i even small, if necessary, we have:

Lemma 4.7. *There exists* $\eta \in (0,1)$ *such that, if* $\boldsymbol{g} \in B(\boldsymbol{f}, (\varepsilon_i)_{i\in\mathbb{Z}})$*, for each* $p \in \boldsymbol{M}$ *we have*:

i. $D_p\boldsymbol{g}(K^u_{\alpha,\boldsymbol{f},p}) \subseteq K^u_{\alpha,\boldsymbol{f},\boldsymbol{g}(p)}$*. Furthermore,* $\|D_p\boldsymbol{g}(v)\| \geq \eta^{-1}\|v\|$ *if* $v \in K^u_{\alpha,\boldsymbol{f},p}$*.*

ii. $D_{\boldsymbol{g}(p)}\boldsymbol{g}^{-1}(K^s_{\alpha,\boldsymbol{f},\boldsymbol{g}(p)}) \subseteq K^s_{\alpha,\boldsymbol{f},p}$*. Furthermore,* $\|D_{\boldsymbol{g}(p)}\boldsymbol{f}^{-1}(v)\| \geq \eta^{-1}\|v\|$ *if* $v \in K^s_{\alpha,\boldsymbol{f},\boldsymbol{g}(p)}$*.*

5. Openness of the Anosov families

A well-known fact is that the set consisting of Anosov diffeomorphisms on a compact Riemannian manifold is open (see, for example, [9]). The purpose of this section is to show the result analogous to Anosov families, that is, we prove that $\mathcal{A}(\mathbf{M})$ is an open subset of $\mathcal{F}(\mathbf{M})$. As we have seen in Section 3, the set consisting of constant families associated to Anosov diffeomorphisms of class C^2 is open in $\mathcal{F}(\mathbf{M})$. On the other hand, let X be a compact metric space, $\phi : X \to X$ a homeomorphism and $A : X \to SL(\mathbb{Z}, d)$ a continuous map such that the linear cocycle F defined by A over ϕ is hyperbolic. Thus, there exists $\varepsilon > 0$ such that, if $B : X \to SL(\mathbb{Z}, d)$ is continuous and $\|A(x) - B(x)\| < \varepsilon$ for all $x \in X$, then the linear cocycle G defined by B over ϕ is hyperbolic (see [11]). This fact shows the openness of the set consisting of Anosov families that are obtained by hyperbolic cocycles. These are particular cases of our result.

First we prove the set consisting of Anosov families satisfying the property of the angles is open and in the end of this work we will show the general case. We will consider $(\varepsilon_i)_{i\in\mathbb{Z}}$ as in Lemma 4.7 and fix $\boldsymbol{g} \in B(\boldsymbol{f}, (\varepsilon_i)_{i\in\mathbb{Z}})$.

Lemma 5.1. *For each* $p \in \boldsymbol{M}$*, take*

(5.1)
$$F^s_p = \bigcap_{n=0}^{\infty} D_{\boldsymbol{g}^n(p)}\boldsymbol{g}^{-n}(\overline{K^s_{\alpha,\boldsymbol{f},\boldsymbol{g}^n(p)}}) \quad \textit{and} \quad F^u_p = \bigcap_{n=0}^{\infty} D_{\boldsymbol{g}^{-n}(p)}\boldsymbol{g}^{n}(\overline{K^u_{\alpha,\boldsymbol{f},\boldsymbol{g}^{-n}(p)}}).$$

Thus, the families F^s_p *and* F^u_p *are* $D\boldsymbol{g}$*-invariant (see Figure 3).*

Proof. By Lemma 4.4 we have for all $p \in \mathbf{M}$, $D_{\boldsymbol{g}(p)}\boldsymbol{g}^{-1}(\overline{K^s_{\alpha,\boldsymbol{f},\boldsymbol{g}(p)}}) \subseteq K^s_{\alpha,\boldsymbol{f},p}$ and $D_pg(\overline{K^u_{\alpha,\boldsymbol{f},p}}) \subseteq K^u_{\alpha,\boldsymbol{f},\boldsymbol{g}(p)}$. Thus

$$D_{\boldsymbol{g}(p)}\boldsymbol{g}^{-1}(F^s_{\boldsymbol{g}(p)}) \subseteq \bigcap_{n=0}^{\infty} D_{\boldsymbol{g}^n(p)}\boldsymbol{g}^{-n}(\overline{K^s_{\alpha,\boldsymbol{f},\boldsymbol{g}^n(p)}}) = F^s_p.$$

On the other hand,

$$D_p\boldsymbol{g}(F^s_p) = D_p\boldsymbol{g}(\overline{K^s_{\alpha,\boldsymbol{f},p}}) \cap \bigcap_{n=1}^{\infty} D_pg(D_{\boldsymbol{g}^n(p)}\boldsymbol{g}^{-n}(\overline{K^s_{\alpha,\boldsymbol{f},\boldsymbol{g}^n(p)}}))$$

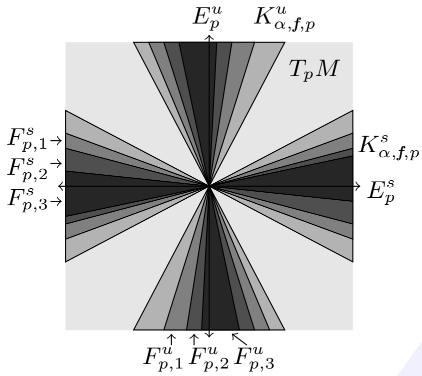

FIGURE 3. $F^r_{p,n} = \bigcap_{k=1}^{n} D\boldsymbol{g}^{\pm k}_{\boldsymbol{g}^{\pm k}(p)}(\overline{K^s_{\alpha,\boldsymbol{f},\boldsymbol{g}^{\pm k}(p)}})$ for $r = s, u$ and $n = 1, 2, 3$.

$$\subseteq \bigcap_{n=0}^{\infty} D_{\boldsymbol{g}^{n+1}(p)}\boldsymbol{g}^{-n}(\overline{K^s_{\alpha,\boldsymbol{f},\boldsymbol{g}^{n+1}(p)}}) = F^s_{\boldsymbol{g}(p)}.$$

Consequently, $D_p\boldsymbol{g}(F^s_p) = F^s_{\boldsymbol{g}(p)}$. Analogously we can prove $D_p\boldsymbol{g}(F^u_p) = F^u_{\boldsymbol{g}(p)}$. □

Inductively we have $D_p\boldsymbol{g}^n(F^s_p) = F^s_{\boldsymbol{g}^n(p)}$ and $D_p\boldsymbol{g}^n(F^u_p) = F^u_{\boldsymbol{g}^n(p)}$, for all $n \geq 1$. Since $F^r_p \subseteq K^r_{\alpha,\boldsymbol{f},p}$ for $r = s, u$, it follows from Lemma 4.7 that, for all $n \geq 1$,

$$\|D_p\boldsymbol{g}^n v\| \geq \frac{1}{\eta^n}\|v\| \text{ for } v \in F^u_p \quad \text{and} \quad \|D_p\boldsymbol{g}^{-n} v\| \geq \frac{1}{\eta^n}\|v\| \text{ for } v \in F^s_p.$$

Lemma 5.2. *F^s_p and F^u_p given in* (5.1) *are vectorial subspaces and furthermore $T_p\boldsymbol{M} = F^s_p \oplus F^u_p$, for each $p \in \boldsymbol{M}$.*

Proof. See Proposition 7.3.3 in [4]. □

Proposition 5.3. *$\boldsymbol{g}$ is an Anosov family and satisfies the property of the angles.*

Proof. From Lemmas 4.7, 5.1 and 5.2 we have that, considering the splitting $T_p\mathbf{M} = F^s_p \oplus F^u_p$, for each $p \in \mathbf{M}$, $\boldsymbol{g}$ has hyperbolic behaviour. We can prove that this splitting is unique (see Lemma 3.1) and depends continuously on p (see Proposition 3.4). Consequently, $\boldsymbol{g}$ is an Anosov family. Finally, since $F^s_p \subseteq K^s_{\alpha,\boldsymbol{f},p}$ and $F^u_p \subseteq K^u_{\alpha,\boldsymbol{f},p}$ for all p and $\alpha < \frac{1-\lambda}{1+\lambda} < 1$, we have that $\boldsymbol{g}$ s.p.a. □

From Proposition 5.3 we obtain the set consisting of Anosov families that s.p.a. is open in $\mathcal{F}(\mathbf{M})$. Finally will show that the set consisting of all the Anosov families is open in $\mathcal{F}(\mathbf{M})$. In order to prove this result, let's see the following facts: suppose that $(\mathbf{M}, \langle\cdot,\cdot\rangle, \boldsymbol{f})$ does not s.p.a. with the Riemannian metric $\langle\cdot,\cdot\rangle$. Thus $(\mathbf{M}, \langle\cdot,\cdot\rangle^\star, \boldsymbol{f})$ is a strictly Anosov family that s. p. a. with the Riemannian metric $\langle\cdot,\cdot\rangle^\star$ obtained in Proposition 3.7. Fix $\varepsilon > 0$ and take $\Delta_i = \frac{1}{\mu_i}(\frac{\lambda+\varepsilon}{\varepsilon}c)^2$ (see (3.7)). Thus,

$$\Delta_i^{-1}\|v\|^\star \leq \|v\| \leq 2\|v\|^\star \quad \text{for all } v \in TM_i,\ i \in \mathbb{Z},$$

where $\|\cdot\|$ and $\|\cdot\|^\star$ are the norms induced by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle^\star$ on $\mathbf{M}$, respectively. From Proposition 5.3 it follows that there exists a sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ such that, if $\boldsymbol{g} = (g_i)_{i\in\mathbb{Z}}$ is a non-stationary dynamical system with $d^\star_{\mathbf{D}_i}(f_i, g_i) < \varepsilon_i$, then $(\mathbf{M}, \langle\cdot,\cdot\rangle^\star, \boldsymbol{g})$ is an Anosov family, where $d^\star_{\mathbf{D}_i}$ is the metric on $\mathbf{D}_i$ induced by the metric $\langle\cdot,\cdot\rangle^\star$ on $\mathbf{M}$. We want to show that each family in some strong basic neighborhood of $\boldsymbol{f}$ is an Anosov family with the metric $\langle\cdot,\cdot\rangle$. This fact is not immediate, since $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle^\star$ are not necessarily uniformly equivalent on $\mathbf{M}$ and the notion of Anosov family depends on the metric on the total space.

Theorem 5.4. *$\mathcal{A}(\boldsymbol{M})$ is open in $\mathcal{F}(\boldsymbol{M})$.*

Proof. If $\boldsymbol{f}$ satisfies the property of the angles, by Proposition 5.3 there exists a strong basic neighborhood $B(\boldsymbol{f}, (\varepsilon_i)_{i\in\mathbb{Z}})$ of $\boldsymbol{f}$ such that, if $\boldsymbol{g} \in B(\boldsymbol{f}, (\varepsilon_i)_{i\in\mathbb{Z}})$ then $\boldsymbol{g}$ is an Anosov family. Suppose that $\boldsymbol{f}$ does not satisfy the property of the angles. From Proposition 5.3 we have there exists a sequence of positive numbers $(\varepsilon_i)_{i\in\mathbb{Z}}$ such that, if $\boldsymbol{g} = (g_i)_{i\in\mathbb{Z}} \in \mathcal{F}(\mathbf{M})$ and $d^\star_{\mathbf{D}_i}(f_i, g_i) < \varepsilon_i$, then $(\mathbf{M}, \langle\cdot,\cdot\rangle^\star, \boldsymbol{g})$ is a strictly Anosov family with constant $\tilde{\lambda} = \eta \in (0, 1)$. For each i, take $\tilde{\varepsilon}_i = \varepsilon_i/\Delta_i$. Notice that if $d_{\mathbf{D}_i}(f_i, g_i) < \tilde{\varepsilon}_i$ then $d_{\mathbf{D}^\star_i}(f_i, g_i) < \varepsilon_i$, for all i. Consequently, if $\boldsymbol{g} \in B(\boldsymbol{f}, (\tilde{\varepsilon}_i)_{i\in\mathbb{Z}})$, then $(\mathbf{M}, \langle\cdot,\cdot\rangle^\star, \boldsymbol{g})$ is an Anosov family. Consider the stable subspace $E^s_{\boldsymbol{g},p}$ of $\boldsymbol{g}$ at p (with respect to the metric $\langle\cdot,\cdot\rangle^\star$). If $v \in E^s_{\boldsymbol{g},p}$, then $v = v_s + v_u$, where $v_s \in E^s_{\boldsymbol{f},p}$ and $v_u \in E^u_{\boldsymbol{f},p}$. Take $\alpha \in (0, N)$, where $N = \min\{\frac{\varepsilon}{c(\lambda+\varepsilon)}, \frac{1-\lambda}{1+\lambda}\}$. Since the stable subspaces of $\boldsymbol{g}$ are contained in the stable α-cones of $\boldsymbol{f}$ and $\|v_s\| \leq \|v_s\|^\star$, it follows from (3.4) that

$$\|v_s\| \leq \|v_s + v_u\| + \|v_u\| \leq \|v_s + v_u\| + \alpha\|v_s\|^\star \leq \|v\| + \alpha\frac{\lambda+\varepsilon}{\varepsilon}c\|v_s\|.$$

Thus $(1 - \alpha\frac{\lambda+\varepsilon}{\varepsilon}c)\|v_s\| \leq \|v\|$ (note that $1 - \alpha\frac{\lambda+\varepsilon}{\varepsilon}c > 0$ because $\alpha < \frac{\varepsilon}{c(\lambda+\varepsilon)}$). Hence

$$\begin{aligned}\|D_p g^n(v)\| &\leq 2\|D_p g^n(v)\|^\star \leq 2\eta^n(\|v_s\|^\star + \|v_u\|^\star) \leq 2\eta^n(1+\alpha)\|v_s\|^\star \\ &\leq 2\eta^n(1+\alpha)\frac{\lambda+\varepsilon}{\varepsilon}c(1 - \alpha\frac{\lambda+\varepsilon}{\varepsilon}c)^{-1}\|v\| = c'\eta^n\|v\|,\end{aligned}$$

where $c' = 2(1+\alpha)\frac{\lambda+\varepsilon}{\varepsilon}c(1 - \alpha\frac{\lambda+\varepsilon}{\varepsilon}c)^{-1}$. Analogously we have $\|D_p\boldsymbol{g}^{-n}(v)\| \leq c'\eta^n\|v\|$ for $v \in E^u_{\boldsymbol{g},p}$. Hence, $(\mathbf{M}, \langle\cdot,\cdot\rangle, \boldsymbol{g})$ is an Anosov family with constants η and c'. □

Note that for the basic strong neighborhoods $B(\boldsymbol{f}, (\varepsilon_i)_{i\in\mathbb{Z}})$ of a system $(f_i)_{i\in\mathbb{Z}}$ the ε_i can be arbitrarily small for $|i|$ large. When there exists $\varepsilon > 0$ such that $\varepsilon_i = \varepsilon$ for all $i \in \mathbb{Z}$, the neighborhood is called *uniform*. As noted above, when $\boldsymbol{f}$ is the constant family associated to an Anosov diffeomorphism, it is possible to find a uniform neighborhood of $\boldsymbol{f}$ whose elements are Anosov families. In general it is not possible to find a uniform neighborhood of an Anosov family such that each family in that neighborhood is Anosov. For example, if the angles between the stable and unstable subspace decay, or if we can not get the inequality (4.2) with a uniform β_i, etc., it is necessary to take the ε_i's ever smaller. In [8] we will give conditions on the families for obtain uniform neighborhoods.

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