

SOME CONDITIONS ON THE FORM OF THIRD ELEMENT FROM DIOPHANTINE PAIRS AND ITS APPLICATION

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ABSTRACT. A set $\{a_1, a_2, \dots, a_m\}$ of positive integers is called a Diophantine m -tuple if $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$. In this paper, we show that the form of third element in Diophantine pairs and develop some results which are needed to prove the extendibility of the Diophantine pair $\{a, b\}$ with some conditions. By using this result, we prove the extendibility of Diophantine pairs $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}\}$ and $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}\}$, where F_n is the n -th Fibonacci number.

1. Introduction

A Diophantine m -tuple is a set which consists of m distinct positive integers satisfying the property that the product of any two of them is one less than a perfect square. If the set which consists of rational numbers satisfy the same property, then we called rational Diophantine m -tuple. Diophantus found the first rational Diophantine quadruple $\{1/16, 33/16, 17/4, 105/16\}$. However, the first Diophantine quadruple $\{1, 3, 8, 120\}$ was found by Fermat. Many famous mathematicians made lots of results related to the problems of Diophantine m -tuple, but still there are many open problems. Especially, the most famous problem is the extendibility of Diophantine m -tuple.

For any Diophantine triple $\{a, b, c\}$ with $a < b < c$, the set $\{a, b, c, d_{\pm}\}$ is a Diophantine quadruple, where

$$d_{\pm} = a + b + c + 2abc \pm 2rst$$

and r, s, t are the positive integers satisfying

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2.$$

A folklore conjecture is that there does not exist a Diophantine quintuple. The stronger version of this conjecture states that if $\{a, b, c, d\}$ is a Diophantine

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quadruple and $d > \max\{a, b, c\}$, then $d = d_+$. These Diophantine quadruples are called regular.

The reason why the extendibility is important is related to the elliptic curves. To extend the Diophantine triple $\{a, b, c\}$ to Diophantine quadruple, we have to solve the equations

$$ax + 1 = \square, \quad bx + 1 = \square, \quad cx + 1 = \square.$$

Hence, we have the equation

$$E : y^2 = (ax + 1)(bx + 1)(cx + 1),$$

which is the elliptic curve by the product of three equations. Then we have always the integer points

$$(0, \pm 1), \quad (d_+, \pm(at + rs)(bs + rt)(cr + st)), \quad (d_-, \pm((at - rs)(bs - rt)(cr - st))),$$

and also $(-1, 0)$ if $1 \in \{a, b, c\}$. The conjecture means it is possible to prove that there are no other integer points on E for some family of Diophantine triples. For example, A. Dujella [5] proved that the elliptic curve

$$E_k : y^2 = ((k - 1)x + 1)((k + 1)x + 1)(4kx + 1)$$

has four integer points

$$(0, \pm 1), \quad (16k^3 - 4k, \pm(128k^6 - 112k^4 - 20k^2 - 1))$$

under assumption that $\text{rank}(E_k(\mathbb{Q})) = 1$. Similar results [7] and [13] were proved for the equation

$$y^2 = (F_{2k}x + 1)(F_{2k+2}x + 1)(F_{2k+4} + 1)$$

and

$$y^2 = (F_{2k+1}x + 1)(F_{2k+3}x + 1)(F_{2k+5} + 1),$$

respectively, where F_n is the n -th Fibonacci number, defined by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Therefore, it is important not only how to prove the extendibility of Diophantine m -tuple but also which case of Diophantine m -tuple is proved.

In 1969, A. Baker and H. Davenport [1] proved that the Diophantine triple $\{1, 3, 8\}$ can not be extended to a Diophantine quintuple, namely, if the set $\{1, 3, 8, d\}$ is a Diophantine quadruple, then $d = 120$. This means the Diophantine triple $\{1, 3, 8\}$ is regular.

Let us consider the Diophantine pair $\{1, 3\}$. If the set $\{1, 3, c\}$ is a Diophantine triple, then we have the equation

$$y^2 - 3x^2 = -2,$$

since $c + 1 = x^2$, $3c + 1 = y^2$. We easily find the form of solutions of the Diophantine equation above is

$$(y + x\sqrt{3}) = (y_0 + x_0\sqrt{3})(2 + \sqrt{3})^k.$$

If (y_0, x_0) belongs to the same class as either of the solutions $(\pm 1, 1)$, then we have the form of third element

$$c = c_k = \frac{1}{6}[(2 + \sqrt{3})(7 + 4\sqrt{3})^k + (2 - \sqrt{3})(7 - 4\sqrt{3})^k - 4].$$

In 1998, Dujella and A. Pethö [9] developed this result, that is, if the set $\{1, 3, c_k, d\}$ is a Diophantine quadruple, then $d = d_- = c_{k-1}$ or $d_+ = c_{k+1}$. This result shows how important the form of third element in Diophantine pair is. Then we have a question such that how can we find the form of third element in Diophantine pairs. The form of third element c_k in the Diophantine pair $\{1, 3\}$ is found by using Theorem 8 in [15]. In fact, for a fixed Diophantine pair $\{a, b\}$ with $a < b$, the theorem shows the form of third element, where $b < 4a$. This theorem was also used in [3], [4], [12].

The recent research of third element in Diophantine m -tuple is Lemma 4.1 in [10] which shows the form of third element in the Diophantine pair $\{a, b\}$, where $a < b \leq 8a$. Unfortunately however, the general form of third elements are unknown until now.

In this paper, we use the following identity

$$F_{k-2}F_{k-1}F_{k+1}F_{k+2} + 1 = F_k^4,$$

which was proved by E. Cesàro. We can make many Diophantine pairs by using above identity. However, we consider only two sets

$$\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}\} \quad \text{and} \quad \{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}\},$$

since the first set satisfy the condition of Lemma 4.1 in [10], namely, $F_{k-1}F_{k+2} < 8F_{k-2}F_{k+1}$, but the second set does not satisfy the condition. Hence, we need to generalize the Lemma 4.1 for the extendibility of Diophantine pairs which do not satisfy the condition $b \leq 8a$. Also, we develop some results which need to prove the extendibility of Diophantine pair $\{a, b\}$ with some conditions. Through these generalized result, we prove the extendibility of $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}\}$ and $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}\}$.

2. Preliminaries

2.1. The bounds of each elements of Diophantine triple

We can find the lower bounds of second element of the Diophantine triple $\{a, b, c\}$ with $a < b$ using the following lemma.

Lemma 2.1 ([11, Lemma 1.3]). *Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d_+ < d$.*

- (1) *If $b < 2a$, then $b > 2.1 \cdot 10^4$.*
- (2) *If $2a \leq b \leq 8a$, then $b > 1.3 \cdot 10^5$.*
- (3) *If $b > 8a$, then $b > 2 \cdot 10^3$.*

Let $\{a, b, c\}$ be a Diophantine triple, and r, s, t be the positive integers satisfying $ab + 1 = r^2$, $ac + 1 = s^2$, $bc + 1 = t^2$. Then we have

$$at^2 - bs^2 = a - b.$$

We easily find the form of solutions of the equation above is

$$(t\sqrt{a} + s\sqrt{b}) = (t_0\sqrt{a} + s_0\sqrt{b})(r + \sqrt{bc})^\nu.$$

If (t_0, s_0) belongs to the same class as either of the solutions $(\pm 1, 1)$, then s can be expressed as $s = s_\nu^\tau$, where $\tau \in \{\pm 1\}$ and

$$s_0 = s_0^\tau = 1, \quad s_1^\tau = r + \tau a, \quad s_{\nu+2}^\tau = 2rs_{\nu+1}^\tau - s_\nu^\tau.$$

Define $c_\nu^\tau = ((s_\nu^\tau)^2 - 1)/a$. Then, we obtain

$$c = c_\nu^\tau = \frac{1}{4ab} [(a + b \pm 2\sqrt{ab})(2ab + 1 + 2r\sqrt{ab})^\nu + (a + b \mp 2\sqrt{ab})(2ab + 1 - 2r\sqrt{ab})^\nu - 2(a + b)].$$

First, we have the form of third element c in the Diophantine triple $\{a, b, c\}$ by the following theorem.

Theorem 2.2 ([15, Theorem 8]). *If $a < b < 4a$, and b are in \mathbb{Z}^+ , and*

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2$$

holds, then $c = c_k^+ = c_k^+(a, b)$ for some k or $c = c_j^- = c_j^-(a, b)$ for some j . The set c_j^- is omitted if $b = a + 2$.

However, it is difficult to apply this theorem in many cases, since the upper bound of b is small. Hence, there were a lot of researches to generalize the result. The following lemma generalizes of the Theorem 2.2.

Lemma 2.3 ([10, Lemma 4.1]). *Let $\{a, b, c\}$ be a Diophantine triple. Assume that $a < b \leq 8a$. Then $c = c_\nu^\tau$ for some ν and τ .*

Next, the following theorem gives us the bound of third element c in the Diophantine triple $\{a, b, c\}$.

Theorem 2.4 ([10, Theorem 1.2]). *Let $\{a, b, c\}$ be a Diophantine triple with $a < b$. Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $d > d_+$ and that $\{a, b, c', c\}$ is not a Diophantine quadruple for any c' with $0 < c' < d_-$, where d_+ and d_- are defined by*

$$d_\pm = a + b + c + 2abc \pm 2rst,$$

respectively.

- (1) *If $b < 2a$, then $c < b^6$.*
- (2) *If $2a \leq b \leq 8a$, then $c < 9.5b^4$.*
- (3) *If $b > 8a$, then $c < b^5$.*

If $c = c_\nu^\tau$, then we can find the upper bound of c more specific by the following theorem.

Theorem 2.5 ([11, Theorem 1.4]). *Suppose that $\{a, b, c_\nu^\tau, d\}$ is a Diophantine quadruple with $d > c_{\nu+1}^\tau$ and that $\{a, b, c', c_\nu^\tau\}$ is not a Diophantine quadruple for any c' with $0 < c' < c_{\nu-1}^\tau$.*

- (1) *If $b < 2a$, then $c \leq c_3^+$.*
- (2) *If $2a \leq b \leq 8a$, then $c \leq c_2^+$.*

2.2. The Properties of solutions of Pell equation

We have to solve the system

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2$$

to extend the Diophantine triple $\{a, b, c\}$ to the Diophantine quadruple $\{a, b, c, d\}$. One can eliminate d to obtain the following system of Pell equations

- (1) $ay^2 - bx^2 = a - b,$
- (2) $az^2 - cx^2 = a - c,$
- (3) $bz^2 - cy^2 = b - c.$

Lemma 2.6 ([6, Lemma 1]). *There exist positive integers i_0, j_0 and integers $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}, i = 1, \dots, i_0, j = 1, \dots, j_0$, with the following properties:*

- (1) $(z_0^{(i)}, x_0^{(i)})$ and $(z_1^{(j)}, y_1^{(j)})$ are solutions of (2) and (3), respectively.
- (2) $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}$ satisfy the following inequalities

$$0 < x_0^{(i)} \leq \sqrt{\frac{a(c-a)}{2(s-1)}} < \sqrt{\frac{s+1}{2}} < \sqrt[4]{ac},$$

$$0 \leq |z_0^{(i)}| \leq \sqrt{\frac{(s-1)(c-a)}{2a}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} < \frac{c}{2},$$

$$0 < y_1^{(j)} \leq \sqrt{\frac{b(c-b)}{2(t-1)}} < \sqrt{\frac{t+1}{2}} < \sqrt[4]{bc},$$

$$0 \leq |z_1^{(j)}| \leq \sqrt{\frac{(t-1)(c-b)}{2b}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}} < \frac{c}{3}.$$

- (3) *If (z, x) and (z, y) are positive integers of (2) and (3), respectively, then there exist $i \in \{1, \dots, i_0\}, j \in \{1, \dots, j_0\}$ and integers $m, n \geq 0$ such that*

$$(4) \quad z\sqrt{a} + x\sqrt{c} = (z_0^{(i)}\sqrt{a} + x_0^{(i)}\sqrt{c})(s + \sqrt{ac})^m,$$

$$(5) \quad z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})(t + \sqrt{bc})^n.$$

From now on, we omit the superscripts (i) and (j) . By (4), we may write $z = v_m$, where

$$(6) \quad v_0 = z_0, \quad v_1 = sz_0 + cx_0, \quad v_{m+2} = 2sv_{m+1} - v_m,$$

and by (5), we may write $z = w_n$, where

$$(7) \quad w_0 = z_1, \quad w_1 = tz_1 + cy_1, \quad w_{n+2} = 2tw_{n+1} - w_n.$$

Lemma 2.7 ([8, Lemma 3]). *If $v_m = w_n$, then $n - 1 \leq m \leq 2n + 1$.*

2.3. Congruence relation between solutions of Pell equations

In this section, we give the congruence relations between v_m and w_n , and properties of initial terms of (6) and (7).

Lemma 2.8 ([6, Lemma 4]). *We have the following properties of v_m and w_n .*

$$\begin{aligned} v_{2m} &\equiv z_0 + 2c[az_0m^2 + sx_0m] \pmod{8c^2}, \\ v_{2m+1} &\equiv sz_0 + c[2asx_0m(m+1) + x_0(2m+1)] \pmod{4c^2}, \\ w_{2n} &\equiv z_1 + 2c[bz_1n^2 + ty_1n] \pmod{8c^2}, \\ w_{2n+1} &\equiv tz_1 + c[2btz_1n(n+1) + y_1(2n+1)] \pmod{4c^2}. \end{aligned}$$

We have a question such that when does $v_m = w_n$ have a solution and if there exists a solution of $v_m = w_n$, then what are the values possible for the solution. The following lemma gives us the answer.

Lemma 2.9 ([8, Lemma 8]). *We have the following results.*

- (1) *If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$. Furthermore, $|z_0| = 1$ or $|z_0| = cr - st$ or $|z_0| < \min\{0.869a^{-5/14}c^{9/14}, 0.972b^{-0.3}c^{0.7}\}$.*
- (2) *If the equation $v_{2m+1} = w_{2n}$ has a solution, then $|z_0| = t, |z_1| = cr - st$ and $z_0z_1 < 0$.*
- (3) *If the equation $v_{2m} = w_{2n+1}$ has a solution, then $|z_0| = cr - st, |z_1| = s$ and $z_0z_1 < 0$.*
- (4) *If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $|z_0| = t, |z_1| = s$ and $z_0z_1 > 0$.*

Furthermore, the solution of $v_m = w_n$ is more specific when $c = c_\nu^\mp \leq c_3^+$ by the following lemma.

Lemma 2.10 ([11, Lemma 3.1]). *Assume that $a < b \leq 8a$.*

(i) *Assume that $b < 3a$. In the case of $c = c_1^-$, we have $v_{2m+1} \neq w_{2n}$, $v_{2m} \neq w_{2n+1}$ and $v_{2m+1} \neq w_{2n+1}$. Moreover, if $v_{2m} = w_{2n}$, then $z_0 = z_1 = 1$.*

(ii) *In the case of $c = c_1^+$, we have $v_{2m+1} \neq w_{2n}$, $v_{2m} \neq w_{2n+1}$ and $v_{2m+1} \neq w_{2n+1}$. Moreover, if $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$.*

(iii) *In the case of $c = c_2^-$, we have $v_{2m+1} \neq w_{2n}$ and $v_{2m+1} \neq w_{2n+1}$. Moreover, we have the following:*

- (1) *If $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$ or $cr - st$.*
- (2) *If $v_{2m} = w_{2n+1}$, then $|z_0| = cr - st$ and $|z_1| = s$ with $z_0z_1 < 0$.*

Furthermore, (2) occurs if and only if (1) with $|z_0| = cr - st$ occurs.

(iv) *In the case of $c \in \{c_2^+, c_3^-, c_3^+\}$, we have $v_{2m+1} \neq w_{2n}$ and $v_{2m} \neq w_{2n+1}$. Moreover, we get the following:*

- (1) *If $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$.*
- (2) *If $v_{2m+1} = w_{2n+1}$, then $|z_0| = t$ and $|z_1| = s$ with $z_0z_1 > 0$.*

2.4. Some theorems for applying the reduction method

From (4), (5) and sum of their conjugate, respectively, we get

$$v_m = \frac{1}{2\sqrt{a}}[(z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m + (z_0\sqrt{a} - x_0\sqrt{c})(s - \sqrt{ac})^m],$$

$$w_n = \frac{1}{2\sqrt{b}}[(z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n + (z_1\sqrt{b} - y_1\sqrt{c})(t - \sqrt{bc})^n].$$

Hence, we transform the equation $v_m = w_n$ into the following inequality.

Lemma 2.11 ([6, Lemma 5]). *Assume that $c > 4b$. If $v_m = w_n$ and $m, n \neq 0$, then*

$$0 < m \log(s + \sqrt{ac}) - n \log(t + \sqrt{bc}) + \log \frac{\sqrt{b}(x_0\sqrt{c} + z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c} + z_1\sqrt{b})} < \frac{8}{3} ac(s + \sqrt{ac})^{-2m}.$$

We use the following theorem and lemma to obtain the upper bound for m .

Theorem 2.12 ([2, p. 20]). *For a linear form*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_l \log \alpha_l \neq 0$$

in logarithms of l algebraic numbers $\alpha_1, \alpha_2, \dots, \alpha_l$ with rational coefficients $\beta_1, \beta_2, \dots, \beta_l$, we have

$$\log |\Lambda| \geq -18(l+1)!l^{l+1}(32d)^{l+2}h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log \beta,$$

where $\beta := \max\{|\beta_1|, \dots, |\beta_l|\}$, $d := [\mathbb{Q}(\alpha_1, \dots, \alpha_l) : \mathbb{Q}]$ and

$$h'(\alpha) = \frac{1}{d} \max\{h(\alpha), |\log \alpha|, 1\}$$

with the standard logarithmic Weil height $h(\alpha)$ of α .

Lemma 2.13 ([9, Lemma 5]). *Suppose that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let $\epsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer.*

(1) *If $\epsilon > 0$, then there is no solution of the inequality*

$$(8) \quad 0 < m\kappa - n + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m \leq M.$$

(2) *Let $r = \lfloor \mu q + \frac{1}{2} \rfloor$. If $p - q + r = 0$, then there is no solution of inequality (8) in integers m and n with*

$$\max \left\{ \frac{\log(3Aq)}{\log B}, 1 \right\} < m \leq M.$$

3. The generalized bound for the second element

We can easily know that the form of third elements of Diophantine triple $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}, c\}$ by using Lemma 2.3. However, we can't apply the Lemma 2.3 in the case of $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c\}$, since

$$8F_{k-2}F_{k-1} < F_{k+1}F_{k+2} < 21F_{k-2}F_{k-1}.$$

Hence, we need to generalize the Lemma 2.3 to find the form of third elements of Diophantine triple $\{a, b, c\}$ when $b > 8a$.

Lemma 3.1. *Let $\{a, b, c\}$ be a Diophantine triple and $a < b \leq 24a$. Suppose that $\{1, 3, a, b\}$ is not a Diophantine quadruple. Then $c = c_v^\tau$ for some n and τ .*

Proof. The proof is same as the proof of Lemma 4.1 in [10], except the case of $c' < b$. If $c' = 0$, then $s' = rs - at = 1$, and $c = c_1^-$ or c_1^+ . Let $r' = s'r - at'$ and $b' = ((r')^2 - 1)/a$. Then, $b' = a + b + c' + 2abc' - 2rs't'$, and thus, [15, Lemma 4] and $b \leq 24a$ together imply that

$$b' < \frac{b}{4ac'} \leq \frac{24a}{4ac'} = \frac{6}{c'}.$$

First, we consider the case of $c' = 1$. Since b' and c' satisfy the condition $b'c' + 1 = b' + 1 = \square$, we have $b' = 0$. For the other cases, we have the following results.

$$b' = \begin{cases} 0 & \text{if } c' \geq 6, \\ 0, 1 & \text{if } c' = 3, 4, 5, \\ 0, 1, 2 & \text{if } c' = 2. \end{cases}$$

We need to check only the case $(b', c') = (1, 3)$ except $b' = 0$ using the condition $b'c' + 1 = \square$. However, this means the set $\{1, 3, a, b\}$ satisfies Diophantine quadruple, which is a contradiction by assumption. In all cases, we obtain $b' = 0$ which yields $c' = a + b - 2r = c_1^-$ and $c = c_2^-$. Hence, we proved the lemma. \square

Assume that $a < b \leq 24a$ and $\{1, 3, a, b\}$ is not a Diophantine quadruple. Then, Lemma 3.1 implies that the positive solutions of the Diophantine equation

$$ay^2 - bx^2 = a - b$$

are given by

$$y\sqrt{a} + x\sqrt{b} = (\pm\sqrt{a} + \sqrt{b})(r + \sqrt{ab})^l$$

with some non-negative integer l . Thus, we may write $x = p_l$, $y = V_l$, where

$$(9) \quad p_0 = 1, \quad p_1 = r \pm a, \quad p_{l+2} = 2rp_{l+1} - p_l,$$

$$(10) \quad V_0 = \pm 1, \quad V_1 = b \pm r, \quad V_{l+2} = 2rV_{l+1} - V_l.$$

From equation (4) and (5), we may also write $x = q_m$, $y = W_n$, where

$$(11) \quad q_0 = x_0, \quad q_1 = sx_0 + az_0, \quad q_{m+2} = 2sq_{m+1} - q_m,$$

$$(12) \quad W_0 = y_1, \quad W_1 = ty_1 + bz_1, \quad W_{n+2} = 2tW_{n+1} - W_n.$$

The result of next lemma is similar to the Lemma 2.10, but the bound of b is further extended.

Lemma 3.2. *Assume that $a < b \leq 24a$ and $\{1, 3, a, b\}$ is not a Diophantine quadruple.*

(i) *In the case of c_1^+ , we have $v_{2m+1} \neq w_{2n}$, $v_{2m} \neq w_{2n+1}$ and $v_{2m+1} \neq w_{2n+1}$. Moreover, if $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$.*

(ii) *In the case of $c = c_2^-$, we have $v_{2m+1} \neq w_{2n}$ and $v_{2m+1} \neq w_{2n+1}$. Moreover, we have the following:*

(1) *If $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$ or $cr - st$.*

(2) *If $v_{2m} = w_{2n+1}$, then $|z_0| = cr - st$ and $|z_1| = s$ with $z_0 z_1 < 0$.*

Furthermore, (2) occurs if and only if (1) with $|z_0| = cr - st$ occurs.

(iii) *In the case of $c \in \{c_2^+, c_3^-, c_3^+\}$, then we have $v_{2m+1} \neq w_{2n}$ and $v_{2m} \neq w_{2n+1}$. Moreover, we get the following:*

• *If $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$.*

• *If $v_{2m+1} = w_{2n+1}$, then $|z_0| = t$ and $|z_1| = s$ with $z_0 z_1 > 0$.*

Proof. The proof proceeds along the same lines as that of [11, Lemma 3.1].

(i) From (10) and (12), we have $y_1^2 \equiv 1 \pmod{b}$ either n is even or odd in W_n . Since $a < b - 2$ and $r < b - 1$, so $c = c_1^+ = a + b + 2r \leq 4b - 7$. Hence

$$0 < y_1^2 \leq \frac{t+1}{2} \leq \frac{\sqrt{b(4b-7)+1}}{2} < b.$$

This means $y_1^2 = 1$, that is, $y_1 = 1$ and $z_1 = \pm 1$. Hence, the case $v_{2m} = w_{2n}$ only occur and $z_0 = z_1$ from Lemma 2.9.

(ii) Suppose that $|z_0| = t$. Then we have $c > 4ab^2$ by upper bound of y . However, this is a contradiction, since

$$c = c_2^- = 4ab \left\{ b - \left(2r - a - \frac{1}{a} \right) - \frac{r-a}{ab} \right\} < 4ab^2.$$

Therefore, we have $|z_0| \neq t$ and this means $v_{2m+1} \neq w_{2n}$ and $v_{2m+1} \neq w_{2n+1}$ by Lemma 2.9. Also, the last case of $|z_0|$ in $v_{2m} = w_{2n}$ occurs when the Diophantine quadruple $\{a, b, c, d\}$ is only irregular. Hence, it suffices to show that any Diophantine quadruple $\{a, b, c, d_0\}$ with $d_0 < c$ is regular when the cases of $c \leq c_3^+$. Assume that the set $\{a, b, c, d_0\}$ with $d_0 < d_-$ is a Diophantine quadruple. First, since $r = \sqrt{ab+1} \leq \sqrt{24a^2+1} < 5a$ and together with Lemma 2.1, we have following inequality

$$c \leq c_3^+ < a^{2.5} b^{3.5} \left(\frac{176}{b} + \frac{16}{b^{0.5}} + \frac{184}{b^{2.5}} + \frac{24}{b^{1.5}} + \frac{39}{b^{3.5}} + \frac{9}{b^{2.5}} \right) < a^{2.5} b^{3.5}.$$

However, we have $c > 16a^{2.5} b^{3.5}$ by Lemma 15 of [14]. Hence, Diophantine quadruple $\{a, b, c, d_0\}$ with $d_0 < c$ can not be irregular. It is easy to see that v_{2m} and w_{2n+1} in (2) are equal to v_{2m} and w_{2n} in (1) with $|z_0| = cr - st$.

(iii) Since we show that if $c \leq c_3^+$, then any Diophantine quadruple $\{a, b, c, d_0\}$ with $d_0 < c$ is regular in (ii), it suffices to show that $\max\{|z_0|, |z_1|\} < cr - st$ when $c \geq c_2^+$. Since $c \geq c_2^+ > 4ab^2 \geq 32ab$, we have

$$cr - st = \frac{c^2 - (a+b)c - 1}{cr + st} > \frac{(1 - a/c - b/c - 1/c^2)c}{2\sqrt{ab}\sqrt{1 + 1/(ab)}} > \frac{c}{2.1\sqrt{ab}}.$$

If $\max\{|z_0|, |z_1|\} \geq cr - st$, then the bounds of $|z_0|$ and $|z_1|$ show that

$$c < \left(\frac{2.1}{\sqrt{2}}\right)^4 ab^2 < 5ab^2.$$

However, we have the following inequalities

$$c \geq c_2^+ > 4ab(a+b+2r) > 4ab^2 \left(\frac{1}{24} + 1 + \frac{1}{\sqrt{6}}\right) > 5.79ab^2,$$

since $b \leq 24a$ implies that $r = \sqrt{ab+1} > b/(2\sqrt{6})$. Hence, this is a contradiction. \square

4. The extendibility of $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}\}$

In this section, we prove the extendibility of the Diophantine pair $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}\}$. From Theorem 2.5, we only need to check the cases of $c \leq c_2^+$.

Lemma 4.1. *Suppose that $m, n \geq 2$.*

(1) *If $v_{2m} = w_{2n}$, then*

$$m \geq \begin{cases} (\sqrt{2F_k+1} - 1)/2, & \text{if } c_1^+ \text{ or } c_2^- \text{ with } |z_0| = cr - st, \\ \sqrt[4]{F_k}/2, & \text{if } c_2^- \text{ with } |z_0| = 1 \text{ or } c_2^+. \end{cases}$$

(2) *If $v_{2m} = w_{2n+1}$ with c_2^- or $v_{2m+1} = w_{2n+1}$ with c_2^+ , then*

$$m \geq \frac{\sqrt{2F_k+1} - 1}{2}.$$

Proof. Since c_i^\pm is divisible by F_k for all $i = 1, 2$ and

- In the case of c_1^+ ,
 $s_1^+ \equiv F_{k-2}F_{k+1} = a \pmod{F_k}$ and
 $t_1^+ \equiv -F_{k-2}F_{k+1} = -a \pmod{F_k}$.
- In the case of c_2^- ,
 $s_2^- = 2F_{k-1}^2F_k^2 - 1 \equiv -1 \pmod{F_k}$ and
 $t_2^- = 2F_k^2(F_{k-1}F_k \pm 1) + 1 \equiv 1 \pmod{F_k}$.
- In the case of c_2^+ ,
 $s_2^+ = 2F_{k-2}F_{k+1}^3 + 1 \equiv -1 \pmod{F_k}$ and
 $t_2^+ = 2F_k^2F_{k+1}^2 - 1 \equiv -1 \pmod{F_k}$.

Hence, we have the following lower bounds from Lemma 2.8.

(1) The case of $v_{2m} = w_{2n}$:

We easily find the relation between v_{2m} and w_{2n} such that

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}.$$

- First, we consider the case of c_1^+ .

Since $F_{k-1}F_{k+2} \equiv -F_{k-2}F_{k+1} \pmod{F_k}$ and $\gcd(F_{k-2}F_{k+1}, F_k) = 1$, so we have

$$\pm(m^2 + n^2 \pm m \pm n) \equiv 0 \pmod{F_k}.$$

This means $2(m^2 + m) \geq m^2 + n^2 \pm m \pm n \geq m^2 + n^2 - m - n > 0$.
Hence $2(m^2 + m) \geq F_k$.

- In the case of c_2^- with $|z_0| = cr - st$, we get the equation

$$\pm am^2 + am \equiv \pm bn^2 + bn \pmod{F_k},$$

since $st \equiv -1 \pmod{F_k}$ and $r \equiv 0 \pmod{F_k}$. Hence, similar to above, we have the result.

- Let us consider the other case of c_2^- , that is, c_2^- with $|z_0| = 1$. We have the equation

$$\pm a(m^2 + n^2) \equiv m + n \pmod{F_k},$$

and squaring both sides, then $a^2(m^2 + n^2)^2 \equiv (m + n)^2 \pmod{F_k}$.
Since $a^2 \equiv 1 \pmod{F_k}$, we have

$$(m^2 + n^2)^2 - (m + n)^2 \geq 0.$$

This means $4m^4 \geq F_k$, so we have the desired result.

- Lastly, in the case of c_2^+ with $|z_0| = 1$, we have

$$\pm a(m^2 + n^2) \equiv m - n \pmod{F_k}.$$

Similar to above, squaring both sides and we have

$$(m^2 + n^2) - (m - n)^2 \equiv 0 \pmod{F_k}.$$

Hence, $4m^4 \geq F_k$, and the result is deduced.

(2) The cases of $v_{2m} = w_{2n+1}$ with c_2^- and $v_{2m+1} = w_{2n+1}$ with c_2^+ :
From Lemma 2.8, we have

$$\pm a(-st)m^2 - astm \equiv \mp 2bstn(n+1) \pmod{F_k}$$

and

$$\pm 2atsm(m+1) \equiv \pm 2bstn(n+1) \pmod{F_k},$$

respectively. So, they become

$$a(m^2 + n^2 + n \pm m) \equiv 0 \pmod{F_k}$$

and

$$\pm a(m^2 + n^2 + m + n) \equiv 0 \pmod{F_k},$$

respectively, and we have the desired result. \square

Let us find the logarithmic inequality for $c = c_1^+$.

Lemma 4.2. *If $v_{2m} = w_{2n}$ with c_1^+ and $m, n \neq 0$, then*

$$\begin{aligned} 0 &< 2m \log(s + \sqrt{ac}) - 2n \log(t + \sqrt{bc}) + \log \frac{\sqrt{b}(x_0\sqrt{c} + z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c} + z_1\sqrt{b})} \\ &< 3.018(s + \sqrt{ac})^{-4m}. \end{aligned}$$

Proof. Put

$$P = \frac{1}{\sqrt{a}}(x_0\sqrt{c} + z_0\sqrt{a})(s + \sqrt{ac})^m, \quad Q = \frac{1}{\sqrt{b}}(y_1\sqrt{c} + z_1\sqrt{b})(t + \sqrt{bc})^n.$$

Then

$$P^{-1} = \frac{\sqrt{a}(x_0\sqrt{c} - z_0\sqrt{a})}{c - a}(s - \sqrt{ac})^m, \quad Q^{-1} = \frac{\sqrt{b}(y_1\sqrt{c} - z_1\sqrt{b})}{c - b}(t - \sqrt{bc})^n.$$

Therefore, the relation $v_m = w_n$ becomes

$$P - \frac{c-a}{a}P^{-1} = Q - \frac{c-b}{b}Q^{-1}.$$

Since $P > 0$, $Q > 0$ and

$$P - Q > \frac{c-a}{a}(Q - P)P^{-1}Q^{-1},$$

it follows that $P > Q$. Furthermore, we have

$$\frac{P-Q}{P} < \frac{c-a}{a}P^{-2} < \frac{1}{a(c-a)} \leq \frac{1}{170},$$

since the case of $\{3, 5\}$ is proved by Filipin, Fujita and Togbé [11]. Hence,

$$\begin{aligned} 0 &< \log \frac{P}{Q} = -\log\left(1 - \frac{P-Q}{P}\right) < \frac{171}{170} \left(\frac{c-a}{a}\right)P^{-2} \\ &< \frac{171}{170} \frac{c-a}{(\sqrt{c}-\sqrt{a})^2} (s + \sqrt{ac})^{-2m}. \end{aligned}$$

Since $\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} < 3$, we obtain the result. \square

4.1. The Theorem of Baker and Wüstholz

Now we apply theorem of Baker and Wüstholz.

(i) First, we consider the equation $v_{2m} = w_{2n}$. We may assume that $k > 2$. Using Lemma 2.11 and Lemma 4.2, and apply Theorem 2.12, then we have $l = 3$, $d = 4$, $\beta = 2m$,

$$\alpha_1 = s + \sqrt{ac}, \quad \alpha_2 = t + \sqrt{bc}, \quad \alpha_3 = \frac{(\sqrt{c} \pm \sqrt{a})\sqrt{b}}{(\sqrt{c} \pm \sqrt{b})\sqrt{a}}.$$

Let α_3' and α_3'' be the conjugates of α_3 whose absolute values are greater than one. Then

$$h'(\alpha_1) = \frac{1}{2} \log(\alpha_1) < \frac{1}{2} \log(2s), \quad h'(\alpha_2) = \frac{1}{2} \log(\alpha_2) < \frac{1}{2} \log(2t),$$

$$h'(\alpha_3) \leq \frac{1}{4} \{ \log(a^2(c-b)^2) + \log(\alpha_3 \alpha_3' \alpha_3'') \}$$

$$= \frac{1}{4} \{ \log(b\sqrt{ab}(\sqrt{c} + \sqrt{a})(\sqrt{c} + \sqrt{b})(c-a)) \} < \log(1.42c),$$

and

$$\log |\Lambda| \geq -18 \cdot 4! 3^4 (32 \cdot 4)^5 \frac{1}{2} \log(2s) \frac{1}{2} \log(2t) \log(1.42c) \cdot \log(24) \cdot \log(2m).$$

Since

$$\log\left(\frac{8}{3} ac(s + \sqrt{ac})^{-4m}\right) < (-2m + 1) \log(4ac)$$

and

$$\log(3.018(s + \sqrt{ac})^{-4m}) < (-2m + 1) \log(4ac),$$

we have

$$(13) \quad \frac{2m-1}{\log(2m)} < 9.556 \cdot 10^{14} \log(2c) \log(1.42c).$$

Let $x = F_k$.

- If $c = c_1^+$, then

$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14} \log(3.17x)^2 \cdot \log(2.67x)^2 \cdot \log(2x+1).$$

Hence, $x < 1.64 \cdot 10^{42}$ and $c_1^+ < 1.35 \cdot 10^{85}$.

- If $c = c_2^-$ with $|z_0| = cr - st$, then

$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14} \log(2x)^6 \cdot \log(1.89x)^6 \cdot \log(2x+1).$$

Hence, $x < 1.72 \cdot 10^{44}$ and $c_2^- < 6.22 \cdot 10^{266}$.

- If $c = c_2^-$ with $|z_0| = \pm 1$, then

$$\sqrt[4]{x} - 1 < 2.389 \cdot 10^{14} \log(2x)^6 \cdot \log(1.89x)^6 \cdot \log x.$$

Hence, $x < 4.38 \cdot 10^{91}$ and $c_2^- < 1.68 \cdot 10^{551}$.

- If $c = c_2^+$, then

$$\sqrt[4]{x} - 1 < 2.389 \cdot 10^{14} \log(2.55x)^6 \cdot \log(2.41x)^6 \cdot \log x.$$

Hence, $x < 4.42 \cdot 10^{91}$ and $c_2^+ < 8.83 \cdot 10^{551}$.

Since $F_k = (\alpha^k - \bar{\alpha}^k)/\sqrt{5}$, where $\alpha = (1 + \sqrt{5})/2 > 1.618$, we have the following inequality from Fibonacci numbers

$$(14) \quad (1.618)^k < (\alpha)^k = \bar{\alpha}^k + \sqrt{5} \cdot F_k.$$

We can find the upper bounds of k and m by using inequalities (13) and (14), respectively.

- If the case of c_1^+ , then $k \leq 203$ and $m \leq 9.04 \cdot 10^{20}$.
- If the case of c_2^- , then $k \leq 440$ and $m \leq 4.07 \cdot 10^{22}$.
- If the case of c_2^+ , then $k \leq 440$ and $m \leq 4.08 \cdot 10^{22}$.

(ii) Let $v_{2m} = w_{2n+1}$ with $n \neq 0$. We have $l = 3$, $d = 4$, $\beta = 2m + 1$,

$$\alpha_1 = s + \sqrt{ac}, \quad \alpha_2 = t + \sqrt{bc}, \quad \alpha_4 = \frac{((sr - ta)\sqrt{c} \pm (cr - st)\sqrt{a})\sqrt{b}}{(r\sqrt{c} \mp s\sqrt{b})\sqrt{a}}.$$

Let α'_4 and α''_4 be the conjugates of α_4 whose absolute values are greater than one. Then

$$\begin{aligned} h'(\alpha_1) &= \frac{1}{2} \log(\alpha_1) < \frac{1}{2} \log(2s), \quad h'(\alpha_2) = \frac{1}{2} \log(\alpha_2) < \frac{1}{2} \log(2t), \\ h'(\alpha_4) &\leq \frac{1}{4} \{\log(a^2(c-b)^2) + \log(\alpha_4 \alpha'_4 \alpha''_4)\} \\ &= \frac{1}{4} \{\log(b\sqrt{ab}((sr - ta)\sqrt{c} + (cr - st)\sqrt{a})(r\sqrt{c} + s\sqrt{b})(c-a))\} < \log(1.42\sqrt{rc}), \end{aligned}$$

since $(sr - ta)\sqrt{c} > (cr - st)\sqrt{a}$ and $sr - ta < r$. Hence, we have

$$(15) \quad \frac{2m-1}{\log(2m+1)} < 9.556 \cdot 10^{14} \log(2c) \log(1.42\sqrt{rc}).$$

If $c = c_2^-$, then

$$\sqrt{x+1} - 2 < 4.778 \cdot 10^{14} \log(1.97x)^6 \cdot \log(1.7x)^7 \cdot \log(x+1),$$

where $x = F_k$. Hence, we have $x < 4.9 \cdot 10^{44}$ and $c_2^- < 3.98 \cdot 10^{269}$. Again, by using the inequality from (14) and (15), we get the bound of $k \leq 215$ and $m \leq 1.11 \cdot 10^{22}$.

(iii) Let $v_{2m+1} = w_{2n+1}$ with $n \neq 0$. We have $l = 3$, $d = 4$, $\beta = 2m + 1$,

$$\alpha_1 = s + \sqrt{ac}, \quad \alpha_2 = t + \sqrt{bc}, \quad \alpha_5 = \frac{(r\sqrt{c} \pm t\sqrt{a})\sqrt{b}}{(r\sqrt{c} \pm s\sqrt{b})\sqrt{a}}.$$

Let α'_5 and α''_5 be the conjugates of α_5 whose absolute values are greater than one. Then

$$\begin{aligned} h'(\alpha_1) &= \frac{1}{2} \log(\alpha_1) < \frac{1}{2} \log(2s), \quad h'(\alpha_2) = \frac{1}{2} \log(\alpha_2) < \frac{1}{2} \log(2t), \\ h'(\alpha_5) &\leq \frac{1}{4} \{\log(a^2(c-b)^2) + \log(\alpha_5 \alpha'_5 \alpha''_5)\} \\ &= \frac{1}{4} \{\log(b\sqrt{ab}(r\sqrt{c} + t\sqrt{a})(r\sqrt{c} + s\sqrt{b})(c-a))\} < \log(1.42\sqrt{rc}). \end{aligned}$$

Hence, we have

$$(16) \quad \frac{2m}{\log(2m+1)} < 9.556 \cdot 10^{14} \log(2c) \log(1.42\sqrt{rc}).$$

If $c = c_2^+$, then

$$\sqrt{x+1} - 1 < 4.778 \cdot 10^{14} \log(2.53x)^6 \cdot \log(2.11x)^7 \cdot \log(x+1),$$

where $x = F_{2k+2}$. Hence, we get $x < 4.95 \cdot 10^{44}$ and $c_2^+ < 1.91 \cdot 10^{270}$. This means $k \leq 215$ and $m \leq 1.12 \cdot 10^{22}$.

4.2. The reduction method

Now dividing logarithmic inequalities from Lemma 2.11 and Lemma 4.2 by $\log \alpha_2$, respectively leads us to the inequalities

$$\begin{aligned} 0 &< m_1\kappa - n_1 + \mu_1 < A_1B^{m_1}, \\ 0 &< m_1\kappa - n_1 + \mu_1 < A_2B^{m_1}, \\ 0 &< m_1\kappa - n_2 + \mu_2 < A_1B^{m_1}, \\ 0 &< m_2\kappa - n_2 + \mu_3 < A_1B^{m_2}, \end{aligned}$$

where $m_1 := 2m$, $m_2 := 2m + 1$, $n_1 := 2n$, $n_2 := 2n + 1$ and

$$\begin{aligned} \kappa &= \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu_1 = \frac{\log \alpha_3}{\log \alpha_2}, \quad \mu_2 = \frac{\log \alpha_4}{\log \alpha_2}, \quad \mu_3 = \frac{\log \alpha_5}{\log \alpha_2}, \\ A_1 &= \frac{(8/3)ac}{\log \alpha_2}, \quad A_2 = \frac{3.018}{\log \alpha_2}, \quad B = \alpha_1^2. \end{aligned}$$

We apply Lemma 2.13 to the logarithmic inequalities with $M_1 := 2m \leq 8.16 \cdot 10^{22}$ and $M_2 := 2m + 1 \leq 2.24 \cdot 10^{22}$. We have to examine $2 \cdot 203 + 2 \cdot 440 + 2 \cdot 440 + 2 \cdot 215 + 2 \cdot 215 = 3026$ cases. The program was developed in **PARI/GP** running with 400 digits. For the computations, if the first convergent such that $q > 6M_i$ with $i = 1, 2$ does not satisfy the condition $\epsilon > 0$, then we use the next convergent until we find the one that satisfies the conditions. Then we have the results as the following Table 1.

TABLE 1. Results from **PARI/GP** running

Case of c	Initial values	Use the next convergent
c_1^+	$z_0 = z_1 = 1$	0 case
	$z_0 = z_1 = -1$	99 cases
c_2^-	$z_0 = z_1 = 1$	330 cases
	$z_0 = z_1 = -1$	352 cases ($k = 89, \dots, 440$)
c_2^-	$z_0 = cr - st, z_1 = -s$	0 case
	$z_0 = st - cr, z_1 = s$	0 case
c_2^+	$z_0 = z_1 = 1$	353 cases ($k = 88, \dots, 440$)
	$z_0 = z_1 = -1$	353 cases ($k = 88, \dots, 440$)
c_2^+	$z_0 = t, z_1 = s$	0 case
	$z_0 = -t, z_1 = -s$	172 cases ($k = 44, \dots, 215$)

We have the upper bounds 13, 12 and 8 of m in the case of c_1^+ , c_2^- with $|z_0| = 1$, and c_2^+ with $|z_0| = 1$, respectively. If we take $M = 2m$ and continue the program again, then we have $m \leq 1$. Other cases, that is, c_2^- with $|z_0| = cr - st, |z_1| = s$, and c_2^+ with $|z_0| = t, |z_1| = s$ have the upper bounds 7 and 8 of

m , respectively. Hence, we take $M = 2m$ or $2m + 1$, and continue the program again, we also get $m \leq 1$. Therefore, we get the following theorem.

Theorem 4.3. *Let $k \geq 3$ be an integer and the set $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}, c, d\}$ be a Diophantine quadruple with $c < d$, then $d = d_+$.*

Corollary 4.4. *The set $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}, c\}$ can be extended only to regular.*

5. The extendibility of $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}\}$

In this section, we show that the set $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c\}$ can be extended to only regular, where $c > \max\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}\}$. We can easily show that the Fibonacci numbers $F_{k-2}F_{k-1}$ and $F_{k+1}F_{k+2}$ do not satisfy the Diophantine quadruple with $\{1, 3\}$, by considering the period of Fibonacci numbers in modulo 4. Hence, we can find the form of third element in the Diophantine triple $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c\}$ by Lemma 3.1.

Lemma 5.1. *Let $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c\}$ be a Diophantine triple. Then $c = c_\nu^\tau$ for some ν and τ .*

The small values of c_ν^τ are listed below.

$$\begin{aligned} c_1^+ &= 4F_k F_{k+1}, \\ c_2^- &= 4F_k^2(4F_k^3 F_{k-1} + 1), \\ c_2^+ &= 4F_k^2(4F_k^3 F_{k+1} - 1), \\ c_3^- &= F_k[64F_{k-2}^2 F_{k-1}^3 F_{k+1}^2 F_{k+2}^2 + 16F_{k-2} F_{k-1} F_k F_{k+1} F_{k+2} \\ &\quad + 96F_{k-2} F_{k-1}^2 F_{k+1} F_{k+2} + 12F_k + 36F_{k-1}], \\ c_3^+ &= F_k[64F_{k-2}^2 F_{k-1}^2 F_{k+1}^3 F_{k+2}^2 + 96F_{k-2} F_{k-1} F_k F_{k+1}^2 F_{k+2} \\ &\quad - 16F_{k-2} F_{k-1}^2 F_{k+1} F_{k+2} + 24F_k + 36F_{k-1}]. \end{aligned}$$

By the Theorem 2.4 and Lemma 3.1, we have the upper bound of third elements c in the Diophantine triple $\{a, b, c\}$ with $a < b \leq 21a$.

Lemma 5.2. *Let $\{a, b\}$ be a Diophantine pair with $a < b \leq 21a$. Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $d > c_{n+1}^\tau$ and that $\{a, b, c', c\}$ is not a Diophantine quadruple for any c' with $0 < c' < c_{n-1}^\tau$. Then $c \leq c_3^+$.*

Proof. Since $c_4^- = 64a^3 b^3(a + b - 2r) + 64a^2 b^2(2a + 2b - 3r) + 80ab(a + b - r) + 8(2a + 2b - r)$, it suffices to show that $64a^3 b^3(a + b - 2r) > b^5$, by Theorem 2.4. We have

$$\frac{c_4^-}{b^5} > \frac{64a^3(a + b - 2r)}{b^2} > \frac{64a^3(a + b - 2r)}{(21a)^2} > 1.16 > 1.$$

Hence, this completes the proof of lemma. \square

Lemma 5.3. *Suppose that $m, n \geq 2$.*

(1) If $v_{2m} = w_{2n}$, then

$$m \geq \begin{cases} (\sqrt[4]{F_k}/2, & \text{if } c_2^- \text{ with } |z_0| = 1 \text{ or } c_2^+, \\ \sqrt{2F_k + 1} - 1)/2, & \text{if the other cases.} \end{cases}$$

(2) If $v_{2m} = w_{2n+1}$ with c_2^- or $v_{2m+1} = w_{2n+1}$ with c_2^+ , c_3^- and c_3^+ , then

$$m \geq \frac{\sqrt{2F_k + 1} - 1}{2}.$$

Proof. We have the following relation between c_ν^r and F_k .

- In the case of c_1^+ : $s_1^+ \equiv a \pmod{F_k}$, $t_1^+ \equiv b \equiv -a \pmod{F_k}$.
- In the case of c_2^- : $s_2^- \equiv -1 \pmod{F_k}$, $t_2^- \equiv 1 \pmod{F_k}$.
- In the case of c_2^+ : $s_2^+ \equiv -1 \pmod{F_k}$, $t_2^+ \equiv 1 \pmod{F_k}$.
- In the case of c_3^- : $s_3^- \equiv a \pmod{F_k}$, $t_3^- \equiv -b \equiv a \pmod{F_k}$.
- In the case of c_3^+ : $s_3^+ \equiv -a \pmod{F_k}$, $t_3^+ \equiv -b \equiv a \pmod{F_k}$.

We can also easily find the relation between v_m and w_n .

(1) The case of $v_{2m} = w_{2n}$:

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{F_k}.$$

(2) The case of $v_{2m} = w_{2n+1}$:

$$\pm a(-st)m^2 - astm \equiv \mp 2bstn(n+1) \pmod{F_k}.$$

(3) The case of $v_{2m+1} = w_{2n+1}$:

$$\pm 2atsm(m+1) \equiv \pm 2bstn(n+1) \pmod{F_k}.$$

Since $F_{k+1}F_{k+2} \equiv -F_{k-2}F_{k-1} \pmod{F_k}$ and $\gcd(F_{k-2}F_{k-1}, F_k) = 1$, we can prove the lemma by the same procedure as in the proof of Lemma 4.1. \square

Lemma 5.4. If $v_{2m} = w_{2n}$ with c_1^+ and $m, n \neq 0$, then

$$\begin{aligned} 0 &< 2m \log(s + \sqrt{ac}) - 2n \log(t + \sqrt{bc}) + \log \frac{\sqrt{b}(x_0\sqrt{c} + z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c} + z_1\sqrt{b})} \\ &< 2.09(s + \sqrt{ac})^{-4m}. \end{aligned}$$

Proof. The proof follows along the same line as that of Lemma 4.2. In this case, we have

$$\frac{P-Q}{P} < \frac{c-a}{a} P^{-2} < \frac{1}{a(c-a)} \leq \frac{1}{23}.$$

Hence,

$$0 < \log \frac{P}{Q} = -\log\left(1 - \frac{P-Q}{P}\right) < \frac{24}{23} \left(\frac{c-a}{a}\right) P^{-2} < \frac{24}{23} \frac{c-a}{(\sqrt{c}-\sqrt{a})^2} (s+\sqrt{ac})^{-2m}.$$

Since $\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} < 2$, we obtain the result. \square

5.1. The Theorem of Baker and Wüstholz

By the theorem of Baker and Wüstholz again, we have the equations (13), (15) and (16) for $v_{2m} = w_{2n}$, $v_{2m} = w_{2n+1}$ and $v_{2m+1} = w_{2n+1}$, respectively, since $2.09(s + \sqrt{ac})^{-4m} < (-2m + 1) \log(4ac)$.

Let $x = F_k$. First, let us consider the cases of the equation $v_{2m} = w_{2n}$. We have the following inequalities for each c_i^\pm , $i = 1, 2, 3$.

- If $c = c_1^+$, then

$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14} \log(4x)^2 \cdot \log(3.38x)^2 \cdot \log(2x+1).$$

Hence, $x < 1.65 \cdot 10^{42}$ and $c_1^+ < 2.18 \cdot 10^{85}$.

- If $c = c_2^-$, then

$$\sqrt[4]{x} - 1 < 2.389 \cdot 10^{14} \log(1.85x)^6 \cdot \log(1.75x)^6 \cdot \log x.$$

Hence, $x < 4.36 \cdot 10^{91}$ and $c_2^- < 1.1 \cdot 10^{551}$.

- If $c = c_2^+$, then

$$\sqrt[4]{x} - 1 < 2.389 \cdot 10^{14} \log(2.04x)^6 \cdot \log(1.93x)^6 \cdot \log x.$$

Hence, $x < 4.38 \cdot 10^{91}$ and $c_2^+ < 2.26 \cdot 10^{551}$.

- If $c = c_3^-$, then

$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14} \log(2.39x)^{10} \cdot \log(2.31x)^{10} \cdot \log(2x+1).$$

Hence, $x < 1.52 \cdot 10^{45}$ and $c_3^- < 2 \cdot 10^{455}$.

- If $c = c_3^+$, then

$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14} \log(2.56x)^{10} \cdot \log(2.47x)^{10} \cdot \log(2x+1).$$

Hence, $x < 1.52 \cdot 10^{45}$ and $c_3^+ < 3.84 \cdot 10^{455}$.

From (14), we get the upper bound of k for each c . Also from (13) and the upper bound of c , we get the upper bound of m .

- If $c = c_1^+$, then $k \leq 203$ and $m \leq 9.08 \cdot 10^{20}$.
- If $c = c_2^-$, then $k \leq 440$ and $m \leq 4.07 \cdot 10^{22}$.
- If $c = c_2^+$, then $k \leq 440$ and $m \leq 4.07 \cdot 10^{22}$.
- If $c = c_3^-$, then $k \leq 217$ and $m \leq 2.76 \cdot 10^{22}$.
- If $c = c_3^+$, then $k \leq 217$ and $m \leq 2.76 \cdot 10^{22}$.

Next, we consider the case of $v_{2m} = w_{2n+1}$ with c_2^\pm with $|z_0| = cr - st$ and $|z_1| = s$. Then we have the inequality from (15).

$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14} \log(1.85x)^6 \cdot \log(1.62x)^7 \cdot \log(x+1).$$

Hence, $x < 2.34 \cdot 10^{44}$ and $c_2^- < 3.29 \cdot 10^{267}$. This means $k \leq 213$ and $m \leq 1.09 \cdot 10^{22}$.

Lastly, consider the case of $v_{2m+1} = w_{2n+1}$. There are three cases c_2^+ , c_3^- and c_3^+ . We have the following inequalities from (16).

- In the case of $c = c_2^+$, then
 $\sqrt{x+1} - 1 < 4.778 \cdot 10^{14} \log(2.04x)^6 \cdot \log(1.76x)^7 \cdot \log(x+1)$.
Hence, $x < 4.91 \cdot 10^{44}$ and $c_2^+ < 5.05 \cdot 10^{269}$.
- In the case of $c = c_3^-$, then
 $\sqrt{x+1} - 1 < 4.778 \cdot 10^{14} \log(2.39x)^{10} \cdot \log(2.14x)^{11} \cdot \log(x+1)$.
Hence, $x < 3.81 \cdot 10^{45}$ and $c_3^- < 1.95 \cdot 10^{459}$.
- In the case of $c = c_3^+$, then
 $\sqrt{x+1} - 1 < 4.778 \cdot 10^{14} \log(2.56x)^{10} \cdot \log(2.28x)^{11} \cdot \log(x+1)$.
Hence, $x < 3.82 \cdot 10^{45}$ and $c_3^+ < 3.86 \cdot 10^{459}$.

Also, using the inequality (14) and upper bounds of c , we have the following upper bound of k and m for each cases.

- In the case of $c = c_2^+$, then $k \leq 215$ and $m \leq 1.11 \cdot 10^{22}$.
- In the case of $c = c_3^-$, then $k \leq 219$ and $m \leq 3.09 \cdot 10^{22}$.
- In the case of $c = c_3^+$, then $k \leq 219$ and $m \leq 3.09 \cdot 10^{22}$.

5.2. The reduction method

We again apply Lemma 2.13 about diving logarithmic inequalities from Lemma 2.11 and Lemma 5.4 by $\log \alpha_2$, that is,

$$\begin{aligned} 0 &< m_1 \kappa - n_1 + \mu_1 < A_1 B^{m_1}, \\ 0 &< m_1 \kappa - n_1 + \mu_1 < A_2 B^{m_1}, \\ 0 &< m_1 \kappa - n_2 + \mu_2 < A_1 B^{m_1}, \\ 0 &< m_2 \kappa - n_2 + \mu_3 < A_1 B^{m_2}, \end{aligned}$$

where $m_1 := 2m$, $m_2 := 2m + 1$, $n_1 := 2n$, $n_2 := 2n + 1$ and

$$\begin{aligned} \kappa &= \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu_1 = \frac{\log \alpha_3}{\log \alpha_2}, \quad \mu_2 = \frac{\log \alpha_4}{\log \alpha_2}, \quad \mu_3 = \frac{\log \alpha_5}{\log \alpha_2}, \\ A_1 &= \frac{(8/3)ac}{\log \alpha_2}, \quad A_2 = \frac{2.09}{\log \alpha_2}, \quad B = \alpha_1^2. \end{aligned}$$

Let $M_1 := 2m \leq 5.52 \cdot 10^{22}$ and $M_2 := 2m + 1 \leq 6.18 \cdot 10^{22}$. We have to examine $2 \cdot 203 + 2 \cdot 440 + 2 \cdot 440 + 2 \cdot 217 + 2 \cdot 217 + 2 \cdot 213 + 2 \cdot 215 + 2 \cdot 219 + 2 \cdot 219 = 4766$ cases. Similar to Section 4.2.2, we have the result as the following Table 2 by using the program **PARI/GP** running with 400 digit. We have the following upper bounds of m in Table 2 for each cases. Use these upper bounds and continue the program again, then we have $m \leq 1$. Hence, we get the following theorem.

Theorem 5.5. *Let $k \geq 3$ be the integer and the set $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c, d\}$ be a Diophantine quadruple with $c < d$, then $d = d_+$.*

Corollary 5.6. *The set $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c\}$ can be extended only to regular.*

TABLE 2. Results from PARI/GP running

Case of c	Initial values	Use the next convergent	Upper bound of m
c_1^+	$z_0 = z_1 = 1$	0 case	15
	$z_0 = z_1 = -1$	0 case	14
c_2^-	$z_0 = z_1 = 1$	353 cases ($k = 88, \dots, 440$)	10
	$z_0 = z_1 = -1$	353 cases ($k = 88, \dots, 440$)	10
c_2^-	$z_0 = cr - st, z_1 = -s$	169 cases ($k = 45, \dots, 213$)	10
	$z_0 = st - cr, z_1 = s$	0 case	10
c_2^+	$z_0 = z_1 = 1$	354 cases ($k = 86, 88, \dots, 440$)	9
	$z_0 = z_1 = -1$	0 case	8
c_2^+	$z_0 = t, z_1 = s$	200 cases ($k = 16, \dots, 215$)	8
	$z_0 = -t, z_1 = -s$	171 cases ($k = 45, \dots, 215$)	8
c_3^-	$z_0 = z_1 = 1$	0 case	6
	$z_0 = z_1 = -1$	0 case	6
c_3^-	$z_0 = t, z_1 = s$	204 cases ($k = 12, \dots, 215$)	6
	$z_0 = -t, z_1 = -s$	0 case	7
c_3^+	$z_0 = z_1 = 1$	174 cases ($k = 44, \dots, 217$)	6
	$z_0 = z_1 = -1$	0 case	6
c_3^+	$z_0 = t, z_1 = s$	0 case	6
	$z_0 = -t, z_1 = -s$	197 cases ($k = 23, \dots, 219$)	6

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