

## DENSITY OF THE HOMOTOPY MINIMAL PERIODS OF MAPS ON INFRA-SOLVMANIFOLDS OF TYPE (R)

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ABSTRACT. We study the homotopical minimal periods for maps on infra-solvmanifolds of type (R) using the density of the homotopical minimal period set in the natural numbers. This extends the result of [10] from flat manifolds to infra-solvmanifolds of type (R). We give some examples of maps on infra-solvmanifolds of dimension three for which the corresponding density is positive.

### 1. Introduction

Let  $f : X \rightarrow X$  be a self-map on a topological space  $X$ . We define the following: The set of *periodic points* of  $f$  with *minimal period*  $n$

$$P_n(f) = \text{Fix}(f^n) - \bigcup_{k < n} \text{Fix}(f^k)$$

and the set of *homotopy minimal periods* of  $f$

$$\text{HPer}(f) = \bigcap_{g \simeq f} \{n \in \mathbb{N} \mid P_n(g) \neq \emptyset\}.$$

The famous Šarkovs'kii theorem characterizes the dynamics (minimal periods) of a map of interval [32]. The set of minimal periods of maps on the circle has been completely described in [2]. This led to a problem of study the set of *homotopy minimal periods* of self-maps. Such an invariant gives an information about rigid dynamics of self-maps. A fundamental question is to determine if the set  $\text{HPer}(f)$  of homotopy minimal periods is empty, finite or infinite. This problem was successfully studied in [18] when the space is a torus of any dimension, and this was extended in [14] (see also [15, 27]) to any nilmanifold,

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and in [12, 28] and [19] to the special solvmanifolds modeled on  $\text{Sol}^3$  and  $\text{Sol}_1^4$  respectively. When  $X$  is the Klein bottle, the same problem was studied in [13, 21, 23, 30], and when  $X$  is an infra-nilmanifold and  $f$  is an expanding map, it was shown in [7, 24, 26, 33] that  $\text{HPer}(f)$  is co-finite.

It is now natural to seek for more information when  $\text{HPer}(f)$  becomes infinite. When  $X$  is a flat manifold, some sufficient conditions on  $X$  and  $f$  for  $\text{HPer}(f)$  to be infinite were found in [10, 29]. For this purpose, the following invariant was considered: The *lower density* of the homotopy minimal periods of  $f$  is defined to be ([10, Definition 1.1] and [16, Remark 3.1.60])

$$\text{DH}(f) = \liminf_{n \rightarrow \infty} \frac{\#(\text{HPer}(f) \cap [0, n])}{n}.$$

From the definition,  $\text{DH}(f) \in [0, 1]$ . If  $\text{HPer}(f)$  is either empty or finite, then  $\text{DH}(f) = 0$ . So, we are interested in the case when  $\text{HPer}(f)$  is infinite. If one picks randomly a natural number,  $\text{DH}(f)$  is a lower bound for the probability of choosing number in  $\text{HPer}(f)$ . Thus, the real number  $\text{DH}(f)$  will bring to us more information about the periods of given map  $f$  when  $\text{HPer}(f)$  is infinite.

The purpose of this paper is to study the lower density of homotopy minimal periods of maps infra-solvmanifolds of type (R). This extends the results in [10] from flat manifolds to infra-solvmanifolds of type (R). We give some examples of maps on infra-solvmanifolds of dimension three for which the corresponding density is positive.

## 2. Infra-solvmanifolds

Let  $S$  be a connected and simply connected solvable Lie group. A discrete subgroup  $\Gamma$  of  $S$  is a *lattice* of  $S$  if  $\Gamma \backslash S$  is compact, and in this case, we say that the quotient space  $\Gamma \backslash S$  is a *special* solvmanifold. Let  $\Pi \subset \text{Aff}(S)$  be a torsion-free finite extension of the lattice  $\Gamma = \Pi \cap S$  of  $S$ . That is,  $\Pi$  fits the short exact sequence:

$$\begin{array}{ccccccc} 1 & \longrightarrow & S & \longrightarrow & \text{Aff}(S) & \longrightarrow & \text{Aut}(S) \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Pi/\Gamma \longrightarrow 1 \end{array}$$

Then  $\Pi$  acts freely on  $S$  and the manifold  $\Pi \backslash S$  is called an *infra-solvmanifold*. The finite group  $\Phi = \Pi/\Gamma$  is the *holonomy group* of  $\Pi$  or  $\Pi \backslash S$ . It sits naturally in  $\text{Aut}(S)$ . Thus every infra-solvmanifold  $\Pi \backslash S$  is finitely covered by the special solvmanifold  $\Gamma \backslash S$ . An infra-solvmanifold  $\Pi \backslash S$  is of type (R) if  $S$  is of *type* (R) or *completely solvable*, i.e., if  $\text{ad } X : \mathfrak{S} \rightarrow \mathfrak{S}$  has only real eigenvalues for all  $X$  in the Lie algebra  $\mathfrak{S}$  of  $S$ . It is known that if  $S$  is of type (R), then  $\exp : \mathfrak{S} \rightarrow S$  is surjective. The abelian groups  $\mathbb{R}^n$  and the connected, simply connected nilpotent Lie groups are of type (R). Hence the flat manifolds and the infra-nilmanifolds are examples of infra-solvmanifolds of type (R).

We first recall the following:

**Lemma 2.1** ([25, Lemma 2.1]). *Let  $S$  and  $S'$  be simply connected solvable Lie groups, and let  $\Pi \subset \text{Aff}(S)$  and  $\Pi' \subset \text{Aff}(S')$  be finite extensions of lattices  $\Gamma = \Pi \cap S$  of  $S$  and  $\Gamma' = \Pi' \cap S'$  of  $S'$ , respectively. Then there exist fully invariant subgroups  $\Lambda \subset \Gamma$  and  $\Lambda' \subset \Gamma'$  of  $\Pi$  and  $\Pi'$  respectively, which are of finite index, so that any homomorphism  $\theta : \Pi \rightarrow \Pi'$  restricts to a homomorphism  $\Lambda \rightarrow \Lambda'$ .*

When the infra-solvmanifolds are of type (R), we have the following second Bieberbach type result.

**Theorem 2.2** ([20, Theorem 2.3]). (1) *Any continuous map  $f : \Pi \backslash S \rightarrow \Pi' \backslash S'$  between infra-solvmanifolds of type (R) has an affine map  $(d, D) : S \rightarrow S'$  as a homotopy lift.*  
 (2) *Any continuous map  $f : \Gamma \backslash S \rightarrow \Gamma' \backslash S'$  between special solvmanifolds of type (R) has a Lie group homomorphism  $D : S \rightarrow S'$  as a homotopy lift.*

*When  $f$  is a homeomorphism,  $D$  can be chosen to be invertible.*

Let  $f : \Pi \backslash S \rightarrow \Pi \backslash S$  be a self-map on the infra-solvmanifold  $\Pi \backslash S$  of type (R) with affine homotopy lift  $(d, D) : S \rightarrow S$ . Since  $\text{HPer}(f)$  is a homotopy invariant, we may assume that  $f$  is induced by the affine map  $(d, D)$ . The map  $f$  induces a homomorphism  $\varphi : \Pi \rightarrow \Pi$  on the group  $\Pi$  of covering transformations of the covering projection  $S \rightarrow \Pi \backslash S$ , which is given by

$$(*) \quad \varphi(\alpha)(d, D) = (d, D)\alpha, \quad \forall \alpha \in \Pi.$$

For any  $(a, A) \in \Phi$ , let  $\varphi(a, A) = (a', A')$ ; then  $A'D = DA$ . Thus the homomorphism  $\varphi$  induces a function  $\bar{\varphi} : \Phi \rightarrow \Phi$  given by  $\bar{\varphi}(A) = A'$  and this function satisfies  $\bar{\varphi}(A)D = DA$  for all  $A \in \Phi$ . However, in general,  $\bar{\varphi}$  is not necessarily a homomorphism.

Recall further that:

**Theorem 2.3** ([11, Theorem 6.1]). *Let  $f : M \rightarrow M$  be a self-map on a compact PL-manifold of dimension  $\geq 3$ . Then  $f$  is homotopic to a map  $g$  with  $P_n(g) = \emptyset$  if and only if  $NP_n(f) = 0$ .*

The infra-solvmanifolds of dimension 1 or 2 are the circle, the torus and the Klein bottle. Theorem 2.3 for dimensions 1 and 2 is verified respectively in [2], [1] and [13, 21, 30]. Immediately we have for any self-map  $f$  on an infra-solvmanifold of any dimension,

$$\text{HPer}(f) = \{k \mid NP_k(f) \neq 0\}.$$

Recalling from [17] that

$$NP_n(f) = (\text{number of irreducible essential orbits of Reidemeister classes of } f^n) \times n,$$

we have

$$\text{HPer}(f) = \{k \mid \exists \text{ an irreducible essential fixed point class of } f^k\}.$$

Recall from [6, Propositions 9.1 and 9.3] the following: Let  $f$  be a map on an infra-solvmanifold  $\Pi \backslash S$  of type (R) induced by an affine map  $(d, D) : S \rightarrow S$ . For any  $\alpha \in \Pi$ ,  $\text{Fix}(\alpha(d, D))$  is an empty set or path connected. Hence every nonempty fixed point class of  $f$  is path connected, and every isolated fixed point class forms an essential fixed point class with index  $\pm \det(I - A_* D_*)$  where  $\alpha = (a, A)$ . When the infra-solvmanifold  $\Pi \backslash S$  is of type (R), the converse also holds. Namely, every essential fixed point class of  $f$  consists of a single element. Remark that  $(d, D)^k$  induces the map  $f^k$ . Any fixed point class of  $f^k$  is of the form  $p(\text{Fix}(\alpha(d, D)^k))$  for some  $\alpha = (a, A) \in \Pi$ . It is essential if and only if it consists of a single element with index  $\pm \det(I - A_* D_*^k)$ . Note further that it is reducible to  $\ell$  if and only if  $\ell \mid k$  and there exists  $\beta \in \Pi$  such that  $p(\text{Fix}(\beta(d, D)^\ell)) \subset p(\text{Fix}(\alpha(d, D)^k))$ , or equivalently, the Reidemeister class  $[\beta]$  of  $f^\ell$  is boosted up to the Reidemeister class  $[\alpha]$  of  $f^k$ . This means that  $[\alpha] = [\beta \varphi^\ell(\beta) \varphi^{2\ell}(\beta) \cdots \varphi^{k-\ell}(\beta)]$  as the Reidemeister class of  $f^k$ . For some  $\gamma \in \Pi$ , we thus have  $\alpha = \gamma(\beta \varphi^\ell(\beta) \varphi^{2\ell}(\beta) \cdots \varphi^{k-\ell}(\beta)) \varphi^k(\gamma)^{-1}$ . Hence

$$\begin{aligned} \alpha &= (\gamma \beta \varphi^\ell(\gamma)^{-1}) (\varphi^\ell(\gamma) \varphi^\ell(\beta) \varphi^{2\ell}(\gamma)^{-1}) \cdots (\varphi^{k-\ell}(\gamma) \varphi^{k-\ell}(\beta) \varphi^k(\gamma)^{-1}) \\ &= \beta' \varphi^\ell(\beta') \varphi^{2\ell}(\beta') \cdots \varphi^{k-\ell}(\beta') \end{aligned}$$

with  $\beta' = \gamma \beta \varphi^\ell(\gamma)^{-1}$ . Consequently, the fixed point class  $p(\text{Fix}(\alpha(d, D)^k))$  is irreducible if and only if for any  $\beta \in \Pi$  and for any  $\ell < k$  with  $\ell \mid k$ ,

$$\alpha(d, D)^k \neq (\beta(d, D)^\ell)^{k/\ell}$$

or

$$\alpha \neq \beta \varphi^\ell(\beta) \varphi^{2\ell}(\beta) \cdots \varphi^{k-\ell}(\beta).$$

For any endomorphism  $D$  on  $S$ , we denote the differential of  $D : S \rightarrow S$  by  $D_* : \mathfrak{S} \rightarrow \mathfrak{S}$ . Now, in conclusion, we can summarize the above observation as follows:

**Theorem 2.4.** *Let  $f : \Pi \backslash S \rightarrow \Pi \backslash S$  be a self-map on the infra-solvmanifold  $\Pi \backslash S$  of type (R) with an affine homotopy lift  $(d, D) : S \rightarrow S$ . Let  $\varphi : \Pi \rightarrow \Pi$  be the homomorphism induced by  $(d, D)$ , i.e.,  $\varphi(\alpha)(d, D) = (d, D)\alpha \forall \alpha \in \Pi$ . Then*

$$\begin{aligned} \text{HPer}(f) &= \left\{ k \mid \begin{array}{l} \exists \alpha = (a, A) \in \Pi \text{ such that } \det(I - A_* D_*^k) \neq 0 \text{ and} \\ \forall \ell < k \text{ with } \ell \mid k, \forall \beta \in \Pi, \\ \alpha(d, D)^k \neq (\beta(d, D)^\ell)^{k/\ell} \end{array} \right\} \\ &= \left\{ k \mid \begin{array}{l} \exists \alpha = (a, A) \in \Pi \text{ such that } \det(I - A_* D_*^k) \neq 0 \text{ and} \\ \forall \ell < k \text{ with } \ell \mid k, \forall \beta \in \Pi, \\ \alpha \neq \beta \varphi^\ell(\beta) \varphi^{2\ell}(\beta) \cdots \varphi^{k-\ell}(\beta) \end{array} \right\}. \end{aligned}$$

In order to generalize the results of [10] from flat manifolds to infra-solvmanifolds of type (R), we need the following observation which is crucial in our discussion.

**Lemma 2.5.** *Let  $\Lambda$  be a lattice of a connected, simply connected solvable Lie group  $S$  of type (R), and let  $K : S \rightarrow S$  be a Lie group homomorphism such that  $K(\Lambda) \subset \Lambda$ . For some choice of a linear basis in the Lie algebra  $\mathfrak{S}$  of  $S$ ,  $K_*$  is an upper block triangular matrix with diagonal blocks integer matrices; in particular  $\det K_*$  is an integer.*

*Proof.* First we assume that  $S$  is nilpotent and thus  $\Lambda$  is a finitely generated torsion-free nilpotent group. The lower central series of  $\Lambda$  is defined inductively via  $\gamma_1(\Lambda) = \Lambda$  and  $\gamma_{i+1}(\Lambda) = [\Lambda, \gamma_i(\Lambda)]$ . The isolator of a subgroup  $H$  of  $\Lambda$  is defined by

$$\sqrt[\wedge]{H} = \{x \in \Lambda \mid x^k \in H \text{ for some } k \geq 1\}.$$

It is known ([31, p. 473], [4, Chap. 1] or [22]) that the sequence

$$\Lambda = \Lambda_1 \supset \Lambda_2 = \sqrt[\wedge]{\gamma_2(\Lambda)} \supset \cdots \supset \Lambda_c = \sqrt[\wedge]{\gamma_c(\Lambda)} \supset \Lambda_{c+1} = 1$$

forms a central series with  $\Lambda_i/\Lambda_{i+1} \cong \mathbb{Z}^{k_i}$ . Now we can choose a generating set

$$\mathbf{a} = \{\mathbf{a}_1, \dots, \mathbf{a}_c\}$$

in such a way that  $\Lambda_i$  is the group generated by  $\Lambda_{i+1}$  and  $\mathbf{a}_i = \{a_{i1}, \dots, a_{i n_i}\}$ . We refer to  $\mathbf{a} = \{\mathbf{a}_1, \dots, \mathbf{a}_c\}$  as a preferred basis of  $\Lambda$ . Under the diffeomorphism  $\log : S \rightarrow \mathfrak{S}$ , the image  $\log \mathbf{a}$  of  $\mathbf{a}$  is a basis of the vector space  $\mathfrak{S}$ . Note also that  $\Lambda_i = \Lambda \cap \gamma_i(S)$  is a lattice of  $\gamma_i(S)$  and a fully invariant subgroup of  $\Lambda$ . Since  $K(\Lambda) \subset \Lambda$ , it follows that  $K(\Lambda_i) \subset \Lambda_i$  and the differential of  $K$  is expressed as a rational matrix with respect to the basis  $\log \mathbf{a}$  of the form

$$\begin{bmatrix} K_{c*} & * & \cdots & * \\ 0 & K_{c-1*} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{1*} \end{bmatrix},$$

where each square matrix  $K_{i*}$  is an integer matrix, see also [24, Lemma 4.2].

Now we go back to the cases where  $S$  is solvable of type (R). According to [34, Remark 8.2],  $\Lambda$  is a positive polycyclic group and  $S$  is its supersolvable completion. Let  $N$  be the maximal connected nilpotent normal subgroup of  $S$ . Then  $\Lambda \cap N$  is the nilradical  $\text{nil}(\Lambda)$  of  $\Lambda$ , which is a lattice of  $N$ , see [34, Proposition 5.1]. Hence we have the following diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \longrightarrow & S & \longrightarrow & S/N \cong \mathbb{R}^s & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \text{nil}(\Lambda) & \longrightarrow & \Lambda & \longrightarrow & \Lambda/\text{nil}(\Lambda) \cong \mathbb{Z}^s & \longrightarrow & 1 \end{array}$$

By the assumption on  $K$ ,  $K$  restricts to a homomorphism  $\kappa : \Lambda \rightarrow \Lambda$ . Thus  $\kappa$  and hence  $K$  in turn restricts to  $\kappa' : \text{nil}(\Lambda) \rightarrow \text{nil}(\Lambda)$  and then induces a homomorphism  $\bar{\kappa} : \mathbb{Z}^s \rightarrow \mathbb{Z}^s$ . We choose a preferred basis of  $\text{nil}(\Lambda)$  under which  $K' : N \rightarrow N$  yields a rational matrix  $K'_*$  with diagonal blocks integer matrices as above. Now we can complete the set of generators of  $\text{nil}(\Lambda)$  to

a set of generators  $\mathbf{a} = \{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_c\}$ , called a *preferred basis*, of  $\Lambda$  so that  $\bar{\kappa}$  induces an integer matrix  $\bar{K}_*$  and so  $\kappa$  induces an upper block triangular matrix

$$K_* = \begin{bmatrix} K'_* & * \\ 0 & \bar{K}_* \end{bmatrix}$$

so that all diagonal blocks are integer matrices and hence  $\det K_*$  is an integer.  $\square$

*Remark 2.6.* Let  $\Lambda$  be a lattice of a connected, simply connected solvable Lie group  $S$  of type (R). In the proof of the above lemma, we can choose a preferred basis (generator)  $\mathbf{a}$  of  $\Lambda$  so that  $\log \mathbf{a}$  is a (linear) basis of the Lie algebra  $\mathfrak{S}$  of  $S$  and if  $K$  is a homomorphism on  $S$  such that  $K(\Lambda) \subset \Lambda$ , then  $K_*$  is an upper block triangular matrix with diagonal blocks integer matrices with respect to the ordered basis  $\log \mathbf{a}$ . We also refer to  $\log \mathbf{a}$  as a *preferred basis* of  $\Lambda$  or  $\mathfrak{S}$ .

### 3. Density of homotopy minimal periods

In this section, we will generalize the main result of [10] from flat manifolds to infra-solvmanifolds of type (R).

Let  $f$  be a self-map on an infra-solvmanifold  $\Pi \backslash S$  of type (R) with holonomy group  $\Phi$ . Let  $f$  have an affine homotopy lift  $(d, D)$ . Recall that  $f$  induces a homomorphism  $\varphi : \Pi \rightarrow \Pi$  satisfying the identity  $(*)$ :  $\varphi(\alpha)(d, D) = (d, D)\alpha$ ,  $\forall \alpha \in \Pi \subset S \rtimes \text{Aut}(S)$ . Let  $\Gamma = \Pi \cap S$ . It is not necessarily true that  $\varphi(\Gamma) \subset \Gamma$ . Using Lemma 2.1, we can choose a lattice  $\Lambda \subset \Gamma$  of  $S$  so that  $\varphi(\Lambda) \subset \Lambda$ . Thus for any  $\lambda = (\lambda, I) \in \Lambda$ , we have  $\varphi(\lambda) = (\varphi(\lambda), I)$  and so

$$(\varphi(\lambda), I)(d, D) = (d, D)(\lambda, I).$$

Evaluating at the identity 1 of  $S$ , we obtain that  $\varphi(\lambda) \cdot d = d \cdot D(\lambda)$ . Consequently, we have that

$$\varphi|_{\Lambda} = \mu(d)D.$$

Furthermore, for any  $(a, A) \in \Pi$ , since  $\Gamma$  is a normal subgroup of  $\Pi$ , we have  $(a, A)(\gamma, I)(a, A)^{-1} \in \Gamma$ ; this implies  $(\mu(a)A)(\Gamma) \subset \Gamma$  and  $(\mu(a)A)(\Lambda) \subset \Lambda$ . Consequently, we have homomorphisms  $\mu(d)D, \mu(a)A : S \rightarrow S$  such that  $(\mu(d)D)(\Lambda) \subset \Lambda$  and  $(\mu(a)A)(\Lambda) \subset \Lambda$ . We have to notice here that it is not necessary to have that  $D(\Lambda), A(\Lambda) \subset \Lambda$ . By Remark 2.6, we can choose a preferred basis  $\mathbf{a}$  of  $\Lambda$  so that  $(\mu(d)D)_* = \text{Ad}(d)D_*$  and  $(\mu(a)A)_* = \text{Ad}(a)A_*$  are upper block triangular rational matrices with diagonal blocks integer matrices with respect to the basis  $\log \mathbf{a}$  of  $\mathfrak{S}$ .

In what follows, we shall denote  $\mu(d)D$  and  $\mu(a)A$  by  $\mathbb{D}$  and  $\mathbb{A}$ , respectively. By Lemma 2.5, the differentials of  $\mathbb{D}$  and  $\mathbb{A}$  induce rational matrices with integer blocks on the diagonal. By considering only integer blocks on the diagonal, we

obtain integer matrices, denoted by  $\mathbb{D}_*$  and  $\mathbb{A}_*$ . Hence,

$$\mathbb{D}_* = \begin{bmatrix} \mathbb{D}_{c_*} & 0 & \cdots & 0 \\ 0 & \mathbb{D}_{c-1_*} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbb{D}_{0_*} \end{bmatrix}.$$

This does not change the determinant and the eigenvalue of the differentials of  $\mathbb{D}$  and  $\mathbb{A}$ . We can call  $\mathbb{D}_*$  and  $\mathbb{A}_*$  *linearizations* of  $D$ ,  $(d, D)$  or  $f$ , and  $\alpha = (a, A) \in \Pi$ , respectively. We denote the free abelian group of all integer linear combinations of the basis vectors in  $\log \mathbf{a} = \{\log \mathbf{a}_c, \log \mathbf{a}_{c-1}, \dots, \log \mathbf{a}_0\}$  by simply  $\mathcal{Z} = \mathcal{Z}_c \oplus \mathcal{Z}_{c-1} \oplus \cdots \oplus \mathcal{Z}_0$ . Then we have  $\mathbb{D}_{i_*}(\mathcal{Z}_i) \subset \mathcal{Z}_i$ ,  $\mathbb{D}_*(\mathcal{Z}) \subset \mathcal{Z}$  and  $\mathbb{A}_*(\mathcal{Z}) \subset \mathcal{Z}$ .

In the following, we provide three lemmas that generalize [10, Lemmas 4.3 and 4.5, Proposition 4.6] from flat manifolds to infra-solvmanifolds of type (R). These are essential in proving our main results.

**Lemma 3.1.** *Let  $M = \Pi \backslash S$  be an infra-solvmanifold of type (R). Let  $f$  be a self-map on  $M$  with an affine homotopy lift  $(d, D)$ . Assume that*

- (1) *any eigenvalue  $\lambda$  of  $\mathbb{D}_*$  of modulus 1 is a root of unity, but not 1;*
- (2)  *$\det \mathbb{D}_* \neq 0, \pm 1$ .*

*Then there exists a positive integer  $N_0$  such that*

$$\left| \det \left( \frac{I - \mathbb{D}_*^{k\ell}}{I - \mathbb{D}_*^\ell} \right) \right| > 1$$

*for all positive integers  $k$  and  $\ell$ , provided their prime divisors are all greater than  $N_0$ .*

*Proof.* Let  $\lambda_1, \dots, \lambda_m$  be the eigenvalues of  $\mathbb{D}_*$  counted with multiplicities. We first show that all  $1 - \lambda_i^\ell$  are nonzero. In fact, if  $1 - \lambda_i^\ell = 0$  for some  $i$ , then  $\lambda_i^\ell = 1$  and so  $\lambda_i$  is a primitive  $\ell_0$ -th root of unity for some  $\ell_0$  where  $1 \leq \ell_0 \mid \ell$ . Since  $\lambda_i \neq 1$ ,  $\ell_0 > 1$ . If  $p$  is a prime divisor of  $\ell_0$ , then it is a prime divisor of  $\ell$  and so  $p > N_0$ . It follows that  $[\mathbb{Q}(\lambda_i) : \mathbb{Q}] \geq p - 1 > m$ . This contradicts the fact that  $[\mathbb{Q}(\lambda_i) : \mathbb{Q}]$  is smaller than the size  $m$  of  $\mathbb{D}_*$ . Thus  $I - \mathbb{D}_*^\ell$  is invertible and  $(I - \mathbb{D}_*^{k\ell}) / (I - \mathbb{D}_*^\ell) = I + \mathbb{D}_*^\ell + \cdots + \mathbb{D}_*^{(k-1)\ell}$  and

$$\det \left( \frac{I - \mathbb{D}_*^{k\ell}}{I - \mathbb{D}_*^\ell} \right) = \prod_{i=1}^m \frac{1 - \lambda_i^{k\ell}}{1 - \lambda_i^\ell}.$$

Let  $N_0 > m + 1$  and let  $\ell$  be a positive integer all of whose prime divisors are greater than  $N_0$ . Assume  $|\lambda_i| = 1$ . By our assumption,  $\lambda_i \neq 1$  and is a root of unity. The above argument shows that  $1 - \lambda_i^\ell \neq 0$  and by the same reasoning  $1 - \lambda_i^{k\ell} \neq 0$ ; thus  $(1 - \lambda_i^{k\ell}) / (1 - \lambda_i^\ell)$  is nonzero and finite for each such  $k$  and  $\ell$ . Hence we can choose a constant  $\delta > 0$  such that for all such  $k$  and  $\ell$

$$\left| \frac{1 - \lambda_i^{k\ell}}{1 - \lambda_i^\ell} \right| > \delta.$$

For  $\lambda_i$  with  $|\lambda_i| \neq 1$ , as  $N_0 \rightarrow \infty$ , we have

$$\left| \frac{1 - \lambda_i^{k\ell}}{1 - \lambda_i^\ell} \right| = |1 + \lambda_i^\ell + \lambda_i^{2\ell} + \cdots + \lambda_i^{k-1}| \rightarrow \begin{cases} 1 & \text{when } |\lambda_i| < 1 \\ \infty & \text{when } |\lambda_i| > 1. \end{cases}$$

By the assumption that  $|\det \mathbb{D}_*| > 1$ , there exists an eigenvalue whose absolute value is bigger than 1. Hence as  $N_0 \rightarrow \infty$  we have

$$\prod_{i=1}^m \left| \frac{1 - \lambda_i^{k\ell}}{1 - \lambda_i^\ell} \right| \rightarrow \infty.$$

Consequently, for  $N_0$  large enough, the lemma is proved.  $\square$

**Lemma 3.2.** *Let  $M = \Pi \setminus S$  be an infra-solvmanifold of type (R). Let  $f$  be a self-map on  $M$  with an affine homotopy lift  $(d, D)$ . Assume that*

- (1) *any eigenvalue  $\lambda$  of  $\mathbb{D}_*$  of modulus 1 is a root of unity, but not  $\pm 1$ ;*
- (2)  *$\det \mathbb{D}_* \neq 0, \pm 1$ .*

*Then there exists a positive integer  $N_1$  such that*

$$|\det(I - \mathbb{D}_*^k)| > \sum_{1 < \ell | k} |\det(I - \mathbb{D}_*^{k/\ell})|$$

*for all positive integers  $k$ , provided all its positive prime divisors are greater than  $N_1$ .*

*Proof.* Let  $\lambda_1, \dots, \lambda_m$  be the eigenvalues of  $\mathbb{D}_*$  counted with multiplicities. From our assumptions and hence from the observations in the proof of Lemma 3.1, we have:

- Since  $\det \mathbb{D}_* \neq 0$ , all  $\lambda_i$  are nonzero.
- If  $|\lambda_i| = 1$  then  $\lambda_i \neq \pm 1$  and  $\lambda_i$  is a root of unity, and  $1 - \lambda_i^k \neq 0$  for all  $k$  whose prime divisors are  $> N_0$  where  $N_0$  is a positive integer chosen in the previous lemma; hence there are constants  $0 < \delta_1 < \delta_2$  such that for all  $\lambda_i$  with  $|\lambda_i| \leq 1$ , we have  $\delta_1 \leq |1 - \lambda_i^k| \leq \delta_2$  for all  $k$  with this property.

For those eigenvalues with  $|\lambda_i| > 1$ , we claim that there is a sufficiently large  $k$  such that

$$\sum_{1 < \ell | k} |1 - \lambda_i^{k/\ell}| < |1 - \lambda_i^k|.$$

Suppose on the contrary that for any  $K > 0$  there is  $k_0 > K$  such that

$$|1 - \lambda_i^{k_0}| \leq \sum_{1 < \ell | k_0} |1 - \lambda_i^{k_0/\ell}|.$$

Then

$$|1 - \lambda_i^{k_0}| \leq \sum_{1 < \ell | k_0} |1 - \lambda_i^{k_0/2}| < \tau(k_0) |1 - \lambda_i^{k_0/2}|,$$



where  $\tau(k)$  is the number of all the divisors of  $k$ . Since  $\tau(k) \leq 2\sqrt{k}$  (see [16, Exercise 3.2.17]), we have

$$2\sqrt{k_0} > |1 + \lambda_i^{k_0/2}| \geq |\lambda_i|^{k_0/2} - 1,$$

which contradicts the obvious fact that  $\lim_{k \rightarrow 0} \sqrt{k}/(|\lambda_i|^{k/2} - 1) = 0$ .

Therefore we can choose  $N_1 \geq N_0$  such that if  $k$  is a positive integer whose prime divisors are  $\geq N_1$ , then

$$\begin{aligned} \sum_{1 < \ell \mid k} |\det(I - \mathbb{D}_*^{k/\ell})| &= \sum_{1 < \ell \mid k} \left( \prod_{i=1}^m |1 - \lambda_i^{k/\ell}| \right) \\ &\leq \prod_{i=1}^m \left( \sum_{1 < \ell \mid k} |1 - \lambda_i^{k/\ell}| \right) \\ &< \prod_{i=1}^m |1 - \lambda_i^k| = |\det(I - \mathbb{D}_*^k)|. \quad \square \end{aligned}$$

**Lemma 3.3.** *Let  $M = \Pi \backslash S$  be an infra-solvmanifold of type (R). Let  $f$  be a self-map on  $M$  with an affine homotopy lift  $(d, D)$ . Assume that*

- (1) *any eigenvalue  $\lambda$  of  $\mathbb{D}_*$  of modulus 1 is a root of unity, but not 1;*
- (2)  *$\det \mathbb{D}_* \neq 0, \pm 1$ .*

*Then there exists a positive integer  $N_2$  such that the equality*

$$\mathcal{Z} = \bigcup_{1 < \ell \mid k} (I + \mathbb{D}_*^\ell + \mathbb{D}_*^{2\ell} + \cdots + \mathbb{D}_*^{k-\ell})(\mathcal{Z})$$

*is impossible for all positive integers  $k$ , provided its positive prime divisors are all greater than  $N_2$ .*

*Proof.* Remark that the proof of Lemma 3.1 shows that there exists a positive integer  $N_0$  such that for all positive integers  $k$  whose prime divisors are greater than  $N_0$ ,  $I - \mathbb{D}_*^\ell$  has nonzero determinant if  $\ell \mid k$ . Since  $\mathbb{D}_*(\mathcal{Z}) \subset \mathcal{Z}$ , we have

$$\begin{aligned} (I - \mathbb{D}_*^k)(\mathcal{Z}) &= (I + \mathbb{D}_*^\ell + \mathbb{D}_*^{2\ell} + \cdots + \mathbb{D}_*^{k-\ell})(I - \mathbb{D}_*^\ell)(\mathcal{Z}) \\ &\subset (I + \mathbb{D}_*^\ell + \mathbb{D}_*^{2\ell} + \cdots + \mathbb{D}_*^{k-\ell})(\mathcal{Z}) \end{aligned}$$

for all  $k$  and  $\ell$  with  $\ell \mid k$ . Thus if we had the equality

$$\mathcal{Z} = \bigcup_{\ell \mid k, 1 < \ell < k} (I + \mathbb{D}_*^\ell + \mathbb{D}_*^{2\ell} + \cdots + \mathbb{D}_*^{k-\ell})(\mathcal{Z})$$

we would have

$$\begin{aligned} \mathcal{Z}/(I - \mathbb{D}_*^k)(\mathcal{Z}) &= \left( \bigcup_{\ell \mid k, \ell \neq 1, k} (I + \mathbb{D}_*^\ell + \mathbb{D}_*^{2\ell} + \cdots + \mathbb{D}_*^{k-\ell})(\mathcal{Z}) \right) / (I - \mathbb{D}_*^k)(\mathcal{Z}) \\ &= \bigcup_{\ell \mid k, \ell \neq 1, k} ((I + \mathbb{D}_*^\ell + \mathbb{D}_*^{2\ell} + \cdots + \mathbb{D}_*^{k-\ell})(\mathcal{Z}) / (I - \mathbb{D}_*^k)(\mathcal{Z})). \end{aligned}$$

We remark that  $\mathcal{Z}$  is a free abelian group and  $I + \mathbb{D}_*^\ell + \mathbb{D}_*^{2\ell} + \cdots + \mathbb{D}_*^{k-\ell}$  defines an injective endomorphism of  $\mathcal{Z}$ . In particular, we have an isomorphism

$$\begin{aligned} & \mathcal{Z}/(I - \mathbb{D}_*^\ell)(\mathcal{Z}) \\ & \cong (I + \mathbb{D}_*^\ell + \mathbb{D}_*^{2\ell} + \cdots + \mathbb{D}_*^{k-\ell})\mathcal{Z}/(I + \mathbb{D}_*^\ell + \mathbb{D}_*^{2\ell} + \cdots + \mathbb{D}_*^{k-\ell})(I - \mathbb{D}_*^\ell)(\mathcal{Z}) \\ & = (I + \mathbb{D}_*^\ell + \mathbb{D}_*^{2\ell} + \cdots + \mathbb{D}_*^{k-\ell})\mathcal{Z}/(I - \mathbb{D}_*^k)(\mathcal{Z}). \end{aligned}$$

This would imply

$$\mathcal{Z}/(I - \mathbb{D}_*^k)(\mathcal{Z}) \cong \bigcup_{\ell \mid k, \ell \neq 1, k} (\mathcal{Z}/(I - \mathbb{D}_*^\ell)(\mathcal{Z}))$$

and hence we would have

$$|\det(I - \mathbb{D}_*^k)| \leq \sum_{\ell \mid k, \ell \neq 1, k} |\det(I - \mathbb{D}_*^\ell)| \leq \sum_{1 < \ell \mid k} |\det(I - \mathbb{D}_*^{\ell})|.$$

This contradicts Lemma 3.2.  $\square$

Now we are ready to state and prove our main results.

**Theorem 3.4.** *Let  $M = \Pi \backslash S$  be an infra-solvmanifold of type (R). Let  $f$  be a self-map on  $M$  with an affine homotopy lift  $(d, D)$ . Let  $\varphi : \Pi \rightarrow \Pi$  be the homomorphism satisfying*

$$\varphi(\alpha)(d, D) = (d, D)\alpha, \quad \forall \alpha \in \Pi.$$

Assume that

- (1) any eigenvalue  $\lambda$  of  $\mathbb{D}_*$  of modulus 1 is a root of unity, but not 1;
- (2)  $\det \mathbb{D}_* \neq 0, \pm 1$ ;
- (3)  $\text{fix}(\bar{\varphi} : \Phi \rightarrow \Phi) = \{I\}$ .

Then there exists an integer  $N$  with the following property: if  $k$  is a positive integer with prime factorization  $k = p_1^{n_1} \cdots p_s^{n_s}$  such that all  $p_i$ 's are greater than  $N$ , then  $k \in \text{HPer}(f)$ .

*Proof.* Choose an integer  $N$  so that  $N \geq \max\{m+1, N_2, \text{order of } \bar{\varphi}\}$ . Let  $k = p_1^{n_1} \cdots p_s^{n_s}$  be a prime factorization of  $k$  such that all  $p_i$ 's are greater than  $N$ . Then we have to show that  $k \in \text{HPer}(f)$ . For this purpose, by Theorem 2.4, we need to find  $\alpha = (a, A) \in \Pi$  satisfying:

- $\det(I - A_* D_*^k) \neq 0$ ,
- $\forall \ell < k$  with  $\ell \mid k$ ,  $\forall \beta \in \Pi$ ,  $\alpha(d, D)^k \neq (\beta(d, D)^\ell)^{k/\ell}$ .

We will show that we can choose  $\alpha = (a, I)$  in  $\Gamma \subset \Pi$ . Recall first that the proof of Lemma 3.1 shows that there exists a positive integer  $N_0$  such that for all positive integers  $k$  whose prime divisors are larger than  $N_0$ ,  $I - \mathbb{D}_*^\ell$  has nonzero determinant if  $\ell \mid k$ . Since  $N \geq N_0$ ,  $\det(I - \mathbb{D}_*^\ell) \neq 0$  for all  $\ell \mid k$ . In particular,  $\det(I - \mathbb{D}_*^k) \neq 0$ . By [25, Lemma 3.3] [9, Theorem 1], we have  $\det(I - D_*^k) = \det(I - \mathbb{D}_*^k) \neq 0$ .

It remains to prove the second condition. We assume on the contrary that for any  $\alpha = (a, I) \in \Gamma$ , there exists  $\ell < k$  with  $\ell \mid k$  and there exist  $\beta = (b, B) \in \Pi$  such that  $\alpha(d, D)^k = (\beta(d, D)^\ell)^{k/\ell}$ , which is equivalent to

$$(\dagger) \quad \alpha = \beta \varphi^\ell(\beta) \varphi^{2\ell}(\beta) \cdots \varphi^{k-\ell}(\beta).$$

Now we recall that since  $D$  is an automorphism,  $\varphi$  is the conjugation by  $(d, D)$ ,  $\varphi|_\Gamma = \mu(d)D$  and  $\bar{\varphi}$  is the conjugation by  $D$ . The matrix part (the holonomy part) of both sides of  $(\dagger)$  yields

$$I = B \bar{\varphi}^\ell(B) \bar{\varphi}^{2\ell}(B) \cdots \bar{\varphi}^{k-\ell}(B).$$

Taking  $\bar{\varphi}^\ell$ , we have

$$I = \bar{\varphi}^\ell(B) \bar{\varphi}^{2\ell}(B) \cdots \bar{\varphi}^{k-\ell}(B) \bar{\varphi}^k(B).$$

Hence

$$\bar{\varphi}^k(B)^{-1} = B^{-1} = \bar{\varphi}^\ell(B) \bar{\varphi}^{2\ell}(B) \cdots \bar{\varphi}^{k-\ell}(B).$$

This gives us  $\bar{\varphi}^k(B) = B$ . By the choice of  $k$ ,  $k$  must be relatively prime to the order  $p$  of  $\bar{\varphi}$ . Choose  $x, y \in \mathbb{Z}$  so that  $kx + py = 1$ . Since  $\bar{\varphi} = \bar{\varphi}^{kx+py} = (\bar{\varphi}^k)^x$ , it follows that  $\bar{\varphi}(B) = B$ . Since  $\text{fix}(\bar{\varphi}) = \{I\}$  by our assumption, we have  $B = I$ . Plugging into  $(\dagger)$ , we have

$$a = b \varphi^\ell(b) \varphi^{2\ell}(b) \cdots \varphi^{k-\ell}(b).$$

Since  $\varphi|_\Gamma = \mu(d)D = \mathbb{D}$ , we have

$$a = b \mathbb{D}^\ell(b) \mathbb{D}^{2\ell}(b) \cdots \mathbb{D}^{k-\ell}(b)$$

for some  $\ell < k$  with  $\ell \mid k$ .

Now we have to show that for any  $\ell < k$  with  $\ell \mid k$

$$\{e \mathbb{D}^\ell(e) \mathbb{D}^{2\ell}(e) \cdots \mathbb{D}^{k-\ell}(e) \mid e \in \Gamma\} \neq \Gamma.$$

Recall in the proof of Lemma 2.5 that  $\Gamma$  has a central series

$$\Gamma = \Gamma_0 \supset \text{nil}(\Gamma) = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_c \supset \Gamma_{c+1} = 1$$

with  $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}^{k_i}$ . Since  $\mathbb{D}(\Gamma_i) \subset \Gamma_i$ , it induces  $\bar{\mathbb{D}}_i : \Gamma_i/\Gamma_{i+1} \rightarrow \Gamma_i/\Gamma_{i+1}$ . Note also that

$$\mathbb{D}_* = \begin{bmatrix} \bar{\mathbb{D}}_c & 0 & \cdots & 0 \\ 0 & \bar{\mathbb{D}}_{c-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\mathbb{D}}_0 \end{bmatrix},$$

where  $\bar{\mathbb{D}}_i$  are integer matrices. Hence some  $\bar{\mathbb{D}}_i$  satisfies the assumptions (1) and (2). By Lemma 3.3, we have

$$\{\bar{e} + \bar{\mathbb{D}}_i^\ell(\bar{e}) + \bar{\mathbb{D}}_i^{2\ell}(\bar{e}) + \cdots + \bar{\mathbb{D}}_i^{k-\ell}(\bar{e}) \mid \bar{e} \in \Gamma_i/\Gamma_{i+1}\} \neq \Gamma_i/\Gamma_{i+1}.$$

This proves our assertion. Hence  $a \in \Gamma$  can be chosen so that

$$a \neq b \varphi^\ell(b) \varphi^{2\ell}(b) \cdots \varphi^{k-\ell}(b).$$

for any  $b \in \Gamma$ . This contradiction proves the second condition.  $\square$

**Corollary 3.5.** *Let  $M = \Pi \backslash S$  be an infra-solvmanifold of type (R) and let  $f$  be a self-map on  $M$  with an affine homotopy lift  $(d, D)$ . Let  $\varphi : \Pi \rightarrow \Pi$  be the homomorphism satisfying*

$$\varphi(\alpha)(d, D) = (d, D)\alpha, \quad \forall \alpha \in \Pi.$$

*Assume that*

- (1) *any eigenvalue  $\lambda$  of  $\mathbb{D}_*$  of modulus 1 is a root of unity, but not 1;*
- (2)  *$\det \mathbb{D}_* \neq 0, \pm 1$ ;*
- (3)  *$\text{fix}(\varphi : \Phi \rightarrow \Phi) = \{I\}$ .*

*Then  $\text{DH}(f)$  is positive.*

*Proof.* By Theorem 3.4, there exists an integer  $N$  with the following property: for any positive integer  $k = p_1^{n_1} \cdots p_s^{n_s}$  with all  $p_i$ 's distinct primes and greater than  $N$ ,  $k \in \text{HPer}(f)$ . Thus

$$\text{HPer}(f) \supset \{k \mid \text{any prime divisor of } k \text{ is } > N\}.$$

Let  $q_1, \dots, q_\ell$  be the all prime numbers which are smaller than or equal to  $N$ . Then the set

$$\{k \mid k \equiv 1 \pmod{q_1 \cdots q_\ell}\}$$

is contained in the set on the right-hand side of the above. For, if  $k \equiv 1 \pmod{q_1 \cdots q_\ell}$  and if  $p$  is a prime divisor of  $k$  with  $p \leq N$ , then  $p = q_j$  for some  $j$ ; thus  $q_j \mid k$  and  $q_j \mid k - 1$  and hence  $q_j = 1$ , a contradiction.

Furthermore, we have that  $N! \mid k - 1$  implies  $q_1 \cdots q_\ell \mid k - 1$ . This shows that

$$\text{HPer}(f) \supset \{k \mid k \equiv 1 \pmod{N!}\}$$

and the set on the right-hand side has density  $1/N!$ . Consequently,

$$\text{DH}(f) \geq 1/N! > 0. \quad \square$$

A special solvmanifold is an infra-solvmanifold with the trivial holonomy group. Hence the third condition of Corollary 3.5 on such a manifold is automatically fulfilled. Immediately we have:

**Corollary 3.6.** *Let  $f$  be a self-map on a special solvmanifold  $M$  with a Lie group homomorphism  $D$  as a homotopy lift. Assume that*

- (1) *any eigenvalue  $\lambda$  of  $\mathbb{D}_*$  of modulus 1 is a root of unity, but not 1;*
- (2)  *$\det \mathbb{D}_* \neq 0, \pm 1$ .*

*Then  $\text{DH}(f)$  is positive.*

#### 4. Computational results

In this section, we will consider some examples on infra-solvmanifolds up to dimension three. For infra-solvmanifolds up to dimension 3, there are only three possibilities for the solvable Lie group  $G$  on which the manifold is modeled. It can be modeled on either the abelian groups  $\mathbb{R}^n (n \leq 3)$ , the 2-step nilpotent Heisenberg group Nil or the 2-step solvable Lie group Sol.

We can find a complete description of  $\text{HPer}(f)$  for maps  $f$  on tori in [2] and [18], and on the Klein bottle in [21]. The remaining infra-solvmanifolds of dimension 3 are three-dimensional flat manifolds, infra-nilmanifolds on Nil and infra-solvmanifolds of Sol.

We will give three examples, one from each remaining manifold. For any self-map  $f$  on the manifold  $\Pi \backslash G$ , let  $\varphi : \Pi \rightarrow \Pi$  be a homomorphism induced by  $f$ . Consider an affine map  $(d, D)$  on  $G$  satisfying (\*). To apply Corollary 3.5, we have to consider the case where  $\mathbb{D} = \mu(d)D$  is invertible. If this is the case, then (\*) says that  $\varphi$  is the conjugation by  $(d, D)$ , that is,  $\varphi(\alpha) = (d, D)\alpha(d, D)^{-1}$ . If  $\alpha = (a, I) \in \Gamma$ , then

$$\varphi(\alpha) = (d, D)(a, I)(d, D)^{-1} = (dD(a)d^{-1}, I) = (\mu(d)D(a), I).$$

Here  $\mu(d)$  is the automorphism on  $G$  obtained by conjugating by the element  $d \in G$ . Thus  $\varphi(\Gamma) \subset \Gamma$  and  $\varphi|_{\Gamma} = \mu(d)D = \mathbb{D}$ , and hence  $\bar{\varphi}$  is the conjugation by  $D$ . In particular  $\bar{\varphi} : \Phi \rightarrow \Phi$  is an isomorphism.

We start with the following easy observation.

**Lemma 4.1.** *Let  $\Phi$  be a group with presentation*

$$\Phi = \langle x, y \mid x^2 = y^2 = 1, xy = yx \rangle,$$

and let  $\psi$  be an isomorphism on  $\Phi$ . Then  $\text{fix}(\psi) = 1$  if and only if  $\psi$  satisfies one of the following:

- $\psi(x) = y, \psi(y) = xy$
- $\psi(x) = xy, \psi(y) = x$

#### 4.1. A flat manifold of dimension three

We have a complete classification of three-dimensional Bieberbach groups. There are six orientable ones and four nonorientable ones, see the book [35, Theorems 3.5.5 and 3.5.9]. Every group has an explicit representation into  $\mathbb{R}^3 \rtimes \text{GL}(3, \mathbb{Z})$  (not into  $\mathbb{R}^4 \rtimes O(4)$ ) in this book. Of course one of them is  $\mathfrak{G}_1 = \mathbb{Z}^3$ . Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and let  $t_i = (e_i, I) \in \mathbb{R}^3 \rtimes \text{GL}(3, \mathbb{Z})$ . Then  $t_1, t_2$  and  $t_3$  generate the subgroup  $\Gamma$  of  $\mathbb{R}^3 \rtimes \text{GL}(3, \mathbb{Z})$ , which is isomorphic to the group of all integer vectors of  $\mathbb{R}^3$ .

Let  $\alpha = (a, A)$ ,  $\beta = (b, B)$  and  $\gamma = (c, C)$  be elements of  $\mathbb{R}^3 \rtimes \text{GL}(3, \mathbb{Z})$ , where

$$a = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, c = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then  $A, B, C$  have order 2 and  $AB = C = BA$ , and

$$\mathfrak{G}_6 = \left\langle t_1, t_2, t_3, \alpha, \beta, \gamma \mid \begin{array}{l} [t_i, t_j] = 1, \gamma\beta\alpha = t_1t_3, \\ \alpha^2 = t_1, \alpha t_2 \alpha^{-1} = t_2^{-1}, \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ \beta t_1 \beta^{-1} = t_1^{-1}, \beta^2 = t_2, \beta t_3 \beta^{-1} = t_3^{-1}, \\ \gamma t_1 \gamma^{-1} = t_1^{-1}, \gamma t_2 \gamma^{-1} = t_2^{-1}, \gamma^2 = t_3 \end{array} \right\rangle.$$

Thus  $\mathfrak{G}_6$  fits the short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \mathfrak{G}_6 \longrightarrow \Phi \longrightarrow 1,$$

where  $\Phi = \langle A, B \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Every element of  $\mathfrak{G}_6$  can be written uniquely in the form  $\alpha^k \beta^m t_3^n$ . We first observe the following: Since  $\gamma\beta\alpha = t_1t_3$ , we have  $\gamma = \alpha\beta^{-1}t_3$ , and

$$\beta^m \alpha^k = \begin{cases} \alpha^k \beta^m & \text{when } (k, m) = (e, e) \\ \alpha^{-k} \beta^m & \text{when } (k, m) = (e, o) \\ \alpha^k \beta^{-m} & \text{when } (k, m) = (o, e) \\ \alpha^{-k} \beta^{-m} t_3 & \text{when } (k, m) = (o, o) \end{cases}$$

and

$$\begin{aligned} (\alpha^k \beta^m t_3^n)^2 &= \alpha^k (\beta^m \alpha^k) \beta^m t_3^{((-1)^{k+m} + 1)n} \\ &= \begin{cases} \alpha^{2k} \beta^{2m} t_3^{2n} & \text{when } (k, m) = (e, e) \\ \beta^{2m} & \text{when } (k, m) = (e, o) \\ \alpha^{2k} & \text{when } (k, m) = (o, e) \\ t_3^{2n-1} & \text{when } (k, m) = (o, o). \end{cases} \end{aligned}$$

Let  $\varphi : \mathfrak{G}_6 \rightarrow \mathfrak{G}_6$  be any homomorphism that induces an isomorphism  $\bar{\varphi}$  on  $\Phi$  satisfying  $\text{fix}(\bar{\varphi}) = \{I\}$ . By Lemma 4.1, we have either  $\bar{\varphi}(\bar{\alpha}) = \bar{\beta}$ ,  $\bar{\varphi}(\bar{\beta}) = \bar{\alpha}\bar{\beta}$  or  $\bar{\varphi}(\bar{\alpha}) = \bar{\alpha}\bar{\beta}$ ,  $\bar{\varphi}(\bar{\beta}) = \bar{\alpha}$ .

In general,  $\varphi$  has the form

$$\varphi(\alpha) = \alpha^{k_1} \beta^{m_1} t_3^{n_1}, \quad \varphi(\beta) = \alpha^{k_2} \beta^{m_2} t_3^{n_2}, \quad \varphi(t_3) = \alpha^{k_3} \beta^{m_3} t_3^{n_3}.$$

Since  $\gamma = \alpha\beta^{-1}t_3$ , a simple calculation shows that

$$\varphi(\gamma) = \alpha^{k_1} \beta^{m_1 - m_2} \alpha^{-k_2 + k_3} \beta^{m_3} t_3^{(-1)^{k_2 + k_3 + m_2 + m_3} (n_1 - n_2) + n_3}.$$

**Case**  $\bar{\varphi}(\bar{\alpha}) = \bar{\beta}$ ,  $\bar{\varphi}(\bar{\beta}) = \bar{\alpha}\bar{\beta}$ .

Then  $k_1$  is even and  $k_2, m_1, m_2$  are odd. So, we have

$$\varphi(\gamma) = \alpha^{k_1 - k_2 + k_3} \beta^{-m_1 + m_2 + m_3} t_3^{n_1 - n_2 + n_3}.$$

Since  $\alpha^2 = t_1, \beta^2 = t_2$  and  $\gamma^2 = t_3$ , a simple calculation shows that

$$\alpha^2 = t_1 \Rightarrow \varphi(t_1) = \varphi(\alpha)^2 = (\alpha^{k_1} \beta^{m_1} t_3^{n_1})^2 = \beta^{2m_1} = t_2^{m_1};$$

$$\begin{aligned}\beta^2 = t_2 &\Rightarrow \varphi(t_2) = \varphi(\beta)^2 = (\alpha^{k_2} \beta^{m_2} t_3^{n_2})^2 = t_3^{2n_2-1}; \\ \gamma^2 = t_3 &\Rightarrow k_3 = 2(k_1 - k_2 + k_3), m_3 = 0, n_3 = 0.\end{aligned}$$

Hence  $\varphi(t_3) = \alpha^{k_3} = t_1^{k_3/2}$ , and it follows that  $\varphi(\Gamma) \subset \Gamma$  and so  $\mathbb{D} = \varphi|_{\Gamma}$  and

$$\mathbb{D}_* = \begin{bmatrix} 0 & 0 & \frac{k_3}{2} \\ m_1 & 0 & 0 \\ 0 & 2n_2 - 1 & 0 \end{bmatrix}.$$

Thus  $\det \mathbb{D}_* = \frac{k_3}{2} m_1 (2n_2 - 1) \neq 0, \pm 1$  if and only if either  $k_3 \neq 0$  or  $k_3 = \pm 2, m_1 = \pm 1, 2n_2 - 1 = \pm 1$ . If  $\mathbb{D}_*$  has an eigenvalue of modulus 1, then  $\det \mathbb{D}_* = \pm 1$ . This shows that the condition (2) of Corollary 3.5 implies the condition (1). Consequently, when  $k_1, k_3, m_3$  are even and  $k_2, m_1, m_2$  are odd, if  $k_3 \neq 0$  or

$$(k_3, m_1, n_2) \notin \{(2, 1, 0), (2, -1, 0), (-2, 1, 0), (-2, -1, 0), \\ (2, 1, 1), (2, -1, 1), (-2, 1, 1), (-2, -1, 1)\},$$

then  $\text{DH}(f) > 0$ .

**Case**  $\bar{\varphi}(\bar{\alpha}) = \bar{\alpha}\bar{\beta}, \bar{\varphi}(\bar{\beta}) = \bar{\alpha}$ .

Then  $k_1, k_2, m_1$  are odd and  $m_2$  is even. So, we have

$$\varphi(\gamma) = \alpha^{k_1+k_2-k_3} \beta^{-m_1+m_2+m_3} t_3^{-n_1+n_2+n_3}.$$

Since  $\alpha^2 = t_1, \beta^2 = t_2$  and  $\gamma^2 = t_3$ , a simple calculation shows that

$$\begin{aligned}\alpha^2 = t_1 &\Rightarrow \varphi(t_1) = \varphi(\alpha)^2 = (\alpha^{k_1} \beta^{m_1} t_3^{n_1})^2 = t_3^{2n_1-1}; \\ \beta^2 = t_2 &\Rightarrow \varphi(t_2) = \varphi(\beta)^2 = (\alpha^{k_2} \beta^{m_2} t_3^{n_2})^2 = \alpha^{2k_2} = t_1^{k_2}; \\ \gamma^2 = t_3 &\Rightarrow k_3 = 0, m_3 = 2(-m_1 + m_2 + m_3), n_3 = 0.\end{aligned}$$

Hence  $\varphi(t_3) = \beta^{m_3} = t_2^{m_3/2}$ , and it follows that  $\varphi(\Gamma) \subset \Gamma$  and so  $\mathbb{D} = \varphi|_{\Gamma}$  and

$$\mathbb{D}_* = \begin{bmatrix} 0 & k_2 & 0 \\ 0 & 0 & \frac{m_3}{2} \\ 2n_1 - 1 & 0 & 0 \end{bmatrix}.$$

Hence  $\det \mathbb{D}_* = k_2 \frac{m_3}{2} (2n_1 - 1) \neq 0, \pm 1$  if and only if either  $m_3 \neq 0$  or  $k_2 = \pm 1, m_3 = \pm 2, 2n_1 - 1 = \pm 1$ . If  $\mathbb{D}_*$  has an eigenvalue of modulus 1, then  $\det \mathbb{D}_* = \pm 1$ . This shows that the condition (2) of Corollary 3.5 implies the condition (1). Consequently, when  $k_1, k_2, m_1$  are odd and  $k_3, m_2, m_3$  are even, if  $m_3 \neq 0$  or

$$(k_3, m_1, n_2) \notin \{(1, 2, 0), (1, -2, 0), (-1, 2, 0), (-1, -2, 0), \\ (1, 2, 1), (1, -2, 1), (-1, 2, 1), (-1, -2, 1)\},$$

then  $\text{DH}(f) > 0$ .

#### 4.2. An infra-nilmanifold modeled on Nil

We will consider a three-dimensional infra-nilmanifold modeled on the Heisenberg group Nil. Recall that

$$\text{Nil} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

For all integers  $k > 0$ , we consider the subgroups  $\Gamma_k$  of Nil:

$$\Gamma_k = \left\{ \begin{bmatrix} 1 & m & -\frac{\ell}{k} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \mid \ell, m, n \in \mathbb{Z} \right\}.$$

These are lattices of Nil and every lattice of Nil is isomorphic to some  $\Gamma_k$ . Letting

$$s_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 1 & 0 & -\frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we obtain a presentation of  $\Gamma_k$

$$\Gamma_k = \langle s_1, s_2, s_3 \mid [s_3, s_1] = [s_3, s_2] = 1, [s_2, s_1] = s_3^k \rangle.$$

Every element of  $\Gamma_k$  can be written uniquely as the form

$$s_2^n s_1^m s_3^\ell = \begin{bmatrix} 1 & m & -\frac{\ell}{k} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark that  $s_1^m s_2^n = s_2^n s_1^m s_3^{-kmn}$ . All possible almost-Bieberbach groups can be found in [3, pp. 799–801] or [5].

Consider an almost Bieberbach group  $\Pi$  given by

$$\Pi = \left\langle s_1, s_2, s_3, \alpha \mid \begin{array}{l} [s_3, s_1] = [s_3, s_2] = 1, [s_2, s_1] = s_3^k, \\ \alpha s_1 \alpha^{-1} = s_2, \alpha s_2 \alpha^{-1} = s_1^{-1} s_2^{-1}, \alpha^3 = s_3^2 \end{array} \right\rangle.$$

This is a 3-dimensional almost Bieberbach group  $\pi_{6,2}$  or  $\pi_{6,3}$  with Seifert bundle type 6.

Let  $\varphi : \Pi \rightarrow \Pi$  be a homomorphism. Every element of  $\Pi$  is of the form  $s_2^n s_1^m s_3^\ell$ ,  $s_2^n s_1^m s_3^\ell \alpha$  or  $s_2^n s_1^m s_3^\ell \alpha^2$ . In order to have an isomorphism  $\bar{\varphi} : \Phi \rightarrow \Phi$  such that  $\text{fix}(\varphi) = \{I\}$ , we must have that  $\bar{\varphi}(\bar{\alpha}) = \bar{\alpha}^2$ . This implies that  $\varphi$  has the form

$$\varphi(s_1) = s_2^{n_1} s_1^{m_1} s_3^{\ell_1}, \quad \varphi(s_2) = s_2^{n_2} s_1^{m_2} s_3^{\ell_2}, \quad \varphi(\alpha) = s_2^{n_3} s_1^{m_3} \alpha^{3\ell_3+2}.$$

Then it can be seen as before that

$$\varphi(s_3^2) = \varphi(\alpha)^3 = s_3^{(3\ell_3+2) - \frac{m_3(m_3+1)}{2}k + (m_3^2 + m_3 n_3 + n_3^2)k - \frac{n_3(n_3+1)}{2}k}.$$

Since  $\varphi(s_3) \in \Gamma_k$ ,  $\varphi(s_3)$  is of the form  $s_2^n s_1^m s_3^\ell$  and so

$$\varphi(s_3^2) = (s_2^n s_1^m s_3^\ell)^2 = s_2^{2n} s_1^{2m} s_3^{2\ell - kmn}.$$



Hence  $\varphi(s_3) = s_3^\ell$ . Furthermore, the relations  $\alpha s_1 \alpha^{-1} = s_2$  and  $\alpha s_2 \alpha^{-1} = s_1^{-1} s_2^{-1}$  are preserved by  $\varphi$ . This induces the conditions  $n_1 = n_2 = -m_2$  and  $m_1 = -2m_2$ . The relation  $[s_2, s_1] = s_3^k$  yields that  $\ell = m_1 n_2 - m_2 n_1 = 3m_2^2$ . Consequently, the integral differential of  $\mathbb{D} = \varphi|_{\Gamma_k}$  with respect to the basis  $\{\log(s_1), \log(s_2), \log(s_3)\}$  of  $\mathfrak{nil}$  is

$$\mathbb{D}_* = \begin{bmatrix} -2m_2 & m_2 & 0 \\ -m_2 & -m_2 & 0 \\ 0 & 0 & 3m_2^2 \end{bmatrix}.$$

Hence  $\det \mathbb{D}_* = (3m_2^2)^2$  and the eigenvalues of  $\mathbb{D}_*$  are  $\ell$  and  $\frac{-3 \pm \sqrt{3}i}{2} m_2$ . No eigenvalues of  $\mathbb{D}_*$  are of modulus 1, and  $\det \mathbb{D}_* = 0$  (i.e.,  $m_2 = 0$ ) or  $\det \mathbb{D}_* \geq 9$ . Consequently if  $m_2 \neq 0$  then  $\text{DH}(f) > 0$ .

### 4.3. Infra-solvmanifolds modeled on Sol

Next we will consider a closed 3-manifold with Sol-geometry. Recall that  $\text{Sol} = \mathbb{R}^2 \rtimes_\phi \mathbb{R}$  where

$$\phi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Then Sol is a connected and simply connected unimodular 2-step solvable Lie group of type (R). It has a faithful representation into  $\text{Aff}(\mathbb{R}^3)$  as follows:

$$\text{Sol} = \left\{ \begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \mid x, y, t \in \mathbb{R} \right\}.$$

Let  $M$  be a closed 3-manifold with Sol-geometry. Then the fundamental group  $\Pi$  of  $M$  is a Bieberbach group of Sol, and  $M = \Pi \backslash \text{Sol}$ . Further,  $\Pi$  can be embedded into  $\text{Aff}(\text{Sol}) = \text{Sol} \rtimes \text{Aut}(\text{Sol})$  so that there is an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \Pi \longrightarrow \Pi/\Gamma \longrightarrow 1,$$

where  $\Gamma = \Pi \cap \text{Sol}$  is a lattice of Sol and  $\Phi = \Pi/\Gamma$  is a finite group, called the holonomy group of  $\Pi$  or  $M$ , which sits naturally into  $\text{Aut}(\text{Sol})$ , see [8]. The lattices  $\Gamma$  of Sol are determined by  $2 \times 2$ -integer matrices  $A$

$$A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

of determinant 1 and trace  $> 2$ , see for example [29, Lemma 2.1]. Namely,

$$\Gamma = \Gamma_A = \langle a_1, a_2, \tau \mid [a_1, a_2] = 1, \tau a_i \tau^{-1} = A(a_i) \rangle = \mathbb{Z}^2 \rtimes_A \mathbb{Z}.$$

Let  $f$  be a self-map on  $\Gamma_A \backslash \text{Sol}$ . By [29, Theorem 2.4], the homomorphism  $\varphi : \Gamma_A \rightarrow \Gamma_A$  induced by  $f$  is determined by

$$\varphi(a_i) = \mathbf{a}^{u_i}, \quad \varphi(\tau) = \mathbf{a}^{\mathbf{p}} \tau^\zeta$$

for some  $\mathbf{u}_i, \mathbf{p} \in \mathbb{Z}^2$  and  $\zeta \in \mathbb{Z}$ . Note that  $\varphi$  extends uniquely to a Lie group homomorphism on Sol. It follows easily that all the possible (integer) matrices  $\mathbb{D}_*$  are of the form

$$\mathbb{D}_* = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{0} \\ 0 & 0 & \zeta \end{bmatrix}.$$

We say that  $\varphi$  is of type (I) if  $\zeta = 1$ ; of type (II) if  $\zeta = -1$ ; of type (III) if  $\zeta \neq \pm 1$ . When  $\varphi$  is of type (III), we have  $\varphi(a_i) = 1$ .

Now we consider the conditions of Corollary 3.6. These eliminate  $\varphi$  of type (I) and (III). If  $\varphi$  of type (II) satisfies the conditions of Corollary 3.6, then  $\text{DH}(f) > 0$ . In fact, it is shown in [28, Theorem 5.1] that such a map has  $\text{HPer}(f) = \mathbb{N} - 2\mathbb{N}$ , and so  $\text{DH}(f) = 1/2$ .

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