

TRANSLATION THEOREMS FOR THE ANALYTIC FOURIER–FEYNMAN TRANSFORM ASSOCIATED WITH GAUSSIAN PATHS ON WIENER SPACE

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ABSTRACT. In this article, we establish translation theorems for the analytic Fourier–Feynman transform of functionals in non-stationary Gaussian processes on Wiener space. We then proceed to show that these general translation theorems can be applied to two well-known classes of functionals; namely, the Banach algebra \mathcal{S} introduced by Cameron and Storvick, and the space $\mathcal{B}_{\mathcal{A}}^{(p)}$ consisting of functionals of the form $F(x) = f(\langle \alpha_1, x \rangle, \dots, \langle \alpha_n, x \rangle)$, where $\langle \alpha, x \rangle$ denotes the Paley–Wiener–Zygmund stochastic integral $\int_0^T \alpha(t) dx(t)$.

1. Introduction

Given a positive real $T > 0$, let $C_0[0, T]$ denote one-parameter Wiener space, that is, the space of all real-valued continuous functions x on the compact interval $[0, T]$ with $x(0) = 0$. Let \mathcal{M} denote the class of all Wiener measurable subsets of $C_0[0, T]$ and let \mathbf{m} denote Wiener measure which is a Gaussian measure on $C_0[0, T]$ with mean zero and covariance function $r(s, t) = \min\{s, t\}$. Then, as is well-known, $(C_0[0, T], \mathcal{M}, \mathbf{m})$ is a complete measure space.

It is well-known that there is no quasi-invariant measure on infinite dimensional linear spaces. Thus, there is no quasi-invariant probability measure on the Wiener space $(C_0[0, T], \mathcal{M}, \mathbf{m})$. Based on such circumstance, numerous constructions and applications of the translation theorem (Cameron–Martin theorem) for integrals on infinite-dimensional spaces have been studied in various research fields in Mathematics and Physics. The most of the results in the

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literature are concentrated on the Wiener space $C_0[0, T]$. Translation theorems for Wiener integrals were given by Cameron and Martin in [3] and by Cameron and Graves in [2]. Translation theorems for analytic Feynman integrals were given by Cameron and Storvick in [4, 7] and translation theorems for analytic Feynman integrals on abstract Wiener and Hilbert spaces were given by Chung and Kang in [13].

On the other hand, the concepts of the generalized Wiener integral and the generalized analytic Feynman integral on $C_0[0, T]$ were introduced by Chung, Park and Skoug in [14]. The generalized Wiener integral was defined by the Wiener integral

$$(1.1) \quad \int_{C_0[0, T]} F(\mathcal{Z}_h(x, \cdot)) \mathbf{m}(dx),$$

where $\mathcal{Z}_h(x, \cdot)$ is the Gaussian paths given by the stochastic integral $\mathcal{Z}_h(x, t) = \int_0^t h(s) dx(s)$ with $h \in L_2[0, T]$. For a precise definition of the process \mathcal{Z}_h , see equation (2.1) below. The Gaussian process $\mathcal{Z}_h : C_0[0, T] \times [0, T] \rightarrow \mathbb{R}$ is generally not stationary in time. Thus the meaning of the generalized Wiener integral (1.1) is an integral of functionals in sample paths of non-stationary Gaussian processes. Since then, the generalized analytic Fourier–Feynman transform (GFFT) associated with the Gaussian paths $\mathcal{Z}_h(x, \cdot)$ also was introduced by Huffman, Park and Skoug in [18], and was further developed in [8, 11, 24].

In this article we establish translation theorems for the GFFT defined on the Wiener space $C_0[0, T]$. We then proceed to show that these general translation theorems for the GFFT can be applied to two well-known classes of functionals; namely, the Banach algebra \mathcal{S} introduced by Cameron and Storvick in [6], and the space $\mathcal{B}_A^{(p)}$ consisting of functionals of the form

$$F(x) = f(\langle \alpha_1, x \rangle, \dots, \langle \alpha_n, x \rangle),$$

where $\langle \alpha, x \rangle$ denotes the Paley–Wiener–Zygmund (PWZ) stochastic integral $\int_0^T \alpha(t) dx(t)$.

2. Preliminaries

In this section, we first present a brief background and some well-known results about the Wiener space $C_0[0, T]$.

A subset B of $C_0[0, T]$ is said to be scale-invariant measurable provided $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $\mathbf{m}(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho \cdot)$ is Wiener-measurable for every $\rho > 0$. If two functionals F and G are equal s-a.e., we write $F \approx G$.

The PWZ stochastic integral [21] plays a key role throughout this article. Let $\{\phi_n\}_{n=1}^\infty$ be a complete orthonormal set in $L_2[0, T]$, each of whose elements is

of bounded variation on $[0, T]$. Then for each $v \in L_2[0, T]$, the PWZ stochastic integral $\langle v, x \rangle$ is defined by the formula

$$\langle v, x \rangle = \lim_{n \rightarrow \infty} \int_0^T \sum_{j=1}^n (v, \phi_j)_2 \phi_j(t) dx(t)$$

for all $x \in C_0[0, T]$ for which the limit exists, where $(\cdot, \cdot)_2$ denotes the L_2 -inner product. For each $v \in L_2[0, T]$, the limit defining the PWZ stochastic integral $\langle v, x \rangle$ is essentially independent of the choice of the complete orthonormal set $\{\phi_n\}_{n=1}^\infty$ and it exists for s-a.e. $x \in C_0[0, T]$. If v is of bounded variation on $[0, T]$, then $\langle v, x \rangle$ equals the Riemann–Stieltjes integral $\int_0^T v(t) dx(t)$ for s-a.e. $x \in C_0[0, T]$, and for all $v \in L_2[0, T]$, $\langle v, x \rangle$ is a Gaussian random variable on $C_0[0, T]$ with mean zero and variance $\|v\|_2^2$. For a more detailed study of the PWZ stochastic integral, see [20, 22].

Given a function h in $L_2[0, T]$ with $\|h\|_2 > 0$, let $\mathcal{Z}_h(x, t)$ be the PWZ stochastic integral

$$(2.1) \quad \mathcal{Z}_h(x, t) = \langle h \chi_{[0, t]}, x \rangle,$$

where $\chi_{[0, t]}$ denotes the indicator function of the set $[0, t]$. Next, let $\beta_h(t) = \int_0^t h^2(u) du$. The stochastic process \mathcal{Z}_h on $C_0[0, T] \times [0, T]$, $(x, t) \mapsto \mathcal{Z}_h(x, t)$, is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0, T]} \mathcal{Z}_h(x, s) \mathcal{Z}_h(x, t) \mathbf{m}(dx) = \beta_h(\min\{s, t\}).$$

In addition, by [25, Theorem 21.1], $\mathcal{Z}_h(\cdot, t)$ is stochastically continuous in t on $[0, T]$. If $h \in L_2[0, T]$ is of bounded variation on $[0, T]$, then for all $x \in C_0[0, T]$, $\mathcal{Z}_h(x, t)$ is continuous in t . Also, for any $h_1, h_2 \in L_2[0, T]$,

$$\int_{C_0[0, T]} \mathcal{Z}_{h_1}(x, s) \mathcal{Z}_{h_2}(x, t) \mathbf{m}(dx) = \int_0^{\min\{s, t\}} h_1(u) h_2(u) du.$$

Of course if $h(t) \equiv 1$ on $[0, T]$, then the process \mathcal{W} on $C_0[0, T] \times [0, T]$ given by $(w, t) \xrightarrow{\mathcal{W}} \mathcal{W}_t(x) = \mathcal{Z}_1(x, t) = x(t)$ is a Wiener process. We note that the coordinate process \mathcal{Z}_1 is stationary in time, whereas the stochastic process \mathcal{Z}_h generally is not. For more detailed studies on the stochastic process \mathcal{Z}_h , see [11, 14, 23].

From [14, Lemma 1], it follows that for each $\varphi \in L_\infty[0, T]$ (resp. $L_2[0, T]$) and each $h \in L_2[0, T]$ (resp. $L_\infty[0, T]$),

$$(2.2) \quad \langle \varphi, \mathcal{Z}_h(x, \cdot) \rangle = \langle \varphi h, x \rangle$$

for s-a.e. $x \in C_0[0, T]$.

We finish this section by stating the Cameron–Martin translation theorem [2, 3].

Theorem 2.1 (Cameron–Martin Translation Theorem). *Let F be Wiener integrable over $C_0[0, T]$ and let $w_v \in C_0[0, T]$ be given by $w_v(t) = \int_0^t v(s)ds$ for some $v \in L_2[0, T]$. Then*

$$(2.3) \quad \int_{C_0[0, T]} F(x+w_v) \mathbf{m}(dx) = \exp \left\{ -\frac{1}{2} \|v\|_2^2 \right\} \int_{C_0[0, T]} F(x) \exp\{\langle v, x \rangle\} \mathbf{m}(dx).$$

3. Analytic Fourier–Feynman transform associated with Gaussian paths

For the definitions and related work involving the generalized analytic Feynman integral and the GFFT associated with the Gaussian paths $\mathcal{Z}_h(x, \cdot)$ (\mathcal{Z}_h -GFFT), see [8, 9, 11, 18, 24].

Let \mathbb{C} , \mathbb{C}_+ and $\tilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part and non-zero complex numbers with nonnegative real part, respectively. For each $\lambda \in \tilde{\mathbb{C}}_+$, $\lambda^{1/2}$ denotes the principal square root of λ ; i.e., $\lambda^{1/2}$ is always chosen to have positive real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in \mathbb{C}_+ .

Definition 3.1. Let h be a function in $L_2[0, T]$ and let F be a \mathbb{C} -valued scale-invariant measurable functional on $C_0[0, T]$ such that

$$\int_{C_0[0, T]} F(\lambda^{-1/2} \mathcal{Z}_h(x, \cdot)) \mathbf{m}(dx) = J(h; \lambda)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(h; \lambda)$ analytic on \mathbb{C}_+ such that $J^*(h; \lambda) = J(h; \lambda)$ for all $\lambda > 0$, then $J^*(h; \lambda)$ is defined to be the generalized analytic Wiener integral (associated with the Gaussian paths $\mathcal{Z}_h(x, \cdot)$) of F over $C_0[0, T]$ with parameter λ , and for $\lambda \in \mathbb{C}_+$ we write

$$(3.1) \quad \int_{C_0[0, T]}^{\text{anw}_\lambda} F(\mathcal{Z}_h(x, \cdot)) \mathbf{m}(dx) = J^*(h; \lambda).$$

Let $q \neq 0$ be a real number and let F be a functional such that

$$\int_{C_0[0, T]}^{\text{anw}_\lambda} F(\mathcal{Z}_h(x, \cdot)) \mathbf{m}(dx)$$

exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral (associated with the Gaussian paths $\mathcal{Z}_h(x, \cdot)$) of F with parameter q and we write

$$(3.2) \quad \int_{C_0[0, T]}^{\text{anf}_q} F(\mathcal{Z}_h(x, \cdot)) d\mathbf{m}(x) = \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[0, T]}^{\text{anw}_\lambda} F(\mathcal{Z}_h(x, \cdot)) \mathbf{m}(dx).$$

The concept of an L_1 analytic Fourier–Feynman transform for functionals on the Wiener space $C_0[0, T]$ was introduced by Brue in [1]. This transform and its properties are similar in many respects to the ordinary Fourier transform of functions on Euclidean spaces. Further work involving the L_2 – L_2 theory and the L_p – $L_{p'}$ theory, $1/p + 1/p' = 1$, includes [5, 15–17, 19].

Next (see [11, 18]) we state the definition of the analytic \mathcal{Z}_h -GFFT.

Definition 3.2. Let h be a function in $L_2[0, T]$. For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0, T]$, let

$$T_{\lambda, h}(F)(y) = \int_{C_0[0, T]}^{\text{anw}\lambda} F(y + \mathcal{Z}_h(x, \cdot)) \mathbf{m}(dx).$$

For $p \in (1, 2]$, we define the L_p analytic \mathcal{Z}_h -GFFT, $T_{q, h}^{(p)}(F)$ of F by the formula,

$$T_{q, h}^{(p)}(F)(y) = \text{l. i. m.}_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} T_{\lambda, h}(F)(y)$$

if it exists; i.e., for each $\rho > 0$,

$$(3.3) \quad \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[0, T]} |T_{\lambda, h}(F)(\rho y) - T_{q, h}^{(p)}(F)(\rho y)|^{p'} \mathbf{m}(dy) = 0,$$

where $1/p + 1/p' = 1$. We define the L_1 analytic \mathcal{Z}_h -GFFT, $T_{q, h}^{(1)}(F)$ of F by the formula

$$(3.4) \quad T_{q, h}^{(1)}(F)(y) = \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} T_{\lambda, h}(F)(y)$$

for s-a.e. $y \in C_0[0, T]$ whenever this limit exists.

We note that for $p \in [1, 2]$, $T_{q, h}^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_{q, h}^{(p)}(F)$ exists and if $F \approx G$, then $T_{q, h}^{(p)}(G)$ exists and $T_{q, h}^{(p)}(G) \approx T_{q, h}^{(p)}(F)$. One can see that for each $h \in L_2[0, T]$, $T_{q, h}^{(p)}(F) \approx T_{q, -h}^{(p)}(F)$, since

$$\int_{C_0[0, T]} F(x) \mathbf{m}(dx) = \int_{C_0[0, T]} F(-x) \mathbf{m}(dx).$$

Moreover, from equations (3.1), (3.2) and (3.4), it follows that

$$(3.5) \quad \int_{C_0[0, T]}^{\text{anf}_q} F(\mathcal{Z}_h(x, \cdot)) \mathbf{m}(dx) = T_{q, k}^{(1)}(F)(0)$$

in the sense that if either side exists, then both sides exist and equality holds.

Remark 3.3. Note that if $h \equiv 1$ on $[0, T]$, then $\mathcal{Z}_h(x, t) = x(t)$ for all $x \in C_0[0, T]$. In this case the generalized analytic Feynman integral given by equation (3.2) with $h \equiv 1$ and the L_p analytic \mathcal{Z}_1 -GFFT, $T_{q, 1}^{(p)}(F)$, agree with the previous definitions of the analytic Feynman integral and the analytic Fourier-Feynman transform, $T_q^{(p)}(F)$, see [5, 7, 15, 19]. Thus we also denote the \mathcal{Z}_1 -GFFT $T_{q, 1}^{(p)}(F)$ of functionals F on $C_0[0, T]$ by $T_q^{(p)}(F)$ throughout this article.

Next we give the definition of our generalized convolution product (GCP) [9].

Definition 3.4. Let F and G be scale-invariant measurable functionals on $C_0[0, T]$. For $\lambda \in \widetilde{\mathbb{C}}_+$ and $h_1, h_2 \in L_2[0, T]$, we define their GCP with respect to $\{\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}\}$ (if it exists) by

$$(3.6) \quad \begin{aligned} & (F * G)_\lambda^{(h_1, h_2)}(y) \\ &= \begin{cases} \int_{C_0[0, T]}^{\text{anw}_\lambda} F(y + \mathcal{Z}_{h_1}(x, \cdot))G(y + \mathcal{Z}_{h_2}(x, \cdot))\mathbf{m}(dx), & \lambda \in \mathbb{C}_+ \\ \int_{C_0[0, T]}^{\text{anf}_q} F(y + \mathcal{Z}_{h_1}(x, \cdot))G(y + \mathcal{Z}_{h_2}(x, \cdot))\mathbf{m}(dx), & \lambda = -iq, q \in \mathbb{R}, q \neq 0. \end{cases} \end{aligned}$$

When $\lambda = -iq$, we denote $(F * G)_\lambda^{(h_1, h_2)}$ by $(F * G)_q^{(h_1, h_2)}$.

Remark 3.5. In [9], a more general definition for the GCP of functionals on $C_0[0, T]$ is presented and fundamental relationships between the \mathcal{Z}_h -GFFT's and the GCP's are investigated.

4. Translation theorems for generalized Fourier–Feynman transform

In [4, 7], Cameron and Storvick showed translation theorems for the analytic Feynman integral of functionals on classical Wiener space and in [13], Chung and Kang established translation theorems for analytic Feynman integrals on abstract Wiener and Hilbert spaces. In Theorems 4.1 and 4.3 below we present translation theorems for the \mathcal{Z}_h -GFFT's of functionals F on the Wiener space $C_0[0, T]$. First we give a translation theorem for the L_1 analytic \mathcal{Z}_h -GFFT.

Theorem 4.1. *Let h_1 be a function in $L_2[0, T]$ (resp. $L_\infty[0, T]$) and let F be a functional on $C_0[0, T]$ such that $F(\mathcal{Z}_h(x, \cdot))$ is Wiener integrable over $C_0[0, T]$. Given $\varphi \in L_\infty[0, T]$ (resp. $L_2[0, T]$), let $w_{\varphi h_1} \in C_0[0, T]$ be defined by*

$$(4.1) \quad w_{\varphi h_1}(t) = \int_0^t \varphi(s)h_1(s)ds.$$

Furthermore, assume that given a non-zero real q , the L_1 analytic \mathcal{Z}_h -GFFT, $T_{q, h_1}^{(1)}(F)$ of F exists. Then for each $h_2 \in L_2[0, T]$ (resp. $L_\infty[0, T]$) and s -a.e. $y \in C_0[0, T]$,

$$(4.2) \quad \begin{aligned} & T_{q, h_1}^{(1)}(F)(y + \mathcal{Z}_{h_2}(w_{\varphi h_1}, \cdot)) \\ &= \exp\left\{\frac{iq}{2}\|\varphi h_2\|_2^2 + iq(\varphi, y)^\sim\right\}(F * R_{q, \varphi})_q^{h_1, h_2}(y), \end{aligned}$$

where the functional $R_{q, \varphi} : C_0[0, T] \rightarrow \mathbb{C}$ is given by

$$(4.3) \quad R_{q, \varphi}(x) = \exp\{-iq\langle \varphi, x \rangle\}.$$

Proof. By the assumption of the existence of $T_{q, h_1}^{(1)}(F)$, we may assume that the analytic \mathcal{Z}_{h_1} -Wiener transform $T_{\lambda, h_1}(F)$ exists for all $\lambda \in \mathbb{C}_+$. We first note that for each $\lambda > 0$,

$$(4.4) \quad \mathcal{Z}_{h_2}(w_{\varphi h_1}, t) + \lambda^{-1/2}\mathcal{Z}_{h_1}(x, t) = \lambda^{-1/2}\mathcal{Z}_{h_1}(x + w_{\lambda^{1/2}\varphi h_2}, t).$$

Next, for $\lambda > 0$, let

$$(4.5) \quad G_{y,h_1}^\lambda(x) = F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)).$$

Using (4.4), (4.5), (2.3) with F and w_v replaced with G_{y,h_1}^λ and $w_{\lambda^{1/2}\varphi h_1}$ respectively, and (2.2) with h replaced with h_2 , it follows that for $\lambda > 0$,

$$(4.6) \quad \begin{aligned} & T_{\lambda,h_1}(F)(y + \mathcal{Z}_{h_2}(w_{\varphi h_1}, \cdot)) \\ &= \int_{C_0[0,T]} F(y + \mathcal{Z}_{h_2}(w_{\varphi h_1}, \cdot) + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \mathbf{m}(dx) \\ &= \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x + w_{\lambda^{1/2}\varphi h_2}, \cdot)) \mathbf{m}(dx) \\ &= \int_{C_0[0,T]} G_{y,h_1}^\lambda(x + w_{\lambda^{1/2}\varphi h_2}) \mathbf{m}(dx) \\ &= \exp\left\{-\frac{1}{2}\|\lambda^{1/2}\varphi h_2\|_2^2\right\} \int_{C_0[0,T]} G_{y,h_1}^\lambda(x) \exp\{\langle \lambda^{1/2}\varphi h_2, x \rangle\} \mathbf{m}(dx) \\ &= \exp\left\{-\frac{\lambda}{2}\|\varphi h_2\|_2^2\right\} \\ &\quad \times \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \exp\{\lambda^{1/2}\langle \varphi, \mathcal{Z}_{h_2}(x, \cdot) \rangle\} \mathbf{m}(dx) \\ &= \exp\left\{-\frac{\lambda}{2}\|\varphi h_2\|_2^2 - \lambda\langle \varphi, y \rangle\right\} \\ &\quad \times \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \exp\{\lambda\langle \varphi, y \rangle + \lambda^{1/2}\langle \varphi, \mathcal{Z}_{h_2}(x, \cdot) \rangle\} \mathbf{m}(dx). \end{aligned}$$

On the other hand, by Definition 3.4, it follows that for $\lambda > 0$,

$$(4.7) \quad \begin{aligned} & (F * R_q)_\lambda^{h_1, h_2}(y) \\ &= \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) R_{q,\varphi}(y + \lambda^{-1/2} \mathcal{Z}_{h_2}(x, \cdot)) \mathbf{m}(dx) \\ &= \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \exp\{-iq\langle \varphi, y + \lambda^{-1/2} \mathcal{Z}_{h_2}(x, \cdot) \rangle\} \mathbf{m}(dx) \\ &= \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \exp\{-iq\langle \varphi, y \rangle - iq\lambda^{-1/2}\langle \varphi, \mathcal{Z}_{h_2}(x, \cdot) \rangle\} \mathbf{m}(dx). \end{aligned}$$

Thus, using (3.4), (4.6), (4.7), and (3.6) with G replaced with $R_{q,\varphi}$, we have that

$$(4.8) \quad \begin{aligned} & T_{q,h_1}^{(1)}(F)(y + \mathcal{Z}_{h_2}(w_{\varphi h_1}, \cdot)) \\ &= \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} T_{\lambda,h_1}(F)(y + \mathcal{Z}_{h_2}(w_{\varphi h_1}, \cdot)) \end{aligned}$$

$$\begin{aligned}
&= \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \exp \left\{ -\frac{\lambda}{2} \|\varphi h_2\|_2^2 - \lambda \langle \varphi, y \rangle \right\} \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \\
&\quad \times \exp \{ \lambda \langle \varphi, y \rangle + \lambda^{1/2} \langle \varphi, \mathcal{Z}_{h_2}(x, \cdot) \rangle \} \mathbf{m}(dx) \\
&= \exp \left\{ \frac{iq}{2} \|\varphi h_2\|_2^2 + iq \langle \varphi, y \rangle \right\} \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \\
&\quad \times \exp \{ \lambda \langle \varphi, y \rangle + \lambda^{1/2} \langle \varphi, \mathcal{Z}_{h_2}(x, \cdot) \rangle \} \mathbf{m}(dx) \\
&= \exp \left\{ \frac{iq}{2} \|\varphi h_2\|_2^2 + iq \langle \varphi, y \rangle \right\} \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \\
&\quad \times \exp \{ -iq \langle \varphi, y \rangle - iq \lambda^{-1/2} \langle \varphi, \mathcal{Z}_{h_2}(x, \cdot) \rangle \} \mathbf{m}(dx) \\
&= \exp \left\{ \frac{iq}{2} \|\varphi h_2\|_2^2 + iq \langle \varphi, y \rangle \right\} \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} (F * R_{q,\varphi})_{\lambda}^{h_1, h_2}(y) \\
&= \exp \left\{ \frac{iq}{2} \|\varphi h_2\|_2^2 + iq \langle \varphi, y \rangle \right\} (F * R_{q,\varphi})_q^{h_1, h_2}(y)
\end{aligned}$$

as desired. \square

Using equation (3.6) with $y = 0$ and with G replaced with $R_{q,\varphi}$, we have the following corollary.

Corollary 4.2. *Let h_1 , F , φ and $w_{\varphi h_1}$ be as in Theorem 4.1. Assume that given a non-zero real q , the \mathcal{Z}_{h_1} -Feynman integral with parameter q of F ,*

$$\int_{C_0[0,T]}^{\text{anf}_q} F(\mathcal{Z}_{h_1}(x, \cdot)) \mathbf{m}(dx)$$

exists. Then for each $h_2 \in L_2[0, T]$ (resp. $L_\infty[0, T]$),

$$\begin{aligned}
(4.9) \quad &\int_{C_0[0,T]}^{\text{anf}_q} F(\mathcal{Z}_{h_1}(x, \cdot) + \mathcal{Z}_{h_2}(w_{\varphi h_1}, \cdot)) \mathbf{m}(dx) \\
&= \exp \left\{ \frac{iq}{2} \|\varphi h_2\|_2^2 \right\} \int_{C_0[0,T]}^{\text{anf}_q} F(\mathcal{Z}_{h_1}(x, \cdot)) \exp \{ -iq \langle \varphi, \mathcal{Z}_{h_2}(x, \cdot) \rangle \} \mathbf{m}(dx).
\end{aligned}$$

Next we present a translation theorem for the L_p analytic \mathcal{Z}_h -GFFT.

Theorem 4.3. *Let $p \in [1, 2]$ be given and let h_1 , F , φ and $w_{\varphi h_1}$ be as in Theorem 4.1. Assume that given a non-zero real q , the L_p analytic \mathcal{Z}_h -GFFT, $T_{q,h_1}^{(p)}(F)$ of F exists. Then for s-a.e. $y \in C_0[0, T]$,*

$$(4.10) \quad T_{q,h_1}^{(p)}(F)(y + \mathcal{Z}_{h_1}(w_{\varphi h_1}, \cdot)) = \exp \left\{ \frac{iq}{2} \|\varphi h_1\|_2^2 + iq \langle \varphi, y \rangle \right\} T_{q,h_1}^{(p)}(FR_{q,\varphi})(y),$$

where the functional $R_{q,\varphi}$ is given by (4.3) above.

Proof. Let $h_1 = h_2$ in (4.8). Then using the fifth expression of (4.8) with $h_1 = h_2$ and the definition of the L_1 analytic \mathcal{Z}_h -GFFT, it follows equation (4.10) with $p = 1$, immediately. But, to obtain equation (4.10) with $p \in (1, 2]$, we have to use the concept of the scale-invariant limit (see (3.3) above) for the proof. Thus we will proceed the proof in the case that $p \in (1, 2]$.

Proceeding as in the proof of Theorem 4.1, we observe that for $\lambda > 0$,

$$\begin{aligned} & T_{\lambda, h_1}(F)(y + \mathcal{Z}_{h_1}(w_{\varphi h_1}, \cdot)) \\ &= \exp \left\{ -\frac{\lambda}{2} \|\varphi h_1\|_2^2 - \lambda \langle \varphi, y \rangle \right\} \\ & \quad \times \int_{C_0[0, T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \exp \{ \lambda \langle \varphi, y \rangle + \lambda^{1/2} \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle \} \mathbf{m}(dx). \end{aligned}$$

We also obtain that for $\lambda > 0$,

$$\begin{aligned} & T_{\lambda, h_1}(FR_{q, \varphi})(y) \\ &= \int_{C_0[0, T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \exp \{ -iq \langle \varphi, y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot) \rangle \} \mathbf{m}(dx) \\ &= \int_{C_0[0, T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \exp \{ -iq \langle \varphi, y \rangle - iq \lambda^{-1/2} \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle \} \mathbf{m}(dx). \end{aligned}$$

On the other hand, using Hölder's inequality, we get that for $\lambda > 0$,

$$\begin{aligned} & \int_{C_0[0, T]} |F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \exp \{ -iq \langle \varphi, y \rangle - iq \lambda^{-1/2} \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle \} \\ & \quad - F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \exp \{ \lambda \langle \varphi, y \rangle + \lambda^{1/2} \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle \}| \mathbf{m}(dx) \\ &= \int_{C_0[0, T]} |F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot))| |\exp \{ -iq \langle \varphi, y \rangle - iq \lambda^{-1/2} \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle \}| \\ & \quad \times |1 - \exp \{ (iq + \lambda) \langle \varphi, y \rangle + (iq \lambda^{-1/2} + \lambda^{1/2}) \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle \}| \mathbf{m}(dx) \\ &= \int_{C_0[0, T]} |F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot))| \\ & \quad \times |1 - \exp \{ (iq + \lambda) \langle \varphi, y \rangle + (iq \lambda^{-1/2} + \lambda^{1/2}) \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle \}| \mathbf{m}(dx) \\ &\leq \left(\int_{C_0[0, T]} |F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot))|^p \mathbf{m}(dx) \right)^{1/p} \\ & \quad \times \left(\int_{C_0[0, T]} |1 - \exp \{ (iq + \lambda) \langle \varphi, y \rangle \right. \\ & \quad \left. + (iq \lambda^{-1/2} + \lambda^{1/2}) \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle \}|^{p'} \mathbf{m}(dx) \right)^{1/p'}. \end{aligned}$$

Note that each factor in the last expression has a limit as $\lambda \rightarrow -iq$ through \mathbb{C}_+ , and that

$$\left(\int_{C_0[0,T]} |1 - \exp\{(iq + \lambda)\langle \varphi, y \rangle + (iq\lambda^{-1/2} + \lambda^{1/2})\langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle\}|^{p'} \mathbf{m}(dx) \right)^{1/p'} \rightarrow 0$$

as $\lambda \rightarrow -iq$ through \mathbb{C}_+ . Hence we have that

$$\begin{aligned} & T_{q,h_1}^{(p)}(F)(y + \mathcal{Z}_{h_1}(w_\varphi, \cdot)) \\ &= \text{l. i. m.}_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} T_{\lambda,h_1}(F)(y + \mathcal{Z}_{h_1}(w_\varphi, \cdot)) \\ &= \text{l. i. m.}_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \exp \left\{ -\frac{\lambda}{2} \|\varphi h_1\|_2^2 - \lambda \langle \varphi, y \rangle \right\} \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \\ &\quad \times \exp\{\lambda \langle \varphi, y \rangle + \lambda^{1/2} \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle\} \mathbf{m}(dx) \\ &= \exp \left\{ \frac{iq}{2} \|\varphi h_1\|_2^2 + iq \langle \varphi, y \rangle \right\} \text{l. i. m.}_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \\ &\quad \times \exp\{\lambda \langle \varphi, y \rangle + \lambda^{1/2} \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle\} \mathbf{m}(dx) \\ &= \exp \left\{ \frac{iq}{2} \|\varphi h_1\|_2^2 + iq \langle \varphi, y \rangle \right\} \text{l. i. m.}_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot)) \\ &\quad \times \exp\{-iq \langle \varphi, y \rangle - iq \lambda^{-1/2} \langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle\} \mathbf{m}(dx) \\ &= \exp \left\{ \frac{iq}{2} \|\varphi h_1\|_2^2 + iq \langle \varphi, y \rangle \right\} \text{l. i. m.}_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} T_{\lambda,h_1}(FR_{q,\varphi})(y) \\ &= \exp \left\{ \frac{iq}{2} \|\varphi h_1\|_2^2 + iq \langle \varphi, y \rangle \right\} T_{q,h_1}^{(p)}(FR_{q,\varphi})(y) \end{aligned}$$

as desired. \square

Using (4.10) with $p = 1$ and $y = 0$, and (3.5), we have the following corollary. But equation (4.11) below also can be obtained from (4.9) with h_2 replaced with h_1 .

Corollary 4.4. *Let h_1 , F , φ and $w_{\varphi h_1}$ be as in Theorem 4.1. Assume that given a non-zero real q , the \mathcal{Z}_{h_1} -Feynman integral with parameter q of F ,*

$$\int_{C_0[0,T]}^{\text{anf}_q} F(\mathcal{Z}_{h_1}(x, \cdot)) \mathbf{m}(dx)$$

exists. Then

$$(4.11) \quad \int_{C_0[0,T]}^{\text{anf}_q} F(\mathcal{Z}_{h_1}(x, \cdot) + \mathcal{Z}_{h_1}(w_{\varphi h_1}, \cdot)) \mathbf{m}(dx)$$

$$= \exp \left\{ \frac{iq}{2} \|\varphi h_1\|_2^2 \right\} \int_{C_0[0,T]}^{\text{anf}_q} F(\mathcal{Z}_{h_1}(x, \cdot)) \exp\{-iq\langle \varphi, \mathcal{Z}_{h_1}(x, \cdot) \rangle\} \mathbf{m}(dx).$$

The Gaussian process given by (2.1) with $h_1 \equiv 1$ is an ordinary Wiener process. Thus we have the following translation theorem for the analytic Feynman integral. This result subsumes a similar result obtained by Cameron and Storvick in [4].

Corollary 4.5. *Setting $h_1 \equiv 1$ in Corollary 4.4 yields the formula*

$$\begin{aligned} & \int_{C_0[0,T]}^{\text{anf}_q} F(x + w_\varphi) \mathbf{m}(dx) \\ (4.12) \quad & \equiv \int_{C_0[0,T]}^{\text{anf}_q} F(\mathcal{Z}_1(x, \cdot) + \mathcal{Z}_1(w_\varphi, \cdot)) \mathbf{m}(dx) \\ & = \exp \left\{ \frac{iq}{2} \|\varphi\|_2^2 \right\} \int_{C_0[0,T]}^{\text{anf}_q} F(x) \exp\{-iq\langle \varphi, x \rangle\} \mathbf{m}(dx) \end{aligned}$$

for all real $q \in \mathbb{R} \setminus \{0\}$, where $w_\varphi(t) = \int_0^t \varphi(s) ds$.

5. Functionals on $C_0[0, T]$

In this section, we give various corollaries which show that Theorems 4.1 and 4.3, giving the translation theorems for the \mathcal{Z}_h -GFFT, are indeed very general theorem since the translation theorems hold for functionals F in large classes. Below we list results of two types.

5.1. Banach algebra \mathcal{S}

We will see that the translation formulas (4.2) and (4.10) hold for the \mathcal{Z}_h -GFFT of functionals in the Banach algebra \mathcal{S} introduced in [6]. The Banach algebra \mathcal{S} consists of functionals expressible in the form

$$(5.1) \quad F(x) = \int_{L_2[0,T]} \exp\{i\langle v, x \rangle\} df(v)$$

for s-a.e. $x \in C_0[0, T]$ where f is an element of $\mathcal{M}(L_2[0, T])$, the space of \mathbb{C} -valued, countably additive (and hence finite) Borel measures on $L_2[0, T]$. Further work on \mathcal{S} , see [7, 8, 11, 18, 24].

Corollary 5.1. *Let $F \in \mathcal{S}$ be given by (5.1), and given $h_1 \in L_\infty[0, T]$ and $\varphi \in L_2[0, T]$, let $w_{\varphi h_1}$ be given by (4.1). Then for all non-zero real q and each $h_2 \in L_\infty[0, T]$, equations (4.2) and (4.9) above hold.*

Proof. This corollary follows from Theorem 4.1 above since, by [18, Theorem 3.1], the \mathcal{Z}_h -GFFT $T_{q,h}^{(1)}(F)$ exists for all $q \in \mathbb{R} \setminus \{0\}$ and each $h \in L_\infty[0, T]$. \square

Corollary 5.2. *Let F, h_1, φ , and $w_{\varphi h_1}$ be as in Corollary 5.1. Then equations (4.10) and (4.11) above hold for all non-zero real q . In particular, setting $h_1 \equiv 1$ in equation (4.11) with F in \mathcal{S} , it follows equation (4.12).*

Remark 5.3. In [7, 8], the authors established equations (4.12) and (4.11) for the functionals F in the Banach algebra \mathcal{S} using the concept of the Radon–Nikodym derivative and a direct calculation.

5.2. Cylinder functionals

Functionals that involve PWZ stochastic integrals are quite common. In this subsection we apply our results to the cylinder functionals F on $C_0[0, T]$ given by

$$(5.2) \quad F(x) = f(\langle g_1, x \rangle, \dots, \langle g_n, x \rangle),$$

where $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is a Lebesgue measurable function and $\{g_1, \dots, g_n\}$ is an independent set of functions in $L_2[0, T]$. It is well-known [12] that the functional F given by (5.2) is Wiener measurable if and only if f is Lebesgue measurable.

Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ is an orthogonal set of non-zero functions in $L_2[0, T]$, each of whose elements is of bounded variation on $[0, T]$. For $p \in [1, \infty)$, let $\mathcal{B}_{\mathcal{A}}^{(p)}$ be the space of all functionals on $C_0[0, T]$ of the form

$$(5.3) \quad F(x) = f(\langle \alpha_1, x \rangle, \dots, \langle \alpha_n, x \rangle)$$

for s-a.e. $x \in C_0[0, T]$ where f is in $L_p(\mathbb{R}^n)$. Let $\mathcal{B}_{\mathcal{A}}^{(\infty)}$ be the space of all functionals having the form (5.3) with f in $C_0(\mathbb{R}^n)$, the space of bounded continuous functions on \mathbb{R}^n that vanish at infinity. It is quite easy to see that if F is in $\mathcal{B}_{\mathcal{A}}^{(p)}$, then F is scale-invariant measurable.

Given an orthogonal set $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ of non-zero functions of bounded variation on $[0, T]$, let $\mathcal{O}_2(\mathcal{A})$ be the class of all functions h in $L_2[0, T]$ such that $\mathcal{A}h = \{\alpha_1 h, \dots, \alpha_n h\}$ is orthonormal in $L_2[0, T]$.

In [10], the authors defined the class $\mathcal{B}_{\mathcal{A}}^{(p)}$, $1 \leq p \leq +\infty$, for the case that \mathcal{A} is simply an orthogonal set of non-zero functions in $L_2[0, T]$. Consequently, in order to ensure the existence of the \mathcal{Z}_h -GFFT of functionals F in $\mathcal{B}_{\mathcal{A}}^{(p)}$, the functions h was taken to be in $\mathcal{O}_2(\mathcal{A}) \cap L_{\infty}[0, T]$. However, using the techniques similar to those used in [10], we can obtain the following theorem because each of the PWZ stochastic integrals $\langle \alpha_j, \mathcal{Z}_h(x, \cdot) \rangle \equiv \langle \alpha_j h, x \rangle$ always exists in our setting for \mathcal{A} and h .

Theorem 5.4. *Let $p \in [1, 2]$ and let $F \in \mathcal{B}_{\mathcal{A}}^{(p)}$ be given by equation (5.3). Then for all non-zero real number q and all $h \in \mathcal{O}_2(\mathcal{A})$, the L_p analytic \mathcal{Z}_h -GFFT, $T_{q,h}^{(p)}(F)(y)$ exists as an element of $\mathcal{B}_{\mathcal{A}}^{(p)}$ and is given by the formula*

$$T_{q,h}^{(p)}(F)(y) = (\Psi_{-iq} f)(\langle \alpha_1, y \rangle, \dots, \langle \alpha_n, y \rangle)$$

for s-a.e. $y \in C_0[0, T]$, where

$$(\Psi_{-iq} f)(\xi_1, \dots, \xi_n) = \left(\frac{-iq}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp \left\{ \frac{iq}{2} \sum_{j=1}^n (u_j - \xi_j)^2 \right\} d\vec{u}.$$

Corollary 5.5. *Let p and F be as in Theorem 5.4, and given $\varphi \in L_\infty[0, T]$ and $h_1 \in \mathcal{O}_2(\mathcal{A})$, let $w_{\varphi h_1}$ be given by (4.1). Then for all $h_2 \in L_2[0, T]$, equations (4.2), (4.9) and (4.10) above hold for all non-zero real q .*

Proof. This corollary follows from Theorems 4.1 and 4.3 since, by [10, Theorems 4.7 and 4.8], $T_{q,h}^{(p)}(F)$ of F in $\mathcal{B}_A^{(p)}$ exist for all non-zero real q , all $h \in L_2[0, T]$ and all $p \in [1, 2]$. \square

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