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AN ARTINIAN POINT-CONFIGURATION QUOTIENT AND THE STRONG LEFSCHETZ PROPERTY

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ABSTRACT. In this paper, we study an Artinian point-configuration quotient having the SLP. We show that an Artinian quotient of points in \mathbb{P}^n has the SLP when the union of two sets of points has a specific Hilbert function. As an application, we prove that an Artinian linear star configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP if \mathbb{X} and \mathbb{Y} are linear starconfigurations in \mathbb{P}^2 of type s and t for $s \ge {t \choose 2} - 1$ and $t \ge 3$. We also show that an Artinian k-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP if \mathbb{X} is a k-configuration of type (1, 2) or (1, 2, 3) in \mathbb{P}^2 , and $\mathbb{X} \cup \mathbb{Y}$ is a basic configuration in \mathbb{P}^2 .

1. Introduction

Ideals of sets of finite points in \mathbb{P}^n have been studied for a long time ([8,9,11]), and in particular we consider an ideal of a special configuration in \mathbb{P}^n , so called a *star-configuration* and a k-*configuration* in \mathbb{P}^n ([1–3, 6, 7, 9–11, 15]). In 2006, Geramita, Migliore, and Sabourin introduced the notion of a star-configuration set of points in \mathbb{P}^2 (see [10]), the name having been inspired by the fact that 10-points in \mathbb{P}^2 , defined by 5 general linear forms in $\Bbbk[x_0, x_1, x_2]$ resembles a star. In this paper, we refer to this as a "linear star-configuration", as more general definition of star-configurations has evolved through the subsequent literature (see [1, 6, 7, 19]). Indeed, a star-configuration in \mathbb{P}^n has been studied to find the dimension of secant varieties to the variety of reducible forms in $R = \Bbbk[x_0, x_1, \ldots, x_n]$, where \Bbbk is a field of characteristic 0 (see [4, 5, 20]).

If R/I is a standard graded Artinian algebra and ℓ is a general linear form, we recall that R/I is said to have the weak Lefschetz property (WLP) if the

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multiplication map by ℓ

$$[R/I]_d \xrightarrow{\times \ell} [R/I]_{d+1}$$

has maximal rank for every $d \ge 0$. Over the years, there have been several papers which have devoted to a classification of possible Artinian quotients having the WLP (see [1,8,9,13,14,16–18,21,22]). The strong Lefschetz property (SLP) says that for every $i \ge 1$ the multiplication map by ℓ^i

$$[R/I]_d \stackrel{\times \ell^i}{\to} [R/I]_{d+i}$$

has maximal rank for every $d \ge 0$ ([13, 14, 17]). In [14] the authors proved that a complete intersection ideal in $\Bbbk[x_0, x_1]$ has the SLP. Moreover, in [13], the authors give a nice description for a graded Artinian ring having the SLP by using the so-called *Jordan type* (see Lemma 2.2). The *Jordan type* is the partition of *n* specifying the lengths of blocks in the Jordan block matrix determined by the multiplication map by ℓ in a suitable \Bbbk -basis for R/I. Here, we apply this result often to show that some Artinian quotients of the ideals of points in \mathbb{P}^n have the SLP.

We use Hilbert functions for many our arguments. Given a homogeneous ideal $I \subset R$, the Hilbert function of R/I, denoted $\mathbf{H}_{R/I}$, is the numerical function $\mathbf{H}_{R/I} : \mathbb{Z}^+ \cup \{0\} \to \mathbb{Z}^+ \cup \{0\}$ defined by

$$\mathbf{H}_{R/I}(i) := \dim_{\mathbb{K}} [R/I]_i = \dim_{\mathbb{K}} [R]_i - \dim_{\mathbb{K}} [I]_i,$$

where $[R]_i$ and $[I]_i$ denote the *i*-th graded component of R and I, respectively. If $I := I_X$ is the defining ideal of a subscheme X in \mathbb{P}^n , then we denote

$$\mathbf{H}_{R/I_{\mathbb{X}}}(i) := \mathbf{H}_{\mathbb{X}}(i) \quad \text{for} \quad i \ge 0,$$

and call it the *Hilbert function* of X.

Let $R = \Bbbk[x_0, x_1, \ldots, x_n]$ be a polynomial ring over a field \Bbbk of characteristic 0. For positive integers r and s with $1 \leq r \leq \min\{n, s\}$, suppose F_1, \ldots, F_s are general forms in R of degrees d_1, \ldots, d_s , respectively. Here s general forms F_1, \ldots, F_s in R means that all subsets of size $1 \leq r \leq \min\{n+1, s\}$ are regular sequences in R, and if $\mathcal{H} = \{F_1, \ldots, F_s\}$ is a collection of distinct hypersurfaces in \mathbb{P}^n corresponding to general F_1, \ldots, F_s respectively, then the hypersurfaces meet properly, by which we mean that the intersection of any rof these hypersurfaces with $1 \leq r \leq \min\{n, s\}$ has codimesion r. We call the variety \mathbb{X} defined by the ideal

$$\bigcap_{1 \le i_1 < \dots < i_r \le s} (F_{i_1}, \dots, F_{i_r})$$

a star-configuration in \mathbb{P}^n of type (r, s). In particular, if X is a star-configuration in \mathbb{P}^n of type (n, s), then we simply call a point star-configuration in \mathbb{P}^n of type s for short. Notice that each *n*-forms F_{i_1}, \ldots, F_{i_n} of *s*-general forms F_1, \ldots, F_s in *R* define $d_{i_1} \cdots d_{i_n}$ points in \mathbb{P}^n for each $1 \leq i_1 < \cdots < i_n \leq s$. Thus the ideal

$$\bigcap_{1 \le i_1 < \dots < i_n \le s} (F_{i_1}, \dots, F_{i_n})$$

defines a finite set \mathbb{X} of points in \mathbb{P}^n with

$$\deg(\mathbb{X}) = \sum_{1 \le i_1 < i_2 < \dots < i_n \le s} d_{i_1} d_{i_2} \cdots d_{i_n}.$$

Furthermore, if F_1, \ldots, F_s are general linear (quadratic, cubic, quartic, quintic, etc) forms in R, then we call \mathbb{X} a linear (quadratic, cubic, quartic, quintic, etc) star-configuration in \mathbb{P}^n of type s, respectively.

To provide some additional focus to this paper, we consider the following questions.

Question 1.1. Let X and Y be finite sets of points in \mathbb{P}^n and $R = \Bbbk[x_0, x_1, \ldots, x_n]$.

- (a) Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the WLP?
- (b) Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the SLP?

Question 1.2. More precisely, let \mathbb{X} and \mathbb{Y} be finite point star configurations in \mathbb{P}^n , or \mathbb{X} be a k-configuration in \mathbb{P}^n such that $\mathbb{X} \cup \mathbb{Y}$ is a basic configuration in \mathbb{P}^n .

- (a) Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the WLP?
- (b) Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the SLP?

In [1], the authors proved that an Artinian linear star-configuration quotient in \mathbb{P}^2 has the WLP, which is a partial answer to Question 1.2 (a). Indeed, it is still true that any finite number of an Artinian linear point star-configuration quotient in \mathbb{P}^n has the WLP. In [8,9], the authors show that Question 1.2(a) is true in general if X is a k-configuration in \mathbb{P}^n and $\mathbb{X} \cup \mathbb{Y}$ is a basic configuration in \mathbb{P}^n with the condition $2\sigma(\mathbb{X}) \leq \sigma(\mathbb{X} \cup \mathbb{Y})$, where

$$\sigma(\mathbb{X}) = \min\{i \mid \mathbf{H}_{\mathbb{X}}(i-1) = \mathbf{H}_{\mathbb{X}}(i)\}.$$

In this paper, we focus on Questions 1.1(b) and 1.2(b). More precisely, we first find a condition in which an Artinian quotient of two sets of points in \mathbb{P}^n has the SLP (see Lemma 2.4 and Proposition 2.5). Next we find some Artinian linear star configuration quotient in \mathbb{P}^2 that has the SLP (see Corollary 2.9). Then, we find an Artinian k-configuration quotient having the SLP (see Proposition 3.4 and Theorem 3.6). Unfortunately, we do not have any counter example of an Artinian quotient $R/(I_X + I_Y)$ of two point sets in \mathbb{P}^n , which does not have the SLP, and thus we expect Question 1.1(a) and (b) are true in general, especially when X and Y are sets of general points in \mathbb{P}^n .

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2. Artinian linear star-configuration quotients in \mathbb{P}^2

In this section, we shall show that an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP if \mathbb{X} and \mathbb{Y} are linear star-configurations in \mathbb{P}^2 of type s and t with $s \ge {t \choose 2} - 1$ and $t \ge 3$, respectively.

We first introduce the following two results of a star-configuration in \mathbb{P}^n in [13,22].

Remark 2.1. Let k be a field of characteristic zero and let $F \in \mathbb{k}[x_0, x_1, \dots, x_n]$ = $R = \bigoplus_{i \ge 0} R_i \ (n \ge 1)$ be a homogeneous polynomial (form) of degree d, i.e., $F \in R_d$. It is well known that in this case each R_i has a basis consisting of *i*-th powers of linear forms. Thus we may write

$$F = \sum_{i=1}^{\prime} \alpha_i L_i^d, \qquad \alpha_i \in \mathbb{k}, \ L_i \in R_1.$$

If k is algebraically closed (which we now assume for the rest of the paper), then each $\alpha_i = \beta_i^d$ for some $\beta_i \in k$ and so we can write

(2.1)
$$F = \sum_{i=1}^{r} (\beta_i L_i)^d = \sum_{i=1}^{r} M_i^d, \quad M_i \in R_1.$$

We call a description of F as in equation (2.1), a Waring Decomposition of F. The least integer r such that F has a Waring Decomposition with exactly r summands is called the Waring Rank (or simply the rank) of F.

Lemma 2.2 ([13]). Assume A is graded and \mathbf{H}_A is unimodal. Then

- (a) A has the WLP if and only if the number of parts of the Jordan type $J_{\ell} = \max{\{\mathbf{H}_A(i)\}}$. (The Sperner number of A);
- (b) ℓ is a strong Lefschetz element of A if and only if $J_{\ell} = \mathbf{H}_{A}^{\vee}$.

Proposition 2.3 ([22, Proposition 2.5]). Let X and Y be linear star-configurations in \mathbb{P}^2 of type s and t, respectively, with $3 \leq t$ and $s \geq \lfloor \frac{1}{2} \binom{t}{2} \rfloor$. Then $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function.

Recall that

 \mathbf{H}_A : h_0 h_1 \cdots h_c

is said to be unimodal if there exists j such that

$$\begin{cases} h_i \le h_{i+1} & (i < j), \\ h_i \ge h_{i+1} & (j \le i). \end{cases}$$

Lemma 2.4. Let \mathbb{X} be a finite set of points in \mathbb{P}^n and let A be an Artinian quotient of the coordinate ring of \mathbb{X} . Assume that $\mathbf{H}_A(i) = \mathbf{H}_{\mathbb{X}}(i)$ for every $0 \le i \le s - 1$ and $A_s = 0$. Then an Artinian ring A has the SLP.

Proof. First, we assume that the Hilbert function of A is of the form

 $\mathbf{H}_A : h_0 \quad h_1 \quad \cdots \quad h_{\sigma-1} \quad h_\sigma \quad \cdots \quad h_{s-1} \quad 0,$

where $h_{\sigma-2} < h_{\sigma-1} = h_{\sigma} = \cdots = h_{s-1}$.

Let ℓ be a general linear form in A_1 . Since ℓ is not a zero divisor of A, we see that the multiplication map by ℓ^{s-1}

$$[R/I_{\mathbb{X}}]_0 = [A]_0 \xrightarrow{\times \ell^{s-1}} [A]_{s-1} = [R/I_{\mathbb{X}}]_{s-1}$$

is injective. Hence we have a string of length s

$$1, \ell, \ldots, \ell^{s-1},$$

and so the Jordan type J_{ℓ} for \mathbf{H}_A is of the form

$$J_{\ell} = (s, \dots).$$

(i) Let i = 1. Then the multiplication map by ℓ^{s-2}

$$[R/I_{\mathbb{X}}]_1 = [A]_1 \xrightarrow{\times \ell^{s-2}} [A]_{s-1} = [R/I_{\mathbb{X}}]_{s-1}$$

is injective. Hence there are $g_1 := (h_1 - h_0) = (h_1 - 1)$ linear forms $F_{1,1}, F_{1,2}, \ldots, F_{1,g_1} \in [A]_1$ such that the h_1 linear forms

$$, F_{1,1}, F_{1,2}, \ldots, F_{1,g_1}$$

are linearly independent. Hence there are g_1 -strings of length (s-1)

$$F_{1,1}, F_{1,1}\ell, \dots, F_{1,1}\ell^{s-2}, \text{ and} F_{1,2}, F_{1,2}\ell, \dots, F_{1,2}\ell^{s-2}, \\ \vdots \\ F_{1,g_1}, F_{1,g_1}\ell, \dots, F_{1,g_1}\ell^{s-2}.$$

(ii) For $1 \le i < \sigma - 1$ and $1 \le j \le i$, define

$$g_j := h_j - h_{j-1}$$

for such j. Assume that there are g_j -forms $F_{j,1}, \ldots, F_{j,g_j} \in [A]_j$ and there are g_j -strings of length (s-j)

such that the $(1 + \sum_{k=1}^{j} g_k)$ -forms

$$\ell^{j}, \underbrace{F_{1,1}\ell^{j-1}, \dots, F_{1,g_{1}}\ell^{j-1}}_{g_{1}\text{-forms}}, \dots, \underbrace{F_{j-1,1}\ell, \dots, F_{j-1,g_{j-1}}\ell}_{g_{j-1}\text{-forms}}, \underbrace{F_{j,1}, \dots, F_{j,g_{j}}}_{g_{j}\text{-forms}}$$

are linearly independent for such j. Since the multiplication map by $\ell^{(s-1)-(i+1)}$

$$[R/I_{\mathbb{X}}]_{i+1} = [A]_{i+1} \xrightarrow{\times \ell^{(s-1)-(i+1)}} [A]_{s-1} = [R/I_{\mathbb{X}}]_{s-1}$$

is injective, there are linearly independent $g_{i+1} := (h_{i+1} - h_i)$ -forms $F_{i+1,1}, \ldots, F_{i+1,g_{i+1}} \in [A]_{i+1}$. Then the following $(1 + \sum_{k=1}^{i+1} g_k)$ -forms

$$\underbrace{F_{i+1}, F_{1,1}\ell^i, \dots, F_{1,g_1}\ell^i}_{g_1 \text{-forms}}, \dots, \underbrace{F_{i-1,1}\ell^2, \dots, F_{i-1,g_{i-1}}\ell^2}_{g_{i-1} \text{-forms}}, \underbrace{F_{i,1}\ell, \dots, F_{i,g_i}\ell}_{g_i \text{-forms}}, \underbrace{F_{i+1,1}, \dots, F_{i+1,g_{i+1}}}_{g_{i+1} \text{-forms}}$$

are linearly independent as well. Hence we have g_{i+1} -strings of length (s-i-1)

$$F_{i+1,1}, F_{i+1,1}\ell, \dots, F_{i+1,1}\ell^{s-i-2}, F_{i+1,2}, F_{i+1,2}\ell, \dots, F_{i+1,2}\ell^{s-i-2}, \\\vdots \\ +1, g_{i+1}, F_{i+1, g_{i+1}}\ell, \dots, F_{i+1, g_{i+1}2}\ell^{s-i-2}$$

It is from (i) \sim (ii) that the Jordan type

 F_i

$$J_{\ell} = (s, \underbrace{s-1, \dots, s-1}_{g_1 \text{-times}}, \dots, \underbrace{s-i, \dots, s-i}_{g_i \text{-times}}, \dots, \underbrace{s-\sigma+1, \dots, s-\sigma+1}_{g_{\sigma-1} \text{-times}}) = \mathbf{H}_A^{\vee}$$

as we wished. Therefore, by Lemma 2.2, an Artinian ring has the SLP, which completes the proof. $\hfill \Box$

The following proposition is immediate from Lemma 2.4.

Proposition 2.5. Let \mathbb{X} and \mathbb{Y} be linear star-configurations in \mathbb{P}^2 of type t and s with $t \geq 2$ and $s \geq \binom{t}{2}$. Then an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

Proof. First, note that the Hilbert functions of $R/I_{\mathbb{X}}$, $R/I_{\mathbb{Y}}$, and $R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}})$ (see Proposition 2.3) are

respectively. Using the exact sequence

$$0 \to R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}}) \to R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \to R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \to 0_{\mathbb{Y}}$$

the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} : 1 \quad 3 \quad \cdots \quad \begin{pmatrix} t-2 \end{pmatrix} - \mathrm{nd} \qquad \begin{pmatrix} s-2 \end{pmatrix} - \mathrm{nd} \qquad \begin{pmatrix} t-2 \end{pmatrix} - \mathrm{nd} \qquad \begin{pmatrix} t$$

and so by Lemma 2.4, an Artinian linear star configuration quotient $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ has the SLP, which completes the proof.

Example 2.6. Let X and Y be linear star-configurations in \mathbb{P}^2 of type 5 and 9, respectively. Note that $9 = \binom{5}{2} - 1$. By Proposition 2.3 the Hilbert function of an Artinian ring $A := R/(I_X + I_Y)$ is

$$(1, 3, 6, 10, 10, 10, 10, 10, 10, 1)$$
.

8-th

 $\ell^8 \in [A]_8,$

i.e., we have a string of length 9

 $1, \ell, \ldots, \ell^8.$

Hence the Jordan type J_{ℓ} is of the form

$$J_{\ell} = (9, \dots).$$

(b) Note that the multiplication map by ℓ^6

$$[A]_1 \stackrel{\times \ell^{\circ}}{\to} [A]_7.$$

is injective, and the multiplication map by ℓ^7

$$[A]_1 \stackrel{\times \ell'}{\to} [A]_8$$

is surjective. Then we can choose a basis $\{\ell, F_{1,1}, F_{1,2}\}$ for $[A]_1$ such that

$$F_{1,1}\ell^6, F_{1,2}\ell^6 \neq 0$$
, and $F_{1,1}\ell^7, F_{1,2}\ell^7 = 0$.

Moreover, since $\{F_{1,1}\ell^6,F_{1,2}\ell^6\}$ is linearly independent, we have 2-strings of length 7

$$F_{1,1}, F_{1,1}\ell, \dots, F_{1,1}\ell^6$$
, and
 $F_{1,2}, F_{1,2}\ell, \dots, F_{1,2}\ell^6$.

(c) Note that the multiplication map by ℓ^5

$$A]_2 \stackrel{\times \ell^\circ}{\to} [A]_7$$

is injective, and the multiplication map by ℓ^6

$$[A]_2 \stackrel{\times \ell^6}{\to} [A]_8$$

is surjective. Then we can choose a basis $\{\ell^2, F_{1,1}\ell, F_{1,2}\ell, F_{2,1}, F_{2,2}, F_{2,3}\}$ for $[A]_2$ such that

$$F_{2,1}\ell^5, F_{2,2}\ell^5, F_{2,3}\ell^5 \neq 0$$
, and $F_{2,1}\ell^6, F_{2,2}\ell^6, F_{2,3}\ell^6 = 0$.

Moreover, since $\{F_{2,1}\ell^5,F_{2,2}\ell^5,F_{2,3}\ell^5\}$ is linearly independent, we have 3-strings of length 6

$$\begin{array}{l} F_{2,1}, F_{2,1}\ell, \dots, F_{2,1}\ell^5, \\ F_{2,2}, F_{2,2}\ell, \dots, F_{2,2}\ell^5, \\ F_{2,3}, F_{2,3}\ell, \dots, F_{2,3}\ell^5. \end{array}$$
 and

(d) Note that the multiplication map by ℓ^4

$$[A]_3 \stackrel{\times \ell^4}{\to} [A]_7$$

is injective, and the multiplication map by ℓ^6

$$[A]_3 \stackrel{\times \ell^*}{\to} [A]_8$$

is surjective. Then we can choose a basis $\{\ell^3, F_{1,1}\ell^2, F_{1,2}\ell^2, F_{2,1}\ell, F_{2,2}\ell, F_{2,3}\ell, F_{3,1}, \dots, F_{3,4}\}$ for $[A]_3$ such that

$$F_{3,1}\ell^4, \dots, F_{3,4}\ell^4 \neq 0$$
, and $F_{3,1}\ell^5, \dots, F_{3,4}\ell^5 = 0$.

Moreover, since $\{F_{3,1}\ell^4, \ldots, F_{3,4}\ell^4\}$ is linearly independent, we have 4-strings of length 5

$$\begin{array}{l} F_{3,1},F_{3,1}\ell,\ldots,F_{3,1}\ell^4,\\ F_{3,2},F_{3,2}\ell,\ldots,F_{3,2}\ell^4,\\ F_{3,3},F_{3,3}\ell,\ldots,F_{3,3}\ell^4, \quad \text{and}\\ F_{3,4},F_{3,4}\ell,\ldots,F_{3,4}\ell^4. \end{array}$$

This shows that the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is

$$J_{\ell} = (9, 7, 7, 6, 6, 6, 5, 5, 5, 5) = \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{X}})}^{\vee}$$

Thus, by Lemma 2.2, an Artinian quotient of two linear star-configurations in \mathbb{P}^2 of type 5 and 9 has the SLP, as we wished.

Example 2.6 motivates the following proposition.

Proposition 2.7. Let \mathbb{X} be a finite set of points in \mathbb{P}^n and let A be an Artinian quotient of the coordinate ring of \mathbb{X} . Assume that $\mathbf{H}_A(i) = \mathbf{H}_{\mathbb{X}}(i)$ for every $0 \le i \le s - 2$ with $A_s = 0$, and the Hilbert function of A is of the form

$$\mathbf{H}_A$$
 : h_0 h_1 \cdots $h_{\sigma-1}$ h_{σ} \cdots $\begin{pmatrix} (s-2)-nd \\ h_{\sigma} \end{pmatrix}$ h_{s-1} (b_s)

where $h_{\sigma-2} < h_{\sigma-1} = h_{\sigma}$ and $h_{s-1} = 1$. Then an Artinian ring A has the SLP.

Proof. We first define

$$q_i := h_i - h_{i-1}$$
 for $i = 1, \dots, \sigma - 1$.

(a) By Waring decomposition, there is a linear form $\ell \in [A]_1$ such that

 $\ell^{s-1} \in [A]_{s-1}.$

In other words, there is a string of length s as

$$1, \ell, \ldots, \ell^{s-1}.$$

Hence Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{\ell} = (s, \dots).$$

(b) Note that the multiplication map by ℓ^{s-3}

$$[R/I_{\mathbb{X}}]_1 = [A]_1 \xrightarrow{\times \ell^{s-s}} [A]_{s-2} = [R/I_{\mathbb{X}}]_{s-2}$$

is injective, and the multiplication map by ℓ^{s-2}

$$[A]_1 \stackrel{\times \ell^{s-2}}{\to} [A]_{s-1}$$

is surjective. Then we can choose a basis $\{\ell,F_{1,1},F_{1,2},\ldots,F_{1,g_1}\}$ for $[A]_1$ such that

$$F_{1,1}\ell^{s-3}, F_{1,2}\ell^{s-3}, \dots, F_{1,g_1}\ell^{s-3} \neq 0$$
, and $F_{1,1}\ell^{s-2}, F_{1,2}\ell^{s-2}, \dots, F_{1,g_1}\ell^{s-2} = 0$.

Moreover, since $\{F_{1,1}\ell^{s-3}, F_{1,2}\ell^{s-3}, \ldots, F_{1,g_1}\ell^{s-3}\}$ is linearly independent, we have g_1 -strings of length (s-2)

$$F_{1,1}, F_{1,1}\ell, \dots, F_{1,1}\ell^{s-3}, F_{1,2}, F_{1,2}\ell, \dots, F_{1,2}\ell^{s-3}, \vdots F_{1,g_1-1}, F_{1,g_1-1}\ell, \dots, F_{1,g_1-1}\ell^{s-3}, and F_{1,g_1}, F_{1,g_1}\ell, \dots, F_{1,g_1}\ell^{s-3}.$$

This means that Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{\ell} = (s, \underbrace{s-2, \dots, s-2}_{q_1 \text{-times}}, \dots)$$

(c) Let $1 \leq i \leq \sigma - 1$. Note that the multiplication map by ℓ^{s-i-2}

$$[R/I_{\mathbb{X}}]_i = [A]_i \xrightarrow{\times \ell^{s-i-2}} [A]_{s-2} = [R/I_{\mathbb{X}}]_{s-2}$$

is injective, and the multiplication map by ℓ^{s-i-1}

$$[R/I_{\mathbb{X}}]_i = [A]_i \xrightarrow{\times \ell^{s-i-1}} [A]_{s-1}$$

is surjective. Then we can choose a basis \mathcal{B}_i

$$\mathcal{B}_{i} = \{\ell^{i}, \underbrace{F_{1,1}\ell^{i-1}, \dots, F_{1,g_{1}}\ell^{i-1}}_{g_{i}-\text{times}}, \underbrace{F_{2,1}\ell^{i-2}, \dots, F_{2,g_{2}}\ell^{i-2}}_{g_{i}-\text{times}}, \dots, \underbrace{F_{i-1,1}\ell, \dots, F_{i-1,g_{i-1}}\ell}_{g_{i}-\text{times}}, \underbrace{F_{i,1}, \dots, F_{i,g_{i}}}_{g_{i}-\text{times}}\}$$

for $[A]_i$ such that

 $F_{i,1}\ell^{s-i-2}, \ldots, F_{i,g_i}\ell^{s-i-2} \neq 0$, and $F_{i,1}\ell^{s-i-1}, \ldots, F_{i,g_i}\ell^{s-i-1} = 0$. Moreover, since $\{F_{i,1}\ell^{s-i-2}, \ldots, F_{i,g_i}\ell^{s-i-2}\}$ is linearly independent, we have g_i -strings of length (s-i-1)

$$\begin{array}{ccccccc} F_{i,1}, F_{i,1}\ell, & \dots, & F_{i,1}^{s-i-2}, \\ F_{i,2}, F_{i,2}\ell, & \dots, & F_{i,2}^{s-i-2}, \\ & & \vdots \\ F_{i,g_1-1}, F_{i,g_1-1}\ell, & \dots, & F_{i,g_1-1}^{s-i-2}, \\ & & F_{i,g_i}, F_{i,g_i}\ell, & \dots, & F_{i,g_i}\ell^{s-i-2}. \end{array}$$
 and

Hence Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{\ell} = (s, \underbrace{s-2, s-2, \dots, s-2}_{g_1 \text{-times}}, \dots, \underbrace{s-i-1, s-i-1, \dots, s-i-1}_{g_i \text{-times}}, \dots)$$

for such i.

It is from (a) ~ (c) that the Jordan type J_{ℓ} of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is

$$J_{\ell} = \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} \vee$$

$$= (s, \underbrace{s-2, s-2, \dots, s-2}_{g_{1}-\text{times}}, \dots, \underbrace{s-i-1, s-i-1, \dots, s-i-1}_{g_{i}-\text{times}}, \dots, \underbrace{s-\sigma, s-\sigma, \dots, s-\sigma}_{g_{\sigma-1}-\text{times}}).$$

Therefore, by Lemma 2.2, an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP, as we wished.

The following two corollaries are immediate from Proposition 2.7.

Corollary 2.8. Let X and Y be finite sets of general points in \mathbb{P}^n with $n \ge 2$ and $s \ge t \ge n$. Assume that

$$\binom{s}{n} \le \deg(\mathbb{X}) < \binom{s+1}{n}, \quad \binom{t}{n} \le \deg(\mathbb{Y}) < \binom{t+1}{n},$$

and

$$\deg(\mathbb{X}) + \deg(\mathbb{Y}) = \binom{s+1}{n} + 1.$$

Then an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

Proof. Since X and Y are finite sets of general points in \mathbb{P}^n , we get that the Hilbert functions of $R/I_{\mathbb{X}}$, $R/I_{\mathbb{Y}}$, and $R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}})$ are

 $\mathbf{H}_{R/(I_{\mathbb{X}}\cap I_{\mathbb{Y}})} : 1 \quad {\binom{1+n}{n}} \quad \cdots \quad {\binom{t-n}{\cdot}}^{\mathsf{th}} \quad {\binom{t+1}{n}} \quad \cdots \quad {\binom{s-n}{\cdot}}^{\mathsf{th}} \quad {\binom{s+1}{n}} = \left[\deg(\mathbb{X}) + \deg(\mathbb{Y}) \right] - 1 \quad \rightarrow,$ respectively. Using the exact sequence

 $(\cdot \cdot \mathbf{D}^{I} | I \circ \mathbf{L}) \cdot \mathbf{D}^{I} | \cdot \mathbf{D}^{I} | \cdot \mathbf{D}^{I} | I$

$$0 \to R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}}) \to R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \to R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \to 0,$$

the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} : 1 \quad 3 \quad \cdots \quad \begin{pmatrix} t-n \end{pmatrix}^{-\mathrm{th}} \quad \deg(\mathbb{Y}) \quad \cdots \quad \deg(\mathbb{Y}) \quad 1 \quad \rightarrow,$$

and so by Proposition 2.7, an Artinian ring $R/(I_X + I_Y)$ has the SLP, which completes the proof.

Corollary 2.9. Let \mathbb{X} and \mathbb{Y} be linear star-configurations in \mathbb{P}^2 of type s and t with $s \geq \binom{t}{2} - 1$ and $t \geq 3$. Then an Artinian linear star-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

Proof. By Proposition 2.5, it holds for $s \ge {t \choose 2}$. So we assume that $s = {t \choose 2} - 1$. First note that

$$\begin{bmatrix} \deg(\mathbb{X}) + \deg(\mathbb{Y}) \end{bmatrix} - \binom{s+1}{2} = \begin{bmatrix} \binom{s}{2} + \binom{t}{2} \end{bmatrix} - \binom{s+1}{2}$$
$$= \begin{bmatrix} \binom{s}{2} + s+1 \end{bmatrix} - \binom{s+1}{2} = 1.$$

Hence the Hilbert functions of R/I_X , R/I_Y , and $R/(I_X \cap I_Y)$ (see Proposition 2.3) are

respectively. Using the exact sequence

$$0 \to R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}}) \to R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \to R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \to 0,$$

the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} \quad : \quad 1 \quad 3 \quad \cdots \quad \begin{pmatrix} (t-2) \text{-nd} \\ t \\ 2 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} s-2) \text{-nd} \\ t \\ 2 \end{pmatrix} \quad 1 \quad \rightarrow$$

and so by Proposition 2.7, an Artinian linear star-configuration quotient $R/(I_X + I_Y)$ has the SLP, as we wished.

3. Artinian k-configuration quotients in \mathbb{P}^2

In this section, we shall introduce another Artinian quotient having the SLP. We first recall a definition of a k-configuration in \mathbb{P}^2 and some preliminary result.

Definition 3.1. A k-configuration of points in \mathbb{P}^2 is a finite set \mathbb{X} of points in \mathbb{P}^2 which satisfy the following conditions: there exist integers $1 \leq d_1 < \cdots < d_m$, and subsets $\mathbb{X}_1, \ldots, \mathbb{X}_m$ of \mathbb{X} , and distinct lines $\mathbb{L}_1, \ldots, \mathbb{L}_m \subseteq \mathbb{P}^2$ such that

(a) $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{X}_i$,

(b) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subset \mathbb{L}_i$ for each $i = 1, \ldots, m$, and

(c) \mathbb{L}_i $(1 < i \leq m)$ does not contain any points of \mathbb{X}_j for all j < i.

In this case, the k-configuration in \mathbb{P}^2 is said to be of type (d_1, \ldots, d_m) .

Recall that a finite complete intersection set of points \mathbb{Z} in \mathbb{P}^n is said to be a *basic configuration* in \mathbb{P}^n (see [11, 12]) if there exist integers r_1, \ldots, r_n and distinct hyperplanes $\mathbb{L}_{ij}(1 \le i \le n, 1 \le j \le r_i)$ such that

 $\mathbb{Z} = \mathbb{H}_1 \cap \cdots \cap \mathbb{H}_n$ as schemes, where $\mathbb{H}_i = \mathbb{L}_{i1} \cup \cdots \cup \mathbb{L}_{ir_i}$.

In this case \mathbb{Z} is said to be of type (r_1, \ldots, r_n) .

Before we prove our main theorem, we first introduce two lemmas.

Lemma 3.2. Let \mathbb{X} be a k-configuration in \mathbb{P}^2 of type (1, 2, ..., d) (see Figure 1), and let \mathbb{L}_i and \mathbb{M}_j be lines in \mathbb{P}^2 defined by linear forms $x_0 - (i-1)x_2$ and $x_1 - (j-1)x_2$ for $1 \leq i, j \leq d-1$, respectively. Then the multiplication map by $L_1 := x_0$

$$R/I_{\mathbb{X}}]_i \stackrel{\times L_1}{\to} [R/I_{\mathbb{X}}]_{i+1}$$

is injective for $i \ge 0$. In particular, for $j \ge 1$, the multiplication map by L_1^j

$$[R/I_{\mathbb{X}}]_i \stackrel{\times L_1^{\prime}}{\to} [R/I_{\mathbb{X}}]_{i+j}$$

is injective for every $i \ge 0$.



Proof. If d = 1, then X is a set of a single point in \mathbb{P}^2 , so it is immediate. Hence we assume that d > 1.

Note that

 $I_{\mathbb{X}} = (L_1 \cdots L_d, M_1 L_2 \cdots L_d, M_1 M_2 L_3 \cdots L_d, \dots, M_1 \cdots M_{d-1} L_d, M_1 M_2 \cdots M_d)$ (see [9,11]) and the Hilbert function of $R/I_{\mathbb{X}}$ is

$$\mathbf{H}_{\mathbb{X}} : 1 \begin{pmatrix} 1+2\\2 \end{pmatrix} \cdots \begin{pmatrix} (d-1)-\mathrm{st}\\2 \end{pmatrix} \begin{pmatrix} d+1\\2 \end{pmatrix} \rightarrow,$$

(see Theorems 2.7 and 3.6 in [9]).

First, it is obvious that the multiplication map by $L_1 := x_0$

$$R/I_{\mathbb{X}}]_i \stackrel{\times L_1}{\to} [R/I_{\mathbb{X}}]_{i+1}$$

is injective for $0 \le i \le d - 2$.

- Let $i = d 1 = j_1 + j_2 + j_3$ with $0 \le j_1, j_2, j_3 \le d$.
- (i) Assume $j_2 = 0$ and

$$x_{0}^{j_{1}}x_{2}^{j_{3}}L_{1} \in [I_{\mathbb{X}}]_{d} = \langle L_{1}\cdots L_{d}, M_{1}L_{2}\cdots L_{d}, M_{1}M_{2}L_{3}\cdots L_{d}, \dots, M_{1}\cdots M_{d-1}L_{d}, M_{1}M_{2}\cdots M_{d}\rangle,$$

that is,

$$x_0^{j_1} x_2^{j_3} L_1 = \alpha_1 L_1 \cdots L_d + \alpha_2 M_1 L_2 \cdots L_d + \alpha_3 M_1 M_2 L_3 \cdots L_d + \cdots$$

$$+\alpha_d M_1 \cdots M_{d-1} L_d + \alpha_{d+1} M_1 M_2 \cdots M_d$$

for some $\alpha_i \in \mathbb{k}$. Let $\varphi_{i,j}$ be a point defined by two linear forms L_i and M_j . Since two linear forms L_1 and M_2 vanish on a point $\varphi_{1,2}$, we get that

$$\alpha_2 = 0.$$

Moreover, since two forms L_1 and M_3 vanish on a point $\wp_{1,3}$, we have

 $\alpha_3 = 0.$

By continuing this procedure, one can show that

$$\alpha_2 = \cdots = \alpha_d = 0.$$

Hence

$$x_0^{j_1} x_2^{j_3} L_1 = \alpha_1 L_1 \cdots L_d + \alpha_{d+1} M_1 M_2 \cdots M_d,$$

that is,

$$L_1 \mid \alpha_{d+1} M_1 M_2 \cdots M_d$$
 and so, $\alpha_{d+1} = 0$.

It follows that

$$x_0^{j_1} x_2^{j_3} L_1 = \alpha_1 L_1 \cdots L_d$$
, and thus, $\alpha_1 = 0$.

(ii) Assume $j_2 > 0$ and

 $x_{0}^{j_{1}}x_{1}^{j_{2}}x_{2}^{j_{3}}L_{1} = \alpha_{1}L_{1}\cdots L_{d} + \alpha_{2}M_{1}L_{2}\cdots L_{d} + \alpha_{3}M_{1}M_{2}L_{3}\cdots L_{d} + \cdots + \alpha_{d}M_{1}\cdots M_{d-1}L_{d} + \alpha_{d+1}M_{1}M_{2}\cdots M_{d}$

for some $\alpha_i \in \mathbb{k}$. Recall that $M_1 := x_1$. Thus

$$M_1 \mid \alpha_1 L_1 \cdots L_d$$
, and hence, $\alpha_1 = 0$.

By the analogous argument as in (i), one can show that

$$\alpha_2 = \dots = \alpha_d = \alpha_{d+1} = 0$$

It is from (i) and (ii) that

$$x_0^{j_1} x_1^{j_2} x_2^{j_3} L_1 \notin [I_{\mathbb{X}}]_d,$$

which means that the multiplication map by L_1

$$[R/I_{\mathbb{X}}]_{d-1} \stackrel{\times L_1}{\to} [R/I_{\mathbb{X}}]_d$$

is injective, and surjective as well. Thus the multiplication map by L_1

$$[R/I_{\mathbb{X}}]_i \stackrel{\times L_1}{\to} [R/I_{\mathbb{X}}]_{i+1}$$

is injective and surjective for every $i \ge d-1$, as we wished.

So it follows that the multiplication map by L_1^j

$$[R/I_{\mathbb{X}}]_i \stackrel{\times L_1^j}{\to} [R/I_{\mathbb{X}}]_{i+j}$$

is injective for every $i \ge 0$. This completes the proof.

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The following lemma is immediate from Proposition 2.7. But we introduce another elementary proof here.

Lemma 3.3. Let \mathbb{X} be a k-configuration in \mathbb{P}^2 of type (1,2) in a basic configuration \mathbb{Z} in \mathbb{P}^2 of type (a,2) with $a \geq 2$, and let $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$, (\mathbb{X} is a set of solid 3-points in \mathbb{Z} in Figure 2). Then an Artinian k-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

•	0	0		0	0	\mathbb{L}_2
•	•	0	• • •	0	0	\mathbb{L}_1
\mathbb{M}_1	\mathbb{M}_2	\mathbb{M}_3		\mathbb{M}_{a-1}	\mathbb{M}_a	
FIGURE 2						

Proof. First, if a = 2, then the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} : 1 \quad 1 \quad 0,$$

(see [12, Theorem 2.1]) and so it follows that $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

Now suppose $a \ge 3$ and assume that \mathbb{L}_i and \mathbb{M}_j are lines defined by linear forms $L_i = x_0 - (i-1)x_2$ and $M_j = x_1 - (j-1)x_2$ for i and j, respectively. Let $\varphi_{i,j}$ be a point defined by two linear forms L_i and M_j . Then

 $I_{\mathbb{X}} = (L_1 L_2, L_1 M_1, M_1 M_2),$

$$I_{\mathbb{Y}} = (L_1 L_2, L_2 M_3 M_4 \cdots M_a, M_2 M_3 M_4 \cdots M_a)$$

(see [9,11]) and an ideal $I_{\mathbb{X}} + I_{\mathbb{Y}}$ has 5-minimal generators, i.e.,

$$M_{\mathbb{X}} + I_{\mathbb{Y}} = (L_1 L_2, L_1 M_1, M_1 M_2, L_2 M_3 M_4 \cdots M_a, M_2 M_3 M_4 \cdots M_a).$$

By [12, Theorem 2.1], the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

 $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$: 1 3 3 ... $\overset{(a-2)-\mathrm{nd}}{3}$ 1 0 \rightarrow .

Note that

1

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}(i) = \mathbf{H}_{R/I_{\mathbb{X}}}(i)$$

for $0 \leq i \leq a - 2$.

(i) Assume $x_0L_1^{a-2} = L_1^{a-1} \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-1}$. Then

$$x_0 L_1^{a-2} = L_1^{a-1} = F_1 L_1 L_2 + F_2 L_1 M_1 + F_3 M_1 M_2 + \beta_1 L_2 M_3 M_4 \cdots M_a + \beta_2 M_2 M_3 M_4 \cdots M_a$$

for some $F_i \in R_{a-3}$ and $\beta_j \in \mathbb{k}$. Since two linear forms L_1 and M_2 vanish on a point $\wp_{1,2}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = 0$ as well. This means that

 $x_0L_1^{a-2} = L_1^{a-1} = F_1L_1L_2 + F_2L_1M_1 + F_3M_1M_2 \in [I_X]_{a-1},$

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $\mathbf{H}_{R/(I_X+I_Y)}$ is of the form

$$J_{L_1} = (a, \dots).$$

(ii) Similarly, it is from Lemma 3.2 that

$$x_1 L_1^{a-3}, x_2 L_1^{a-3} \notin [I_{\mathbb{X}}]_{a-2} = [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-2}.$$

Furthermore, it is obvious that two forms $x_1L_1^{a-3}, x_2L_1^{a-3}$ are linearly independent in $[R/(I_{\mathbb{X}} + I_{\mathbb{Y}})]_{a-2} = [R/I_{\mathbb{X}}]_{a-2}$. So it is from (i) and (ii) that the Jordan type J_{L_1} of $\mathbf{H}_{R/(I_{\mathbb{X}} + I_{\mathbb{Y}})}$ is

$$J_{L_1} = \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}^{\vee} = (a, a-2, a-2).$$

Therefore, by Lemma 2.2, an Artinian k-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

The following proposition can be obtained using Proposition 2.7. However, we also introduce a different proof here.

Proposition 3.4. Let \mathbb{X} be a \Bbbk -configuration of type (1,2) contained in a basic configuration \mathbb{Z} in \mathbb{P}^2 of type (a,b) with $2 \leq b \leq a$. Define $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$, (\mathbb{X} is a set of solid 3-points in Figure 3). Then an Artinian \Bbbk -configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.



Proof. First, if a = b = 2, then it is immediate. If $a \ge 3$ and b = 2, by Lemma 3.3 it holds.

Now suppose $a \ge b \ge 3$ and assume that \mathbb{L}_i is a line defined by a linear form $L_i = x_0 - (i-1)x_2$ and \mathbb{M}_j is a line defined by a linear form $M_j = x_1 - (j-1)x_2$ for i and j. Let $\wp_{i,j}$ be a point defined by two linear forms L_i and M_j . Then it is from [9,11] that

$$I_{\mathbb{X}} = (L_1L_2, L_1M_1, M_1M_2),$$
 and
 $I_{\mathbb{Y}} = (L_1L_2 \cdots L_b, L_2L_3 \cdots L_bM_3 \cdots M_a, L_3 \cdots L_bM_2M_3 \cdots M_a, M_1M_2 \cdots M_a).$
Then an ideal $I_{\mathbb{X}} + I_{\mathbb{Y}}$ has 5-minimal generators, i.e.,

 $I_{\mathbb{X}} + I_{\mathbb{Y}} = (L_1 L_2, L_1 M_1, M_1 M_2, L_2 L_3 \cdots L_b M_3 \cdots M_a, L_3 \cdots L_b M_2 M_3 \cdots M_a),$ and by [12, Theorem 2.1] the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$$
 : 1 3 3 \cdots 3 $\overset{(a+b-4)\text{-st}}{3}$ 1 0 \rightarrow

(i) Assume
$$x_0 L_1^{a+b-4} = L_1^{a+b-3} \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a+b-3}$$
. Then
 $x_0 L_1^{a+b-4} = L_1^{a+b-3} = F_1 L_1 L_2 + F_2 L_1 M_1 + F_3 M_1 M_2$
 $+ \beta_1 L_2 L_3 \cdots L_b M_3 \cdots M_a + \beta_2 L_3 \cdots L_b M_2 M_3 \cdots M_a$

for some $F_i \in R_{a+b-5}$ and $\beta_j \in \mathbb{k}$. Since two linear forms L_1 and M_2 vanish on a point $\wp_{1,2}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = 0$ as well. This means that

$$x_0 L_1^{a+b-4} = L_1^{a+b-3} = F_1 L_1 L_2 + F_2 L_1 M_1 + F_3 M_1 M_2 \in [I_{\mathbb{X}}]_{a+b-3},$$

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $\mathbf{H}_{R/(I_X+I_Y)}$ is of the form

$$J_{L_1} = (a+b-2,\dots).$$

(ii) Similarly, it is from Lemma 3.2 that the following 3-forms

$$x_0L_1^{a+b-5}, x_1L_1^{a+b-5}, x_2L_1^{a+b-5}$$

are linearly independent. In particular, the following 2-forms

$$x_1L_1^{a+b-5}, x_2L_1^{a+b-5}$$

are linearly independent. Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is

$$J_{L_1} = \mathbf{H}_{R/(I_{\mathbb{X}} + I_{\mathbb{Y}})}^{\vee} = (a + b - 2, a + b - 4, a + b - 4).$$

It is from (i) and (ii) with Lemma 2.2 that an Artinian k-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP, which completes the proof.

We now slightly extend the previous result.

Lemma 3.5. Let \mathbb{X} be a k-configuration of type (1, 2, 3) in a basic configuration \mathbb{Z} in \mathbb{P}^2 of type (a, 3) with $a \geq 3$ such that $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$, (\mathbb{X} is a set of solid 6-points in Figure 4). Then an Artinian k-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

• • • • • • •
$$\mathbb{L}_{2}$$

• • • • • • \mathbb{L}_{2}
• • • • • \mathbb{L}_{2}
• • • • • \mathbb{L}_{2}
 $\mathbb{M}_{1} \ \mathbb{M}_{2} \ \mathbb{M}_{3} \ \mathbb{M}_{4} \ \cdots \ \mathbb{M}_{a}$
FIGURE 4

Proof. If a = 3, then in Proposition 3.4, \mathbb{Z} is a basic configuration of type (3, 3) and hence, \mathbb{Y} is a set of 6 points, Lemma holds. So we suppose that a > 3. First note that the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$$
 : 1 3 6 \cdots 6 3 1 0.

We assume that \mathbb{L}_i is a line defined by a linear form $L_i = x_0 - (i-1)x_2$ and \mathbb{M}_j is a line defined by a linear form $M_j = x_1 - (j-1)x_2$ for i and j. Let $\wp_{i,j}$ be a point defined by two linear forms L_i and M_j . Then

$$I_{\mathbb{X}} = (L_1 L_2 L_3, L_1 L_2 M_1, L_1 M_1 M_2, M_1 M_2 M_3), \text{ and} I_{\mathbb{Y}} = (L_1 L_2 L_3, L_2 L_3 M_4 \cdots M_a, L_3 M_3 M_4 \cdots M_a, M_2 M_3 \cdots M_a).$$

So an ideal $I_{\mathbb{X}} + I_{\mathbb{Y}}$ has 7-minimal generators, i.e.,

$$I_{\mathbb{X}} + I_{\mathbb{Y}} = (L_1 L_2 L_3, L_1 L_2 M_1, L_1 M_1 M_2, M_1 M_2 M_3, L_2 L_3 M_4 \cdots M_a, L_3 M_3 M_4 \cdots M_a, M_2 M_3 \cdots M_a).$$

Note that

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}(i) = \mathbf{H}_{R/I_{\mathbb{X}}}(i)$$

for $0 \leq i \leq a - 2$.

(i) Assume $x_0L_1^{a-1} = L_1^a \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_a$. Then

$$x_0 L_1^{a-1} = L_1^a = F_1 L_1 L_2 L_3 + F_2 L_1 L_2 M_1 + F_3 L_1 M_1 M_2 + F_4 M_1 M_2 M_3 + \beta_1 L_2 L_3 M_4 \cdots M_a + \beta_2 L_3 M_3 M_4 \cdots M_a + \beta_3 M_2 M_3 \cdots M_a$$

for some $F_i \in R_{a-3}$ and $\beta_j \in \mathbb{k}$. Since two linear forms L_1 and M_3 vanish on a point $\wp_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$x_0L_1^{a-1} = L_1^a = F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \in [I_{\mathbb{X}}]_a,$$

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+1,\ldots).$$

(ii) By the analogous argument as in (i), one can show that

$$x_1 L_1^{a-2}, x_2 L_1^{a-2} \notin [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-1}.$$

We now suppose that

$$\alpha x_1 L_1^{a-2} + \beta x_2 L_1^{a-2} \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-1}$$

for some $\alpha, \beta \in k$. Then

$$\alpha x_1 L_1^{a-2} + \beta x_2 L_1^{a-2}$$

= $F_1 L_1 L_2 L_3 + F_2 L_1 L_2 M_1 + F_3 L_1 M_1 M_2 + F_4 M_1 M_2 M_3$
+ $\beta_1 L_2 L_3 M_4 \cdots M_a + \beta_2 L_3 M_3 M_4 \cdots M_a + \beta_3 M_2 M_3 \cdots M_a$

for some $F_i \in R_{a-3}$ and $\beta_j \in \mathbb{k}$. Since two linear forms L_1 and M_3 vanish on a point $\wp_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$\begin{aligned} &\alpha x_1 L_1^{a-2} + \beta x_2 L_1^{a-2} \\ &= F_1 L_1 L_2 L_3 + F_2 L_1 L_2 M_1 + F_3 L_1 M_1 M_2 + F_4 M_1 M_2 M_3 \in [I_{\mathbb{X}}]_{a-1}. \end{aligned}$$

By Lemma 3.2, we get that

$$\alpha x_1 + \beta x_2 = 0, \quad \text{i.e.,} \quad \alpha = \beta = 0,$$

which implies that two forms

$$x_1L_1^{a-2}, x_2L_1^{a-2}$$

are linearly independent. Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+1, a-1, a-1, \dots).$$

(iii) It is from Lemma 3.2 that

$$x_1^2 L_1^{a-4}, x_1 x_2 L_1^{a-4}, x_2^2 L_1^{a-4} \notin [I_{\mathbb{X}}]_{a-2} = [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-2}$$

and the following set of 6-forms

$$\{x_0L_1^{a-3}, x_1L_1^{a-3}, x_2L_1^{a-3}, x_1^2L_1^{a-4}, x_1x_2L_1^{a-4}, x_2^2L_1^{a-4}\}$$

= $\{x_0^2L_1^{a-4}, x_0x_1L_1^{a-4}, x_0x_2L_1^{a-4}, x_1^2L_1^{a-4}, x_1x_2L_1^{a-4}, x_2^2L_1^{a-4}\}$

is linearly independent. In particular, the 3-forms

$$x_1^2 L_1^{a-4}, x_1 x_2 L_1^{a-4}, x_2^2 L_1^{a-4}$$

are linearly independent. Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+1, a-1, a-1, a-3, a-3, a-3).$$

It is from (i) ~ (iii) that the Jordan type J_{L_1} is

$$J_{L_1} = \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{V}})}^{\vee} = (a+1, a-1, a-1, a-3, a-3, a-3).$$

Therefore, by Lemma 2.2, an Artinian k-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

Theorem 3.6. Let \mathbb{X} be a \Bbbk -configuration of type (1, 2, 3) in a basic configuration \mathbb{Z} in \mathbb{P}^2 of type (a, b) with $a \ge 4$ and $b \ge 3$, and let $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$, (\mathbb{X} is a set of solid 6-points in Figure 5). Then an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.



Proof. If b = 3, then, by Lemma 3.5, it holds. So we suppose that b > 3. Note that, by [12, Theorem 2.1], the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$$
 : 1 3 6 \cdots 6 3 1 0.

We assume that \mathbb{L}_i is a line defined by a linear form $L_i = x_0 - (i-1)x_2$ and \mathbb{M}_j is a line defined by a linear form $M_j = x_1 - (j-1)x_2$ for i and j. Let $\varphi_{i,j}$ be a point defined by two linear forms L_i and M_j . Then

$$I_{\mathbb{X}} = (L_1 L_2 L_3, L_1 L_2 M_1, L_1 M_1 M_2, M_1 M_2 M_3), \text{ and}$$

$$I_{\mathbb{Y}} = (L_1 L_2 \cdots L_b, L_2 \cdots L_b M_4 \cdots M_a, L_3 \cdots L_b M_3 \cdots M_a, L_4 \cdots L_b M_2 \cdots M_a, M_1 M_2 M_3 \cdots M_a).$$

So an ideal $I_{\mathbb{X}}+I_{\mathbb{Y}}$ has 7-minimal generators, i.e.,

$$I_{\mathbb{X}} + I_{\mathbb{Y}} = (L_1 L_2 L_3, \ L_1 L_2 M_1, \ L_1 M_1 M_2, \ M_1 M_2 M_3, \\ L_2 \cdots L_b M_4 \cdots M_a, \ L_3 \cdots L_b M_3 \cdots M_a, \ L_4 \cdots L_b M_2 \cdots M_a).$$

Note that

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}(i) = \mathbf{H}_{R/I_{\mathbb{X}}}(i)$$

for $0 \le i \le a+b-5$.

(i) Assume
$$x_0 L_1^{a+b-4} = L_1^{a+b-3} \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a+b-3}$$
. Then
 $x_0 L_1^{a+b-4} = L_1^{a+b-3} = F_1 L_1 L_2 L_3 + F_2 L_1 L_2 M_1 + F_3 L_1 M_1 M_2 + F_4 M_1 M_2 M_3$
 $+ \beta_1 L_2 \cdots L_b M_4 \cdots M_a + \beta_2 L_3 \cdots L_b M_3 \cdots M_a$
 $+ \beta_3 L_4 \cdots L_b M_2 \cdots M_a$

for some $F_i \in R_{a+b-6}$ and $\beta_j \in k$. Since two linear forms L_1 and M_3 vanish on a point $\wp_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$x_0 L_1^{a+b-4} = L_1^{a+b-3}$$

$$=F_1L_1L_2L_3+F_2L_1L_2M_1+F_3L_1M_1M_2+F_4M_1M_2M_3\in [I_{\mathbb{X}}]_{a+b-3}$$

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+b-2,\dots)$$

(ii) By the analogous argument as in (i), one can show that

$$x_1 L_1^{a+b-5}, x_2 L_1^{a+b-5} \notin [I_{\mathbb{X}}]_{a+b-4} = [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a+b-4}$$

We now suppose that the following 3-forms

$$\alpha x_0 L_1^{a+b-5} + \beta x_1 L_1^{a+b-5} + \beta x_2 L_1^{a+b-5} \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a+b-4}$$

for some $\alpha, \beta, \gamma \in \mathbb{k}$, that is,

$$\alpha x_0 L_1^{a+b-5} + \beta x_1 L_1^{a+b-5} + \beta x_2 L_1^{a+b-5}$$

= $F_1 L_1 L_2 L_3 + F_2 L_1 L_2 M_1 + F_3 L_1 M_1 M_2 + F_4 M_1 M_2 M_3$

$$+\beta_1 L_2 \cdots L_b M_4 \cdots M_a + \beta_2 L_3 \cdots L_b M_3 \cdots M_a + \beta_3 L_4 \cdots L_b M_2 \cdots M_a$$

for some $F_i \in R_{a+b-6}$ and $\beta_j \in \mathbb{k}$. Since two linear forms L_1 and M_3 vanish on a point $\wp_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$\alpha x_0 L_1^{a+b-5} + \beta x_1 L_1^{a+b-5} + \beta x_2 L_1^{a+b-5}$$

= $F_1 L_1 L_2 L_3 + F_2 L_1 L_2 M_1 + F_3 L_1 M_1 M_2 + F_4 M_1 M_2 M_3 \in [I_{\mathbb{X}}]_{a+b-5}$

Hence, Lemma 3.2, $\alpha = \beta = \gamma = 0$, as we wished. This implies that the 3-forms

-4.

$$x_0L_1^{a+b-5}, x_1L_1^{a+b-5}, x_2L_1^{a+b-5}$$

are linearly independent. In particular, the 2-forms

$$x_1 L_1^{a+b-5}, x_2 L_1^{a+b-5}$$

are linearly independent. Hence the Jordan type of $\mathbf{H}_{R/(I_X+I_Y)}$ is of the form $I_{T-1} = (a+b-2, a+b-4, a+b-4, \dots).$

$$J_{L_1} = (a + b - 2, a + b - 4, a + b - 4, ...$$

(iii) It is from Lemma 3.2 that the following 6-forms

$$x_0^2 L_1^{a+b-7}, x_0 x_1 L_1^{a+b-7}, x_0 x_2 L_1^{a+b-7}, x_1^2 L_1^{a+b-7}, x_1 x_2 L_1^{a+b-7}, x_2^2 L_1^{a+b-7}$$

are linearly independent. In particular, the following 3-forms

$$x_1^2 L_1^{a+b-7}, x_1 x_2 L_1^{a+b-7}, x_2^2 L_1^{a+b-7}$$

are linearly independent. Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a + b - 2, a + b - 4, a + b - 4, a + b - 6, a + b - 6, a + b - 6).$$

It is from (i) ~ (iii) that the Jordan type J_{L_1} is

 $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}^{\vee}=(a+b-2,a+b-4,a+b-4,a+b-6,a+b-6,a+b-6).$

Therefore, by Lemma 2.2, an Artinian &-configuration quotient $R/(I_{\&} + I_{\&})$ has the SLP, which completes the proof of this theorem.

Remark 3.7. Theorem 3.6 has been proved if X is a k-configuration in \mathbb{P}^2 of type (1,2) or (1,2,3) in a basic configuration in \mathbb{P}^2 . However, if X is a k-configuration in \mathbb{P}^2 of type $(1,2,\ldots,d)$ in a basic configuration in \mathbb{P}^2 with $d \ge 4$, then it cannot be proved by the same method as in the proof of Theorem 3.6.

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