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APPLICATIONS OF THE COUPLED FIXED POINT THEOREM TO THE NONLINEAR MATRIX EQUATIONS

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Abstract. In this article we consider certain types of nonlinear matrix equations including the stochastic rational Riccati equation and show the existence and uniqueness of the positive definite solution by using Bhaskar-Lakshmikantham's coupled fixed point theorem.

1. Introduction

The fixed point theory has been studied widely in nonlinear analysis, and its results have been developed in a variety of areas in mathematics and engineering. In recent years, there has been a lot of interest in establishing fixed point theorems on ordered metric spaces with a contractive condition which holds for all points that are related by partial ordering. This trend was initiated by Ran and Reurings in [15] where they extended the Banach contraction principle in partially ordered sets with some applications to matrix equations.

Bhaskar and Lakshmikantham [3] introduced the notions of a mixed monotone mapping and a coupled fixed point, and proved some coupled fixed point theorems for mixed mappings in ordered metric spaces. Afterwards, Lakshmikantham and Ćirić [10] established coupled coincidence and coupled fixed point theorems. Many different kinds of coupled fixed point theorems with applications have been developed; see the literatures [2, 7, 10, 13]. In this article we mainly focus on solving the certain nonlinear matrix equations using the coupled fixed point theorem.

In [1] Berzig, Duan, and Samet have studied the positive definite solution to the nonlinear matrix equation

$$X = Q - A^* X^{-1} A + B^* X^{-1} B, \tag{1.1}$$

where Q is an $n \times n$ positive definite Hermitian matrix, A, B are arbitrary $n \times n$ matrices. This is a special stochastic rational Riccati equation arisen in stochastic control theory. Also, the special cases of (1.1) such as $X = Q - A^* X^{-1} A$ and $X = Q + B^* X^{-1} B$ have been studied (see [8, 14]).

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Furthermore, many researchers have considered the following nonlinear matrix equation generalizing the equations

$$X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q, \tag{1.2}$$

where $0 < |\delta_i| < 1$ and A_i's are arbitrary $n \times n$ matrices for all $i = 1, \ldots, m$, and suggested numerical methods for finding a solution [5, 6, 12]. Huang, Huang, Tsai [9] and Duan, Liao, Tang [5] have shown the existence and uniqueness of the positive definite solution of the equation (1.2) by using Hilbert's projective metric and fixed point theorems for monotone and mixed monotone operators, respectively. Moreover, Lim [11] has shown it by proving that the map $F(X) = Q + \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i$ is a strict contraction for the Thompson metric with the contraction coefficient less than or equal to $\delta := \max\{|\delta_i|\}_{i=1}^m$.

In this article, we investigate the matrix equation generalized the equations (1.1) and (1.2)

$$X = Q \pm \sum_{i=1}^{m} A_i X^{\pm p_i} A_i^* \mp \sum_{j=1}^{n} B_j X^{\pm q_j} B_j^*, \tag{1.3}$$

where $p_i, q_j \in (0, 1]$, and A_i's and B_j's are arbitrary $n \times n$ matrices for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We show the existence and uniqueness of the positive definite solution to the nonlinear matrix equation (1.3) using Bhaskar-Lakshmikantham's coupled fixed point theorem. We also discuss some simple cases of the equation (1.3) including two equations (1.1) and (1.2).

2. Preliminaries

Throughout this paper, we denote by $\mathcal{H}(n)$ the set of all $n \times n$ Hermitian matrices. For $A, B \in \mathcal{H}(n)$, $A \leq B (A < B)$ means that $B - A$ is positive semidefinite (positive definite, respectively). Moreover, $X \in [A, B]$ means that $A \leq X \leq B$, and $X \in [A, \infty)$ means that $X \geq A$. We denote by $\|A\|$ and $\|A\|_{\mathrm{tr}}$ the spectral norm and trace norm, respectively, that is,

$$\|A\| = \max\{\sigma_j(A) : j = 1, \ldots, n\},$$
$$\|A\|_{\mathrm{tr}} = \sum_{j=1}^{n} \sigma_j(A),$$

where $\sigma_j(A), j = 1, \ldots, n$ are the singular values of A.

The following lemmas will be useful later.

Lemma 2.1. *Let $A, B \in \mathcal{H}(n)$ such that $A \geq O$ and $B \geq O$. Then*

$$0 \leq \mathrm{tr}(AB) \leq \|A\| \,\mathrm{tr}(B).$$

Lemma 2.2 ([4])**.** *If $A, B > O$ with $A \leq B$, then $A^t \leq B^t$ for any $0 \leq t \leq 1$ and $A^{-1} \geq B^{-1}$.*

Lemma 2.3 ([4]). *Let P and Q be positive definite matrices of the same order with $P, Q \geq aI$, where $a > 0$. Then for every unitarily invariant norm $|||\cdot|||$ and $0 < t \leq 1$*

$$|||P^t - Q^t||| \leq ta^{t-1}|||P - Q|||,$$
$$|||P^{-t} - Q^{-t}||| \leq ta^{-(t+1)}|||P - Q|||.$$

Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a given mapping. We say that F has the *mixed monotone property* if for any $x, y \in X$

$$x_1, x_2 \in X,\ \ x_1 \preceq x_2 \ \Rightarrow\ F(x_1, y) \preceq F(x_2, y),$$
$$y_1, y_2 \in X,\ \ y_1 \preceq y_2 \ \Rightarrow\ F(x, y_1) \succeq F(x, y_2).$$

We say that (x, y) is a *coupled fixed point* of F if $x = F(x, y)$ and $y = F(y, x)$.

The following fixed point theorem is the key of the main result.

Theorem 2.4 ([3]). *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d such that (X, d) is complete. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $\delta \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \leq \frac{\delta}{2}[d(x, u) + d(y, v)]$$

for any $(x, y), (u, v) \in X \times X$ with $x \succeq u$ and $y \preceq v$. We suppose that there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Then

(a) *F has a coupled fixed point $(\bar{x}, \bar{y}) \in X \times X$; and*
(b) *the sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$ converge to $\bar{x}$ and $\bar{y}$, respectively.*

In addition, suppose that every pair of elements has a lower bound and an upper bound, then

(c) *F has a coupled fixed point $(\bar{x}, \bar{y}) \in X \times X$;*
(d) *$\bar{x} = \bar{y}$; and*
(e) *we have the following estimate*

$$\max\{d(x_n, \bar{x}), d(y_n, \bar{x})\} \leq \frac{\delta^n}{2(1-\delta)}[d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)].$$

3. Solving $X = Q + \sum_{i=1}^{m} A_i X^{p_i} A_i^* - \sum_{j=1}^{n} B_j X^{q_j} B_j^*$

In this section, we consider the following matrix equation

$$X = Q + \sum_{i=1}^{m} A_i X^{p_i} A_i^* - \sum_{j=1}^{n} B_j X^{q_j} B_j^*, \tag{3.4}$$

where $p_i, q_j \in (0, 1]$.

For $a, b > 0$ the following assumptions are considered:

1. $aI + b_M \sum_{j=1}^{n} B_j B_j^* \leq Q \leq bI - b_M \sum_{i=1}^{m} A_i A_i^*$,
2. $\sum_{i=1}^{m} \|A_i A_i^*\| < \frac{a}{2a_M}$, $\sum_{j=1}^{n} \|B_j B_j^*\| < \frac{a}{2a_M}$,

where

$$a_M := \max\{a^{p_i}, a^{q_j} : 1 \leq i \leq m, 1 \leq j \leq n\},$$
$$b_M := \max\{b^{p_i}, b^{q_j} : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Theorem 3.1. *Under the assumptions 1 and 2, we have*

(I) *Equation* (3.4) *has a unique solution* $\bar{X} \in [aI, bI]$, *and*
(II) *the sequences* $\{X_k\}$ *and* $\{Y_k\}$ *defined by*

$$\begin{cases} X_0 = aI, \\ X_{k+1} = Q + \sum_{i=1}^{m} A_i X_k^{p_i} A_i^* - \sum_{j=1}^{n} B_j Y_k^{q_j} B_j^*, \end{cases}$$
$$\begin{cases} Y_0 = bI, \\ Y_{k+1} = Q + \sum_{i=1}^{m} A_i Y_k^{p_i} A_i^* - \sum_{j=1}^{n} B_j X_k^{q_j} B_j^*, \end{cases}$$

converge to $\bar{X}$, *that is,*

$$\lim_{k\to\infty} \|X_k - \bar{X}\|_{\mathrm{tr}} = \lim_{k\to\infty} \|Y_k - \bar{X}\|_{\mathrm{tr}} = 0.$$

Furthermore, the error estimation is given by

$$\max\{\|X_k - \bar{X}\|_{\mathrm{tr}}, \|Y_k - \bar{X}\|_{\mathrm{tr}}\} \leq \frac{\delta^k}{1-\delta} \max\{\|X_1 - X_0\|_{\mathrm{tr}}, \|Y_1 - Y_0\|_{\mathrm{tr}}\},$$

where $0 < \delta < 1$.

Proof. For $X, Y \in \mathcal{H}(n)$ let

$$F(X, Y) = Q + \sum_{i=1}^{m} A_i X^{p_i} A_i^* - \sum_{j=1}^{n} B_j Y^{q_j} B_j^*.$$

By Lemma 2.2 F is a continuous mapping having the mixed monotone property.

We claim that $F([aI, bI]^2) \subseteq [aI, bI]$. Let $X, Y \in [aI, bI]$, that is, $aI \leq X, Y \leq bI$. This implies by assumption 1 that

$$F(X, Y) \leq Q + \sum_{i=1}^{m} A_i X^{p_i} A_i^* \leq Q + \sum_{i=1}^{m} b^{p_i} A_i A_i^* \leq Q + b_M \sum_{i=1}^{m} A_i A_i^* \leq bI,$$

and

$$F(X, Y) \geq Q - \sum_{j=1}^{n} B_j Y^{q_j} B_j^* \geq Q - \sum_{j=1}^{n} b^{q_j} B_j B_j^* \geq Q - b_M \sum_{j=1}^{n} B_j B_j^* \geq aI.$$

Let $X, Y, U, V \in [aI, bI]$ such that $X \geq U$ and $Y \leq V$. Then

$$\begin{aligned}
&\|F(X,Y) - F(U,V)\|_{\mathrm{tr}} \\
&= \left\| \sum_{i=1}^{m} A_i (X^{p_i} - U^{p_i}) A_i^* + \sum_{j=1}^{n} B_j (V^{q_j} - Y^{q_j}) B_j^* \right\|_{\mathrm{tr}} \\
&\leq \sum_{i=1}^{m} \|A_i (X^{p_i} - U^{p_i}) A_i^*\|_{\mathrm{tr}} + \sum_{j=1}^{n} \left\|B_j (V^{q_j} - Y^{q_j}) B_j^*\right\|_{\mathrm{tr}} \\
&\leq \sum_{i=1}^{m} \|A_i A_i^*\| \, \|X^{p_i} - U^{p_i}\|_{\mathrm{tr}} + \sum_{j=1}^{n} \left\|B_j B_j^*\right\| \, \|V^{q_j} - Y^{q_j}\|_{\mathrm{tr}} \\
&\leq \left(\sum_{i=1}^{m} p_i a^{p_i - 1} \|A_i A_i^*\| \right) \|X - U\|_{\mathrm{tr}} + \left(\sum_{j=1}^{n} q_j a^{q_j - 1} \left\|B_j B_j^*\right\| \right) \|V - Y\|_{\mathrm{tr}} \\
&\leq \frac{a_M}{a} \left[\left(\sum_{i=1}^{m} \|A_i A_i^*\| \right) \|X - U\|_{\mathrm{tr}} + \left(\sum_{j=1}^{n} \left\|B_j B_j^*\right\| \right) \|V - Y\|_{\mathrm{tr}} \right].
\end{aligned}$$

The first inequality follows from the triangle inequality, the second from Lemma 2.1, the third from Lemma 2.3, and the last from the following inequality: for any $t \in \{p_i, q_j\}$

$$0 < t a^{t-1} \leq \frac{a^t}{a} \leq \frac{a_M}{a}.$$

This implies that

$$\|F(X,Y) - F(U,V)\|_{\mathrm{tr}} \leq \frac{\delta}{2} \left[\|X - U\|_{\mathrm{tr}} + \|V - Y\|_{\mathrm{tr}} \right],$$

where

$$\delta = \frac{2a_M}{a} \max \left\{ \sum_{i=1}^{m} \|A_i A_i^*\|, \sum_{j=1}^{n} \left\|B_j B_j^*\right\| \right\}.$$

From condition 2 we can easily see that $0 \leq \delta < 1$.

Taking $X_0 = aI$ and $Y_0 = bI$ we can show from condition 1 that $X_0 \leq F(X_0, Y_0)$ and $Y_0 \geq F(Y_0, X_0)$. On the other hand, for every $X, Y \in \mathcal{H}(n)$ there exist a greatest lower bound and a least upper bound. Thus, (I) and (II) follow immediately from Theorem 2.4, and $\bar{X}$ is the unique solution to Equation (3.4) in $[aI, bI]$. □

The following results are immediate consequences of Theorem 3.1.

Corollary 3.2. *Consider the matrix equation* (3.4) *with* $Q = I$. *Suppose that*

$$\sum_{i=1}^{m} \|A_i A_i^*\| \leq \min \left\{ \frac{b-1}{b_M}, \frac{a}{2a_M} \right\} \quad \text{and} \quad \sum_{j=1}^{n} \|B_j B_j^*\| \leq \min \left\{ \frac{1-a}{b_M}, \frac{a}{2a_M} \right\}.$$

Then items (I) *and* (II) *of Theorem 3.1 hold.*

Corollary 3.3. *Consider the matrix equation* (3.4) *with unitary matrices* A_i *and* B_j *for all* i *and* j. *Suppose that*

1. $(a + nb_M)I \leq Q \leq (b - mb_M)I$, *and*
2. $a_M < \min\left\{\frac{a}{2m}, \frac{a}{2n}\right\}$.

Then items (I) *and* (II) *of Theorem 3.1 hold.*

The following is the case when $p_i = q_j = t \in (0,1]$ for all i, j, and $A_i = B_j = O$ for all i, j except one term.

Corollary 3.4. *Consider the matrix equation*

$$X = Q + AX^t A^* - BX^t B^*, \tag{3.5}$$

where $t \in (0,1]$. *Suppose that*

1. $aI + b^t BB^* \leq Q \leq bI - b^t AA^*$, *and*
2. $\|AA^*\| < \frac{a^{1-t}}{2}$, $\|BB^*\| < \frac{a^{1-t}}{2}$.

Then items (I) *and* (II) *of Theorem 3.1 hold.*

4. Solving $X = Q - \sum_{i=1}^{m} A_i X^{-p_i} A_i^* + \sum_{j=1}^{n} B_j X^{-q_j} B_j^*$

In this section, we consider the following matrix equation

$$X = Q - \sum_{i=1}^{m} A_i X^{-p_i} A_i^* + \sum_{j=1}^{n} B_j X^{-q_j} B_j^*, \tag{4.6}$$

where $p_i, q_j \in (0,1]$.

For $a, b > 0$ the following assumptions are considered:

1. $aI + \frac{1}{a_m}\sum_{i=1}^{m} A_i A_i^* \leq Q \leq bI - \frac{1}{a_m}\sum_{j=1}^{n} B_j B_j^*$,
2. $\sum_{i=1}^{m} \|A_i A_i^*\| < \frac{a_m a}{2}$, $\sum_{j=1}^{n} \|B_j B_j^*\| < \frac{a_m a}{2}$,

where

$$a_m := \min\{a^{p_i}, a^{q_j} : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Theorem 4.1. *Under the assumptions 1 and 2,*

(I) *Equation* (4.6) *has a unique solution* $\hat{X} \in [aI, \infty)$, *and*
(II) *the sequences* $\{X_k\}$ *and* $\{Y_k\}$ *defined by*

$$\begin{cases} X_0 = aI, \\ X_{k+1} = Q - \sum_{i=1}^{m} A_i X_k^{-p_i} A_i^* + \sum_{j=1}^{n} B_j Y_k^{-q_j} B_j^*, \end{cases}$$

$$\begin{cases} Y_0 = bI, \\ Y_{k+1} = Q - \sum_{i=1}^{m} A_i Y_k^{-p_i} A_i^* + \sum_{j=1}^{n} B_j X_k^{-q_j} B_j^*, \end{cases}$$

converge to $\hat{X}$*, that is,*

$$\lim_{k\to\infty} \|X_k - \hat{X}\|_{\mathrm{tr}} = \lim_{k\to\infty} \|Y_k - \hat{X}\|_{\mathrm{tr}} = 0.$$

Furthermore, the error estimation is given by

$$\max\{\|X_k - \hat{X}\|_{\mathrm{tr}}, \|Y_k - \hat{X}\|_{\mathrm{tr}}\} \leq \frac{\delta^k}{1-\delta} \max\{\|X_1 - X_0\|_{\mathrm{tr}}, \|Y_1 - Y_0\|_{\mathrm{tr}}\},$$

where $0 < \delta < 1$*.*

Proof. For $X, Y \in \mathcal{H}(n)$ let

$$G(X,Y) = Q - \sum_{i=1}^{m} A_i X^{-p_i} A_i^* + \sum_{j=1}^{n} B_j Y^{-q_j} B_j^*.$$

By Lemma 2.2, G is a continuous mapping having the mixed monotone property.

We claim that $G([aI,\infty)^2) \subseteq [aI,\infty)$. Let $X, Y \in [aI,\infty)$, that is, $X, Y \geq aI$. This implies by assumption 1 that

$$G(X,Y) \geq Q - \sum_{i=1}^{m} A_i X^{-p_i} A_i^* \geq Q - \sum_{i=1}^{m} a^{-p_i} A_i A_i^* \geq Q - a_m^{-1} \sum_{i=1}^{m} A_i A_i^* \geq aI.$$

Let $X, Y, U, V \in [aI,\infty)$ such that $X \geq U$ and $Y \leq V$. Then

$$\begin{aligned} &\|G(X,Y) - G(U,V)\|_{\mathrm{tr}} \\ &= \left\| \sum_{i=1}^{m} A_i (U^{-p_i} - X^{-p_i}) A_i^* + \sum_{j=1}^{n} B_j (Y^{-q_j} - V^{q_j}) B_j^* \right\|_{\mathrm{tr}} \\ &\leq \sum_{i=1}^{m} \left\| A_i (U^{-p_i} - X^{-p_i}) A_i^* \right\|_{\mathrm{tr}} + \sum_{j=1}^{n} \left\| B_j (Y^{-q_j} - V^{-q_j}) B_j^* \right\|_{\mathrm{tr}} \\ &\leq \sum_{i=1}^{m} \|A_i A_i^*\| \left\| U^{-p_i} - X^{-p_i} \right\|_{\mathrm{tr}} + \sum_{j=1}^{n} \left\| B_j B_j^* \right\| \left\| Y^{-q_j} - V^{-q_j} \right\|_{\mathrm{tr}} \\ &\leq \left(\sum_{i=1}^{m} p_i a^{-p_i - 1} \|A_i A_i^*\| \right) \|U - X\|_{\mathrm{tr}} + \left(\sum_{j=1}^{n} q_j a^{-q_j - 1} \left\| B_j B_j^* \right\| \right) \|Y - V\|_{\mathrm{tr}} \\ &\leq \frac{1}{a_m a} \left[\left(\sum_{i=1}^{m} \|A_i A_i^*\| \right) \|U - X\|_{\mathrm{tr}} + \left(\sum_{j=1}^{n} \left\| B_j B_j^* \right\| \right) \|Y - V\|_{\mathrm{tr}} \right]. \end{aligned}$$

The last follows from the following inequality: for any $t \in \{p_i, q_j\}$

$$0 < ta^{-t-1} \le \frac{1}{a^t a} \le \frac{1}{a_m a}.$$

This implies that

$$\|G(X,Y) - G(U,V)\|_{\mathrm{tr}} \le \frac{\delta}{2}\left[\|X-U\|_{\mathrm{tr}} + \|V-Y\|_{\mathrm{tr}}\right],$$

where

$$\delta = \frac{2}{a_m a}\max\left\{\sum_{i=1}^{m}\|A_iA_i^*\|, \sum_{j=1}^{n}\left\|B_jB_j^*\right\|\right\}.$$

From condition 2 we can easily see that $0 \le \delta < 1$.

Taking $X_0 = aI$ and $Y_0 = bI$ we can show from condition 1 that $X_0 \le G(X_0, Y_0)$ and $Y_0 \ge G(Y_0, X_0)$. On the other hand, for every $X, Y \in \mathcal{H}(n)$ there exist a greatest lower bound and a least upper bound. Thus, (I) and (II) follow immediately from Theorem 2.4, and $\hat{X}$ is the unique solution to Equation (4.6) in $[aI, \infty)$. □

The following results are immediate consequences of Theorem 4.1.

Corollary 4.2. *Consider the matrix equation* (4.6) *with* $Q = I$. *Suppose that*

1. $\sum_{i=1}^{m} A_iA_i^* \le a_m(1-a)I$, $\sum_{j=1}^{n} B_jB_j^* \le a_m(b-1)I$, *and*
2. $\sum_{i=1}^{m}\|A_iA_i^*\| < \frac{a_m a}{2}$, $\sum_{j=1}^{n}\left\|B_jB_j^*\right\| < \frac{a_m a}{2}$.

Then items (I) *and* (II) *of Theorem 4.1 hold.*

Corollary 4.3. *Consider the matrix equation* (4.6) *with unitary matrices* A_i *and* B_j *for all* i *and* j. *Suppose that*

1. $\left(a + \frac{m}{a_m}\right)I \le Q \le \left(b - \frac{n}{a_m}\right)I$, *and*
2. $a_m > \max\left\{\frac{2m}{a}, \frac{2n}{a}\right\}$.

Then items (I) *and* (II) *of Theorem 4.1 hold.*

The following is the case when $p_i = q_j = t \in (0,1]$ for all i, j, and $A_i = B_j = O$ for all i, j except one term.

Corollary 4.4. *Consider the matrix equation*

$$X = Q - AX^{-t}A^* + BX^{-t}B^*, \tag{4.7}$$

where $t \in (0,1]$. *Suppose that*

1. $aI + a^{-t}AA^* \le Q \le bI - a^{-t}BB^*$, *and*
2. $\|AA^*\| < \frac{a^{1+t}}{2}$, $\|BB^*\| < \frac{a^{1+t}}{2}$.

Then items (I) *and* (II) *of Theorem 4.1 hold.*

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