J. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/JKMS.j160753
pISSN: 0304-9914 / eISSN: 2234-3008

CLOSURE PROPERTY AND TAIL PROBABILITY ASYMPTOTICS FOR RANDOMLY WEIGHTED SUMS OF DEPENDENT RANDOM VARIABLES WITH HEAVY TAILS

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Abstract. In this paper we study the closure property and probability tail asymptotics for randomly weighted sums $S_n^\Theta = \Theta_1 X_1 + \cdots + \Theta_n X_n$ for long-tailed random variables $X_1, \ldots, X_n$ and positive bounded random weights $\Theta_1, \ldots, \Theta_n$ under similar dependence structure as in [26]. In particular, we study the case where the distribution of random vector $(X_1, \ldots, X_n)$ is generated by an absolutely continuous copula.

1. Introduction

Let $X_1, \ldots, X_n$ be real-valued random variables (r.v.s) with corresponding distributions $F_1, \ldots, F_n$ and let $\Theta_1, \ldots, \Theta_n$ be arbitrarily dependent positive bounded r.v.s, independent of $X_1, \ldots, X_n$. Denote the randomly weighted sum by

$$S_n^\Theta := \Theta_1 X_1 + \cdots + \Theta_n X_n. \tag{1.1}$$

The primary interest of this paper is to focus on the following two questions. First is the closure property of the sum S_n^Θ, where the primary (heavy-tailed) r.v.s $X_1, \ldots, X_n$ possess some general dependence structure. More precisely, the question is the following: given that distributions $F_1, \ldots, F_n$ are from the long-tailed distribution class (denoted by $\mathscr{L}$, see Section 2), whether the distribution function (d.f.) of sum S_n^Θ belongs to the same class $\mathscr{L}$? Second question we address here, is the asymptotic equivalence of the tail probabilities $\mathrm{P}(S_n^\Theta > x)$ and $\mathrm{P}(S_n^{\Theta+} > x)$, where $S_n^{\Theta+} := \Theta_1 X_1^+ + \cdots + \Theta_n X_n^+$, i.e., for a given dependence structure among the heavy-tailed r.v.s $X_1, \ldots, X_n$, whether it holds that

$$\mathrm{P}(S_n^\Theta > x) \sim \mathrm{P}(S_n^{\Theta+} > x) \tag{1.2}$$

Received November 25, 2016; Revised March 18, 2017; Accepted June 19, 2017.
2010 *Mathematics Subject Classification.* 60E99, 62E20, 62H20.
Key words and phrases. randomly weighted sum, long-tail distribution, copula, FGM copula.

for $x \to \infty$? Relation (1.2) is not only of theoretical interest but also has practical implications as it allows, for large x, to replace the sum of real-valued r.v.s by much easier to handle sum of r.v.s concentrated on $[0, \infty)$.

The first problem in the case $\Theta_1 = \cdots = \Theta_n = 1$ reduces to the question of convolution closure for the class $\mathscr{L}$, which was studied by Embrechts and Goldie ([5], Theorem 3(b)) when $n = 2$ (in fact, they proved the closure property for more general class $\mathscr{L}_\gamma$) and by Ng et al. [17]. The closure property for some other heavy-tailed classes was considered in [2, 6, 8, 12, 18, 23, 24]. The closure property for *randomly weighted* sums S_n^Θ was studied in [3,26]. The probability tail asymptotics for sums S_n^Θ of independent heavy tailed r.v.s $X_1, \dots, X_n$ with $\Theta_1, \dots, \Theta_n$ being nonnegative bounded r.v.s were investigated in [3, 18–20, 25], among others; some dependence among $X_1, \dots, X_n$ was allowed in [4, 7, 11, 13, 21], etc. We note that both mentioned questions are closely related: the proof of asymptotic equivalence (1.2) is based on the uniform closure property (see Lemma 3.1 and Remark 5.1 below).

Recently, Yang et al. [26] considered the randomly weighted sum S_2^Θ under the following dependence structure between real-valued r.v.s X_1 and X_2:

$$
\begin{aligned}
\mathrm{P}(X_2 > x | X_1 = y) &\sim h_1(y)\overline{F_2}(x), \\
\mathrm{P}(X_1 > x | X_2 = y) &\sim h_2(y)\overline{F_1}(x), \quad x \to \infty,
\end{aligned} \tag{1.3}
$$

uniformly in $y \in \mathbb{R}$, where $h_k : \mathbb{R} \mapsto (0, \infty)$, $k = 1, 2$, are measurable functions. Such a dependence structure, proposed in [1], can be easily checked for some well-known bivariate copulas, allowing both positive and negative dependence, see, e.g., [1], [14], [26]. The main result of [26] is the following theorem.

Theorem 1.1 ([26]). *Assume that X_1, X_2 are real-valued r.v.s with distributions $F_k \in \mathscr{L}$, $k = 1, 2$, satisfying relation* (1.3)*; Θ_1, Θ_2 are arbitrarily dependent, but independent of X_1, X_2, and such that $\mathrm{P}(a \le \Theta_k \le b) = 1$, $k = 1, 2$, with some constants $0 < a \le b < \infty$. Then the distribution of S_2^Θ is in $\mathscr{L}$ and relation* (1.2) *holds.*

The goal of the present paper is to extend the result on the closure property and tail asymptotics of randomly weighted sums S_n^Θ under similar dependence structure to (1.3) for *any* $n \ge 2$. Also, we study the case where the distribution of random vector $(X_1, \dots, X_n)$ is generated by an absolutely continuous copula. In particular, we show that, if the distribution of $(X_1, \dots, X_n)$ is generated by the FGM copula, $F_k \in \mathscr{L} \cap \mathscr{D}$ (see Section 2), $k = 1, \dots, n$, and $\mathrm{P}(0 < \Theta \le b) = 1$, $k = 1, \dots, n$, then the probabilities $\mathrm{P}(S_n^\Theta > x)$ and $\mathrm{P}(S_n^{\Theta +} > x)$ are asymptotically equivalent to $\sum_{k=1}^n \mathrm{P}(\Theta_k X_k > x)$.

The rest of the paper is organized as follows. Section 2 presents the main results of the paper. Their proofs are given in Section 3. Section 4 focuses to the dependence generated by a copula, and, particularly, by the FGM copula. Auxiliary results are given in Section 5.

2. Main results

Throughout this paper, all limit relationships hold for x tending to ∞ unless stated otherwise. For two positive functions $u(x)$ and $v(x)$, we write $u(x) \sim v(x)$ if $\lim u(x)/v(x) = 1$; write $u(x) \lesssim v(x)$ if $\limsup u(x)/v(x) \le 1$. For a real number x, write $x^+ = \max\{x, 0\}$. The indicator function of an event A is denoted by $\mathbb{I}_A$. For any distribution F, define its tail distribution by $\overline{F} = 1-F$.

A distribution F is called long-tailed, denoted by $F \in \mathscr{L}$, if $\overline{F}(x+y) \sim \overline{F}(x)$ holds for every fixed y; is called dominatedly varying-tailed, denoted by $F \in \mathscr{D}$, if $\limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x) < \infty$ for any $y \in (0,1)$; is said to have a consistently varying tail, denoted by $F \in \mathscr{C}$, if $\lim_{y\nearrow 1} \limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x) = 1$. A d.f. F supported on $[0,\infty)$ belongs to the class $\mathscr{S}$ (is subexponential) if $\lim_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} = 2$, where $F_1 * F_2$ denotes the convolution of F_1 with F_2. In the case where d.f. F is concentrated on $\mathbb{R}$, we write $F \in \mathscr{S}$ if $F^+(x) = F(x)\mathbf{1}_{\{x\ge 0\}}$ belongs to $\mathscr{S}$.

Let $n \ge 2$ be an integer. Consider the real-valued r.v.s $X_1, \dots, X_n$ with corresponding distributions $F_1, \dots, F_n$, such that $\overline{F_k}(x) > 0$ for $k = 1, \dots, n$, and assume the following dependence structures.

Assumption A. For each $k = 2, \dots, n$ relation

$$\mathrm{P}(X_k > x | X_1 = y_1, \dots, X_{k-1} = y_{k-1}) \;\sim\; g_k(y_1, \dots, y_{k-1})\overline{F_k}(x) \tag{2.1}$$

holds uniformly for $(y_1, \dots, y_{k-1}) \in \mathbb{R}^{k-1}$, i.e.,

$$\lim_{x\to\infty} \sup_{(y_1,\dots,y_{k-1})\in\mathbb{R}^{k-1}} \left| \frac{\mathrm{P}(X_k > x | X_1 = y_1, \dots, X_{k-1} = y_{k-1})}{g_k(y_1, \dots, y_{k-1})\overline{F_k}(x)} - 1 \right| = 0,$$

where $g_k \colon \mathbb{R}^{k-1} \mapsto \mathbb{R}_+ := (0, \infty)$, $k = 2, \dots, n$, are measurable functions.

Assumption B. For each $k = 2, \dots, n$ relation

$$\mathrm{P}\Big(\sum_{i=1}^{k-1} w_i X_i > x | X_k = y \Big) \;\sim\; h_k^{(w)}(y)\mathrm{P}\Big(\sum_{i=1}^{k-1} w_i X_i > x \Big) \tag{2.2}$$

holds uniformly for $y \in \mathbb{R}$ and $\overline{w}_{k-1} := (w_1, \dots, w_{k-1}) \in [a,b]^{k-1}$, with some positive constants $0 < a \le b < \infty$, i.e.,

$$\lim_{x\to\infty} \sup_{y\in\mathbb{R}} \sup_{\overline{w}_{k-1}\in[a,b]^{k-1}} \left| \frac{\mathrm{P}\big(\sum_{i=1}^{k-1} w_i X_i > x | X_k = y\big)}{h_k^{(w)}(y)\mathrm{P}\big(\sum_{i=1}^{k-1} w_i X_i > x\big)} - 1 \right| = 0,$$

where $h_k^{(w)} \equiv h_k(w_1, \dots, w_{k-1}, \cdot) \colon \mathbb{R} \mapsto \mathbb{R}_+$, $k = 1, \dots, n$, are measurable functions.

If, for some $i \in \{1, \dots, k-1\}$, $y_i = y_i^*$ in (2.1) is not possible value of X_i, i.e., $\mathrm{P}(X_i \in \Delta) = 0$ for some open interval containing y_i^*, then the conditional probability in Assumption A is understood as unconditional and therefore $g_k(y_1, \dots, y_i^*, \dots, y_{k-1}) = 1$ for such y_i. The same agreement holds for (2.2).

Clearly, the uniformity in (2.1) and (2.2) implies that $\mathrm{E}g_k(X_1,\dots,X_{k-1}) = \mathrm{E}h_k^{(w)}(X_k) = 1$ for $k = 2,\dots,n$.

Our first main result is the following theorem.

Theorem 2.1. *Let $X_1,\dots,X_n$ be real-valued r.v.s satisfying Assumptions A, B, and let $\Theta_1,\dots,\Theta_n$ be random weights, independent of $X_1,\dots,X_n$, such that $\mathrm{P}(a \le \Theta_k \le b) = 1$, $k = 1,\dots,n$. If $F_k \in \mathscr{L}$ for all $k = 1,\dots,n$, then d.f. $\mathrm{P}(S_n^{\Theta} \le x)$ belongs to $\mathscr{L}$.*

In order to obtain our second main result we have to strengthen the assumption of dependence from Assumptions A, B to the following:

Assumption C. For arbitrary nonempty sets of indices $I = \{k_1,\dots,k_m\} \subset \{1,2,\dots,n\}$ and $J = \{r_1,\dots,r_p\} \subset \{1,2,\dots,n\}\backslash I$, relation

$$\mathrm{P}\Big(\sum_{k\in I} w_k X_k > x \big| X_{r_1} = y_{r_1},\dots,X_{r_p} = y_{r_p}\Big)$$
$$\sim h_{I,J}^{(w)}(y_{r_1},\dots,y_{r_p})\mathrm{P}\Big(\sum_{k\in I} w_k X_k > x\Big)$$

holds uniformly for $(y_{r_1},\dots,y_{r_p}) \in \mathbb{R}^p$ and $(w_{k_1},\dots,w_{k_m}) \in [a,b]^m$, $0 < a \le b < \infty$, with some measurable function $h_{I,J}^{(w)}\colon \mathbb{R}^p \mapsto \mathbb{R}_+$, such that $h_{I,J}^{(w)}(y_{r_1},\dots,y_{r_p})$ is bounded uniformly in $w_k \in [a,b], k \in I$ and $(y_{r_1},\dots,y_{r_p}) \in \mathbb{R}^p$.

Clearly, Assumption C implies both Assumptions A and B with $g_k(y_1,\dots,y_{k-1}) \equiv h^{(w)}_{\{k\},\{1,\dots,k-1\}}(y_1,\dots,y_{k-1})$ and $h_k^{(w)}(y) \equiv h^{(w)}_{\{1,\dots,k-1\},\{k\}}(y)$, $k = 2,\dots,n$.

Theorem 2.2. *Let $X_1,\dots,X_n$ be real-valued r.v.s satisfying Assumption C and let $\Theta_1,\dots,\Theta_n$ be random weights, independent of $X_1,\dots,X_n$, such that $\mathrm{P}(a \le \Theta_k \le b) = 1$, $k = 1,\dots,n$. If $F_k \in \mathscr{L}$ for all $k = 1,\dots,n$, then*

$$\mathrm{P}(S_n^{\Theta} > x) \sim \mathrm{P}(S_n^{\Theta+} > x) \ \sim \ \mathrm{P}(M_n^{\Theta} > x), \tag{2.3}$$

where $M_n^{\Theta} := \max\{S_1^{\Theta},\dots,S_n^{\Theta}\}$.

Remark 2.1. In the case $n = 2$, conjunction of Assumptions A and B coincides with Assumption C, which is the same as condition (1.3). Thus, Theorems 2.1–2.2 generalize the result in Theorem 1.1.

Remark 2.2. If conditions of Theorem 2.2 are satisfied and $X_1,\dots,X_n$ are independent, then relations (2.3) were proved by Wang ([21], Lemma 4) and Chen et al. ([3], Theorem 2.1); moreover, the interval $[a,b]$ can be extended to $(0,b]$ if, additionally, Θ_k's are positively associated (see Theorem 2.2 in [3]).

Remark 2.3. Note that, in general, equivalence relations in (2.3) can not be extended to

$$\mathrm{P}(S_n^{\Theta} > x) \sim \sum_{i=1}^{n} \mathrm{P}(\Theta_i X_i > x).$$

Let $n = 2$, $\Theta_1 = \Theta_2 = 1$ and let X_1, X_2 be independent r.v.s. According to [12], $F_1 \in \mathscr{S}$ and $F_2 \in \mathscr{S}$ does not imply that convolution of F_1 and F_2 is in $\mathscr{S}$, unless $F_1 = F_2$. Hence, both convolution closure and property $\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) + \overline{F_2}(x)$ do not hold in $\mathscr{S}$. Therefore, equivalence relation $\mathrm{P}(X_1 + X_2 > x) \sim \mathrm{P}(X_1 > x) + \mathrm{P}(X_2 > x)$ is not valid in $\mathscr{L}$ since $\mathscr{S} \subset \mathscr{L}$, see also discussion in [2].

3. Proofs of main results

3.1. Proof of Theorem 2.1

The proof of Theorem 2.1 is essentially based on the uniform closure property of the sum $S_n^w := w_1 X_1 + \cdots + w_n X_n$: if Assumptions A and B are satisfied and each $F_k \in \mathscr{L}$, then the distribution of sum S_n^w is uniformly in $\mathscr{L}$ too, in the sense of the following lemma.

Lemma 3.1. *Let $X_1, \ldots, X_n$ (with $n \geq 2$) be the real-valued r.v.s with corresponding distributions $F_1, \ldots, F_n$ and let Assumptions A, B hold. If $F_k \in \mathscr{L}$, $k = 1, \ldots, n$, then for any $K > 0$ the relation*

$$\mathrm{P}\big(S_n^w > x - K\big) \sim \mathrm{P}\big(S_n^w > x\big) \tag{3.1}$$

holds uniformly for $\overline{w}_n = (w_1, \ldots, w_n) \in [a, b]^n$.

Proof. It is sufficient to prove that

$$\limsup_{x\to\infty} \sup_{\overline{w}_n \in [a,b]^n} \frac{\mathrm{P}(S_n^w > x - K)}{\mathrm{P}(S_n^w > x)} \leq 1. \tag{3.2}$$

By Remark 2.1, relation (3.1) holds for $n = 2$ (see Lemma 3.1 in [26]). Suppose that relation (3.2) holds for some $n = N \geq 2$, i.e.,

$$\mathrm{P}(S_N^w > x - K) \sim \mathrm{P}(S_N^w > x) \tag{3.3}$$

with above uniformity. We will prove that (3.2) holds for $n = N + 1$. This will prove the statement of the lemma.

Let $\epsilon \in (0, 1)$ be an arbitrary constant. Since $F_{N+1} \in \mathscr{L}$, we have that

$$\frac{\mathrm{P}(X_{N+1} > x - K)}{\mathrm{P}(X_{N+1} > x)} \leq 1 + \epsilon \tag{3.4}$$

if $x \geq x_1 > 0$. Also, condition (2.1) implies that

$$\begin{aligned}(1-\epsilon)\overline{F}_{N+1}(x) g_{N+1}(y_1, \ldots, y_N) &\leq \mathrm{P}(X_{N+1} > x | X_1 = y_1, \ldots, X_N = y_N) \\ &\leq (1+\epsilon)\overline{F}_{N+1}(x) g_{N+1}(y_1, \ldots, y_N)\end{aligned} \tag{3.5}$$

for all $y_i \in \mathbb{R}$, $i = 1, \ldots, N$ and $x \geq x_2 \geq x_1$.

If $x \geq \max\{bx_2, x_2\}$, then

(3.6)
$$
\begin{aligned}
&\frac{\mathrm{P}(S^w_{N+1} > x - K)}{\mathrm{P}(S^w_{N+1} > x)} \\
&= \frac{(\int_{\mathcal{D}_1} + \int_{\mathcal{D}_2})\mathrm{P}(w_{N+1}X_{N+1} > x - K - \sum_{i=1}^N w_i y_i | X_1 = y_1, \ldots, X_N = y_N)\mathrm{d}F_{X_1,\ldots,X_N}(y_1,\ldots,y_N)}{(\int_{\mathcal{D}_3} + \int_{\mathcal{D}_4})\mathrm{P}(w_{N+1}X_{N+1} > x - \sum_{i=1}^N w_i y_i | X_1 = y_1, \ldots, X_N = y_N)\mathrm{d}F_{X_1,\ldots,X_N}(y_1,\ldots,y_N)} \\
&=: \frac{I_{11}(x) + I_{12}(x)}{I_{21}(x) + I_{22}(x)} \;\leq\; \max\left\{\frac{I_{11}(x)}{I_{21}(x)}, \frac{I_{12}(x)}{I_{22}(x)}\right\},
\end{aligned}
$$
where
$$
\begin{aligned}
\mathcal{D}_1 &:= \{(y_1, \ldots, y_N) \colon \sum_{i=1}^N w_i y_i \leq x - bx_2 - K\}, \\
\mathcal{D}_2 &:= \{(y_1, \ldots, y_N) \colon \sum_{i=1}^N w_i y_i > x - bx_2 - K\}, \\
\mathcal{D}_3 &:= \{(y_1, \ldots, y_N) \colon \sum_{i=1}^N w_i y_i \leq x - bx_2\}, \\
\mathcal{D}_4 &:= \{(y_1, \ldots, y_N) \colon \sum_{i=1}^N w_i y_i > x - bx_2\}.
\end{aligned}
$$
Since $x \geq bx_2$, $x \geq x_2 \geq x_1$, relations (3.4), (3.5) imply that

(3.7)
$$
\begin{aligned}
&\sup_{\overline{w}_{N+1} \in [a,b]^{N+1}} \frac{I_{11}(x)}{I_{21}(x)} \\
&\leq \frac{1+\epsilon}{1-\epsilon} \sup_{\overline{w}_{N+1} \in [a,b]^{N+1}} \frac{\int_{\mathcal{D}_1} \mathrm{P}(w_{N+1}X_{N+1} > x - K - \sum_{i=1}^N w_i y_i) g_{N+1}(y_1, \ldots, y_N)\mathrm{d}F_{X_1,\ldots,X_N}(y_1, \ldots, y_N)}{\int_{\mathcal{D}_1} \mathrm{P}(w_{N+1}X_{N+1} > x - \sum_{i=1}^N w_i y_i) g_{N+1}(y_1, \ldots, y_N)\mathrm{d}F_{X_1,\ldots,X_N}(y_1, \ldots, y_N)} \\
&\leq \frac{1+\epsilon}{1-\epsilon} \sup_{\overline{w}_{N+1} \in [a,b]^{N+1}} \sup_{(y_1,\ldots,y_N) \in \mathcal{D}_1} \frac{\mathrm{P}(w_{N+1}X_{N+1} > x - K - \sum_{i=1}^N w_i y_i)}{\mathrm{P}(w_{N+1}X_{N+1} > x - \sum_{i=1}^N w_i y_i)} \\
&\leq \frac{1+\epsilon}{1-\epsilon} \sup_{z \geq x_2} \frac{\mathrm{P}(X_{N+1} > z - K)}{\mathrm{P}(X_{N+1} > z)} \;\leq\; \frac{(1+\epsilon)^2}{1-\epsilon}.
\end{aligned}
$$
On the other hand, condition (2.2) implies that
$$
\begin{aligned}
(1-\epsilon) h^{(w)}_{N+1}(y_{N+1}) \mathrm{P}(S^w_N > x) &\leq \mathrm{P}(S^w_N > x | X_{N+1} = y_{N+1}) \\
&\leq (1+\epsilon) h^{(w)}_{N+1}(y_{N+1}) \mathrm{P}(S^w_N > x) \qquad (3.8)
\end{aligned}
$$
for all $y_{N+1} \in \mathbb{R}$, $\overline{w}_N \in [a,b]^N$ and $x \geq x_3$. Hence,
$$
\begin{aligned}
I_{22}(x) &= \mathrm{P}\Big(S^w_N > x - bx_2, S^w_{N+1} > x\Big) \\
&\geq \mathrm{P}\Big(S^w_N > x, S^w_{N+1} > x\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{P}(S_N^w > x, X_{N+1} \geq 0) + \mathrm{P}(S_N^w + w_{N+1}X_{N+1} > x, X_{N+1} < 0)\\
&= \int_{[0,\infty)} \mathrm{P}(S_N^w > x | X_{N+1} = y_{N+1}) \mathrm{d}F_{N+1}(y_{N+1})\\
&\quad + \int_{(-\infty,0)} \mathrm{P}(S_N^w > x - w_{N+1}y_{N+1} | X_{N+1} = y_{N+1}) \mathrm{d}F_{N+1}(y_{N+1})\\
&\geq (1-\epsilon) \int_{[0,\infty)} \mathrm{P}(S_N^w > x) h_{N+1}^{(w)}(y_{N+1}) \mathrm{d}F_{N+1}(y_{N+1})\\
&\quad + (1-\epsilon) \int_{(-\infty,0)} \mathrm{P}(S_N^w > x - w_{N+1}y_{N+1}) h_{N+1}^{(w)}(y_{N+1}) \mathrm{d}F_{N+1}(y_{N+1})\\
&= (1-\epsilon)\mathrm{P}(S_N^w > x)\mathrm{E}h_{N+1}^{(w)}(X_{N+1})\mathbb{1}_{\{X_{N+1}\geq 0\}}\\
&\quad + (1-\epsilon) \int_{(-\infty,0)} \mathrm{P}(S_N^w > x - w_{N+1}y_{N+1}) h_{N+1}^{(w)}(y_{N+1}) \mathrm{d}F_{N+1}(y_{N+1}) \qquad (3.9)
\end{aligned}
$$

for all $\overline{w}_{N+1} \in [a,b]^{N+1}$ and $x \geq x_3$. Here, $\mathrm{E}h_{N+1}^{(w)}(X_{N+1})\mathbb{1}_{\{X_{N+1}\geq 0\}} > 0$ because of heavy tailedness of F_{N+1}. Similarly, under (3.8),

(3.10)

$$
\begin{aligned}
&I_{12}(x)\\
&= \mathrm{P}(S_{N+1}^w > x - K, S_N^w > x - bx_2 - K)\\
&\leq \mathrm{P}(S_{N+1}^w > x - K, S_N^w > x - K) + \mathrm{P}(x - bx_2 - K < S_N^w \leq x - K)\\
&= \mathrm{P}(S_N^w > x - K, X_{N+1} \geq 0) + \mathrm{P}(S_N^w + w_{N+1}X_{N+1} > x - K, X_{N+1} < 0)\\
&\quad + \ \mathrm{P}(x - bx_2 - K < S_N^w \leq x - K)\\
&\leq (1+\epsilon)\mathrm{P}(S_N^w > x - K)\mathrm{E}h_{N+1}^{(w)}(X_{N+1})\mathbb{1}_{\{X_{N+1}\geq 0\}}\\
&\quad + (1+\epsilon) \int_{(-\infty,0)} \mathrm{P}(S_N^w > x - K - w_{N+1}y_{N+1}) h_{N+1}^{(w)}(y_{N+1}) \mathrm{d}F_{N+1}(y_{N+1})\\
&\quad + \mathrm{P}(S_N^w > x - bx_2 - K) - \mathrm{P}(S_N^w > x - K)
\end{aligned}
$$

for $x \geq x_3$ and all $\overline{w}_{N+1} \in [a,b]^{N+1}$.

Relations (3.9), (3.10) imply that

$$
\begin{aligned}
&\limsup_{x\to\infty} \sup_{\overline{w}_{N+1}\in[a,b]^{N+1}} \frac{I_{12}(x)}{I_{22}(x)}\\
&\leq \frac{1}{1-\epsilon} \limsup_{x\to\infty} \sup_{\overline{w}_N\in[a,b]^N} \left(\frac{\mathrm{P}(S_N^w > x - bx_2 - K)}{\mathrm{P}(S_N^w > x)} - \frac{\mathrm{P}(S_N^w > x - K)}{\mathrm{P}(S_N^w > x)} \right)\\
&\quad + \frac{1+\epsilon}{1-\epsilon} \max\Bigg\{ \limsup_{x\to\infty} \sup_{\overline{w}_N\in[a,b]^N} \frac{\mathrm{P}(S_N^w > x - K)}{\mathrm{P}(S_N^w > x)},\\
&\quad \limsup_{x\to\infty} \sup_{\overline{w}_N\in[a,b]^N} \sup_{y_{N+1}<0} \frac{\mathrm{P}(S_N^w > x - w_{N+1}y_{N+1} - K)}{\mathrm{P}(S_N^w > x - w_{N+1}y_{N+1})} \Bigg\}.
\end{aligned}
$$

From induction hypothesis (3.3) we obtain that

$$\limsup_{x\to\infty} \sup_{\overline{w}_{N+1}\in[a,b]^{N+1}} \frac{I_{12}(x)}{I_{22}(x)} \le \frac{1+\epsilon}{1-\epsilon}. \tag{3.11}$$

Hence, by (3.6), (3.7), (3.11), we get

$$\limsup_{x\to\infty} \sup_{\overline{w}_{N+1}\in[a,b]^{N+1}} \frac{\mathrm{P}(S^w_{N+1} > x-K)}{\mathrm{P}(S^w_{N+1} > x)} \le \frac{(1+\epsilon)^2}{1-\epsilon}.$$

The arbitrariness of $\epsilon > 0$ implies inequality (3.2) for $n = N+1$. □

It is easy to see that the result in Lemma 3.1 can be reformulated replacing "for any constant $K > 0$" by "for some infinitely increasing positive function $K(x)$" (see, e.g., the arguments in [27]). Thus we have:

Corollary 3.1. *Assume the conditions in Lemma 3.1. Then, for some infinitely increasing positive function $K(x)$, it holds that*

$$\mathrm{P}\big(S^w_n > x \pm K(x)\big) \sim \mathrm{P}\big(S^w_n > x\big) \tag{3.12}$$

uniformly for $\overline{w}_n \in [a,b]^n$.

Proof of Theorem 2.1. Using Lemma 3.1, we obtain that for any $K > 0$

$$\begin{aligned}\mathrm{P}(S^\Theta_n > x-K) &= \int\cdots\int_{[a,b]^n} \mathrm{P}(S^w_n > x-K)\mathrm{P}(\Theta_1\in \mathrm{d}w_1,\dots,\Theta_n\in \mathrm{d}w_n)\\ &\sim \int\cdots\int_{[a,b]^n} \mathrm{P}(S^w_n > x)\mathrm{P}(\Theta_1\in \mathrm{d}w_1,\dots,\Theta_n\in \mathrm{d}w_n)\\ &= \mathrm{P}(S^\Theta_n > x).\end{aligned}$$

□

3.2. Proof of Theorem 2.2

The proof of Theorem 2.2 is based on the following lemma. Set $S^w_n := \sum_{k=1}^n w_k X_k$, $S^{w+}_n := \sum_{k=1}^n w_k X_k^+$ and $M^w_n := \max\{S^w_1,\dots,S^w_n\}$.

Lemma 3.2. *Let $X_1,\dots,X_n$ $(n \ge 2)$ be real-valued r.v.s with corresponding distributions $F_1,\dots,F_n$, such that each $F_k \in \mathscr{L}$. Then, under Assumption C,*

$$\mathrm{P}(S^w_n > x) \sim \mathrm{P}(S^{w+}_n > x) \sim \mathrm{P}(M^w_n > x)$$

uniformly for $\overline{w}_n \in [a,b]^n$.

Proof. Since $S^w_n \le M^w_n \le S^{w+}_n$, we only need to prove that

$$\mathrm{P}(S^{w+}_n > x) \lesssim \mathrm{P}(S^w_n > x). \tag{3.13}$$

Obviously, for positive x, it holds

$$\begin{aligned}\mathrm{P}(S^{w+}_n > x) &= \mathrm{P}(S^w_n > x) + \mathrm{P}(S^{w+}_n > x, S^w_n \le x)\\ &= \mathrm{P}(S^w_n > x) + \sum_I \mathrm{P}(S^{w+}_n > x, S^w_n \le x, \mathcal{A}_I(X))\end{aligned}$$

$$=: \mathrm{P}(S_n^w > x) + \sum_I p_I, \tag{3.14}$$

where the sum $\sum\limits_I$ is taken over all nonempty subsets $I \subset \{1, 2, \ldots, n\}$ and

$$\mathcal{A}_I(X) := \Big\{ \bigcap_{k \in I} \{X_k \geq 0\} \Big\} \bigcap \Big\{ \bigcap_{k \in I^c} \{X_k < 0\} \Big\}.$$

Let $I = \{k_1, \ldots, k_m\}$ be a fixed subset of indices with nonempty $I^c = \{r_1, \ldots, r_{n-m}\}$. Set $l := n - m$ and write

$$\begin{aligned}
p_I &= \mathrm{P}\Big(\sum_{k \in I} w_k X_k > x, \sum_{k \in I} w_k X_k + \sum_{r \in I^c} w_r X_r \leq x, X_k \geq 0, k \in I; X_r < 0, r \in I^c \Big) \\
&\leq \mathrm{P}\Big(\sum_{k \in I} w_k X_k > x, \sum_{k \in I} w_k X_k + \sum_{r \in I^c} w_r X_r \leq x, X_r < 0, r \in I^c \Big) \\
&= \mathrm{P}\Big(\sum_{k \in I} w_k X_k > x, X_r < 0, r \in I^c \Big) - \mathrm{P}\Big(\sum_{k \in I} w_k X_k + \sum_{r \in I^c} w_r X_r > x, X_r < 0, r \in I^c \Big) \\
&\leq \int\limits_{(-\infty,0)} \cdots \int\limits_{(-\infty,0)} \mathrm{P}\Big(\sum_{k \in I} w_k X_k > x \big| X_r = y_r, r \in I^c \Big) \mathrm{d}F_{X_{r_1}, \ldots, X_{r_l}}(y_{r_1}, \ldots, y_{r_l}) \\
&\quad - \int\limits_{(-\infty,0)} \cdots \int\limits_{(-\infty,0)} \mathrm{P}\Big(\sum_{k \in I} w_k X_k > x - b \sum_{r \in I^c} y_r \big| X_r = y_r, r \in I^c \Big) \mathrm{d}F_{X_{r_1}, \ldots, X_{r_l}}(y_{r_1}, \ldots, y_{r_l}) \\
&\leq C \Bigg(\int\limits_{(-\infty,0)} \cdots \int\limits_{(-\infty,0)} \pi_I'(x, y_r, r \in I^c) \mathrm{d}F_{X_{r_1}, \ldots, X_{r_l}}(y_{r_1}, \ldots, y_{r_l}) \\
&\qquad - \int\limits_{(-\infty,0)} \cdots \int\limits_{(-\infty,0)} \pi_I''(x, y_r, r \in I^c) \mathrm{d}F_{X_{r_1}, \ldots, X_{r_l}}(y_{r_1}, \ldots, y_{r_l}) \Bigg) \\
&=: C p_I',
\end{aligned}$$

where

$$\pi_I'(x, y_r, r \in I^c) := \frac{\mathrm{P}\big(\sum_{k \in I} w_k X_k > x \big| X_r = y_r, r \in I^c \big)}{h_{I,I^c}^{(w)}(y_{r_1}, \ldots, y_{r_l})},$$

$$\pi_I''(x, y_r, r \in I^c) := \frac{\mathrm{P}\big(\sum_{k \in I} w_k X_k > x - b \sum_{r \in I^c} y_r \big| X_r = y_r, r \in I^c \big)}{h_{I,I^c}^{(w)}(y_{r_1}, \ldots, y_{r_l})},$$

and where we have used that, by Assumption C,

$$\sup_{w_k \in [a,b], k \in I} \; \sup_{(y_{r_1}, \ldots, y_{r_l}) \in \mathbb{R}^l} h_{I,I^c}^{(w)}(y_{r_1}, \ldots, y_{r_l}) \leq \text{Const} \; < \; \infty.$$

According to the Fatou lemma, Assumption C and Lemma 3.1,

$$\limsup_{x \to \infty} \sup_{w_k \in [a,b], k \in I} \frac{p_I'}{\mathrm{P}\big(\sum_{k \in I} w_k X_k > x \big)}$$

$$\leq \int_{(-\infty,0)} \cdots \int_{(-\infty,0)} \limsup_{x\to\infty} \sup_{w_k\in[a,b],k\in I} \frac{\pi'_I(x, y_r, r\in I^c)}{\mathrm{P}(\sum_{k\in I} w_k X_k > x)}\, \mathrm{d}F_{X_{r_1},\dots,X_{r_l}}(y_{r_1},\dots,y_{r_l})$$
$$- \int_{(-\infty,0)} \cdots \int_{(-\infty,0)} \liminf_{x\to\infty} \inf_{w_k\in[a,b],k\in I} \frac{\pi''_I(x, y_r, r\in I^c)}{\mathrm{P}(\sum_{k\in I} w_k X_k > x)}\, \mathrm{d}F_{X_{r_1},\dots,X_{r_l}}(y_{r_1},\dots,y_{r_l})$$
$$= 0.$$

Since $p_I \leq \mathrm{Const} p'_I$, for each subset I in (3.14) we obtain that

$$\limsup_{x\to\infty} \sup_{\overline{w}_n\in[a,b]^n} \frac{p_I}{\mathrm{P}\big(\sum_{k\in I} w_k X_k > x\big)} = 0.$$

This, together with (3.14), implies

$$\liminf_{x\to\infty} \inf_{\overline{w}_n\in[a,b]^n} \frac{\mathrm{P}(S_n^w > x)}{\mathrm{P}(S_n^{w+} > x)}$$
$$\geq 1 - \sum_I \limsup_{x\to\infty} \sup_{\overline{w}_n\in[a,b]^n} \frac{p_I}{\mathrm{P}(S_n^{w+} > x)}$$
$$= 1 - \sum_I \limsup_{x\to\infty} \sup_{\overline{w}_n\in[a,b]^n} \frac{p_I}{\mathrm{P}\big(\sum_{k\in I} w_k X_k > x\big)} = 1.$$

Thus, relation (3.13) holds and the lemma is proved. □

Proof of Theorem 2.2. Similarly, as in the case of Theorem 2.1, the proof follows immediately from Lemma 3.2. □

4. The case of dependence described through copula

In this section we demonstrate how the functions g_k, $h_k^{(w)}$ and $h_{I,J}^{(w)}$, appearing in Assumptions A, B and C, can be found when the dependence structure among $X_1, \dots, X_n$ is generated by an n-dimensional absolutely continuous copula $C(v_1, \dots, v_n)$.

4.1. General copula dependence

Assume that the distribution of vector $(X_1, \dots, X_n)$ is given by

(4.1)
$$\mathrm{P}(X_1 \leq x_1, \dots, X_n \leq x_n) = C(F_1(x_1), \dots, F_n(x_n)),\ (x_1, \dots, x_n) \in [-\infty, \infty]^n,$$

where $C(v_1, \dots, v_n)$ is some absolutely continuous copula function with corresponding positive copula density $c(v_1, \dots, v_n)$. Assume that marginal distributions $F_1, \dots, F_n$ are absolutely continuous with corresponding positive densities $f_1, \dots, f_n$.

Consider first the case of Assumptions A and B.

Let $C_k(v_1, \dots, v_k) := C(v_1, \dots, v_k, 1, \dots, 1)$, where $k = 2, \dots, n$, be k-dimensional marginal copulas. Also write $C_1(v_1) = v_1$. Let the corresponding copula densities be $c_k(v_1, \dots, v_k)$, $k = 1, \dots, n$. Denote $\widetilde{C}_k(v_1, \dots, v_k) :=$

$C_{k-1}(v_1,\dots,v_{k-1}) - C_k(v_1,\dots,v_k)$ and let

$$\widetilde{c}_k(v_1,\dots,v_k) := \frac{\partial^{k-1}\widetilde{C}_k(v_1,\dots,v_k)}{\partial v_1 \dots \partial v_{k-1}}. \tag{4.2}$$

Further, we introduce the following assumption: for any $k = 2,\dots,n$, there exists positive limit

$$\bar{c}_k(v_1,\dots,v_{k-1},1-) := \lim_{v\searrow 0} \frac{\widetilde{c}_k(v_1,\dots,v_{k-1},1-v)}{v} \tag{4.3}$$

uniformly for $(v_1,\dots,v_{k-1}) \in [0,1]^{k-1}$.

Denote $X_1^*,\dots,X_n^*$ the corresponding independent copies of r.v.s $X_1,\dots,X_n$ and set $S_k^{w*} := w_1X_1^* + \dots + w_kX_k^*$, $k = 1,\dots,n$.

Proposition 4.1. *Assume that the distribution of random vector* $(X_1,\dots,X_n)$ *is given by* (4.1) *with some absolutely continuous copula* $C(v_1,\dots,v_n)$ *and absolutely continuous marginal distributions* $F_1,\dots,F_n$ *with corresponding positive densities* $f_1,\dots,f_n$. *Then Assumption A is equivalent to* (4.3) *and in this case functions* g_k, $k = 2,\dots,n$ *are given by*

$$g_k(y_1,\dots,y_{k-1}) = \frac{\bar{c}_k(F_1(y_1),\dots,F_{k-1}(y_{k-1}),1-)}{c_{k-1}(F_1(y_1),\dots,F_{k-1}(y_{k-1}))}. \tag{4.4}$$

Furthermore, Assumption B is equivalent to the existence of positive limits

$$h_k^{(w)}(y) := \lim_{x\to\infty} \frac{\mathrm{E}c_k(F_1(X_1^*),\dots,F_{k-1}(X_{k-1}^*),F_k(y))\mathbb{1}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{E}c_{k-1}(F_1(X_1^*),\dots,F_{k-1}(X_{k-1}^*))\mathbb{1}_{\{S_{k-1}^{w*}>x\}}} \tag{4.5}$$

uniformly for $\overline{w}_{k-1} \in [a,b]^{k-1}$, $y \in \mathbb{R}$ *and* $k = 2,\dots,n$.

Proof. Denote the k-dimensional density of vector $(X_1,\dots,X_k)$ by $f_{X_1,\dots,X_k}$. Clearly,

$$f_{X_1,\dots,X_k}(y_1,\dots,y_k) = c_k(F_1(y_1),\dots,F_k(y_k))f_1(y_1)\cdots f_k(y_k), \tag{4.6}$$

which is positive for all k by the positivity of copula density c and marginal densities $f_1,\dots,f_n$. Hence,

$$\begin{aligned} &\mathrm{P}(X_k > x | X_1 = y_1,\dots,X_{k-1} = y_{k-1}) \\ &= \frac{\partial^{k-1}\mathrm{P}(X_k > x, X_1 \le y_1,\dots,X_{k-1} \le y_{k-1})}{\partial y_1 \dots \partial y_{k-1}} \frac{1}{f_{X_1,\dots,X_{k-1}}(y_1,\dots,y_{k-1})} \\ &= \frac{\widetilde{c}_k(F_1(y_1),\dots,F_{k-1}(y_{k-1}),F_k(x))}{c_{k-1}(F_1(y_1),\dots,F_{k-1}(y_{k-1}))}, \end{aligned} \tag{4.7}$$

which follows from (4.6) and equality

$$\begin{aligned} &\frac{\partial^{k-1}\mathrm{P}(X_k > x, X_1 \le y_1,\dots,X_{k-1} \le y_{k-1})}{\partial y_1 \dots \partial y_{k-1}} \\ &= \widetilde{c}_k(F_1(y_1),\dots,F_{k-1}(y_{k-1}),F_k(x))f_1(y_1)\dots f_{k-1}(y_{k-1}). \end{aligned}$$

The last equality holds by (4.2).

By (4.7), Assumption A is equivalent to

$$\lim_{x\to\infty} \frac{\widetilde{c}_k(F_1(y_1),\ldots,F_{k-1}(y_{k-1}),F_k(x))}{\overline{F_k}(x)} = g_k(y_1,\ldots,y_{k-1})c_{k-1}(F_1(y_1),\ldots,F_{k-1}(y_{k-1}))$$

for some positive functions g_k, uniformly for $(y_1,\ldots,y_{k-1}) \in \mathbb{R}^{k-1}$, $k = 2,\ldots,n$. But the last relation is equivalent to (4.3). Thus, (4.4) holds.

Let's deal with Assumption B. Since F_k is absolutely continuous, we have

$$\mathrm{P}\big(S^w_{k-1} > x | X_k = y\big) = \frac{\partial \mathrm{P}(S^w_{k-1} > x, X_k \le y)}{\partial y}\,\frac{1}{f_k(y)}. \tag{4.8}$$

It is easy to see that

$$\begin{aligned}\frac{\partial \mathrm{P}(S^w_{k-1} > x, X_k \le y)}{\partial y} &= f_k(y)\int_{\sum_{i=1}^{k-1} w_i u_i > x} c_k(F_1(u_1),\ldots,F_{k-1}(u_{k-1}),F_k(y)) \\ &\qquad f_1(u_1)\cdots f_{k-1}(u_{k-1})\mathrm{d}u_1\cdots\mathrm{d}u_{k-1} \\ &= f_k(y)\mathrm{E}c_k(F_1(X_1^*),\ldots,F_{k-1}(X^*_{k-1}),F_k(y))\mathbb{1}_{\{S^{w*}_{k-1}>x\}}.\end{aligned}$$

Hence, by (4.8) and equality

$$\mathrm{P}(S^w_{k-1} > x) = \mathrm{E}c_{k-1}(F_1(X_1^*),\ldots,F_{k-1}(X^*_{k-1}))\mathbb{1}_{\{S^{w*}_{k-1}>x\}},$$

we obtain

$$\mathrm{P}\big(S^w_{k-1} > x | X_k = y\big) = \frac{\mathrm{E}c_k(F_1(X_1^*),\ldots,F_{k-1}(X^*_{k-1}),F_k(y))\mathbb{1}_{\{S^{w*}_{k-1}>x\}}}{\mathrm{E}c_{k-1}(F_1(X_1^*),\ldots,F_{k-1}(X^*_{k-1}))\mathbb{1}_{\{S^{w*}_{k-1}>x\}}}\,\mathrm{P}\big(S^w_{k-1} > x\big).$$

This implies the second statement of proposition. □

Next we formulate the similar result in the case of Assumption C. For any (not necessarily nonempty) subsets $I = \{k_1,\ldots,k_m\}$, $J = \{r_1,\ldots,r_p\} \subset \{1,\ldots,n\}\backslash I$ denote by $c_{I,J}(v_k, k\in I, v_r, r \in J)$ the copula density corresponding to random vector $(X_{k_1},\ldots,X_{k_m},X_{r_1},\ldots,X_{r_p})$, i.e.,

$$f_{X_{k_1},\ldots,X_{k_m},X_{r_1},\ldots,X_{r_p}}(y_{k_1},\ldots,y_{k_m},y_{r_1},\ldots,y_{r_p}) = c_{I,J}(F_k(y_k), k\in I, F_r(y_r), r\in J)\prod_{k\in I} f_k(y_k)\prod_{r\in J} f_r(y_r),$$

and let $c_I := c_{I,\varnothing}$, $c_J := c_{\varnothing,J}$.

Proposition 4.2. *Assume that the distribution of random vector* $(X_1,\ldots,X_n)$ *is given by* (4.1) *with some absolutely continuous copula* $C(v_1,\ldots,v_n)$ *and absolutely continuous marginal distributions* $F_1,\ldots,F_n$. *Then Assumption C is* *equivalent to the existence of positive, uniformly bounded limits*

$$h_{I,J}^{(w)}(y_{r_1},\dots,y_{r_p}) := \frac{1}{c_J(F_r(y_r), r\in J)} \lim_{x\to\infty} \frac{\mathrm{E}c_{I,J}(F_k(X_k^*), k\in I, F_r(y_r), r\in J)\mathbb{1}_{\{\sum_{k\in I} w_k X_k^* > x\}}}{\mathrm{E}c_I(F_k(X_k^*), k\in I)\mathbb{1}_{\{\sum_{k\in I} w_k X_k^* > x\}}}$$

which hold uniformly for $w_k \in [a,b]$, $k\in I$, $y_r\in\mathbb{R}$, $r\in J$ and all nonempty sets of indices $I\subset\{1,\dots,n\}$ and $J\subset\{1,\dots,n\}\backslash I$.

Proof. The proof is similar to that of Proposition 4.1. We have

$$\mathrm{P}\Big(\sum_{k\in I} w_k X_k > x \big| X_r = y_r, r\in J\Big) = \frac{\partial^p \mathrm{P}(\sum_{k\in I} w_k X_k > x, X_r \le y_r, r\in J)}{\partial y_{r_1}\dots\partial y_{r_p}} \; \frac{1}{f_{X_{r_1},\dots,X_{r_p}}(y_{r_1},\dots,y_{r_p})},$$

where

$$\frac{\partial^p \mathrm{P}(\sum_{k\in I} w_k X_k > x, X_r \le y_r, r\in J)}{\partial y_{r_1}\dots\partial y_{r_p}} = \prod_{r\in J} f_r(y_r) \int\limits_{\sum_{k\in I} w_k u_k > x} c_{I,J}(F_k(u_k), k\in I, F_r(y_r), r\in J) \prod_{k\in I} f_k(u_k)\mathrm{d}u_{k_1}\cdots\mathrm{d}u_{k_m}$$

and $f_{X_{r_1},\dots,X_{r_p}}(y_{r_1},\dots,y_{r_p}) = c_J(F_r(y_r), r\in J)\prod_{r\in J} f_r(y_r)$. Now the proof follows observing that

$$\mathrm{P}\Big(\sum_{k\in I} w_k X_k > x\Big) = \mathrm{E}c_I(F_k(X_k^*), k\in I)\mathbb{1}_{\{\sum_{k\in I} w_k X_k^* > x\}}.$$

□

4.2. The case of FGM copula

In this subsection, we consider the case where $C(v_1,\dots,v_n)$ is n-dimensional Farley–Gumbel–Morgenstern (FGM) copula, given by

$$C(v_1,\dots,v_n) = \prod_{i=1}^n v_i\Big(1 + \sum_{1\le l<m\le n} \theta_{lm}(1-v_l)(1-v_m)\Big), \tag{4.9}$$

where $(v_1,\dots,v_n)\in[0,1]^n$ and real numbers θ_{lm} are chosen such that $C(v_1,\dots,v_n)$ is a proper n-dimensional copula. For example, if $n=3$, the conditions can be summarized as follows: $\theta_{12}+\theta_{13}+\theta_{23}\ge -1$, $\theta_{13}+\theta_{23}-\theta_{12}\le 1$, $\theta_{12}+\theta_{23}-\theta_{13}\le 1$, $\theta_{12}+\theta_{13}-\theta_{23}\le 1$. In this case,

$$C_k(v_1,\dots,v_k) = \prod_{i=1}^k v_i\Big(1 + \sum_{1\le l<m\le k} \theta_{lm}(1-v_l)(1-v_m)\Big), \quad k=2,\dots,n,$$

and the corresponding copula densities are given by

$$c_k(v_1,\dots,v_k) = 1 + \sum_{1\le l<m\le k} \theta_{lm}(1-2v_l)(1-2v_m), \quad k=2,\dots,n. \tag{4.10}$$

Everywhere below we assume the parameters θ_{lm} to be such that $c_n(v_1,\dots,v_n) > 0$ for all $(v_1,\dots,v_n) \in [0,1]^n$. Obviously, this implies that $c_k(v_1,\dots,v_k) > 0$ for all $(v_1,\dots,v_k) \in [0,1]^k$ and $k=2,\dots,n$.

Next, we make the following assumption:

Assumption D. For each $k=1,\dots,n-1$ there exists limit

$$\lim_{x\to\infty} \frac{\overline{F}_k(x/w_k)}{\overline{F}_1(x/w_1)+\dots+\overline{F}_{n-1}(x/w_{n-1})} =: a_k^{(w)} \in (0,1]$$

uniformly for $\overline{w}_{n-1} \in [a,b]^{n-1}$.

To illustrate Assumption D, suppose that $F_1,\dots,F_n$ are such that $\overline{F_i}(x) \sim c_i L(x) x^{-\alpha}$, $\alpha \ge 0$, with some positive constants c_i, $i=1,\dots,n$, and slowly varying function $L(x)$. Then Assumption D is satisfied and

$$a_k^{(w)} = \frac{c_k}{c_1(w_1/w_k)^\alpha + \dots + c_{n-1}(w_{n-1}/w_k)^\alpha}.$$

On the other hand, if $a=b$ and $\overline{F}_i(x) \sim c_i\overline{G}(x)$, $i=1,\dots,n$, where $\overline{G}(x)>0$ for all x, then

$$a_k^{(w)} = \frac{c_k}{c_1+\dots+c_{n-1}}.$$

Next we will derive the expressions for functions g_k and $h_k^{(w)}$, omitting the case of function $h_{I,J}^{(w)}$, for which the corresponding expression is complicated and does not carry much interest.

For a distribution F, denote $\widetilde{F} := 1-2F = 2\overline{F}-1$.

Proposition 4.3. *Assume $n\ge 2$ and let $X_1,\dots,X_n$ be real-valued r.v.s whose distribution is generated by FGM copula in* (4.9)*, marginal distributions $F_1,\dots,F_n$ are absolutely continuous and $F_i \in \mathscr{L}\cap\mathscr{D}$, $i=1,\dots,n$. Then*

$$g_k(y_1,\dots,y_{k-1}) = 1 - \frac{\sum_{1\le l\le k-1}\theta_{lk}\widetilde{F}_l(y_l)}{c_{k-1}(F_1(y_1),\dots,F_{k-1}(y_{k-1}))}, \quad k=2,\dots,n.$$

If $n\ge 3$ and Assumption D holds, then

$$h_k^{(w)}(y) = 1 - \widetilde{F}_k(y)\sum_{1\le l\le k-1}\theta_{lk}a_{l,k-1}^{(w)}, \quad k=3,\dots,n,$$

where $a_{l,k-1}^{(w)} := a_l^{(w)}/(a_1^{(w)}+\dots+a_{k-1}^{(w)})$.

Proof. We apply Proposition 4.1. Obviously,

$$\widetilde{C}_k(v_1,\dots,v_k) = (1-v_k)C_{k-1}(v_1,\dots,v_{k-1}) - v_1\cdots v_k(1-v_k)\sum_{1\le l\le k-1}\theta_{lk}(1-v_l),$$

implying that $\tilde{c}_k(v_1,\dots,v_k)$ in (4.2) is

$$\tilde{c}_k(v_1,\dots,v_k) = (1-v_k)c_{k-1}(v_1,\dots,v_{k-1}) - \ v_k(1-v_k)\sum_{1\le l\le k-1}\theta_{lk}(1-2v_l).$$

Hence, condition (4.3) is satisfied (uniformly in $(v_1,\dots,v_{k-1})\in[0,1]^{k-1}$) and

$$\begin{aligned}\bar{c}_k(v_1,\dots,v_{k-1},1-) &= \lim_{v\searrow 0}\Big(c_{k-1}(v_1,\dots,v_{k-1}) - (1-v)\sum_{1\le l\le k-1}\theta_{lk}(1-2v_l)\Big)\\ &= c_{k-1}(v_1,\dots,v_{k-1}) - \sum_{1\le l\le k-1}\theta_{lk}(1-2v_l).\end{aligned}$$

Therefore, by (4.4),

$$g_k(y_1,\dots,y_{k-1}) = 1 - \frac{\sum_{1\le l\le k-1}\theta_{lk}(1-2F_l(y_l))}{c_{k-1}(F_1(y_1),\dots,F_{k-1}(y_{k-1}))}.$$

Consider now function $h_k^{(w)}(y)$. For $k=2,\dots,n$ we have

$$h_k^{(w)}(y) = \lim_{x\to\infty}\frac{\varphi_k^{(w)}(x,y)}{\varphi_{k-1}^{(w)}(x)},$$

where, by (4.5) and (4.10),

$$\begin{aligned}\varphi_k^{(w)}(x,y) &:= \mathrm{E}c_k(F_1(X_1^*),\dots,F_{k-1}(X_{k-1}^*),F_k(y))\mathbb{1}_{\{S_{k-1}^{w*}>x\}}\\ &= \mathrm{P}(S_{k-1}^{w*}>x) + \sum_{1\le l<m\le k-1}\theta_{lm}\mathrm{E}\widetilde{F}_l(X_l^*)\widetilde{F}_m(X_m^*)\mathbb{1}_{\{S_{k-1}^{w*}>x\}}\\ &\quad + \ \widetilde{F}_k(y)\sum_{1\le l\le k-1}\theta_{lk}\mathrm{E}\widetilde{F}_l(X_l^*)\mathbb{1}_{\{S_{k-1}^{w*}>x\}},\\ \varphi_{k-1}^{(w)}(x) &:= \mathrm{E}c_{k-1}(F_1(X_1^*),\dots,F_{k-1}(X_{k-1}^*))\mathbb{1}_{\{S_{k-1}^{w*}>x\}}\\ &= \mathrm{P}(S_{k-1}^{w*}>x) + \sum_{1\le l<m\le k-1}\theta_{lm}\mathrm{E}\widetilde{F}_l(X_l^*)\widetilde{F}_m(X_m^*)\mathbb{1}_{\{S_{k-1}^{w*}>x\}}.\end{aligned}$$

Rewrite now

$$\frac{\varphi_k^{(w)}(x,y)}{\varphi_{k-1}^{(w)}(x)} = 1 + \widetilde{F}_k(y)b_k^{(w)}(x),$$

where

$$b_k^{(w)}(x) := \frac{\sum\limits_{1\le l\le k-1}\theta_{lk}\mathrm{E}\widetilde{F}_l(X_l^*)\mathbb{1}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{P}(S_{k-1}^{w*}>x) + \sum\limits_{1\le l<m\le k-1}\theta_{lm}\mathrm{E}\widetilde{F}_l(X_l^*)\widetilde{F}_m(X_m^*)\mathbb{1}_{\{S_{k-1}^{w*}>x\}}}.$$

It remains to prove that, uniformly in $\overline{w}_{k-1}\in[a,b]^{k-1}$,

$$b_k^{(w)}(x) \to -\sum_{1\le l\le k-1}\theta_{lk}a_{l,k-1}^{(w)} =: b_k^{(w)}, \quad k=3,\dots,n. \tag{4.11}$$

Rewrite

$$b_k^{(w)}(x) = \frac{2\sum\limits_{1\le l\le k-1}\theta_{lk}\mathrm{E}\overline{F_l}(X_l^*)\mathbb{1}_{\{S_{k-1}^{w*}>x\}} - \mathrm{P}(S_{k-1}^{w*}>x)\sum\limits_{1\le l\le k-1}\theta_{lk}}{2\sum\limits_{1\le l<m\le k-1}\theta_{lm}\mathrm{E}Y_{lm}^*\mathbb{1}_{\{S_{k-1}^{w*}>x\}} + \mathrm{P}(S_{k-1}^{w*}>x) + \mathrm{P}(S_{k-1}^{w*}>x)\sum\limits_{1\le l<m\le k-1}\theta_{lm}},$$

where $Y_{lm}^* := 2\overline{F_l}(X_l^*)\overline{F_m}(X_m^*) - \overline{F_l}(X_l^*) - \overline{F_m}(X_m^*)$. The desired convergence (4.11) will follow if we show that

$$\mathrm{E}\overline{F_l}(X_l^*)\mathbb{1}_{\{S_{k-1}^{w*}>x\}} \sim \frac{1}{2}\,(1-a_{l,k-1}^{(w)})\mathrm{P}(S_{k-1}^{w*}>x), \quad l=1,\dots,k-1, \tag{4.12}$$

$$\mathrm{E}Y_{lm}^*\mathbb{1}_{\{S_{k-1}^{w*}>x\}} \sim -\frac{1}{2}\,\mathrm{P}(S_{k-1}^{w*}>x), \quad 1\le l<m\le k-1, \tag{4.13}$$

uniformly in $\overline{w}_{k-1}\in[a,b]^{k-1}$.

To show (4.12), take $Y_i = X_i^*$, $a_i(x)\equiv\overline{F_i}(x)$ in Corollary 5.1 below and note that condition (5.16) is satisfied:

$$\mathrm{E}\overline{F_i}(X_i^*)\mathbb{1}_{\{X_i^*>x\}} = \overline{F_j}(x)\int_x^\infty \frac{\overline{F_i}(y)}{\overline{F_j}(x)}\,\mathrm{d}F_i(y) \;=\; o(\overline{F_j}(x)), \;\; j\ne i,$$

because, by Assumption D, $\overline{F_i}(x)\sim c_{ij}\overline{F_j}(x)$ with some positive constant c_{ij}. Combining Corollary 5.1, Proposition 5.1(i) and using that $\mathrm{E}\overline{F_l}(X_l^*)=1/2$ for all $l=1,\dots,n$ (since distribution F_l has positive density), we get

$$\begin{aligned}\lim_{x\to\infty}\frac{\mathrm{E}\overline{F_l}(X_l^*)\mathbb{1}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{P}(S_{k-1}^{w*}>x)} &= \mathrm{E}\overline{F_l}(X_l^*)\lim_{x\to\infty}\frac{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)-\overline{F}_l(x/w_l)}{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)}\\ &= \frac{1}{2}\,(1-a_{l,k-1}^{(w)}), \quad l=1,\dots,k-1,\end{aligned}$$

uniformly in $\overline{w}_{k-1}\in[a,b]^{k-1}$ (note that $0<a_{l,k-1}^{(w)}<1$ because $\sum_{l=1}^{k-1}a_{l,k-1}^{(w)}=1$ and $a_{l,k-1}^{(w)}>0$, $k\ge 3$). Thus, we get (4.12).

The proof of relation (4.13) is similar. If $k>3$, then, by Corollary 5.1,

$$\begin{aligned}&\lim_{x\to\infty}\frac{\mathrm{E}Y_{lm}^*\mathbb{1}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{P}(S_{k-1}^{w*}>x)}\\ &= \lim_{x\to\infty}\frac{\mathrm{E}(2\overline{F_l}(X_l^*)\overline{F_m}(X_m^*)-\overline{F_l}(X_l^*)-\overline{F_m}(X_m^*))\mathbb{1}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{P}(S_{k-1}^{w*}>x)}\\ &= 2\mathrm{E}\overline{F_l}(X_l^*)\mathrm{E}\overline{F_m}(X_m^*)\lim_{x\to\infty}\frac{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)-\overline{F_l}(x/w_l)-\overline{F_m}(x/w_m)}{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)}\\ &\quad-\;\mathrm{E}\overline{F_l}(X_l^*)\lim_{x\to\infty}\frac{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)-\overline{F_l}(x/w_l)}{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)}\\ &\quad-\;\mathrm{E}\overline{F_m}(X_m^*)\lim_{x\to\infty}\frac{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)-\overline{F_m}(x/w_m)}{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)} \;=\; -\frac{1}{2}\end{aligned}$$

uniformly in $\overline{w}_{k-1} \in [a,b]^{k-1}$. The case $k=3$ in (4.13) easily follows from arguments above and (5.17). The proof is complete. □

Consider now the tail asymptotics of the sum $S_n^\Theta = \Theta_1 X_1 + \cdots + \Theta_n X_n$ in the case when the distribution of vector $(X_1, \ldots, X_n)$ is generated by the FGM copula in (4.9). The next proposition shows that in the case of primary distributions from class $\mathscr{L} \cap \mathscr{D}$, the probabilities $\mathrm{P}(S_n^\Theta > x)$ and $\mathrm{P}(S_n^{\Theta+} > x)$ asymptotically are the same and are both asymptotically equivalent to $\mathrm{P}(\Theta_1 X_1 > x) + \cdots + \mathrm{P}(\Theta_n X_n > x)$ even in the case where the positive weights Θ_k are not bounded from zero. This result follows from Theorem 1 in [21] proved in the case of the so-called pairwise strong quasi-asymptotically independence (pSQAI) structure, introduced by Geluk and Tang [9]. Recall that r.v.s $X_1, \ldots, X_n$ are pSQAI if, for any $i \neq j$,

$$\lim_{x_i \wedge x_j \to \infty} \mathrm{P}(|X_i| > x_i | X_j > x_j) = 0. \tag{4.14}$$

It easy to see that the FGM distribution given by (4.9) satisfies (4.14) (see, e.g., [9]).

Proposition 4.4. *Suppose that $n \geq 2$ and $X_1, \ldots, X_n$ are real-valued r.v.s with corresponding distributions $F_1, \ldots, F_n$, such that $F_k \in \mathscr{L} \cap \mathscr{D}$, $k = 1, \ldots, n$. Let the distribution of vector $(X_1, \ldots, X_n)$ is generated by the FGM copula* (4.9). *If $\mathrm{P}(0 < \Theta_k \leq b) = 1$, $k = 1, \ldots, n$, for some $b \in (0, \infty)$, then*

$$\begin{aligned} \mathrm{P}(S_n^\Theta > x) &\sim \mathrm{P}(S_n^{\Theta+} > x) \sim \mathrm{P}(M_n^\Theta > x) \\ &\sim \mathrm{P}\big(\max_{k=1,\ldots,n} \Theta_k X_k > x\big) \sim \sum_{k=1}^n \mathrm{P}(\Theta_k X_k > x). \end{aligned} \tag{4.15}$$

Remark 4.1. The proof of relations in (4.15) is based essentially on two facts: first, the fact that the distribution of the product ΘX, where Θ and X are independent r.v.s with $0 < \Theta \leq b$ a.s. and $F_X \in \mathscr{L} \cap \mathscr{D}$, is again in $\mathscr{L} \cap \mathscr{D}$ (see Lemmas 3.9 and 3.10 in [18]); second, the result as in (4.15) but with products $\Theta_k X_k$ replaced by the (dependent) r.v.s Y_k, such that $F_{Y_k} \in \mathscr{L} \cap \mathscr{D}$, $k = 1, \ldots, n$. Alternatively, the relation in (4.15) can be derived replacing the Θ_k's by w_k's and then proving the corresponding relations *uniformly* with respect to $\overline{w}_n = (w_1, \ldots, w_n)$. For instance, using Proposition 5.1(ii) and representation

$$\mathrm{P}(S_n^w > x) = \mathrm{P}(S_n^{w*} > x) + \sum_{1 \leq l < m \leq n} \theta_{lm} \int\limits_{w_1 y_1 + \cdots + w_n y_n > x} \mathrm{d}H_{lm}(y_1, \ldots, y_n),$$

where $S_n^{w*} := w_1 X_1^* + \cdots + w_n X_n^*$ and $H_{lm}(y_1, \ldots, y_n) := F_1(y_1) \cdots F_n(y_n)$ $\overline{F_l}(y_l) \overline{F_m}(y_m)$, or directly applying (5.1) below to the pSQAI r.v.s, we have that for the FGM copula case it holds

$$\mathrm{P}(S_n^w > x) \sim \mathrm{P}(S_n^{w*} > x) \sim \sum_{k=1}^n \overline{F}_k(x/w_k)$$

uniformly for $\overline{w}_n \in [a,b]^n$. Hence

$$\begin{aligned}
&\mathrm{P}(S_n^{\Theta} > x)\\
&\sim \int\!\!\cdots\!\!\int_{[a,b]^n} \big(\mathrm{P}(w_1X_1 > x) + \cdots + \mathrm{P}(w_nX_n > x)\big)\mathrm{P}(\Theta_1 \in \mathrm{d}w_1, \ldots, \Theta_n \in \mathrm{d}w_n)\\
&= \mathrm{P}(\Theta_1 X_1 > x) + \cdots + \mathrm{P}(\Theta_n X_n > x).
\end{aligned}$$

Obviously, the last approach leads to a weaker result as it requires the restriction $\Theta_k \in [a,b] \subset (0,b]$, $k = 1,\ldots,n$, unless the d.f.s $F_1,\ldots,F_n$ are in the class $\mathscr{C}$, see Proposition 5.1(ii).

5. Auxiliary results

In this section we present some useful statements, which are used proving the corresponding results in Section 4.2.

Proposition 5.1. *Suppose that $Y_1,\ldots,Y_n$ are real-valued independent r.v.s with corresponding distributions $F_{Y_1},\ldots,F_{Y_n}$.*

(i) *If $F_{Y_k} \in \mathscr{L} \cap \mathscr{D}$, $k = 1,\ldots,n$, then*

$$\mathrm{P}(w_1Y_1 + \cdots + w_nY_n > x) \sim \sum_{k=1}^{n} \overline{F}_{Y_k}(x/w_k) \tag{5.1}$$

uniformly for $\overline{w}_n \in [a,b]^n$, where $0 < a \le b < \infty$.

(ii) *If $F_{Y_k} \in \mathscr{C}$, $k = 1,\ldots,n$, then relation (5.1) holds uniformly for $\overline{w}_n \in (0,b]^n$, $0 < b < \infty$.*

Proof. (i) The proof of this fact follows from Theorem 2.1 in [13] (note that Li's result also holds for more general, pSQAI, dependence structure, see (4.14)).

(ii) Denote $S^w_{Y,n} := w_1Y_1 + \cdots + w_nY_n$ and write for any $\delta \in (0,1)$ and $x > 0$

$$\begin{aligned}
\mathrm{P}(S^w_{Y,n} > x) \ge\ & \sum_{k=1}^{n} \mathrm{P}(S^w_{Y,n} > x, w_kY_k > x + \delta x)\\
&- \sum_{1 \le i < j \le n} \mathrm{P}(w_iY_i > x + \delta x, w_jY_j > x + \delta x)\\
=:\ & p_1^w(x) - p_2^w(x).
\end{aligned}$$

Obviously,

$$p_2^w(x) \le \Big(\sum_{k=1}^{n} \overline{F}_{Y_k}(x/w_k)\Big)^2 = o\Big(\sum_{k=1}^{n} \overline{F}_{Y_k}(x/w_k)\Big) \tag{5.2}$$

uniformly in $\overline{w}_n \in (0,b]^n$. For $p_1^w(x)$ we have

$$p_1^w(x) \ge \sum_{k=1}^{n} \mathrm{P}(S^w_{Y,n} - w_kY_k > -\delta x, w_kY_k > x + \delta x)$$

$$= \sum_{k=1}^{n} \mathrm{P}(w_k Y_k > x + \delta x) - \sum_{k=1}^{n} \mathrm{P}(S_{Y,n}^{w} - w_k Y_k \le -\delta x, w_k Y_k > x + \delta x)$$
$$=: p_{11}^{w}(x) - p_{12}^{w}(x).$$

Here,

$$\liminf_{x\to\infty} \inf_{\overline{w}_n \in (0,b]^n} \frac{p_{11}^{w}(x)}{\sum_{k=1}^{n} \overline{F}_{Y_k}(x/w_k)} \ge \liminf_{x\to\infty} \inf_{\overline{w}_n \in (0,b]^n} \min_{1\le k\le n} \frac{\overline{F}_{Y_k}((1+\delta)x/w_k)}{\overline{F}_{Y_k}(x/w_k)}, \tag{5.3}$$

where, for any $k = 1, \dots, n$,

$$\liminf_{x\to\infty} \inf_{w_k \in (0,b]} \frac{\overline{F}_{Y_k}((1+\delta)x/w_k)}{\overline{F}_{Y_k}(x/w_k)} \ge \lim_{x\to\infty} \inf_{z\ge x/b} \frac{\overline{F}_{Y_k}((1+\delta)z)}{\overline{F}_{Y_k}(z)}$$
$$= \liminf_{x\to\infty} \frac{\overline{F}_{Y_k}((1+\delta)x)}{\overline{F}_{Y_k}(x)} \longrightarrow 1 \quad \text{if} \quad \delta \searrow 0 \tag{5.4}$$

by the definition of class $\mathscr{C}$. We get from (5.3)–(5.4) that

$$\lim_{\delta\searrow 0} \liminf_{x\to\infty} \inf_{\overline{w}_n \in (0,b]^n} \frac{p_{11}^{w}(x)}{\sum_{k=1}^{n} \overline{F}_{Y_k}(x/w_k)} \ge 1. \tag{5.5}$$

For the term $p_{12}^{w}(x)$ we get

$$p_{12}^{w}(x) \le \sum_{k=1}^{n} \mathrm{P}(S_{Y,n}^{w} - w_k Y_k \le -\delta x) \mathrm{P}(w_k Y_k > x)$$
$$\le \mathrm{P}(b(Y_1^- + \cdots + Y_n^-) \le -\delta x) \sum_{k=1}^{n} \overline{F}_{Y_k}(x/w_k) \;=\; o(1) \sum_{k=1}^{n} \overline{F}_{Y_k}(x/w_k) \tag{5.6}$$

uniformly in $\overline{w}_n \in (0,b]^n$. (5.2), (5.5) and (5.6) imply

$$\liminf_{x\to\infty} \inf_{\overline{w}_n \in (0,b]^n} \frac{\mathrm{P}(S_{Y,n}^{w} > x)}{\sum_{k=1}^{n} \overline{F}_{Y_k}(x/w_k)} \ge \liminf_{x\to\infty} \inf_{\overline{w}_n \in (0,b]^n} \frac{p_{1}^{w}(x)}{\sum_{k=1}^{n} \overline{F}_{Y_k}(x/w_k)} \;\ge\; 1.$$

In order to show the upper asymptotic bound in (5.1), write

$$\mathrm{P}(S_{Y,n}^{w} > x) = \mathrm{P}\Big(S_{Y,n}^{w} > x, \bigcup_{i<j} \{w_i Y_i > \delta x/(n-1), w_j Y_j > \delta x/(n-1)\}\Big)$$
$$+ \mathrm{P}\Big(S_{Y,n}^{w} > x, \bigcap_{i<j} \big\{\{w_i Y_i \le \delta x/(n-1)\} \cup \{w_j Y_j \le \delta x/(n-1)\}\big\}\Big)$$
$$\le \sum_{i<j} \mathrm{P}(w_i Y_i > \delta x/(n-1)) \mathrm{P}(w_j Y_j > \delta x/(n-1))$$
$$+ \mathrm{P}\Big(\bigcup_{k=1}^{n} \{w_k Y_k > (1-\delta)x\}\Big)$$
$$\le \Big(\sum_{i=1}^{n} \mathrm{P}(w_i Y_i > \delta x/(n-1))\Big)^2 + \sum_{k=1}^{n} \mathrm{P}(w_k Y_k > (1-\delta)x)$$
$$=: r_1^{w}(x) + r_2^{w}(x), \tag{5.7}$$

where we have used that for any sets $A_1, \dots, A_n$ it holds $\bigcap_{1\le i<j\le n}\{A_i \bigcup A_j\} \subset \bigcup_{i=1}^n \bigcap_{j\ne i} A_j$. It is easy to see that $r_1^w(x) = o(1)\sum_{k=1}^n \overline{F}_{Y_k}(x/w_k)$ and, by the definition of class $\mathscr{C}$,

$$\lim_{\delta\searrow 0}\limsup_{x\to\infty}\sup_{\overline{w}_n\in(0,b]^n}\frac{r_2^w(x)}{\sum_{k=1}^n \overline{F}_{Y_k}(x/w_k)} \le 1.$$

This and (5.7) completes the proof of proposition. □

Remark 5.1. Uniform asymptotic relation (5.1) was investigated earlier in a number of papers. Tang and Tsitsiashvili [19] obtained this relation for independent r.v.s with common subexponential d.f. and weights $\overline{w}_n \in [a,b]^n$, $0 < a \le b < \infty$. Subexponential r.v.s (independent or dependent) were also investigated in [28], [21], [10]. Liu et al. [16] and Wang et al. [22] proved relation (5.1) for identically distributed r.v.s from class $\mathscr{L}\cap\mathscr{D}$ allowing some dependence among primary variables with weights $\overline{w}_n \in [a,b]^n$. Li [13] showed that this uniform equivalence holds for nonidentically distributed (with some dependence) r.v.s from the class $\mathscr{C}$ or $\mathscr{L}\cap\mathscr{D}$ and $\overline{w}_n \in [a,b]^n$.

Proposition 5.2. *Suppose that $Y_1, Y_2, \dots$ are real-valued independent r.v.s with corresponding distributions $F_{Y_1}, F_{Y_2}, \dots$ and $a_i\colon(-\infty,\infty)\to[0,\infty)$, $i = 1, 2$, are measurable functions.*

(i) *If $0 < \mathrm{E}a_1(Y_1) < \infty$, $F_{Y_i} \in \mathscr{L}\cap\mathscr{D}$, $i = 2,\dots,k$, where $k \ge 2$ is an arbitrary integer, and*

$$\mathrm{E}a_1(Y_1)\mathbb{1}_{\{Y_1>x\}} = o(\overline{F_{Y_2}}(x) + \dots + \overline{F_{Y_k}}(x)), \tag{5.8}$$

then, uniformly for $\overline{w}_k \in [a,b]^k$, $0 < a \le b < \infty$, it holds

$$\begin{aligned}\mathrm{E}a_1(Y_1)\mathbb{1}_{\{w_1Y_1+\dots+w_kY_k>x\}} &\sim \mathrm{E}a_1(Y_1)\mathrm{P}(w_2Y_2+\dots+w_kY_k>x)\\ &\sim \mathrm{E}a_1(Y_1)\big(\overline{F_{Y_2}}(x/w_2)+\dots+\overline{F_{Y_k}}(x/w_k)\big);\end{aligned} \tag{5.9}$$

(ii) *if $0 < \mathrm{E}a_i(Y_i) < \infty$, $F_{Y_i} \in \mathscr{D}$, $i = 1, 2$, and*

$$\mathrm{E}a_i(Y_i)\mathbb{1}_{\{Y_i>x\}} = o(\overline{F_{Y_j}}(x)),\ i,j = 1,2,\ i\ne j, \tag{5.10}$$

then

$$\mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{w_1Y_1+w_2Y_2>x\}} = o(\overline{F_{Y_1}}(x/w_1) + \overline{F_{Y_2}}(x/w_2)) \tag{5.11}$$

uniformly for $\overline{w}_2 \in (0,b]^2$.

(iii) *if $0 < \mathrm{E}a_i(Y_i) < \infty$, $i = 1, 2$, $F_{Y_i} \in \mathscr{L}\cap\mathscr{D}$, $i = 3,\dots,k$, where $k \ge 3$ is an arbitrary integer, and*

$$\mathrm{E}a_i(Y_i)\mathbb{1}_{\{Y_i>x\}} = o(\overline{F_{Y_3}}(x) + \dots + \overline{F_{Y_k}}(x)),\ i = 1,2, \tag{5.12}$$

then, uniformly for $\overline{w}_k \in [a,b]^k$, $0 < a \le b < \infty$, it holds

$$\begin{aligned}&\mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{w_1Y_1+\dots+w_kY_k>x\}}\\ &\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)(\overline{F_{Y_3}}(x/w_3) + \dots + \overline{F_{Y_k}}(x/w_k)).\end{aligned} \tag{5.13}$$

Proof. (i) By Corollary 3.1 we can choose some positive function $K_1(x)$, $K_1(x) \le x$ such that $K_1(x) \nearrow \infty$ and

$$\mathrm{P}(w_2Y_2 + \cdots + w_kY_k > x \pm K_1(x)) \sim \mathrm{P}(w_2Y_2 + \cdots + w_kY_k > x) \tag{5.14}$$

uniformly for $w_2, \dots, w_k \in [a, b]$. Next, write

$$\begin{aligned} &\mathrm{E}a_1(Y_1)\mathbb{1}_{\{w_1Y_1+\cdots+w_kY_k>x\}} \\ &= \mathrm{E}a_1(Y_1)\mathbb{1}_{\{w_1Y_1+\cdots+w_kY_k>x\}}\big(\mathbb{1}_{\{w_1|Y_1|\le K_1(x)\}} + \mathbb{1}_{\{w_1|Y_1|>K_1(x)\}}\big) \\ &=: i_1(x) + i_2(x). \end{aligned}$$

By (5.14) we have

$$\begin{aligned} &\limsup_{x\to\infty} \sup_{\overline{w}_k\in[a,b]^k} \frac{i_1(x)}{\mathrm{E}a_1(Y_1)\mathrm{P}(w_2Y_2+\cdots+w_kY_k>x)} \\ &\le \limsup_{x\to\infty} \sup_{\overline{w}_k\in[a,b]^k} \frac{\mathrm{P}(w_2Y_2+\cdots+w_kY_k>x-K_1(x))}{\mathrm{P}(w_2Y_2+\cdots+w_kY_k>x)} = 1. \end{aligned}$$

This, together with Proposition 5.1(i), yields

$$i_1(x) \lesssim \mathrm{E}a_1(Y_1)(\overline{F_{Y_2}}(x/w_2) + \cdots + \overline{F_{Y_k}}(x/w_k))$$

uniformly in $\overline{w}_k \in [a, b]^k$.

For the lower bound, due to (5.14) and Proposition 5.1(i), we can write

$$\begin{aligned} i_1(x) &\ge \mathrm{E}a_1(Y_1)\mathbb{1}_{\{w_2Y_2+\cdots+w_kY_k>x+K_1(x),w_1|Y_1|\le K_1(x)\}} \\ &= \mathrm{E}a_1(Y_1)\mathbb{1}_{\{w_1|Y_1|\le K_1(x)\}}\mathrm{P}(w_2Y_2+\cdots+w_kY_k>x+K_1(x)) \\ &\sim \mathrm{E}a_1(Y_1)\mathrm{P}(w_2Y_2+\cdots+w_kY_k>x) \\ &\sim \mathrm{E}a_1(Y_1)(\overline{F_{Y_2}}(x/w_2)+\cdots+\overline{F_{Y_k}}(x/w_k)) \end{aligned}$$

uniformly in $\overline{w}_k \in [a, b]^k$.

It remains to show that $i_2(x) = o(\overline{F_{Y_2}}(x/w_2) + \cdots + \overline{F_{Y_k}}(x/w_k))$. Write

$$\begin{aligned} i_2(x) &\le \mathrm{E}a_1(Y_1)(\mathbb{1}_{\{w_1Y_1>x/2\}} + \mathbb{1}_{\{w_2Y_2+\cdots+w_kY_k>x/2\}})\mathbb{1}_{\{w_1|Y_1|>K_1(x)\}} \\ &\le \mathrm{E}a_1(Y_1)\mathbb{1}_{\{Y_1>x/(2b)\}} \\ &\quad + \mathrm{E}a_1(Y_1)\mathbb{1}_{\{|Y_1|>K_1(x)/b\}}\mathrm{P}(w_2Y_2+\cdots+w_kY_k>x/2). \end{aligned}$$

Hence, by assumption (5.8), Proposition 5.1(i) and the definition of class $\mathscr{D}$ we get

$$\begin{aligned} i_2(x) &\lesssim o\big(\overline{F_{Y_2}}(x/(2b)) + \cdots + \overline{F_{Y_k}}(x/(2b))\big) + o(1)\big(\overline{F_{Y_2}}(x/(2w_2)) \\ &\qquad + \cdots + \overline{F_{Y_k}}(x/(2w_k))\big) \\ &= o\big(\overline{F_{Y_2}}(x/w_2) + \cdots + \overline{F_{Y_k}}(x/w_k)\big) \end{aligned}$$

uniformly in $\overline{w}_k \in [a, b]^k$.

(ii) We have by (5.10) and $F_{Y_i} \in \mathscr{D}$, $i = 1, 2$, that

$$\begin{aligned} &\mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{w_1Y_1+w_2Y_2>x\}} \\ &\le \mathrm{E}a_2(Y_2)\mathrm{E}a_1(Y_1)\mathbb{1}_{\{Y_1>x/(2w_1)\}} + \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)\mathbb{1}_{\{Y_2>x/(2w_2)\}} \end{aligned}$$

$$= \mathrm{E}a_2(Y_2)o(\overline{F_{Y_2}}(x/(2w_1))) + \mathrm{E}a_1(Y_1)o(\overline{F_{Y_1}}(x/(2w_2)))$$
$$= o(\overline{F_{Y_1}}(x/w_1) + \overline{F_{Y_2}}(x/w_2))$$

uniformly for $\overline{w}_2 \in (0,b]^2$.

(iii) Choose $K_2(x) > 0$ such that $K_2(x) \le x$, $K_2(x) \nearrow \infty$ and

$$\mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x \pm K_2(x)) \sim \mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x) \tag{5.15}$$

uniformly for $w_3, \ldots, w_k \in [a,b]$. Now, split

$$\begin{aligned}
&\mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{w_1Y_1+\cdots+w_kY_k>x\}} \\
&= \mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{w_1Y_1+\cdots+w_kY_k>x\}}(\mathbb{1}_{\{|w_1Y_1+w_2Y_2|\le K_2(x)\}} \\
&\quad + \mathbb{1}_{\{|w_1Y_1+w_2Y_2|>K_2(x)\}}) \\
&=: k_1(x) + k_2(x).
\end{aligned}$$

Similarly to case (i), we can show that

$$\begin{aligned}
k_1(x) &\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)(\overline{F_{Y_3}}(x/w_3) + \cdots + \overline{F_{Y_k}}(x/w_k)), \\
k_2(x) &= o(\overline{F_{Y_3}}(x/w_3) + \cdots + \overline{F_{Y_k}}(x/w_k)).
\end{aligned}$$

Indeed, by (5.15) and Proposition 5.1(i),

$$\begin{aligned}
k_1(x) &\le \mathrm{E}a_1(Y_1)a_2(Y_2)\mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x - K_2(x)) \\
&\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)\mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x) \\
&\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)(\overline{F_{Y_3}}(x/w_3) + \cdots + \overline{F_{Y_k}}(x/w_k)), \\
k_1(x) &\ge \mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{|w_1Y_1+w_2Y_2|\le K_2(x)\}}\mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x + K_2(x)) \\
&\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)\mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x) \\
&\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)(\overline{F_{Y_3}}(x/w_3) + \cdots + \overline{F_{Y_k}}(x/w_k))
\end{aligned}$$

uniformly for $\overline{w}_k \in [a,b]^k$, where we have used that

$$\begin{aligned}
&\mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{|w_1Y_1+w_2Y_2|>K_2(x)\}} \\
&\le \mathrm{E}a_1(Y_1)\mathbb{1}_{\{b|Y_1|>K_2(x)/2\}}\mathrm{E}a_2(Y_2) \\
&\quad + \mathrm{E}a_2(Y_2)\mathbb{1}_{b|Y_2|>K_2(x)/2\}}\mathrm{E}a_1(Y_1) \ \to\ 0.
\end{aligned}$$

For $k_2(x)$ we have

$$\begin{aligned}
k_2(x) &\le \mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{w_1Y_1+w_2Y_2>x/2\}} \\
&\quad + \mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{|w_1Y_1+w_2Y_2|>K_2(x)\}}\mathrm{P}\Big(\sum_{i=3}^{k} w_iY_i > x/2\Big) \\
&=: k_{21}(x) + k_{22}(x),
\end{aligned}$$

where, by assumption (5.12), Proposition 5.1(i) and the definition of class $\mathscr{D}$,

$$k_{21}(x) \le \mathrm{E}a_2(Y_2)\mathrm{E}a_1(Y_1)\mathbb{1}_{\{w_1Y_1>x/4\}} + \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)\mathbb{1}_{\{w_2Y_2>x/4\}}$$

$$= \mathrm{E}a_2(Y_2)o\Big(\sum_{i=3}^{k}\overline{F_{Y_i}}(x/(4w_1))\Big) + \mathrm{E}a_1(Y_1)o\Big(\sum_{i=3}^{k}\overline{F_{Y_i}}(x/(4w_2))\Big)$$
$$= o\Big(\sum_{i=3}^{k}\overline{F_{Y_i}}(x/w_i)\Big)$$

and

$$k_{22}(x) = o(1)\sum_{i=3}^{k}\overline{F_{Y_i}}(x/(2w_i))$$

uniformly for $\overline{w}_k \in [a,b]^k$. The proof is complete. □

Corollary 5.1. *Assume that $k \geq 2$ and $Y_1,\dots,Y_k$ are real-valued independent r.v.s, such that $F_{Y_i} \in \mathscr{L}\cap\mathscr{D}$, $i = 1,\dots,k$. Let $a_i\colon (-\infty,\infty) \to [0,\infty)$, $i = 1,\dots,k$, be measurable functions such that $0 < \mathrm{E}a_i(Y_i) < \infty$ for each i and let*

$$\mathrm{E}a_i(Y_i)\mathbb{1}_{\{Y_i>x\}} = o(\overline{F_{Y_j}}(x)),\ i,j = 1,\dots,k,\ i \neq j. \tag{5.16}$$

Then, uniformly for $\overline{w}_k \in [a,b]^k$, for all $l = 1,\dots,k$ it holds

$$\mathrm{E}a_l(Y_l)\mathbb{1}_{\{w_1Y_1+\cdots+w_kY_k>x\}} \sim \mathrm{E}a_l(Y_l)\sum_{\substack{j=1\\ j\neq l}}^{k}\overline{F_{Y_j}}(x/w_j),$$

and for all l,m, $1 \leq l < m \leq k$, it holds

$$\mathrm{E}a_l(Y_l)a_m(Y_m)\mathbb{1}_{\{w_1Y_1+\cdots+w_kY_k>x\}} = \begin{cases} o\big(\overline{F_{Y_1}}(x/w_1) + \overline{F_{Y_2}}(x/w_2)\big), & k = 2,\\ \mathrm{E}a_l(Y_l)\mathrm{E}a_m(Y_m)\sum\limits_{\substack{j=1\\ j\neq l, j\neq m}}^{k}\overline{F_{Y_j}}(x/w_j)(1+o(1)), & k \geq 3.\end{cases} \tag{5.17}$$

Proof. Observe that (5.16) with $i = 1$ implies all three conditions (5.8), (5.10), (5.12) with $i = 1$. Then the statement follows straightforwardly from Proposition 5.2. □

Acknowledgement. We are grateful to the referee for his/her useful comments and suggestions leading to an improvement of the paper.

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