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ON COATOMIC MODULES AND LOCAL COHOMOLOGY MODULES WITH RESPECT TO A PAIR OF IDEALS

Tran Tuan Nam and Nguyen Minh Tri

Abstract. In this paper, we show some results on the vanishing and the finiteness of local cohomology modules with respect to a pair of ideals. We also prove that $\operatorname{Supp}(H^{\dim M-1}_{I,J}(M)/JH^{\dim M-1}_{I,J}(M))$ is a finite set.

1. Introduction

Throughout this paper, R is a Noetherian commutative (with non-zero identity) ring and I, J are two ideals of R. It is well-known that the local cohomology theory of Grothendieck is an important tool in commutative algebra and algebraic geometry. In [8], Takahashi, Yoshino and Yoshizawa introduced the module $H^i_{I,J}(M)$ as a generalization of the ordinary local cohomology module $H^i_I(M)$. For an R-module M, the (I, J)-torsion submodule of M is $\Gamma_{I,J}(M) = \{x \in M \mid I^n x \subseteq Jx$ for some positive integer $n\}$. They denoted by $H^i_{I,J}$ the i-th right derived functor of the functor $\Gamma_{I,J}$. It is clear that when $J = 0$, the functor $H^i_{I,0}$ coincides with the usual local cohomology functor H^i_I.

When M is a finitely generated R-module, many properties of $H^i_{I,J}(M)$ have been studied in [2], [3], [4], [5], [6] and [8]. We now improve some results of those papers in the case M is a coatomic module. An R-module M is called coatomic if every proper submodule of M is contained in a maximal submodule of M. The coatomic modules were introduced and studied by H. Zöschinger in [9]. In [1] the authors studied some properties of the local cohomology modules $H^i_I(M)$ concerning to coatomic modules. An important result on coatomic modules was shown in [9, Satz 2.4] which says that: Let $(R, \mathfrak{m})$ be a local ring and M an R-module. The following statements are equivalent:

(i) M is a coatomic module;
(ii) There is an integer $t \geq 1$ such that $\mathfrak{m}^t M$ is finitely generated;
(iii) There is an integer $t \geq 1$ such that $M/(0 :_M \mathfrak{m}^t)$ is finitely generated.

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The purpose of this paper is to show some results on the vanishing and the finiteness of local cohomology modules with respect to a pair of ideals $H^i_{I,J}(M)$. Some equivalent conditions of (I,J)-torsion modules when M is a coatomic R-module are shown in Theorem 2.1. Theorem 2.3 shows that if M is a coatomic module of $\dim M > 0$ or a minimax module of $d = \dim M > 1$ over a local ring $(R,\mathfrak{m})$, then the module $H^d_{I,J}(M)$ is artinian and $\mathrm{Att}(H^d_{I,J}(M)) = \{\mathfrak{p} \in \mathrm{Supp}(M)\cap V(J) \mid \mathrm{cd}(I,J,R/\mathfrak{p}) = d\}$. We will see in Proposition 2.4 that if M is a coatomic R-module with $\dim M > 0$ and t is a non-negative integer such that $\mathrm{Supp}(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < t$, then $H^i_{I,J}(M)$ is artinian for all $0 < i < t$. An important result of this paper is Theorem 2.6 which shows that $H^i_{I,J}(M)$ is finitely generated or coatomic for all $i \geq t$ if and only if $H^i_{I,J}(M) = 0$ for all $i \geq t$. When studying the finiteness of support of local cohomology modules with respect to an ideal, M. Aghapournahr and L. Melkersson in [1], Saremi in [7] showed that $\mathrm{Supp}(H^{\dim M-1}_I(M))$ is a finite set. Now, we prove in Theorem 2.13 that in a semi-local ring, the set $\mathrm{Supp}(H^{\dim M-1}_{I,J}(M)/JH^{\dim M-1}_{I,J}(M))$ is finite.

2. Main results

Let I, J be two ideals of R. In [8], the authors denoted by $W(I,J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some } n \gg 1\}$ and $\tilde{W}(I,J) = \{\mathfrak{a} \lhd R \mid I^n \subseteq \mathfrak{a} + J \text{ for some } n \gg 1\}$. An R-module M is called (I,J)-torsion if $M = \Gamma_{I,J}(M)$. When M is a finitely generated R-module, it follows from [8, 1.9] that M is (I,J)-torsion if and only if M/JM is I-torsion. We have the first result on the equivalent conditions of (I,J)-torsion modules when M is a coatomic R-module.

Theorem 2.1. *Let $(R,\mathfrak{m})$ be a local ring and M a coatomic R-module. The following statements are equivalent*:

(i) *M is (I,J)-torsion*;
(ii) *M/JM is I-torsion*;
(iii) *$H^i_{I,J}(M) = 0$ for all $i > 0$.*

Proof. (i) ⇒ (ii). Trivial.

(i) ⇒ (iii). It follows from [8, 1.13(1)].

(ii) ⇒ (i). Assume that M/JM is an I-torsion R-module. Since M is coatomic, by [9, Satz 2.4] there exists a positive integer t such that $M/(0:_M \mathfrak{m}^t)$ is a finitely generated R-module. Let $N = 0 :_M \mathfrak{m}^t$. It is clear that N is $\mathfrak{m}$-torsion and then N is (I,J)-torsion. We see that

$$\frac{M/N}{J(M/N)} \cong \frac{M}{JM+N} \cong \frac{M/JM}{JM+N/JM}.$$

By the assumption, we can conclude that $(M/N)/J(M/N)$ is I-torsion. Since M/N is finitely generated, we have by [8, 1.9] that M/N is (I,J)-torsion. Now, combining the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

with [8, 1.8 (2)] we get that M is (I,J)-torsion.

(iii) $\Rightarrow$ (i). By [9, Satz 2.4], there is an integer t such that $M/(0:_M \mathfrak{m}^t)$ is finitely generated. If $M = 0:_M \mathfrak{m}^t$, then M is an $\mathfrak{m}$-torsion R-module. Since $(R,\mathfrak{m})$ is a local ring, we see that M is (I,J)-torsion. Now assume that $M/(0:_M \mathfrak{m}^t) \neq 0$. This implies that $\mathfrak{m} \in \operatorname{Supp}(M/(0:_M \mathfrak{m}^t))$. From the short exact sequence

$$0 \to 0:_M \mathfrak{m}^t \to M \to M/(0:_M \mathfrak{m}^t) \to 0$$

we have

$$\operatorname{Supp}(M) = \operatorname{Supp}(M/(0:_M \mathfrak{m}^t)) \cup \operatorname{Supp}(0:_M \mathfrak{m}^t) = \operatorname{Supp}(M/(0:_M \mathfrak{m}^t))$$

and

$$H^i_{I,J}(M) \cong H^i_{I,J}(M/(0:_M \mathfrak{m}^t))$$

for all $i > 0$. By the assumption, $H^i_{I,J}(M/(0:_M \mathfrak{m}^t)) = 0$ for all $i > 0$. Therefore, we can conclude that

$$\operatorname{Supp}(M/(0:_M \mathfrak{m}^t)) \subseteq W(I,J)$$

by [8, 4.2] and the proof is complete. □

Now, if M is a minimax R-module, then we have a similar result. We recall that an R-module M is minimax if there is a finitely generated submodule N of M such that M/N is artinian. Minimax modules were first introduced and studies by H. Zöschinger in [10].

Proposition 2.2. *Let $(R,\mathfrak{m})$ be a local ring and M a minimax R-module. The following statements are equivalent*:

(i) *M is (I,J)-torsion.*
(ii) *M/JM is I-torsion.*

Proof. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (i). Since M is a minimax R-module, there exists a finitely generated submodule N of M such that M/N is artinian. From the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

we have the following exact sequence

$$\cdots \to \operatorname{Tor}^R_1(R/J, M/N) \to N/JN \to M/JM \to (M/N)/J(M/N) \to 0.$$

Since M/N is an artinian R-module, we have $\operatorname{Supp}(M/N) \subseteq \{\mathfrak{m}\}$ and then M/N is I-torision and (I,J)-torsion. Therefore $\operatorname{Tor}^R_1(R/J, M/N)$ is an I-torsion R-module. It follows from the hypothesis that N/JN is I-torsion. Since N is finitely generated, we have by [8, 1.9] that N is an (I,J)-torsion R-module. By [8, 1.8(2)], we imply that M is (I,J)-torsion. □

In [3, 2.1], if M is a finitely generated R-module over a local ring $(R, \mathfrak{m})$ with $\dim M = d$, then $H^d_{I,J}(M)$ is artinian. We now prove that these properties hold for the larger class of coatomic modules of minimax modules instead of the class of finitely generated modules.

Theorem 2.3. *Let $(R, \mathfrak{m})$ be a local ring and M a coatomic R-module with $d = \dim M > 0$ or a minimax R-module with $d = \dim M > 1$. Then $H^d_{I,J}(M)$ is artinian and*

$$\operatorname{Att}(H^d_{I,J}(M)) = \{\mathfrak{p} \in \operatorname{Supp}(M) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = d\},$$

where $\operatorname{cd}(I, J, M) = \sup\{n \mid H^n_{I,J}(M) \neq 0\}$.

Proof. At first we assume that M is a coatomic R-module. Then there is an integer $k \geq 1$ such that $M/(0 :_M \mathfrak{m}^k)$ is finitely generated by [9, Satz 2.4] and

$$H^i_{I,J}(M) \cong H^i_{I,J}(M/(0 :_M \mathfrak{m}^k))$$

for all $i > 0$. Since $\dim M > 0$, we can conclude that

$$\operatorname{Supp}(M) = \operatorname{Supp}(M/(0 :_M \mathfrak{m}^k))$$

and

$$\dim M = \dim M/(0 :_M \mathfrak{m}^k).$$

From [3, 2.1], $H^d_{I,J}(M/(0 :_M \mathfrak{m}^k))$ is artinian and then $H^d_{I,J}(M)$ is also artinian. Now we have by [2, 2.1],

$$\begin{aligned}\operatorname{Att}(H^d_{I,J}(M)) &= \operatorname{Att}(H^d_{I,J}(M/(0 :_M \mathfrak{m}^k)))\\ &= \{\mathfrak{p} \in \operatorname{Supp}(M/(0 :_M \mathfrak{m}^k)) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = d\}\\ &= \{\mathfrak{p} \in \operatorname{Supp}(M) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = d\}.\end{aligned}$$

In the case M is a minimax R-module. There exists a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where N is finitely generated and A is artinian. Since $\dim M > 0$ and A is artinian, we have $\operatorname{Supp}(M) = \operatorname{Supp}(N)$ and $\dim M = \dim N$. By applying the functor $\Gamma_{I,J}(-)$ to the above exact sequence, we obtain an exact sequence

$$0 \to H^0_{I,J}(N) \to H^0_{I,J}(M) \to H^0_{I,J}(A) \to H^1_{I,J}(N) \to H^1_{I,J}(M) \to 0$$

and

$$H^i_{I,J}(N) \cong H^i_{I,J}(M)$$

for all $i \geq 2$. Since N is a finitely generated R-module, we have by [3, 2.1] that $H^d_{I,J}(N)$ is artinian and then so is $H^d_{I,J}(M)$. By using [2, 2.1] again, we have

$$\begin{aligned}\operatorname{Att}(H^d_{I,J}(M)) &= \operatorname{Att}(H^d_{I,J}(N))\\ &= \{\mathfrak{p} \in \operatorname{Supp}(N) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = d\}\\ &= \{\mathfrak{p} \in \operatorname{Supp}(M) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = d\}\end{aligned}$$

and the proof is complete. □

Note that, if M is a minimax R-module with $\dim M = 1$, then we see that

$$\operatorname{Att}(H^1_{I,J}(M)) \subseteq \{\mathfrak{p} \in \operatorname{Supp}(M) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = 1\}.$$

It should be mentioned that the above result is not true when $\dim M = 0$. The example is similar to [1, 3.5]. On the other hand, if R is not a local ring and $\dim M = 0$, then $H^0_{I,J}(M)$ is not artinian. Let $R = \mathbb{Z}, M = (\mathbb{Z}_2)^{\mathbb{N}}$ and $I = 2\mathbb{Z}, J = 4\mathbb{Z}$. We see that $\dim M = 0$ and $H^0_{I,J}(M) = M$ is not artinian.

We see in [6, 2.8] that $H^i_{I,J}(M)$ is artinian for all $i < t$ if M is a minimax module such that $\operatorname{Supp}(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < t$. Now, we consider in the case M is a coatomic module.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a local ring, M a coatomic R-module with $\dim M > 0$. Assume that t is a non-negative integer such that $\operatorname{Supp}(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < t$. Then $H^i_{I,J}(M)$ is artinian for all $0 < i < t$.*

Proof. It follows from the proof of Theorem 2.3, there is an integer $k \geq 1$ such that $M/(0 :_M \mathfrak{m}^k)$ is finitely generated and $\operatorname{Supp}(M) = \operatorname{Supp}(M/(0 :_M \mathfrak{m}^k))$. By the hypothesis we see that $\operatorname{Supp}(H^i_{I,J}(M/(0 :_M \mathfrak{m}^k)) \subseteq \{\mathfrak{m}\}$ for all $i < t$. Since finitely generated modules are minimax modules, we have by [6, 2.8] that $H^i_{I,J}(M/(0 :_M \mathfrak{m}^k))$ is artinian for all $i < t$. Note that $H^i_{I,J}(M/(0 :_M \mathfrak{m}^k)) \cong H^i_{I,J}(M)$ for all $i > 0$ and which completes the proof. □

When M is a finitely generated module, in [8] we see that $H^i_{I,J}(M) = 0$ for all $i > \dim M/JM$. Now, we give an extension of this property in the case M is a coatomic R-module.

Proposition 2.5. *Let $(R, \mathfrak{m})$ be a local ring and M a coatomic R-module. The following statements hold:*

(i) *If $J \neq R$, then $H^i_{I,J}(M) = 0$ for all $i > \dim M/JM$.*
(ii) *Suppose that $\sqrt{I + J} = \mathfrak{m}$. Then $\sup\{n \mid H^n_{I,J}(M) \neq 0\} = \dim M/JM$.*

Proof. (i) If $\dim M/JM = -1$, then $M = JM$. Since M is coatomic, there is an integer $t \geq 1$ such that $\mathfrak{m}^t M$ is finitely generated by [9, Satz 2.4]. This implies that M is finitely generated since $M = J^t M \subseteq \mathfrak{m}^t M$. Therefore $M = 0$ by Nakayama's Lemma.

Now suppose that $\dim M/JM \geq 0$. By the assumption on M, there exists an integer $t \geq 1$ such that $M/(0 :_M \mathfrak{m}^t)$ is finitely generated. Let $N = 0 :_M \mathfrak{m}^t$, now the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

gives rise a long exact sequence

$$\cdots \to N/JN \stackrel{\alpha}{\to} M/JM \to (M/N)/J(M/N) \to 0.$$

Note that

$$\operatorname{Supp}(\operatorname{Im} \alpha) \subseteq \operatorname{Supp}(N/JN) \subseteq \operatorname{Supp}(N) \subseteq \{\mathfrak{m}\}.$$

This implies that $\dim(\operatorname{Im}\alpha) \leq 0$. If $M = N$, then we can easily check the claim. So in the remainder of the proof, we may and do assume that $N \subsetneq M$. Now from the short exact sequence

$$0 \to \operatorname{Im}\alpha \to M/JM \to (M/N)/J(M/N) \to 0$$

we get

$$\dim M/JM = \dim(M/N)/J(M/N).$$

Since M/N is a finitely generated R-module, we have $H^i_{I,J}(M/N) = 0$ for all $i > \dim M/JM$ by [8, 4.3]. Now the conclusion follows from the isomorphism $H^i_{I,J}(M) \cong H^i_{I,J}(M/N)$ for all $i > 0$.

(ii) Combining [8, 4.5] with the isomorphism $H^i_{I,J}(M) \cong H^i_{I,J}(M/N)$ for all $i > 0$, we get the assertion. □

We are going to state and prove one of main results of this paper. The following theorem is a generalization of [1, 3.9] which shows a relationship on the vanishing, the finiteness and the coatomicness of $H^i_{I,J}(M)$.

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R-module and t a positive integer. The following statements are equivalent*:

(i) $H^i_{I,J}(M) = 0$ *for all* $i \geq t$;
(ii) $H^i_{I,J}(M)$ *is finitely generated for all* $i \geq t$;
(iii) $H^i_{I,J}(M)$ *is coatomic for all* $i \geq t$.

Proof. (i) ⇒ (ii) ⇒ (iii). Trivial.

(iii) ⇒ (i). The proof is by induction on $\dim M$. Let $n = \dim M$. If $n = 0$, then $H^i_{I,J}(M) = 0$ for all $i > 0$.

Let $n > 0$, it follows from [8, 1.13] that

$$H^i_{I,J}(M) \cong H^i_{I,J}(M/\Gamma_{I,J}(M))$$

for all $i > 0$. Denote by $\overline{M} = M/\Gamma_{I,J}(M)$, it is clear that $\overline{M}$ is (I, J)-torsion-free. This implies that $\overline{M}$ is $\mathfrak{a}$-torsion-free for all $\mathfrak{a} \in \tilde{W}(I, J)$. In particular, $\overline{M}$ is $\mathfrak{m}$-torsion-free and there is an element $x \in \mathfrak{m}$ which is regular on $\overline{M}$. Now, the short exact sequence

$$0 \to \overline{M} \stackrel{.x}{\to} \overline{M} \to \overline{M}/x\overline{M} \to 0$$

induces a long exact sequence

$$\cdots \to H^i_{I,J}(\overline{M}) \stackrel{.x}{\to} H^i_{I,J}(\overline{M}) \to H^i_{I,J}(\overline{M}/x\overline{M}) \to \cdots .$$

By the assumption, $H^i_{I,J}(\overline{M}/x\overline{M})$ is coatomic for all $i \geq t$. Since $\dim(\overline{M}/x\overline{M}) < \dim(\overline{M}) \leq n$ and $\overline{M}$ is a finitely generated R-module, it follows from the inductive hypothesis that $H^i_{I,J}(\overline{M}/x\overline{M}) = 0$ for all $i \geq t$. Now the long exact sequence yields

$$H^i_{I,J}(\overline{M}) = xH^i_{I,J}(\overline{M})$$

for all $i \geq t$. Note that coatomic modules satisfy Nakayama's Lemma. Thus $H^i_{I,J}(\overline{M}) = 0$ for all $i \geq t$, and the proof is complete. □

We may improve these results as follows.

Corollary 2.7. *Let* $(R, \mathfrak{m})$ *be a local ring,* M *a coatomic* R*-module and* t *a positive integer. The following statements are equivalent*:

(i) $H^i_{I,J}(M) = 0$ *for all* $i \geq t$;
(ii) $H^i_{I,J}(M)$ *is finitely generated for all* $i \geq t$;
(iii) $H^i_{I,J}(M)$ *is coatomic for all* $i \geq t$.

Proof. Since M is a coatomic R-module, there is an integer $k \geq 1$ such that $M/(0 :_M \mathfrak{m}^k)$ is finitely generated by [9, Satz 2.4]. Therefore, we have the isomorphisms

$$H^i_{I,J}(M) \cong H^i_{I,J}(M/(0 :_M \mathfrak{m}^k))$$

for all $i > 0$. The assertion follows immediate from 2.6. □

Corollary 2.8. *Let* $(R, \mathfrak{m})$ *be a local ring,* M *a minimax* R*-module and* $t > 1$ *a positive integer. The following statements are equivalent*:

(i) $H^i_{I,J}(M) = 0$ *for all* $i \geq t$;
(ii) $H^i_{I,J}(M)$ *is finitely generated for all* $i \geq t$;
(iii) $H^i_{I,J}(M)$ *is coatomic for all* $i \geq t$.

Proof. (i) ⇒ (ii) ⇒ (iii). Trivial. We now prove (iii) ⇒ (i). Since M is a minimax R-module, there is a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where N is finitely generated and A is artinian. By applying the functor $\Gamma_{I,J}(-)$ to the above exact sequence, we get a long exact sequence

$$0 \to H^0_{I,J}(N) \to H^0_{I,J}(M) \to H^0_{I,J}(A) \to H^1_{I,J}(N) \to H^1_{I,J}(M) \to 0$$

and

$$H^i_{I,J}(N) \cong H^i_{I,J}(M)$$

for all $i \geq 2$. By the hypothesis, $H^i_{I,J}(N)$ is coatomic for all $i \geq t$. It follows from 2.6 that $H^i_{I,J}(N) = 0$ for all $i \geq t$ and which completes the proof. □

Corollary 2.9. *Let* $(R, \mathfrak{m})$ *be a local ring and* M *a finitely generated* R*-module with* $\mathrm{cd}(I, J, M) > 0$. *Then* $H^{\mathrm{cd}(I,J,M)}_{I,J}(M)$ *is not finitely generated.*

Combining [8, 4.5] with 2.9, we have an immediate consequence.

Corollary 2.10. *Let* $(R, \mathfrak{m})$ *be a local ring,* M *a finitely generated* R*-module with* $\dim(M/JM) > 0$ *and* $\sqrt{I + J} = \mathfrak{m}$. *Then* $H^{\dim M/JM}_{I,J}(M)$ *is not finitely generated.*

In [5, Theorem 2], if M is a finitely generated with finite dimension and t is a positive integer such that $H^i_{I,J}(M) = 0$ for all $i > t$, then $H^t_{I,J}(M)/\mathfrak{a}H^t_{I,J}(M) = 0$ for all $\mathfrak{a} \in \tilde{W}(I, J)$. This property will be extended in the case M is a coatomic module.

Proposition 2.11. *Let $(R, \mathfrak{m})$ be a local ring and M a coatomic R-module. Suppose that t is a positive integer such that $H^i_{I,J}(M) = 0$ for all $i > t$. Then $H^t_{I,J}(M)/\mathfrak{a}H^t_{I,J}(M) = 0$ for all $\mathfrak{a} \in \tilde{W}(I, J)$.*

Proof. Since M is a coatomic R-module, there is an integer $k \geq 1$ such that $M/(0 :_M \mathfrak{m}^k)$ is finitely generated. The proof above gives

$$H^t_{I,J}(M) \cong H^t_{I,J}(M/(0 :_M \mathfrak{m}^k)).$$

Hence, the assertion follows from [5, Theorem 2]. □

Corollary 2.12. *Let $(R, \mathfrak{m})$ be a local ring and M a coatomic R-module. Assume that $\operatorname{cd}(I, J, M) > 0$ and K is a proper submodule of $H^{\operatorname{cd}(I,J,M)}_{I,J}(M)$. Then $H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/K$ is not a coatomic R-module.*

Proof. Suppose that the conclusion is false. It follows from the definition of coatomic modules, there exists a submodule L of $H^{\operatorname{cd}(I,J,M)}_{I,J}(M)$ such that we have a short exact sequence

$$0 \to L/K \to H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/K \to R/\mathfrak{m} \to 0.$$

Let $\mathfrak{a} \in \tilde{W}(I, J)$, by applying the functor $R/\mathfrak{a} \otimes_R -$ to the above exact sequence, there is a following exact sequence

$$\cdots \to L/\mathfrak{a}L + K \to H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/\mathfrak{a}H^{\operatorname{cd}(I,J,M)}_{I,J}(M) + K \to R/\mathfrak{m} \to 0.$$

Note that $H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/\mathfrak{a}H^{\operatorname{cd}(I,J,M)}_{I,J}(M) + K$ is a homomorphic image of

$$H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/\mathfrak{a}H^{\operatorname{cd}(I,J,M)}_{I,J}(M).$$

Consequently, we can conclude that $H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/\mathfrak{a}H^{\operatorname{cd}(I,J,M)}_{I,J}(M) + K = 0$ by 2.11. This implies that $R/\mathfrak{m} = 0$ which is a contradiction. □

Next, we will consider the dimension of $H^i_{I,J}(M)$ and the support of $H^{d-1}_{I,J}(M)$ where $d = \dim M$. In [1, 3.3] or [7, 2.3], when studying the local cohomology modules with respect to an ideal, the authors showed that $\dim H^i_I(M) \leq d - i$ and $\operatorname{Supp}(H^{d-1}_I(M))$ is a finite set. The proof of next theorem is based on these results.

Theorem 2.13. *Let M be a finitely generated R-module with $d = \dim M < \infty$. Then*

(i) $\dim H^i_{I,J}(M) \leq d - i$.
(ii) *If R is a semi-local ring, then* $\operatorname{Supp}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M))$ *is finite.*

Proof. (i) Our proof starts with the observation that

$$H^i_{I,J}(M) = \varinjlim_{\mathfrak{a} \in \tilde{W}(I,J)} H^i_{\mathfrak{a}}(M).$$

This implies that

$$\operatorname{Supp}(H^i_{I,J}(M)) \subseteq \bigcup_{\mathfrak{a}\in \tilde{W}(I,J)} \operatorname{Supp}(H^i_{\mathfrak{a}}(M)).$$

From [7, 2.3], $\dim(H^i_{\mathfrak{a}}(M)) \leq \dim M - i$ for all $\mathfrak{a} \in \tilde{W}(I,J)$. We conclude that $\dim(H^i_{I,J}(M)) \leq \dim M - i$.

(ii) We prove by induction on $d = \dim M$. It is nothing to prove when $d = 0$. If $d = 1$, we see that $H^0_{I,J}(M)$ is finitely generated. Since $\dim(H^0_{I,J}(M)) \leq 1$ by (i), it follows that

$$\operatorname{Supp}(H^0_{I,J}(M)) \subseteq \operatorname{Min}(H^0_{I,J}(M)) \cup \operatorname{Max}(R).$$

Since $H^0_{I,J}(M)$ is finitely generated, we can conclude that $\operatorname{Supp}(H^0_{I,J}(M))$ is finite. Let $d > 1$, we now assume that the statement is true for all non-zero finitely generated modules with dimension less than $\dim M$. Now the short exact sequence

$$0 \to \Gamma_J(M) \to M \to M/\Gamma_J(M) \to 0$$

induces a long exact sequence

$$\cdots H^{d-1}_{I,J}(\Gamma_J(M)) \xrightarrow{f} H^{d-1}_{I,J}(M) \xrightarrow{g} H^{d-1}_{I,J}(M/\Gamma_J(M)) \xrightarrow{h} H^{d}_{I,J}(\Gamma_J(M)) \cdots .$$

It follows from [8, 2.5] that $H^i_{I,J}(\Gamma_J(M)) \cong H^i_I(\Gamma_J(M))$ for all $i \geq 0$. On the other hand $\dim \Gamma_J(M) \leq \dim M$, so in the view of [7, 2.5] we see that $\operatorname{Supp}(H^{d-1}_{I,J}(\Gamma_J(M)))$ is finite. This implies that $\operatorname{Supp}(\operatorname{Im} f)$ is finite. Since $H^d_{I,J}(\Gamma_J(M))$ is artinian, the support of $\operatorname{Im} h$ is finite. We now have two short exact sequences

$$0 \to \operatorname{Im} f \to H^{d-1}_{I,J}(M) \to \operatorname{Im} g \to 0$$

and

$$0 \to \operatorname{Im} g \to H^{d-1}_{I,J}(M/\Gamma_J(M)) \to \operatorname{Im} h \to 0.$$

By applying the functor $R/J \otimes_R -$ to above short exact sequences, we obtain the following exact sequences

$$\cdots \to \operatorname{Im} f/J \operatorname{Im} f \to H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M) \to \operatorname{Im} g/J \operatorname{Im} g \to 0$$

and

$$\cdots \to \operatorname{Tor}^R_1(R/J, \operatorname{Im} h) \to \operatorname{Im} g/J \operatorname{Im} g \to$$
$$\to H^{d-1}_{I,J}(M/\Gamma_J(M))/JH^{d-1}_{I,J}(M/\Gamma_J(M)) \to \operatorname{Im} h/J \operatorname{Im} h \to 0.$$

The proof is complete by showing that

$$\operatorname{Supp}(H^{d-1}_{I,J}(M/\Gamma_J(M))/JH^{d-1}_{I,J}(M/\Gamma_J(M)))$$

is finite. Let $\overline{M} = M/\Gamma_J(M)$, we see that $\overline{M}$ is J-torsion free. Then there is an element $x \in J$ which is $\overline{M}$-regular. Now the short exact sequence

$$0 \to \overline{M} \xrightarrow{.x} \overline{M} \to \overline{M}/x\overline{M} \to 0$$

induces the following exact sequence

$$\cdots \to H^{d-2}_{I,J}(\overline{M}/x\overline{M}) \to H^{d-1}_{I,J}(\overline{M}) \xrightarrow{.x} H^{d-1}_{I,J}(\overline{M}) \to \cdots.$$

This gives us an exact sequence

$$H^{d-2}_{I,J}(\overline{M}/x\overline{M})/JH^{d-2}_{I,J}(\overline{M}/x\overline{M}) \to (0:_{H^{d-1}_{I,J}(\overline{M})} x)/J(0:_{H^{d-1}_{I,J}(\overline{M})} x) \to 0.$$

Since $\dim(\overline{M}/x\overline{M}) \leq d-1$, we get by the inductive hypothesis that

$$\operatorname{Supp}(H^{d-2}_{I,J}(\overline{M}/x\overline{M})/JH^{d-2}_{I,J}(\overline{M}/x\overline{M}))$$

is finite and then so is $\operatorname{Supp}((0:_{H^{d-1}_{I,J}(\overline{M})} x)/J(0:_{H^{d-1}_{I,J}(\overline{M})} x))$. Since $x \in J$, it follows that the homomorphism

$$(0:_{H^{d-1}_{I,J}(\overline{M})} x)/J(0:_{H^{d-1}_{I,J}(\overline{M})} x) \to H^{d-1}_{I,J}(\overline{M})/JH^{d-1}_{I,J}(\overline{M})$$

is surjective. Therefore $\operatorname{Supp}(H^{d-1}_{I,J}(\overline{M})/JH^{d-1}_{I,J}(\overline{M}))$ is finite, and the proof is complete. □

Corollary 2.14. *Let M be a finitely generated R-module with finite dimension $d = \dim M$. Then*

$$\operatorname{Supp}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \subseteq \operatorname{Ass}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \cup \operatorname{Max}(R).$$

Proof. It follows from 2.13 that $\dim(H^{d-1}_{I,J}(M)) \leq 1$, we see that

$$\dim(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \leq 1.$$

Therefore $\operatorname{Supp}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M))$ contains minimal prime ideals of

$$\operatorname{Ass}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M))$$

and maximal ideals, which completes the proof. □

In the case R is not a semi-local ring, we will see that $\operatorname{Supp}(H^{\dim M-1}_{I,J}(M))$ is not finite.

Example 2.15. Let $R = M = \mathbb{Z}$ and $I = 2\mathbb{Z}$, $J = 4\mathbb{Z}$. We see that $\dim M = 1$ and M is (I,J)-torsion. However, $\operatorname{Supp}(H^0_{I,J}(M)) = \operatorname{Spec}(\mathbb{Z})$ is an infinite set.

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Tran Tuan Nam
Department of Mathematics-Informatics
Ho Chi Minh University of Pedagogy
Ho Chi Minh city, Vietnam
E-mail address: namtuantran@gmail.com

Nguyen Minh Tri
Department of Natural Science Education
Dong Nai University
Dong Nai, Vietnam
and
Faculty of Mathematics & Computer Science
University of Science
VNU-HCM, Ho Chi Minh City, Vietnam
E-mail address: triminhng@gmail.com