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GLOBAL SOLUTIONS FOR THE $\overline{\partial}$-PROBLEM ON NON PSEUDOCONVEX DOMAINS IN STEIN MANIFOLDS

Sayed Saber

Abstract. In this paper, we prove basic a priori estimate for the $\overline{\partial}$-Neumann problem on an annulus between two pseudoconvex submanifolds of a Stein manifold. As a corollary of the result, we obtain the global regularity for the $\overline{\partial}$-problem on the annulus. This is a manifold version of the previous results on pseudoconvex domains.

1. Introduction

Let X be a Stein manifold of dimension $n \geq 3$. Let Ω_1 and Ω_2 be two open pseudoconvex submanifolds with smooth boundary in X such that $\overline{\Omega}_2 \Subset \Omega_1 \Subset X$. Assume that $\Omega = \Omega_1 \setminus \overline{\Omega}_2$. In this paper, we prove the basic a priori estimate for the $\overline{\partial}$-Neumann problem on Ω. Also, we study the global boundary regularity of the $\overline{\partial}$-equation, $\overline{\partial}u = f$, on Ω. The existence and regularity properties of the solution to the $\overline{\partial}$-equation are important problems in several complex variables. Our method is to use the $\overline{\partial}$-Neumann problem with weights which was used by Kohn [9], Hörmander [7] to solve the $\overline{\partial}$-problem on weakly pseudo-convex domains. In the case of an annulus, some of the important known results are the following:

(1) If Ω_1 and Ω_2 are both strictly pseudo-convex and $n \geq 3$, then Ω satisfies condition $z(q)$ and the $\overline{\partial}$-Neumann problem satisfies the subelliptic $\frac{1}{2}$ estimate (see Kohn [9], Hörmander [7] and Folland and Kohn [6]).

(2) If Ω_1 and Ω_2 are pseudoconvex domains with real analytic boundaries in $\mathbb{C}^n$ and $0 < q < n-1$, then it is proved by Dirridj and Fornaess [5] that the subelliptic estimate holds for the $\overline{\partial}$-Neumann problem on the annulus $\Omega = \Omega_1 \setminus \overline{\Omega}_2$.

(3) If Ω_1 and Ω_2 are pseudoconvex domains with smooth boundaries in $\mathbb{C}^n$, the closed range property and global boundary regularity for $\overline{\partial}$ were studied by Shaw [12] for $1 \leq q \leq n-2$ with $n \geq 3$ on the annulus $\Omega = \Omega_1 \setminus \overline{\Omega}_2$. The critical case when $q = n-1$ was established in Shaw [13].

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(4) Ahn and Zampieri [2], studied the $\overline{\partial}$-problem on an annulus between an internal p-pseudoconcave and an external q-pseudoconvex domains in $\mathbb{C}^n$.

(5) If Ω_1 and Ω_2 are two strictly q-convex domains with smooth boundaries in Stein manifold for some bidegree, Khidr and Abdelkader [8] studied global boundary regularity for $\overline{\partial}$ on the annulus $\Omega = \Omega_1 \diagdown \overline{\Omega}_2$.

(6) If Ω_1 and Ω_2 are pseudoconvex submanifolds which satisfy property (P), Cho [4] obtained the global boundary regularity for $\overline{\partial}$ on the annulus $\Omega = \Omega_1 \diagdown \overline{\Omega}_2$.

(7) If Ω_1 is a weakly q-convex and Ω_2 a weakly $(n-q-1)$-convex in an n-dimensional complex manifold X such that $b\Omega_1$ and $b\Omega_2$ satisfy property (P), Saber [11] obtained the global boundary regularity for $\overline{\partial}$ on the annulus $\Omega = \Omega_1 \diagdown \overline{\Omega}_2$.

This paper is arranged as follows. In Section 2, we give the background that are used in the later sections. In Section 3, we prove the basic a priori estimate (3.1). In Section 4, based on the estimate (3.1), one can prove global regularity for $\overline{\partial}$. Moreover, if f is $\overline{\partial}$-closed (p,q)-form, $0 < q < n-1$, which is C^∞ on $\overline{\Omega}$, then the canonical solution u of $\overline{\partial} u = f$ is smooth on $\overline{\Omega}$.

2. Background

Let X be a complex manifold of dimension n with a Hermitian metric g. Let $\Omega \Subset X$ be an open submanifold with smooth boundary $b\Omega$ and defining function ρ. Denote by $L_1, L_2, \ldots, L_n$ a C^∞ special boundary coordinate chart in a small neighborhood U of $z_0 \in b\Omega$, i.e., $L_i \in T^{1,0}$ and $\langle L_i, L_j\rangle = \delta_{ij}$ on U with L_i tangential on $U \cap b\Omega$ for $1 \le i \le n-1$, that is, $L_i(\rho) = 0$ for $1 \le i \le n-1$ and $L_n(\rho) = 1$. Then $\overline{L}_1, \overline{L}_2, \ldots, \overline{L}_n$, the conjugate of $L_1, L_2, \ldots, L_n$, form an orthonormal basis of $T^{0,1}$ on U. The dual basis of $(1,0)$ forms are $\omega^1, \ldots, \omega^n$ with $\omega^n = \partial\rho$. Let $\left(\frac{\partial^2\rho(z)}{\partial z_i \overline{\partial} z_j}\right)_{i,j=1}^{n-1}$ be the matrix of the Levi form $\partial\overline{\partial}\rho(z)$ in the complex tangential direction at z. Let $C^\infty(\Omega)$ be the space of C^∞-function on Ω.

We shall fix the function $\lambda \in C^\infty(\overline{\Omega})$ and let t be any nonnegative real number and we write

$$\lambda_{ij} = \langle L_i \wedge \overline{L}_j, \partial\overline{\partial}\lambda\rangle,\ i,j = 1,2,\ldots,n.$$

Let $C^\infty_{p,q}(X)$ be the space of (p,q) complex-valued differential forms of class C^∞ on X, where $0 \le p \le n$, $0 \le q \le n$. Then any (p,q)-form $f \in C^\infty_{p,q}(X)$ can be expressed as $f = \sum_{I,J}' f_{I,J} dz^I \wedge d\bar{z}^J$, where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are multiindices and $dz^I = dz_1 \wedge \cdots \wedge dz_p$, $d\bar{z}^J = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$. The notation $\sum'$ means the summation over strictly increasing multiindices. Denote by $C^\infty_{p,q}(\overline{\Omega}) = \left\{ f|_{\overline{\Omega}};\ f \in C^\infty_{p,q}(X)\right\}$ the subspace of $C^\infty_{p,q}(\Omega)$ whose elements can be extended smoothly up to the boundary. Let $\mathcal{D}(X)$ be the space of C^∞-functions with compact support in X. We say that a form $f \in C^\infty_{p,q}(X)$ has compact support in X if its coefficients belongs to $\mathcal{D}(X)$. The subspace of $C^\infty_{p,q}(X)$ which has compact support in X is denoted by $\mathcal{D}_{p,q}(X)$. For $f \in C^\infty_{p,q}(\Omega)$ and $g \in \mathcal{D}_{p,q-1}(\Omega)$, the formal adjoint operator ϑ of $\overline{\partial} : C^\infty_{p,q-1}(\Omega) \longrightarrow C^\infty_{p,q}(\Omega)$, with respect to $\langle \cdot, \cdot \rangle$, is defined by:

$$\langle \overline{\partial} g, f \rangle = \langle g, \vartheta f \rangle.$$

Thus, ϑ can be expressed by

$$\vartheta f = (-1)^{p-1} {\sum_{I,K}}' \sum_{k=1}^{n} \frac{\partial f_{I\overline{k}\,\overline{K}}}{\partial \overline{z}^k}\, dz^I \wedge d\overline{z}^K, \ |K| = q-1.$$

Denote by $L^2(\Omega)$ the space of square integrable functions on Ω with respect to the Lebesgue measure in X. For each nonnegative integer s, $W^s(\Omega)$ is the space of all the distributions u in $L^2(\Omega)$ such that

$$D^\alpha u \in L^2(\Omega), \ \mid \alpha \mid \leq s,$$

where α is a multiindex and $\mid \alpha \mid = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The Sobolev s-norm $\| \ \|_{W^s}$ is defined by

$$\|f\|_{W^s} = \int_\Omega \sum_{|\alpha| \leq s} |D^\alpha f|^2 dx < \infty.$$

Indeed $W^s(\Omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm $\|.\|_{W^s}$. The closure of $\mathcal{D}(\Omega)$ with respect to the same topology is denoted by $W^s_0(\Omega)$. The Sobolev norm $\|f\|_{W^{-1}}$ of order -1 for forms f on Ω is defined by

$$\|f\|_{W^{-1}} = \sup_{g \in W^1_0(\Omega)} \frac{|\langle f, g \rangle|}{\|g\|_{W^1}}.$$

The norm $\| \ \|_{W^{-1}}$ is weaker than the norm $\| \ \|$ in the sense that any sequence of functions which is bounded in the norm $\| \ \|$ has a subsequence which is convergent in the norm $\| \ \|_{W^{-1}}$. Use $W^s_{p,q}(\Omega)$ to denote the space of (p, q) forms with coefficients in $W^s(\Omega)$.

Denote by $L^2_{p,q}(\Omega)$ the space of (p, q)-forms with coefficients in $L^2(\Omega)$. For $f, g \in L^2_{p,q}(\Omega)$, the inner product $\langle f, g \rangle$ and the norm $\|f\|$ are denoted by:

$$\langle f, g \rangle = \int_\Omega f \wedge \star \overline{g} \text{ and } \| \ f \ \|^2 = \langle f, f \rangle,$$

where $\star$ is the Hodge star operator. For $t \geq 0$, denote by $L^2_{p,q}(\Omega, t\lambda)$ the space of (p, q)-forms with coefficients in $L^2(\Omega)$ with respect to the weighted function $e^{-t\lambda}$. For $f, g \in L^2_{p,q}(\Omega, t\lambda)$, we denote the inner product $\langle f, g \rangle_t$ and the norm $\|f\|_t$ by:

$$\langle f, g \rangle_t = \int_\Omega f \wedge \star \overline{g}\ e^{-t\lambda} \text{ and } \|f\|^2_t = \langle f, f \rangle_t.$$

In that case $\langle f, g \rangle_t$ denotes $\langle f, g \rangle_{t\lambda}$, that is, we use subscripts t instead of $t\lambda$. Note that since λ is bounded on $\overline{\Omega}$, the two norms $\| \ \|$ and $\| \ \|_t$ are equivalent.

Define a Hermitian form $Q^t(u,u)$ from $\mathcal{D}_{p,q}(\Omega)\times\mathcal{D}_{p,q}(\Omega)$ to $\mathbb{C}$ by

$$Q^t(u,u)=\|\overline{\partial}u\|_t^2+\|\overline{\partial}_t^\star u\|_t^2+\|u\|_t^2.$$

Let $\overline{\partial}:\operatorname{dom}\overline{\partial}\subset L^2_{p,q}(\Omega,t\lambda)\longrightarrow L^2_{p,q+1}(\Omega,t\lambda)$ be the maximal closure of the Cauchy-Riemann operator and $\overline{\partial}_t^\star$ be its Hilbert space adjoint. Recall that $\operatorname{dom}\overline{\partial}^\star=\operatorname{dom}\overline{\partial}_t^\star$. The $\overline{\partial}$-Neumann operator $N^t=N^t_{p,q}:L^2_{p,q}(\Omega,t\lambda)\longrightarrow L^2_{p,q}(\Omega,t\lambda)$, is defined as the inverse of the restriction of $\square^t$ to $(\ker\square^t)^\perp$, where $\square^t=\overline{\partial}\,\overline{\partial}_t^\star+\overline{\partial}_t^\star\overline{\partial}$ is the weighted Laplace Beltrami operator. The space of the weighted harmonic (p,q)-forms $\mathcal{H}_t$ is defined by

$$\mathcal{H}_t=\{u\in\mathcal{D}_{p,q}(\Omega):\overline{\partial}u=\overline{\partial}_t^\star u=0\}.$$

3. The basic a priori estimate

In this section, we prove the basic a priori estimate (3.1). The estimate is similar (but weaker) to the basic estimate obtained by Hörmander in [7] on pseudoconvex domains. A complex manifold X is said to be Stein manifold if there exists an exhaustion function $\mu\in C^2(X,\mathbb{R})$ such that $i\partial\overline{\partial}\mu>0$ on X.

Theorem 3.1. *Let X be a Stein manifold of dimension n. Let Ω_1 and Ω_2 be two open pseudoconvex submanifolds with smooth boundary in X such that $\overline{\Omega}_2\Subset\Omega_1\Subset X$. Assume that $\Omega=\Omega_1\diagdown\overline{\Omega}_2$. Let ρ be a defining function of Ω near $b\Omega_1$ and λ be a smooth function on $\overline{\Omega}$ such that $\lambda=\mu$ in a neighborhood of $b\Omega_1$ and $\lambda=-\mu$ in a neighborhood of $b\Omega_2$. Then, for $1\le q\le n-2$, $n\ge 3$, there exists $c,T>0$ such that for every $t\ge T$ there exists $C_t>0$ such that*

$$t\|u\|_t^2\le c\,Q^t(u,u)+C_t\|u\|^2_{W^{-1}} \tag{3.1}$$

for $u\in\mathcal{D}_{p,q}(\Omega)$.

Proof. By using a partition of unity $\{\xi_i\}_{i=1}^m$, $\sum_{i=1}^m\xi_i^2=1$, it suffices to prove the estimate (3.1) when u is supported in a small neighborhood U. If $\overline{U}\subset\Omega$, then by the ellipticity of Q^t in the interior of Ω we have

$$\|u\|^2_{W^1}\le c'\,Q^t(u,u)\ \text{ for }\ u\in\mathcal{D}_{p,q}(U).$$

Thus by a well-known inequality in Sobolev space (see, for example, Section 4.2 in Straube [14], page 86 and Proposition 3.1 in Shaw [12]; page 261, inequality (3.3)), we have

$$\|u\|_t^2\le c'\,\|u\|^2_{W^1}+C'_t\,\|u\|^2_{W^{-1}} \tag{3.2}$$

which imply (3.1), when $\overline{U}\cap b\Omega=\varnothing$ and $u\in\mathcal{D}_{p,q}(U)$.

If u is supported in a neighborhood U of $b\Omega_1$, since Ω is pseudoconvex at $b\Omega_1$ and $\lambda=\mu$ is strongly plurisubharmonic on U (shrink U if necessary). Following Hörmander [7], it follows that

$$t\int_{U\cap\Omega_1}{\sum_{I,J}}'\,|u_{I,J}|^2\,e^{-t\lambda}dV\le c'\,Q^t(u,u)$$

for $c' > 0$ and for $u \in \mathcal{D}_{p,q}(U \cap \Omega_1)$ with $1 \le q \le n-1$. Thus, there exists $C'_t > 0$ such that

$$t \int_{U\cap\Omega_1} {\sum_{I,J}}' |u_{I,J}|^2 e^{-t\lambda} dV \le c'\, Q^t(u,u) + C'_t \, \|u\|^2_{W^{-1}} \tag{3.3}$$

for $u \in \mathcal{D}_{p,q}(U \cap \Omega_1)$ with $1 \le q \le n-1$.

Let $S_{\delta_1} = \{z \in X : -\delta_1 < \rho(z) \leqslant 0\}$, where δ_1 a positive number (depend on t) small enough. Since $b\Omega_1$ is compact, by a finite covering $\{U_\nu\}_{\nu=1}^m$ of $b\Omega_1$ by neighborhoods U_ν as in (3.3), we have

$$t \int_{S_{\delta_1}} {\sum_{I,J}}' |u_{I,J}|^2 e^{-t\lambda} dV \le c'\, Q^t(u,u) + C'_t \, \|u\|^2_{W^{-1}} \tag{3.4}$$

when u is supported in the strip S_{δ_1}.

Now since Ω is psudoconcave at $b\Omega_2$. Thus we only have to prove (3.1) when u is supported in a neighborhood U such that $\overline{U} \cap b\Omega_2 \neq \varnothing$. Following Ahn [1], for every integer q with $0 \le q \le n-1$, there exists a neighborhood U of z_0 and a suitable positive constant C such that

$$\begin{aligned}
&2(\|\overline{\partial} u\|_t^2 + \|\overline{\partial}_t^\star u\|_t^2) + C\|u\|_t^2 \\
&\ge \frac{1}{2} {\sum_{I,J}}' \left[\sum_{j \ge q+1} \|\overline{L}_j u_{I,J}\|_t^2 + \sum_{j \le q} \|\delta_j^t u_{I,J}\|_t^2 \right] \\
&\quad + {\sum_{I,K}}' \sum_{j,k} \int_{U\cap b\Omega_2} \rho_{jk} u_{I,jK} \overline{u}_{I,kK}\, e^{-t\lambda} dS \\
&\quad - {\sum_{I,J}}' \sum_{j \le q} \int_{U\cap b\Omega_2} \rho_{jj}\, |u_{I,J}|^2 \, e^{-t\lambda} dS \\
&\quad + {\sum_{I,K}}' \sum_{j,k} \int_{U\cap \Omega_2} \lambda_{jk}\, u_{I,jK} \overline{u}_{I,kK}\, e^{-t\lambda} dV \\
&\quad - {\sum_{I,J}}' \sum_{j \le q} \int_{U\cap \Omega_2} \lambda_{jj}\, u_{I,J}\; \overline{u}_{I,J}\, e^{-t\lambda} dV
\end{aligned} \tag{3.5}$$

for $u \in \mathcal{D}_{p,q}(U \cap \Omega_2)$, where $\delta_j^t = e^{t\lambda} L_j(e^{-t\lambda})$. Since

$$\begin{aligned}
&{\sum_{I,K}}' \sum_{j,k=1}^{n-1} \rho_{jk} u_{I,jK} \overline{u}_{I,kK} - {\sum_{I,J}}' \sum_{j=1}^{n-1} \rho_{jj} |u_{I,J}|^2 \\
&= {\sum_{I,K}}' \sum_{j,k=1}^{n-1} \left(\rho_{jk} - \sum_{l=1}^{n-1} \rho_{ll}\, \delta_{jk} \right) u_{I,jK} \overline{u}_{I,kK}.
\end{aligned}$$

Assume that $(\rho_{jk})_{j,k=1}^{n-1}$ is diagonal, then $\left(\rho_{jk} - \sum_{l=1}^{n-1} \rho_{ll} \delta_{jk}\right)_{j,k=1}^{n-1}$ is also diagonal and the diagonal elements are negative value of $n-2$ sums of eigenvalues of the Levi form. Since Ω is psudoconcave at $b\Omega_2$. For each $z \in b\Omega_2$, we may diagonalize $(\rho_{jk})_{j,k=1}^{n-1}$ under a unitary transformation and the positive semi-definiteness is invariant under such transformation. Thus

$$\left(\rho_{jk} - \frac{1}{q}\left(\sum_{j=1}^{n-1}\rho_{jj}\right)\delta_{jk}\right)_{j,k=1}^{n-1}$$

is positive semidefinite in $U \cap b\Omega_2$. Then, for $1 \leq q \leq n-2$, we have

$$\sum_{I,K}{}' \sum_{j,k=1}^{n-1} \rho_{jk} u_{I,jK} \overline{u}_{I,kK} - \sum_{I,J}{}' \sum_{j=1}^{n-1} \rho_{jj} |u_{I,J}|^2 \geq 0 \ \text{ for each } z \in U \cap b\Omega_2. \tag{3.6}$$

We write

$$\begin{aligned} &\sum_{I,K}{}' \sum_{j,k=1}^{n} \int_{U\cap\Omega_2} \lambda_{jk}\, u_{I,jK}\, \overline{u}_{I,kK}\; e^{-t\lambda} dV \\ &- \sum_{I,J}{}' \left(\sum_{j=1}^{n-1} \int_{U\cap\Omega_2} \lambda_{jj}\right) |u_{I,J}|^2\; e^{-t\lambda} dV = X_1 + X_2, \end{aligned}$$

where

$$\begin{aligned} X_1 = &\sum_{I,K}{}' \sum_{j=n \text{ or } k=n} \int_{U\cap\Omega_2} \lambda_{jk}\, u_{I,jK}\, \overline{u}_{I,kK}\; e^{-t\lambda} dV \\ &+ \sum_{\substack{I,K \\ n\in K}}{}' \sum_{j,k=1}^{n-1} \int_{U\cap\Omega_2} \lambda_{jk}(z) u_{I,jK}\, \overline{u}_{I,kK}\; e^{-t\lambda} dV \\ &- \sum_{\substack{I,J \\ n\in J}}{}' \sum_{j=1}^{n-1} \int_{U\cap\Omega_2} \lambda_{jj}(z) |u_{I,J}|^2\; e^{-t\lambda} dV, \end{aligned}$$

and

$$\begin{aligned} X_2 = &\sum_{\substack{I,K \\ n\notin K}}{}' \sum_{j,k=1}^{n-1} \int_{U\cap\Omega_2} \lambda_{jk}(z) u_{I,jK}\, \overline{u}_{I,kK}\; e^{-t\lambda} dV \\ &- \sum_{\substack{I,J \\ n\notin J}}{}' \sum_{j=1}^{n-1} \int_{U\cap\Omega_2} \lambda_{jj}(z) |u_{I,J}|^2\; e^{-t\lambda} dV. \end{aligned}$$

Take the coordinate functions $z_1, z_2, \ldots, z_n$ about z_0. Then in $z_1, z_2, \ldots, z_n$ coordinates, $A = \left(\frac{\partial^2 \mu}{\partial z_j \partial \overline{z}_k}\right)(z_0)$, $1 \leq j,k \leq n-1$ is an Hermitian matrix and there exists a unitary matrix $P = (P_{jk})_{1\leq j,k\leq 1}$ such that $P^*AP = A$, where

$A = (\lambda_j)_{j=1}^{n-1}$ a diagonal matrix whose entries λ_j are eigenvalues of A. Set

$$\omega_j = \sum_{k=1}^{n-1} \overline{P}_{kj} z_k,\ j = 1, \ldots, n, \text{ and } \omega_n = z_n.$$

Then

$$\lambda_{jk}^2(z_0) = \left(\frac{\partial^2 \mu}{\partial z_j \partial \overline{z}_k}\right)(z_0) = \lambda_j \delta_{jk},\ 1 \le j, k \le n-1.$$

Every term in X_1 has the form $(\lambda_{jk}\, u_{I,J}, u_{I,L})$, whenever $n \in J$ or $n \in L$. Applying (3.2) to those J containing n, we have

$$|\langle \lambda_{jk}\, u_{I,J}, u_{I,L}\rangle_t| \le \|\lambda_{jk}\, u_{I,J}\|_t \|u_{I,L}\|_t \le c'\, \|u_{I,J}\|^2_{W^1} + C'_t\, \|u_{I,J}\|^2_{W^{-1}} + \|u_{I,L}\|^2_t.$$

Thus it follows that

$$X_1 \ge -c' {\sum_{\substack{I,J\\ n\in J}}}' \|u_{I,J}\|^2_{W^1} - C'_t\, \|u\|^2_{W^{-1}} - \|u\|^2_t.$$

Let

$$R(u,u)(z) = {\sum_{\substack{I,K\\ n\notin K}}}' \sum_{j,k=1}^{n-1} \lambda_{jk} u_{I,jK}\, \overline{u}_{I,kK} - {\sum_{\substack{I,J\\ n\notin J}}}' \sum_{j=1}^{n-1} \lambda_{jj}(z) |u_{I,J}|^2.$$

Then

$$\begin{aligned}
R(u,u)(z_0) &= {\sum_{\substack{I,K\\ n\notin K}}}' \sum_{j,k=1}^{n-1} \lambda_{jk}(z_0) u_{I,jK}\, \overline{u}_{I,kK} - {\sum_{\substack{I,J\\ n\notin J}}}' \sum_{j=1}^{n-1} \lambda_{jj}(z_0) |u_{I,J}|^2 \\
&= {\sum_{\substack{I,K\\ n\notin K}}}' \sum_{j,k=1}^{n-1} \left(-\left(\frac{\partial^2 \mu}{\partial z_j \partial \overline{z}_k}\right)(z_0)\right) u_{I,jK}\, \overline{u_{I,kK}} \\
&\quad - {\sum_{\substack{I,J\\ n\notin J}}}' \sum_{j=1}^{n-1} \left(-\left(\frac{\partial^2 \mu}{\partial z_j \partial \overline{z}_j}\right)(z_0)\right) |u_{I,J}|^2 \\
&= -{\sum_{\substack{I,J\\ j\in J\\ n\notin J}}}' \lambda_j |u_{I,J}|^2 + {\sum_{\substack{I,J\\ n\notin J}}}' \sum_{j=1}^{n-1} \lambda_j |u_{I,J}|^2 \\
&= {\sum_{\substack{I,J\\ j\notin J\\ n\notin J}}}' \lambda_j |u_{I,J}|^2 \ge d {\sum_{\substack{I,J\\ n\notin J}}}' |u_{I,J}|^2,
\end{aligned}$$

where d the smallest eigenvalues of A at the point $z \in U \cap \overline{\Omega}_2$. Then $d(z) \ge d_0 > 0$ for some positive number d_0 and all $z \in U \cap b\Omega_2$. Thus for $n \ge 3$ and

$0 < q < n-1$, if we shrink U sufficiently, by continuity of the second derivatives of λ, we have

$$X_2 \geq d_0 \sideset{}{'}\sum_{\substack{I,J \\ n \notin J}} \|u_{I,J}\|_t^2 .$$

Then we obtain

$$\begin{aligned} &\sideset{}{'}\sum_{I,K} \sum_{j,k=1}^{n} \int_{U\cap\Omega_2} \lambda_{jk}\, u_{I,jK}\, \overline{u}_{I,kK}\, e^{-t\lambda} dV \\ &- \sideset{}{'}\sum_{I,J} \left(\sum_{j=1}^{n-1} \int_{U\cap\Omega_2} \lambda_{jj} \right) |u_{I,J}|^2\, e^{-t\lambda} dV \\ \geq{}& d_0 \sideset{}{'}\sum_{\substack{I,J \\ n\notin J}} \|u_{I,J}\|_t^2 - c' \sideset{}{'}\sum_{\substack{I,J \\ n\in J}} \|u_{I,J}\|_{W^1}^2 - C_t' \|u\|_{W^{-1}}^2 - \|u\|_t^2. \end{aligned} \tag{3.7}$$

By substituting (3.6) and (3.7) into (3.5), we obtain

$$\begin{aligned} &2\left(\|\overline{\partial} u\|_t^2 + \|\overline{\partial}_t^\star u\|_t^2 \right) + C\|u\|_t^2 + C_t' \|u\|_{W^{-1}}^2 \\ &\geq \frac{1}{2} \sideset{}{'}\sum_{I,J} \left[\left\|\overline{L}_n u_{I,J}\right\|_t^2 + \sum_{j=1}^{n} \left\|\delta_j^t u_{I,J}\right\|_t^2 \right] \\ &\quad + d_0 \sideset{}{'}\sum_{\substack{I,J \\ n\notin J}} \|u_{I,J}\|_t^2 - c' \sideset{}{'}\sum_{\substack{I,J \\ n\in J}} \|u_{I,J}\|_{W^1}^2 . \end{aligned} \tag{3.8}$$

If $j = n$ or $k = n$ we have $u_{I,jK} = 0$ or $u_{I,kK} = 0$ on the boundary. Since $u_{I,J}$ vanishes on the boundary when $n \in J$, by performing the same manipulation as (4.3.6) in Chen and Shaw [3], we have

$$\left\|\overline{L}_j u_{I,J}\right\|_t^2 = \left\|\delta_j^t u_{I,J}\right\|_t^2 - \langle \lambda_{jj} u_{I,J}, u_{I,J} \rangle_t + O\left(\left\|\overline{L} u_{I,J}\right\|_t \|u_{I,J}\|_t \right),$$

where $j = 1, 2, \ldots, n$. Using the inequality (3.2), we have for $n \in J$

$$\begin{aligned} \|u_{I,J}\|_{W^1}^2 &= \sum_{j=1}^{n} \left\|\overline{L}_j u_{I,J}\right\|_t^2 + \sum_{j=1}^{n} \left\|\delta_j^t u_{I,J}\right\|_t^2 + \|u_{I,J}\|_t^2 \\ &\leq 4\left(\left\|\overline{L}_n u_{I,J}\right\|_t^2 + \sum_{j=1}^{n-1} \left\|\delta_j^t u_{I,J}\right\|_t^2\right) + C_t' \|u_{I,J}\|_{W^{-1}}^2 , \end{aligned} \tag{3.9}$$

where C is a constant depending only on t. By combining (3.8) and (3.9) we easily obtain

$$4Q^t(u,u) + C_t' \|u\|_{W^{-1}}^2 \geq \left(\frac{1}{4} - c' \right) \sideset{}{'}\sum_{\substack{I,J \\ n\in J}} \|u_{I,J}\|_{W^1}^2 + d_0 \sideset{}{'}\sum_{\substack{I,J \\ n\notin J}} \|u_{I,J}\|_t^2 . \tag{3.10}$$

By an interpolation theorem in Sobolev space for $n \in J$, we have

$$\|u_{I,J}\|_t^2 \le t^{-1} \|u_{I,J}\|_{W^1}^2 + C_t' \|u_{I,J}\|_{W^{-1}}^2$$

to those J containing n and put into (3.10), we obtain

$$t \int_{U\cap\Omega_2} \sum_{I,J}{}' |u_{I,J}|^2 e^{-t\lambda} dV \le c' \, Q^t(u,u) + C_t' \|u\|_{W^{-1}}^2 \tag{3.11}$$

for $u \in \mathcal{D}_{p,q}(U \cap \Omega_2)$ with $1 \le q \le n-1$.

Let $S_{\delta_2} = \{z \in X : 0 \le \rho(z) < \delta_2\}$, where δ_2 a positive number (depend on t) small enough. Since $b\Omega_2$ is compact, by a finite covering $\{U_\nu\}_{\nu=1}^m$ of $b\Omega_2$ by neighborhoods U_ν as in (3.11), we have

$$t \int_{S_{\delta_2}} \sum_{I,J}{}' |u_{I,J}|^2 e^{-t\lambda} dV \le c' \, Q^t(u,u) + C_t' \|u\|_{W^{-1}}^2 \tag{3.12}$$

when u is supported in the strip S_{δ_2}.

Let $S_\delta = S_{\delta_1} \cup S_{\delta_2}$, where $\delta = \min\{\delta_1, \delta_2\}$. Then by using (3.4) and (3.12), we obtain

$$t \int_{S_\delta} |u_{I,J}|^2 e^{-t\lambda} \, dV \le c' \, Q^t(u,u) + C_t' \|u\|_{W^{-1}}^2 . \tag{3.13}$$

Now, we estimate the integral over $\Omega \backslash S_\delta$. Choose $\gamma_\delta \in \mathcal{D}(\Omega)$ so that $\gamma_\delta(z) = 1$ whenever $\rho(z) \le -\delta$ and $z \in \Omega \backslash S_\delta$. By an interpolation theorem in Sobolev space, we have for a constant $s > 0$ still to be determined we have the inequality

$$\|\gamma_\delta u\|_t^2 \le s \, \|\gamma_\delta u\|_{W^1}^2 + \frac{1}{s} \|\gamma_\delta u\|_{W^{-1}}^2 .$$

On the other hand, since Q^t is elliptic, by Gårding's inequality, there is a constant C_2 depending only on the diameter of the domain Ω such that

$$\begin{aligned} \|\gamma_\delta u\|_{W^1}^2 &\le C_2 \left(Q^t(\gamma_\delta u, \gamma_\delta u) + \|\gamma_\delta u\|_t^2 \right) \\ &\le 2C_2 \Big(\|\gamma_\delta(\overline{\partial} u)\|_t^2 + \|\gamma_\delta(\overline{\partial}^\star u)\|_t^2 + \|[\gamma_\delta, \overline{\partial}] \, u)\|_t^2 + \|[\gamma_\delta, \overline{\partial}^\star] \, u\|_t^2 + \|\gamma_\delta u\|_t^2 \Big). \end{aligned}$$

Since the sum of the commutator terms is bounded by $C_3 \|u\|^2$ for some constant C_3 dependent of δ, we obtain the inequality

$$\|\gamma_\delta u\|_t^2 \le 2C_2 s \, Q^t(u,u) + 2C_2 C_3 s \|u\|_t^2 + \frac{1}{s} \|u\|_{W^{-1}}^2 . \tag{3.14}$$

By combining (3.13) and (3.14), we obtain

$$\begin{aligned} t \|u\|_t^2 &\le t \int_{S_\delta} |u|_t^2 \, dV + t \|\gamma_\delta u\|_t^2 \\ &\le C_1 \, Q^t(u,u) + C_\varepsilon' \|u\|_{W^{-1}}^2 + 2C_2 st \, Q^t(u,u) \\ &\quad + 2C_2 C_3 st \|u\|_t^2 + \frac{t}{s} \|u\|_{W^{-1}}^2 \\ &= (C_1 + 2C_2 st) \, Q^t(u,u) + 2C_2C_3 st \|u\|_t^2) + (C_t' + \frac{t}{s}) \|u\|_{W^{-1}}^2 . \end{aligned}$$

Now, we choose small s and large t so that $2C_2C_3s < \frac{1}{2}$ and so that $\frac{C_1}{t} + 2C_2s < \frac{c}{2}$. Then, we obtain the estimate

$$\|u\|_t^2 \leq c\, Q^t(u,u) + C_t\, \|u\|_{W^{-1}}^2 \,,$$

where $C_t = 2(\frac{C'_t}{t} + \frac{1}{s})$. □

Remark 3.1. It is easy to observe that (3.1) implies:

$$t\, \|u\|_t^2 \leq c\, Q^t(u,u) + C_t\, \|u\|_{W^{-1}}^2$$

for $u \in Dom(\square)$.

Lemma 3.2. *Let Ω be an "annulus" as in Theorem 3.1 with smooth boundary. Let $\{U_j\}_{j=1}^N$ be a finite covering of $b\Omega$ by a local patching. If a basic a priori estimate (3.1) hold in each U_j :*

$$t\, \|u\|_t^2 \leq c\, Q^t(u,u) + C_t\, \|u\|_{W^{-1}}^2$$

for $u \in C^\infty_{p,q}(\overline{\Omega} \cap U_j) \cap dom\, \overline{\partial}^\star_t$. Then we have global basic a priori estimate (3.1).

Proof. Let $\{\zeta_j\}_{j=0}^N$ be a partition of the unity such that $\zeta_0 \in \mathcal{D}_{p,q}(\Omega)$, $\zeta_j \in \mathcal{D}_{p,q}(U_j)$, $j = 1, 2, \ldots, N$ and $\sum_{j=0}^N \zeta_j^2 = 1$ on $\overline{\Omega}$. where $\{U_j\}_{j=1,\ldots,N}$ is a covering of $b\Omega$.

For $u \in \mathcal{D}_{p,q}(\Omega)$ we wish to prove (3.1). From the interior elliptic regularity of $Q^t(u,u)$ we have

$$\|\zeta_0 u\|_{W^1}^2 \leq Q^t(\zeta_0 u, \zeta_0 u).$$

On the other hand, by an interpolation theorem in Sobolev space, we have

$$\|\zeta_0 u\|_t^2 \lesssim c\, \|\zeta_0 u\|_{W^1}^2 + C_t \|\zeta_0 u\|_{W^{-1}}^2.$$

It follows

$$\begin{aligned}\|\zeta_0 u\|_t^2 &\lesssim c\, Q^t(\zeta_0 u, \zeta_0 u) + C_t \|\zeta_0 u\|_{W^{-1}}^2 \\ &\lesssim c\, Q^t(u,u) + C_t \|u\|_{W^{-1}}.\end{aligned}$$

Similarly, for $j = 1, \ldots, N$, using the hypothesis, we have

$$\begin{aligned}\|\zeta_j u\|_t^2 &\lesssim c\, Q^t(\zeta_j u, \zeta_j u) + C_t \|\zeta_j u\|_{W^{-1}}^2 \\ &\lesssim c\, Q^t(u,u) + C_t \|u\|_{W^{-1}}^2.\end{aligned}$$

Summing up over j, we get the proof of the lemma. □

4. Global regularity up to the boundary

As an immediate consequence of the basic estimate (3.1) is the following results:

Lemma 4.1. *Let Ω be an "annulus" as in Theorem 3.1 with smooth boundary. Then, for a sufficiently large t and for $1 \le q \le n-2$, $n \ge 3$, we have*

(1) *$\mathcal{H}_t$ is finite dimensional.*
(2) *The Laplacian $\Box^t$ has closed range in $L^2_{p,q}(\Omega)$.*
(3) *The operator $\overline{\partial}$ has closed range in $L^2_{p,q}(\Omega)$ and $L^2_{p,q+1}(\Omega)$.*
(4) *The operator $\overline{\partial}^\star$ has closed range in $L^2_{p,q}(\Omega)$ and $L^2_{p,q-1}(\Omega)$.*
(5) *There exists $C > 0$ such that for all $u \in \mathcal{D}^{p,q}(\Omega)$ with $u \perp \mathcal{H}_t$, we have*

$$\|u\|_t^2 \le C(\|\overline{\partial} u\|_t^2 + \|\vartheta_t u\|_t^2). \tag{4.1}$$

Proof. Inequality (3.1) implies that from every sequence $\{u_\nu\}_{\nu=1}^\infty$ in $\operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}_t^\star$ with $\|u_\nu\|_t$ bounded and $\overline{\partial}\, u_\nu \longrightarrow 0$, $\overline{\partial}_t^* u_\nu \longrightarrow 0$, one can extract a subsequence which converges in (weighted) $L^2_{p,q}(\Omega)$. It suffices to find a subsequence which converges in $W^{-1}_{p,q}(\Omega)$ (using that $L^2_{p,q}(\Omega) \hookrightarrow W^{-1}_{p,q}(\Omega)$ is compact); (3.1) implies that such a subsequence is cauchy (hence convergent) in $L^2_{p,q}(\Omega)$. General Hilbert space theory (Hörmander [7]; Theorems 1.1.3 and 1.1.2) now gives that $\mathcal{H}_t$ is finite dimensional and that $\overline{\partial} : L^2_{p,q}(\Omega) \longrightarrow L^2_{p,q+1}(\Omega)$ and $\overline{\partial}_t^\star : L^2_{p,q}(\Omega) \longrightarrow L^2_{p,q-1}(\Omega)$ have closed range.

To prove (4.1), we assume that (4.1) does not hold and deduce a contradiction. If for every $\nu \in \mathbb{N}$ there exists a $u_\nu \perp \mathcal{H}_t$, then $\|u_\nu\|_t = 1$ such that

$$\|u_\nu\|_t^2 \ge \nu(\|\overline{\partial} u_\nu\|_t^2 + \|\vartheta_t u_\nu\|_t^2). \tag{4.2}$$

Combining this and (3.1), we have

$$\|u_\nu\|_t^2 \le C_t \|u_\nu\|_{W^{-1}}^2$$

which implies u_ν converges in L^2 to u where $u \perp \mathcal{H}_t$. By (4.2) we have that $u \in \mathcal{H}_t$, a contradiction. Thus (4.1) must hold for all $u \perp \mathcal{H}_t$. $\Box$

As an immediate consequence of the basic estimate (4.1) are the following theorems whose proof can be found in Hörmander [7].

Theorem 4.2. *Let Ω be an "annulus" as in Theorem 3.1 with smooth boundary. Then, for $1 \le q \le n-2$, $n \ge 3$, the range of $\Box^t$ is closed and there exists a bounded linear operator N^t for sufficiently large $t > 0$ satisfies the following properties*:

(i) *range$(N^t) \subset$ dom$(\Box^t)$, $N^t \Box^t = I$ on dom$(\Box^t)$,*
(ii) *for $f \in L^2_{p,q}(\Omega)$, we have $u = \overline{\partial}\,\overline{\partial}_t^\star N^t f \oplus \overline{\partial}_t^\star \overline{\partial} N^t f$,*
(iii) *$\overline{\partial} N^t = N^t \overline{\partial}$, and $\overline{\partial}_t^\star N^t = N^t\, \overline{\partial}_t^\star$,*
(iv) *for all $f \in L^2_{p,q}(\Omega)$, we have the estimates*

$$\|N^t f\|_t \le c\|f\|_t,$$

$$\|\overline{\partial} N^t f\|_t + \|\overline{\partial}_t^\star N^t f\|_t \le \sqrt{c}\|f\|_t.$$

(v) *If $f \in ker(\Box_t)$, then $\overline{\partial}_t^\star N^t f$ gives the solution u_t to the equation $\overline{\partial} u_t = f$ of minimal $u_t \in L^2_{p,q-1}(\Omega)$-norm.*

(vi) *If* $f \in ker(\Box_t)$, *then* $\overline{\partial} N^t f$ *gives the solution* u_t *to the equation* $\overline{\partial}_t^{\star} u_t = f$ *of minimal* $u_t \in L^2_{p,q+1}(\Omega)$*-norm.*

By Theorem 4.2(ii) and the density of $C^\infty_{p,q}(\overline{\Omega})$ in $W^s_{p,q}(\Omega)$, the following is immediate.

Theorem 4.3. *Let* Ω *be an "annulus" as in Theorem 3.1 with smooth boundary. If* $f \in C^\infty_{p,q}(\overline{\Omega})$ *with* $1 \leq q \leq n-2$, $n \geq 3$ *and* $N^t f \in C^\infty_{p,q}\overline{\Omega})$, *then for any nonnegative integer* s *there exist constants* C_s *and* T_s *such that*

$$\|N^t f\|_{W^s} \leq C_s \|f\|_{W^s}, \text{ for every } t > T_s. \tag{4.3}$$

Proof. The proof is the same as in [9]. □

Using the elliptic regularization method which was used in [9], one can pass from the a priori estimates (3.1) to actual estimates and we can prove the following theorem:

Theorem 4.4. *Let* Ω *be an "annulus" as in Theorem 3.1 with smooth boundary. For every integer* $s \geq 0$ *and real* $t > T > 0$ *the weighted* $\overline{\partial}$*-Neumann operator* N^t *is bounded from* $W^s_{p,q}(\Omega)$ *into itself for* $1 \leq q \leq n-2$, $n \geq 3$.

By Theorem 4.3 and the density of $C^\infty_{p,q}(\overline{\Omega})$ in $W^s_{p,q}(\Omega)$, the following is immediate.

Corollary 1. *Let* Ω *be an "annulus" as in Theorem 3.1 with smooth boundary. If* $f \in W^s_{p,q}(\Omega)$, $s = 0, 1, 2, 3, \ldots$ *satisfies* $\overline{\partial} f = 0$, *where* $1 \leq q \leq n-2$, $n \geq 3$, *then there exits* $u \in W^s_{p,q-1}(\Omega)$ *so that* $\overline{\partial} u = f$ *on* Ω *with estimate*

$$\|u\|_{W^s} \leq C_s \|f\|_{W^s}.$$

Theorem 4.5. *Let* Ω *be an "annulus" as in Theorem 3.1 with smooth boundary. Then, for* $f \in C^\infty_{p,q}(\overline{\Omega})$, *with* $\overline{\partial} f = 0$, $1 \leq q \leq n-2$, $n \geq 3$, *there exists* $u \in C^\infty_{p,q-1}(\overline{\Omega})$ *such that* $\overline{\partial} u = f$.

Proof. The proof is the same as in [10]. □

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Sayed Saber
Mathematics Department
Faculty of Science
Beni-Suef University
Egypt
E-mail address: sayedkay@yahoo.com