J. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/JKMS.j160556
pISSN: 0304-9914 / eISSN: 2234-3008

ON 2-GENERATING INDEX OF FINITE DIMENSIONAL LEFT-SYMMETRIC ALGEBRAS

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Abstract. In this paper, we introduce the notion of generating index $\mathcal{I}_1(A)$ (2-generating index $\mathcal{I}_2(A)$, resp.) of a left-symmetric algebra A, which is the maximum of the dimensions of the subalgebras generated by any element (any two elements, resp.). We give a classification of left-symmetric algebras with $\mathcal{I}_1(A) = 1$ and $\mathcal{I}_2(A) = 2, 3$ resp., and show that all such algebras can be constructed by linear and bilinear functions. Such algebras can be regarded as a generalization of those relating to the integrable (generalized) Burgers equation.

1. Introduction

Left-symmetric algebras form a class of important non-associative algebras. This kind of algebras were first introduced by Cayley in [6], where they were used to describe some properties of rooted tree algebras. However, only very little attention had been paid to this subject until Vinberg applied them to study convex homogeneous cones in [20] and Koszul exploited them to investigate affine manifolds in [14]. Later, they were used by Gerstenhaber to solve the problem of deformation of associative rings in [11]. From that time they have appeared in many different fields of mathematics and mathematical physics and hence they are known under many different names, such as pre-Lie algebras, Vinberg algebras, Koszul algebras, Gerstenhaber algebras, or quasi-associative algebras. In [5], Burde gave a survey of the fields in which left-symmetric algebras play an important role, such as vector fields, vertex algebras, operad theory and so on. Also they have strong relation with classical Yang-Baxter equations ([7], [8], [12]) and Rota-Baxter operators ([15], [16]).

Meanwhile, a great deal of mathematical effort has been made to studying the relationship between integrable evolution multicomponent PDE's and ODE's and some special kinds of non-associative algebras ([13], [17], [18], [19]). It turned out that left-symmetric algebras are closely related to multicomponent Burgers equations. In [1], Bai constructed a class of left-symmetric

Received August 24, 2016.
2010 *Mathematics Subject Classification.* 17B05, 17B60, 17D99.
Key words and phrases. left-symmetric algebra, generating index, non-associative algebra, linear function.

algebras using linear functions and symmetric bilinear functions:

$$x * y = f(x)y + g(y)x + h(x,y)c, \quad \forall x, y \in \mathfrak{g}, \tag{1.1}$$

where c is a fixed nonzero element, f, g are linear functions and h is a non-zero symmetric bilinear function. Such algebras can be regraded as a generalization of a class of left-symmetric algebras relating to the integrable (generalized) Burgers equation (see [18], [19]):

$$U_t = U_{xx} + 2U * U_x + (U * (U * U)) - ((U * U) * U). \tag{1.2}$$

Note that the above mentioned left-symmetric algebras share the same property that any two elements generate a subalgebra (containing a fixed element c) of dimension no more than 3. Therefore, it is natural to consider what kinds of left-symmetric algebras can be generated by two elements. More generally, if a left-symmetric algebra is not generated by two elements, what is the maximal subalgebra generated by two elements. Thus for a finite-dimensional left-symmetric algebra A, we call the maximum of the dimensions of the subalgebras generated by any two elements in A the 2-generating index, and denote it as $\mathcal{I}_2(A)$. Similarly, we call the maximum of the dimensions of the subalgebras generated by any element in A the generating index, and denote it as $\mathcal{I}(A) = \mathcal{I}_1(A)$. In fact, Bai constructed some special left-symmetric structures with 2-generating index 2 or 3 under some additional conditions. We are interested in a more general class of left-symmetric algebras of 2-generating index 2 or 3, that is, left-symmetric algebras with $\mathcal{I}_2(A) \le 3$.

In this paper, we assume that $\dim A \ge 4$. The reason is that Bai has classified left-symmetric algebras with $\dim A \le 3$ in [2] and [3]. The problem will be divided into two cases (see Theorem 5.1): either $\mathcal{I}_1(A) = 1$; or $\mathcal{I}_1(A) = 2$, and there exists an element $c_{x,y}$ such that $\langle x, y\rangle \subset \operatorname{span}\{x, y, c_{x,y}\}$ for any $x, y \in A$. The former case is a special case of those with $\mathcal{I}_2(A) \le 3$. If A is a left-symmetric algebra with $\mathcal{I}_1(A) = 1$, then it is natural to assume

$$x^2 := x * x = f(x)x, \quad \forall\, x \in A,$$

where $f : A \to \mathbb{F}$ is a function with $f(0) = 0$. The function f must be linear and the classification of such left-symmetric algebras follows by investigating the properties of the function f (Section 3). Note that the set of left-symmetric algebras with $\mathcal{I}_2(A) = 2$ are included in that of left-symmetric algebras with $\mathcal{I}_1(A) = 1$. In this case, we have

$$x * y = f(x,y)x + g(x,y)y,$$

where f, g are functions on $A \times A$. In [1], Bai classified such left-symmetric algebras with the additional assumption that f, g are linear functions on A, which is redundant as we will see in Theorem 4.1. For the reader's convenience, we reproduce the result of classification of this case in Theorem 4.2.

For the later case, we have

$$x * y = f(x)y + g(y)x + h(x,y)c_{x,y}, \quad \forall\, x, y \in A.$$

It turns out that $c_{x,y}$ is independent of the choice of $x, y \in A$ (Theorem 5.1). This implies that f, g are linear functions on A and h is a bilinear function on $A \times A$ (Theorem 5.2), which is an essential step towards our classification of such algebras (Theorem 5.3). Our main results are summarized as follows:

Theorem 1.1. *Let A be a finite dimensional left-symmetric algebra with $\mathcal{I}_1(A) = 1$. Then A is isomorphic to one of the following algebras.*

(1) *A is a two-step nilpotent algebra, i.e., $(x * y) * z = x * (y * z) = 0,\ \forall\ x, y, z \in A$. Furthermore, such algebras are defined by two-step nilpotent Lie algebras with the bracket $x * y = \frac{1}{2}[x, y]$.*

(2) *There exists a basis $\{e_1, \dots, e_n\}$ in A such that $A = \langle e_1 \rangle \oplus \langle e_2, \dots, e_n \rangle$, where $\langle e_2, \dots, e_n \rangle$ is an ideal of A. Furthermore, we have $e_1 * e_1 = e_1$, $e_i * e_k = 0$, $i, k = 2, \dots, n$, and there exists $2 \le j \le n$, such that*

$$\begin{array}{lll} e_1 * e_k = e_k, & e_k * e_1 = 0, & \textit{for } \ 2 \le k \le j, \\ e_1 * e_l = 0, & e_l * e_1 = e_l, & \textit{for } \ j < l \le n. \end{array}$$

Theorem 1.2. *Let A be a finite-dimensional left-symmetric algebra with $\mathcal{I}_2(A) = 3$ and $\mathcal{I}_1(A) = 2$. Then A is isomorphic to one of the following left-symmetric algebras in Table 1.*

Table 1: The classification of $\mathcal{I}_2(A) = 3$ and $\mathcal{I}_1(A) = 2$

	f	g	h	Characteristic matrix
A_1	0	0	E_{nn}	$\begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & e_n \end{pmatrix}$
$A_2(H)$ $(H \in \mathbb{F}^{(n-1)\times(n-1)})$	0	0	$\begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} He_n & 0 \\ 0 & 0 \end{pmatrix}$
A_3	0	0	E_{1n}	$\begin{pmatrix} 0 & 0 & \cdots & 0 & e_n \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$
B_1	0	g_1	$E_{1n} + E_{nn}$	$\begin{pmatrix} e_1 & 0 & \cdots & 0 & e_n \\ e_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{n-1} & 0 & \cdots & 0 & 0 \\ e_n & 0 & \cdots & 0 & e_n \end{pmatrix}$
$B_2(\alpha, H)$ $(\alpha \in \mathbb{F}^{n-2})$ $(H \in \mathbb{F}^{(n-2)\times(n-2)})$	0	g_1	$\begin{pmatrix} 0 & 0 & -1 \\ \alpha & H & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e_1 & 0 & -e_n \\ e_i + \alpha e_n & He_n & 0 \\ e_n & 0 & 0 \end{pmatrix}$
$B_3(\lambda)$ $(\lambda \neq 0)$	0	g_1	λE_{1n}	$\begin{pmatrix} e_1 & 0 & \cdots & 0 & \lambda e_n \\ e_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_n & 0 & \cdots & 0 & 0 \end{pmatrix}$

$B_4(k)$ $(k<n)$	0	g_1	E_{1k}	$\begin{pmatrix} e_1 & 0 & \cdots & e_n & \cdots & 0 \\ e_2 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ e_n & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$ $(k>1)$ $\begin{pmatrix} e_1+e_n & 0 & \cdots & 0 \\ e_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_n & 0 & \cdots & 0 \end{pmatrix}$ $(k=1)$
$B_5(r)$ $(1<r<n)$	0	g_1	$-E_{1n}-E_{n1} + \sum_{i=2}^{r} E_{ii}$	$\begin{pmatrix} e_1 & 0 & 0 & -e_n \\ e_i & e_nI_r & 0 & 0 \\ e_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
B_6	0	g_1	$E_{11}+E_{1n}$ $-E_{n1}$	$\begin{pmatrix} e_1+e_n & 0 & \cdots & 0 & e_n \\ e_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{n-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$
$B_7(\lambda)$	0	g_1	$\lambda E_{1n}-E_{n1}$	$\begin{pmatrix} e_1 & 0 & \cdots & 0 & \lambda e_n \\ e_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{n-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$
C_1	f_1	0	$-E_{1n}+E_{nn}$	$\begin{pmatrix} e_1 & \cdots & e_{n-1} & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & e_n \end{pmatrix}$
$C_2(\alpha, H)$ $(\alpha \in \mathbb{F}^{n-2})$ $(H \in \mathbb{F}^{(n-2)\times(n-2)})$	f_1	0	$\begin{pmatrix} 0 & 0 & 1 \\ \alpha & H & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e_1 & \cdots & 2e_n \\ \alpha e_n & He_n & 0 \\ 0 & \cdots & 0 \end{pmatrix}$
$C_3(\lambda)$ $(\lambda \neq 0,1)$	f_1	0	λE_{1n}	$\begin{pmatrix} e_1 & \cdots & e_{n-1} & (1+\lambda)e_n \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}$
$C_4(k)$ $(k<n)$	f_1	0	E_{1k}	$\begin{pmatrix} e_1 & \cdots & e_k+e_n & \cdots & e_n \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$ $(k>1)$ $\begin{pmatrix} e_1+e_n & e_2 & \cdots & e_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ $(k=1)$
$C_5(r)$ $(1<r<n)$	f_1	0	$E_{1n}+E_{n1} + \sum_{i=2}^{r} E_{ii}$	$\begin{pmatrix} e_1 & \cdots & e_{n-1} & 2e_n \\ 0 & e_nI_r & 0 & 0 \\ e_n & 0 & 0 & 0 \end{pmatrix}$
C_6	f_1	0	$E_{11}+E_{n1}$ $-E_{1n}$	$\begin{pmatrix} e_1+e_n & e_2 & \cdots & e_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_n & 0 & \cdots & 0 & 0 \end{pmatrix}$

$C_7(\lambda)$ $(\lambda \neq 0)$	f_1	0	$E_{n1} + \lambda E_{1n}$	$\begin{pmatrix} e_1 & \cdots & e_{n-1} & (1+\lambda)e_n \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ e_n & \cdots & 0 & 0 \end{pmatrix}$
$C_8(r)$ $(r < n)$	f_n	0	$\sum_{i=1}^{r} E_{ii} + E_{nn}$	$\begin{pmatrix} e_n I_r & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & 0 \\ e_1 & \cdots & e_{n-1} & 2e_n \end{pmatrix}$
$C_9(\lambda)$ $(\lambda \neq 0, 1)$	f_n	0	λE_{nn}	$\begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ e_1 & \cdots & e_{n-1} & (1+\lambda)e_n \end{pmatrix}$
$D_1(\lambda)$ $(\lambda \neq 0)$	$-\lambda f_n$	g_n	λE_{nn}	$\begin{pmatrix} 0 & \cdots & 0 & e_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & e_{n-1} \\ -\lambda e_1 & \cdots & -\lambda e_{n-1} & e_n \end{pmatrix}$
D_2	$f_1 + f_n$	g_n	$-E_{1n} - E_{nn}$	$\begin{pmatrix} e_1 & \cdots & e_{n-1} & e_1 \\ 0 & \cdots & 0 & e_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & e_{n-1} \\ e_1 & \cdots & e_{n-1} & e_n \end{pmatrix}$

Remark 1.3. In the above tables, f_i and g_i are the linear functions defined by $f_i(e_j) = g_i(e_j) = \delta_{ij}$.

Remark 1.4. In the above tables, there are some relations between different types of left-symmetric algebras, which can be described as follows:

- $A_2(H_1) \cong A_2(H_2)$ if and only if there exists an invertible $(n-1)\times(n-1)$ matrix T such that $TH_2T' = H_1$.
- $B_2(\alpha_1, H_1) \cong B_2(\alpha_2, H_2)$ if and only if there exists an invertible $(n-2) \times (n-2)$ matrix T such that $TH_2T' = H_1$ and $T\alpha_2 = \alpha_1$.
- $C_2(\alpha_1, H_1) \cong C_2(\alpha_2, H_2)$ if and only if there exists an invertible $(n-2) \times (n-2)$ matrix T such that $TH_2T' = H_1$ and $T\alpha_2 = \alpha_1$.

Our paper is organized as follows. In Section 2, we present some definitions and notation. In Section 3, we classify left-symmetric algebras with $\mathcal{I}_1(A) = 1$. In Section 4, we study left-symmetric algebras with $\mathcal{I}_2(A) = 2$. In Section 5, we consider left-symmetric algebras with $\mathcal{I}_1(A) = 2$ and $\mathcal{I}_2(A) = 3$.

2. Preliminaries

In this section, we recall the definition of left-symmetric algebras and introduce the notions of generating index of a left-symmetric algebra.

Let A be a vector space over an algebraically closed field $\mathbb{F}$ of characteristic 0 with a bilinear product $(x, y) \mapsto x * y$. Then A is said to be a *left-symmetric algebra*, if

$$(x * y) * z - x * (y * z) = (y * x) * z - y * (x * z), \tag{2.1}$$

or equivalently,

$$(x,y,z) = (y,x,z), \tag{2.2}$$

for any $x, y, z \in A$, where $(x,y,z) = (x*y)*z - x*(y*z)$.

If A is a left-symmetric algebra, then the operation

$$[x,y] = x*y - y*x \tag{2.3}$$

is skew-symmetric and satisfies the Jacobi identity. Thus every left-symmetric algebra has an underlying Lie algebra structure. Conversely, if $\mathfrak{g}$ is a Lie algebra over $\mathbb{F}$, then a left-symmetric operation satisfying (2.1) and (2.3) will be called a compatible left-symmetric algebra structure on $\mathfrak{g}$.

Inspired by the work of Fang [9] and Bai [4] , where the authors introduced the notion of generating index of finite-dimensional Lie algebras and n-Lie algebras, it is natural to give the following definition.

Definition 2.1. Let A be a finite-dimensional left-symmetric algebra. The 2-*generating index* $\mathcal{I}_2(A)$ of A is the maximum of the dimensions of the subalgebras generated by any two elements in A.

The generating index of a left-symmetric algebra defined by (1.1) is no more than 3, which is a key point for our further study. For further discussion, the following notion is useful.

Definition 2.2. Let A be a finite dimensional left-symmetric algebra. The *generating index* $\mathcal{I}(A) = \mathcal{I}_1(A)$ of A is the maximum of the dimensions of the subalgebras generated by any element in A.

The algebras that we consider in this paper are finite dimensional and over an algebraically closed field $\mathbb{F}$ of characteristic 0, and we will use the following notation frequently throughout this paper.

(1) $\langle e_1, \dots, e_m \rangle$: the subalgebra of A generated by $e_1, \dots, e_m$.
(2) span $\{e_1, \dots, e_m\}$: the linear subspace of A spanned by $e_1, \dots, e_m$.
(3) L_x, R_y: the left and right multiplications, i.e., $L_x y = xy, R_x y = yx$ for all $x, y, \in A$.
(4) E_{ij}: $n \times n$ matrix with 1 in the (i,j) position and zero elsewhere.
(5) Let $\{e_1, \dots, e_{n-1}, e_n\}$ be a basis of left-symmetric algebra A. Then A is determined by its characteristic matrix

$$\begin{pmatrix} e_1 * e_1 & e_1 * e_2 & \cdots & e_1 * e_n \\ e_2 * e_1 & e_2 * e_2 & \cdots & e_2 * e_n \\ \vdots & \vdots & \ddots & \vdots \\ e_n * e_1 & e_n * e_2 & \cdots & e_n * e_n \end{pmatrix}.$$

3. Left-symmetric algebras with $\mathcal{I}_1(A) = 1$

In this section, we classify left-symmetric algebras with generating index 1. Let A be a left-symmetric algebra over $\mathbb{F}$ with $\mathcal{I}_1(A) = 1$. Then for any $x \in A$,

x^2 is a multiple of x. We may assume that

$$x^2 = f(x)x, \ \forall\, x \in A,$$

where $f : A \to \mathbb{F}$ is a function with $f(0) = 0$.

The rest of the proof of Theorem 1.1 consists of several lemmas.

Lemma 3.1. *The function f is linear.*

Proof. For any $k \in \mathbb{F}$, we have

$$f(kx)kx = (kx)^2 = k^2x^2 = k^2 f(x)x,$$

which implies that

$$f(kx) = kf(x). \tag{3.1}$$

Since $\dim A \geq 4$, for any linearly independent elements $x, y \in A$, we have

$$(x \pm y)^2 = f(x \pm y)(x \pm y),$$

and

$$(x \pm y)^2 = x^2 + y^2 \pm (x * y + y * x) = f(x)x + f(y)y \pm (x * y + y * x).$$

It follows that

$$f(x+y)(x+y) + f(x-y)(x-y) = 2(f(x)x + f(y)y).$$

This immediately implies that

$$2f(x) = f(x+y) + f(x-y), \ 2f(y) = f(x+y) - f(x-y).$$

Hence,

$$f(x+y) = f(x) + f(y).$$

By (3.1), the above assertion is also true when x, y are linearly dependent. Therefore f is linear. □

Lemma 3.2. *For any $x, y, z \in A$, we have*

$$x * y + y * x = f(y)x + f(x)y, \tag{3.2}$$

$$(x * y) * x = f(y * x)x. \tag{3.3}$$

Proof. Since f is linear, for any $x, y \in A$ we have

$$\begin{aligned} x * y + y * x &= (x+y) * (x+y) - x * x - y * y \\ &= f(x+y)(x+y) - f(x)x - f(y)y \\ &= f(y)x + f(x)y. \end{aligned}$$

By (2.1) and (3.2), we have

$$\begin{aligned} (x * y) * x &= x * (y * x) + (y * x) * x - y * (x * x) \\ &= f(x)(y * x) + f(y * x)x - y * (f(x)x) \\ &= f(y * x)x. \end{aligned}$$

□

Lemma 3.3. *If $f=0$, then we have $(x*y)*z = x*(y*z) = 0$ for any $x, y, z \in A$.*

Proof. Since $f=0$, we have $x*y+y*x=0$ by (3.2). It follows that

$$[x, y] = x*y - y*x = 2x*y.$$

By the Jacobi identity and (2.1), we have

$$x*(y*z) + y*(z*x) + z*(x*y) = 0,$$

and

$$\begin{aligned}(x*y)*z-(y*x)*z &= x*(y*z)-y*(x*z)\\ &= -(y*x)*z,\end{aligned}$$

which implies that $(x*y)*z=0$. Similarly, we have $x*(y*z)=0$ for any $x, y, z \in A$. □

Note that the left-symmetric algebra we considered in the above has two-step nilpotent Lie algebra as its adjacent Lie algebra. And every two-step nilpotent Lie algebra is the adjacent Lie algebra of a left-symmetric algebra of the above case.

Remark 3.4. The classification of two-step nilpotent Lie algebras are obtained in [10].

If $f \neq 0$, then there exists a basis $\{e_1, \dots, e_n\}$ in A, such that $f(e_i) = \delta_{i1}, i = 1, \dots, n$.

Lemma 3.5. *The subspace* $\ker f$ *is an ideal of A.*

Proof. It is easy to see that $\ker f \subseteq \langle e_2, \dots, e_n\rangle$. If $\ker f \neq \langle e_2, \dots, e_n\rangle$, then we may assume that $e_2 * e_3 = e_1$.

By (3.2) and Lemma 3.3, we have

$$e_2 = e_1*e_2+e_2*e_1 = (e_2*e_3)*e_2 - e_2*(e_3*e_2) = 0,$$

which is a contradiction, thus $\ker f = \langle e_2, \dots, e_n\rangle$.

For any $i = 2, \dots, n$, we can write $e_i*e_1 = a_1e_1+x$ for some $x \in \langle e_2, \dots, e_n\rangle$ and $a_1 \in \mathbb{F}$. Since $f(e_i*e_1) = f(a_1e_1)+f(x) = a_1$, it follows that $e_i*e_1 = f(e_i*e_1)e_1 + x$.

By (3.2), we get $e_1*e_i+e_i*e_1 = e_i$. Thus

$$\begin{aligned}(e_i*e_1)*e_i &= f(e_i*e_1)e_1*e_i + x*e_i\\ &= f(e_i*e_1)(e_i - f(e_i*e_1)e_1 - x) + x*e_i.\end{aligned} \tag{3.4}$$

On the other hand, by (3.3), we obtain

$$(e_i*e_1)*e_i = f(e_1*e_i)e_i = -f(e_i*e_1)e_i. \tag{3.5}$$

Combining (3.4) with (3.5), we get

$$(f(e_i*e_1))^2e_1 = 2f(e_i*e_1)e_i - f(e_i*e_1)x + x*e_i \in \ker f, \tag{3.6}$$

which implies that

$$f(e_i * e_1) = 0.$$

Thus $e_i * e_1 \in \langle e_2, \dots, e_n\rangle,\ e_1 * e_i = e_i - e_i * e_1 \in \langle e_2, \dots, e_n\rangle$. Therefore, $\ker f$ is an ideal. □

Corollary 3.6. *$(e_1 * x) * e_1 = 0$ for any $x \in \ker f$.*

Proof. Since $\ker f$ is an ideal, we have $(e_1 * x) * e_1 \in \ker f$ for any $x \in \ker f$. By (3.3), we have $(e_1 * x) * e_1 = f(x * e_1)e_1$, which implies that $(e_1 * x) * e_1 = 0$. □

Lemma 3.7. *If $f \neq 0$, then there exists a basis $\{e_1, \dots, e_n\}$ in A such that $A = \langle e_1\rangle \oplus \langle e_2, \dots, e_n\rangle$, where $\langle e_2, \dots, e_n\rangle$ is an ideal of A. More precisely,*

(1) *There exists $2 \leq j \leq n$, such that $e_1 * e_k = e_k$, $e_k * e_1 = 0$, $2 \leq k \leq j$, $e_1 * e_k = 0$, and $e_k * e_1 = e_k$, $j < k \leq n$;*
(2) *$e_1 * e_1 = e_1$ and $e_i * e_k = 0$ for any $i, k = 2, \dots, n$.*

Proof. By (3.2) and Corollary 3.6, we have

$$(L_{e_1} + R_{e_1})|_{\ker f} = \mathrm{id}\,|_{\ker f}, \tag{3.7}$$

and

$$(R_{e_1} \circ L_{e_1})|_{\ker f} = 0. \tag{3.8}$$

This implies that

$$L_{e_1}^2 = L_{e_1},\ R_{e_1}^2 = R_{e_1},\ L_{e_1} \circ R_{e_1} = 0. \tag{3.9}$$

Since both L_{e_1} and R_{e_i} are idempotent and commute with each other, (1) follows easily.

Since $f(e_i) = \delta_{i1}$, we have $e_i * e_i = \delta_{i1}e_i$ by (3.2). For $i, k \geq 2$, we have

$$(e_1 * e_i) * e_k - (e_i * e_1) * e_k = e_1 * (e_i * e_k) - e_i * (e_1 * e_k). \tag{3.10}$$

Thus $e_1 * (e_i * e_k) = 2e_i * e_k$ for any $i, k \leq j$. Then we obtain

$$e_i * e_k = 0,\ \forall\ i, k \leq j.$$

Similarly, for any $i, k > j$, we have $e_1 * (e_i * e_k) = -e_i * e_k$. Therefore

$$e_i * e_k = 0,\ \forall\ i, k > j.$$

On the other hand, for any $i \leq j < k$, we have $e_1 * (e_i * e_k) = e_i * e_k$. Thus

$$e_i * e_k \in \langle e_2, \dots, e_j\rangle,\ \forall\ i \leq j < k.$$

Similarly, for any $i > j \geq k$, we have $e_1 * (e_i * e_k) = 0$. Therefore

$$e_i * e_k \in \langle e_{j+1}, \dots, e_n\rangle,\ \forall\ i > j \geq k.$$

By the observation above, we see that $e_s * e_t \in \langle e_2, \dots, e_j\rangle$ and $e_t * e_s \in \langle e_{j+1}, \dots, e_n\rangle$ for any $1 < s \leq j < t$. Since $e_s * e_t = -e_t * e_s$ for any $s, t > 1$, it is sufficient to show

$$e_s * e_t \in \langle e_2, \dots, e_j\rangle \cap \langle e_{j+1}, \dots, e_n\rangle = \{0\}.$$

Summarizing what we have proved, we have

$$e_i * e_k = 0, \ \forall\, i, k = 2, \ldots, n. \tag{3.11}$$

Now one can easily check that

$$(e_i * e_j) * e_k - (e_j * e_i) * e_k = e_i * (e_j * e_k) - e_j * (e_i * e_k)$$

for any $e_i, e_j, e_k \in A$. □

4. Left-symmetric algebras with $\mathcal{I}_2(A) = 2$

Since $\dim A \geq 4$, it is clear that if $\mathcal{I}_2(A) = 2$, then $\mathcal{I}_1(A) = 1$ and $\langle x, y\rangle = \mathrm{span}\,\{x, y\}$ for any $x, y \in A$. Thus it is natural to assume that $x*y = f(x, y)x + g(x, y)y$, where f, g are two functions on $A \times A$. In [1], Bai classified such algebras under the condition that f, g are linear functions, which is redundant as we show in this section.

Theorem 4.1. *Let A be a finite-dimensional left-symmetric algebra of dimension $n \geq 3$. If $\mathcal{I}_2(A) = 2$, then there exist linear functions $f, g \in A^*$ such that*

$$x * y = f(y)x + g(x)y, \quad \forall\, x, y \in A.$$

Proof. Since $\langle x, y\rangle = \mathrm{span}\,\{x, y\}$ for any $x, y \in A$, for x, y linearly independent, we have

$$x * y = f(x, y)x + g(x, y)y, \tag{4.1}$$

where $f(x, y)$ and $g(x, y)$ are functions on

$$S = \{(x, y) \in A \times A \,|\, x, y \text{ are linearly independent}\}.$$

For $x_1, x_2 \in A$ such that $(x_i, y) \in S$, choose $z \in A$ such that both x_1, y, z and x_2, y, z are linearly independent. From $(x_i + z) * y = x_i * y + z * y$, we see that

$$f(x_i + z, y)(x_i + z) + g(x_i + z, y)y = f(x_i, y)x_i + g(x_i, y)y + f(z, y)z + g(z, y)y.$$

It implies that

$$\begin{cases} (1) \quad f(x_i + z, y) = f(x_i, y) = f(z, y), \\ (2) \quad g(x_i + z, y) = g(x_i, y) + g(z, y). \end{cases} \tag{4.2}$$

By (4.2)-(1), we have $f(x_1, y) = f(z, y) = f(x_2, y)$. Thus we may define a function $f : A \to \mathbb{F}$ such that $f(0) = 0$ and $f(y) = f(x, y)$ for any $(x, y) \in S$. Similarly, we may define a function g such that $g(0) = 0$ and $g(x) = g(x, y)$ for any $(x, y) \in S$. By (4.2)-(2), we have $g(x+y) = g(x)+g(y)$ for $(x, y) \in S$. The same identity also holds for f.

Therefore, for $(x, y) \in S$, we have

$$x * y = f(y)x + g(x)y.$$

Now, since for any $k \in \mathbb{F}, (x, y) \in S$, $(kx) * y = k(x * y) = x * (ky)$, we have

$$f(y)kx + g(kx)y = k(f(y)x + g(x)y) = f(ky)x + g(x)ky.$$

This shows that

$$f(kx) = kf(x), \quad g(kx) = kg(x).$$

For $x, y \in A$ linearly dependent, say $x = ky$ for some $k \in F$, we have

$$f(x+y) = f((k+1)y) = (k+1)f(y) = kf(y) + f(y) = f(x) + f(y).$$

Thus $f \in A^*$. Similarly, g is a linear function. □

Therefore, left-symmetric algebras with $\mathcal{I}_2(A) = 2$ are all defined by linear functions. Furthermore, one may easily show that (4.1) defines a left-symmetric algebra if and only if $f = 0$ or $g = 0$. Thus we have the classification of such left-symmetric algebras as described by Bai in [1].

Theorem 4.2 ([1, P. 4, Corollary 2.2])**.** *Let A be a left-symmetric algebra. If $\mathcal{I}_2(A) = 2$, then A is isomorphic to one of the followings.*

(1) *There exists a basis $\{e_1, \ldots, e_n\}$ in A such that $L_{e_1} = \mathrm{Id}$, $L_{e_i} = 0$, $i = 2, 3, \ldots, n$, where Id is the identity transformation.*
(2) *There exists a basis $\{e_1, \ldots, e_n\}$ in A such that $R_{e_1} = \mathrm{Id}$, $R_{e_i} = 0$, $i = 2, 3, \ldots, n$.*
(3) *A is a trivial algebra, that is, all products are zero.*

5. Left-symmetric algebras with $\mathcal{I}_2(A) = 3$

In this section, we investigate left-symmetric algebras of dimension ≥ 4 with $\mathcal{I}_2(A) = 3$. In general, we can assume that $x*y = f(x,y)x + g(x,y)y + h(x,y)c_{x,y}$ for any $x, y \in A$, where f, g, h are three functions on $A \times A$ and $c_{x,y} \in A$ is dependent of the choice of x, y. In [1], Bai classified some of these algebras by assuming that $c_{x,y}$ is a fixed element for any $x, y \in A$, f, g are linear functions and $h(x,y)$ is a symmetric bilinear function. But we can prove that $c_{x,y}$ is a fixed element for any $x, y \in A$ and f, g are linear functions, h is a bilinear function as long as $\dim A \geq 4$.

Set

$$L = \{x \in A \,|\, x^2 \in \mathrm{span}\,\{x\}\}, \quad L' = \{x \in A \,|\, x \notin L\}.$$

Theorem 5.1. *Let A be a finite-dimensional left-symmetric algebra of dimension ≥ 4 with $\mathcal{I}_2(A) \leq 3$. Then one of the following assertions holds*:

(1) $\mathcal{I}_1(A) = 1$;
(2) $\mathcal{I}_1(A) = 2$ *and there exists an element c in A such that $\langle x, y\rangle \subseteq \mathrm{span}\,\{x, y, c\}$ for any $x, y \in A$.*

Proof. Since $\mathcal{I}_2(A) \leq 3$, we have $\mathcal{I}_1(A) \leq 2$. If $\mathcal{I}_1(A) = 1$, it is easy to check, by Theorem 1.1, that $\mathcal{I}_2(A) \leq 3$.

Now consider the case of $\mathcal{I}_1(A) = 2$. Obviously, there exists an element $x \in A$ such that $x^2 \notin \mathrm{span}\,\{x\}$ and we can choose an element $y \in L'$ such that $y \notin \langle x\rangle$. Furthermore, we can also find an element $z \in L'$ such that $z \notin \langle x, y\rangle$ (such z exists since there exist infinite many $t \in \mathbb{F}$ such that $z + tx \in L'$ for any $x \in L'$ and $z \in L$). Now, it is easy to see that $\langle x, x^2\rangle$, $\langle y, y^2\rangle$, $\langle z, z^2\rangle$

are 2-dimensional subalgebras and we have $\langle x,y\rangle = \langle x,x^2,y,y^2\rangle$, $\langle x,z\rangle = \langle x,x^2,z,z^2\rangle$, and $\langle y,z\rangle = \langle y,y^2,z,z^2\rangle$. Since $\mathcal{I}_2(A)=3$, there exists an element $c_{x,y}$ such that $\langle x,x^2\rangle \cap \langle y,y^2\rangle = \operatorname{span}\{c_{x,y}\}$. Similarly, we have $\langle x,x^2\rangle \cap \langle z,z^2\rangle = \operatorname{span}\{c_{x,z}\}$ and $\langle y,y^2\rangle \cap \langle z,z^2\rangle = \operatorname{span}\{c_{y,z}\}$ for some $c_{x,z}, c_{y,z} \in A$.

Claim: $\dim(\operatorname{span}\{c_{x,y}, c_{x,z}, c_{y,z}\}) = 1$.

First we assume that $\dim(\operatorname{span}\{c_{x,y}, c_{x,z}, c_{y,z}\}) = 3$. Then it is easily seen that $\langle x,x^2\rangle = \operatorname{span}\{c_{x,y}, c_{x,z}\}$, $\langle y,y^2\rangle = \operatorname{span}\{c_{x,y}, c_{y,z}\}$, and $\langle z,z^2\rangle = \operatorname{span}\{c_{x,z}, c_{y,z}\}$. This implies that

$$\langle x,x^2,y,y^2,z,z^2\rangle = \operatorname{span}\{c_{x,y}, c_{x,z}, c_{y,z}\}, \tag{5.1}$$

which is absurd since the dimension of the left-hand side is greater than 3.

Now assume that $\dim(\operatorname{span}\{c_{x,y}, c_{x,z}, c_{y,z}\}) = 2$, where $c_{x,y}, c_{x,z}$ are linearly independent. Then $c_{y,z} \in \operatorname{span}\{c_{x,y}, c_{x,z}\} \subseteq \langle x,x^2\rangle$, and we have

$$\langle y,y^2\rangle \cap \langle z,z^2\rangle = \operatorname{span}\{c_{y,z}\} \subseteq \langle x,x^2\rangle \cap \langle z,z^2\rangle = \operatorname{span}\{c_{x,z}\}.$$

Therefore, $c_{x,z}, c_{y,z}$ are linearly dependent. Similarly, $c_{x,y}, c_{y,z}$ are linearly dependent, which is a contradiction. Thus we have $\dim(\operatorname{span}\{c_{x,y}, c_{x,z}, c_{y,z}\}) = 1$, as claimed.

Hence there exists an element $c \in A$ such that $c \in \langle x\rangle = \operatorname{span}\{x,x^2\}$ for any $x \in L'$. It follows that $\langle x,y\rangle \subseteq \operatorname{span}\{x,y,c\}$ for any $x,y \in L'$.

For any $x \in L', a \in L$, there exists some nonzero $k \in \mathbb{F}$ such that $x+ka, x-ka \in L'$. Then $(x+ka)^2 \in \operatorname{span}\{x+ka, c\}$ and $x^2+(ka)^2$ is a linear combination of x,a,c. It implies that $a*x+x*a \in \operatorname{span}\{x,a,c\}$. Moreover, since $(x+ka)*(x-ka)$ is a linear combination of $(x+ka),(x-ka),c$, we can also get $a*x-x*a \in \operatorname{span}\{x,a,c\}$. This implies that $a*x, x*a \in \operatorname{span}\{a,x,c\}$. Therefore, $\langle a,x\rangle \subseteq \operatorname{span}\{a,x,c\}$.

Now assume that $a,b \in L$. For any $x \in L'$, there exists some nonzero $k \in \mathbb{F}$ such that $a+kx, a-kx \in L'$. Then $a*b = (\frac{a+kx}{2} + \frac{a-kx}{2})*b$. Thus $\frac{a+kx}{2}*b \in \operatorname{span}\{\frac{a+kx}{2}, b, c\} \subseteq \operatorname{span}\{a,x,b,c\}$ and $\frac{a-kx}{2}*b \in \operatorname{span}\{\frac{a-kx}{2}, b, c\} \subseteq \operatorname{span}\{a,x,b,c\}$. Since $a*b$ is independent of the choice of $x \in L'$ and $\mathcal{I}_2(A)=3$, we have $a*b \in \operatorname{span}\{a,b,c\}$ for any $a,b \in L$. Hence $\langle a,x\rangle \subseteq \operatorname{span}\{a,x,c\}$. □

In the following, we only need to explore finite-dimensional left-symmetric algebras with $\mathcal{I}_2(A)=3$ and $\mathcal{I}_1(A)=2$.

Theorem 5.2. *Let A be a finite-dimensional left-symmetric algebra with $\mathcal{I}_2(A) = 3$ and $\mathcal{I}_1(A)=2$. Then there exist linear functions $f(x), g(x)$ and a bilinear function $h(x,y)$ such that*

$$x*y = f(x)y + g(y)x + h(x,y)c \tag{5.2}$$

for any $x,y \in A$.

Proof. Since $\langle x,y\rangle \subseteq \operatorname{span}\{x,y,c\}$ for x,y,c linearly independent, we have

$$x*y = f(x,y)x + g(x,y)y + h(x,y)c, \tag{5.3}$$

where $f(x, y), g(x, y)$ and $h(x, y)$ are functions on

$$S = \{(x, y) \mid x, y \in A, x, y, c \text{ are linearly independent}\}.$$

Let $(x_1, y), (x_2, y) \in S$. Then we can choose z such that x_i, z, y, c are linearly independent. From $(x_i + z) * y = x_i * y + z * y$, it follows that

$$\begin{aligned} &cf(x_i + z, y)(x_i + z) + g(x_i + z, y)y + h(x_i + z, y)c \\ = &f(x_i, y)x_i + g(x_i, y)y + h(x_i, y)c + f(z, y)z + g(z, y)y + h(z, y)c. \end{aligned} \tag{5.4}$$

Therefore,

$$\begin{cases} (1) & f(x_i + z, y) = f(x_i, y) = f(z, y), \\ (2) & g(x_i + z, y) = g(x_i, y) + g(z, y), \\ (3) & h(x_i + z, y) = h(x_i, y) + h(z, y). \end{cases} \tag{5.5}$$

By (5.5)-(1), we have $f(x_1, y) = f(z, y) = f(x_2, y)$. Then we may write $f(x, y) = f(y)$ for any $(x, y) \in S$ and define $f(0) = 0$.

Similarly, we have

$$\begin{cases} (1) & g(y, x_i + z) = g(y, x_i) = g(y, z), \\ (2) & f(y, x_i + z) = f(y, x_i) + f(y, z), \\ (3) & h(y, x_i + z) = h(y, x_i) + h(y, z). \end{cases} \tag{5.6}$$

Thus we may also write $g(x, y) = g(x)$ for any $(x, y) \in S$ and define $g(0) = 0$. Therefore, for $(x, y) \in S$, we have

$$x * y = f(y)x + g(x)y + h(x, y)c.$$

Now, for any $k \in \mathbb{F}, (x, y) \in S$, using the equation $(kx) * y = k(x * y) = x * (ky)$, we have

$$\begin{aligned} f(y)kx + g(kx)y + h(kx, y)c &= k(f(y)x + g(x)y + h(x, y)c) \\ &= f(ky)x + g(x)ky + h(x, ky)c. \end{aligned}$$

It implies that

$$\begin{cases} (1) & f(kx) = kf(x), \\ (2) & g(kx) = kg(x), \\ (3) & h(kx, y) = kh(x, y) = h(x, ky). \end{cases} \tag{5.7}$$

For any $(x, y) \in S$, we have $(x + c, y) \in S$ and hence define

$$\begin{cases} f(c) = f(x + c) - f(x), \\ g(c) = g(x + c) - g(x), \\ h(c, y) = h(x + c, y) - h(x, y), \\ h(y, c) = h(y, x + c) - h(y, x), \\ h(c, c) = h(x + c, c) - h(x, c). \end{cases}$$

It is easy to check that the above definitions are independent of the choice of $(x, y) \in S$. Therefore, by (5.5), (5.6) and (5.7), f, g are linear functions, $h(x, y)$ is a bilinear function, and

$$x * y = f(y)x + g(x)y + h(x, y)c$$

for any $x, y \in S$, or $x \in \text{span}\{c\}$, or $y \in \text{span}\{c\}$. Thus we only need to check that the above equality holds for any $x, y \in A$.

For x, y, c linearly dependent and $x, y \notin \text{span}\{c\}$, choose an element z such that $(x, z), (x, y + z) \in S$. Then

$$\begin{aligned} x * y &= x * (y + z) - x * z \\ &= f(y + z)x + g(x)(y + z) + h(x, y + z)c - (f(z)x + g(x)z + h(x, z)c) \\ &= f(y)x + g(x)y + h(x, y)c, \end{aligned}$$

which completes the proof of the theorem. □

Theorem 5.3. *Let A be a finite-dimensional left-symmetric algebra with $\mathcal{I}_2(A) = 3$ and $\mathcal{I}_1(A) = 2$. Then A is isomorphic to one of the left-symmetric algebras in Table 1.*

Proof. Note that $h \neq 0$, otherwise, $\mathcal{I}_2(A) = 2$. We write $h_{ij} = h(e_i, e_j)$, where $\{e_1, \ldots, e_n = c\}$ is a basis of A.

Now by (2.1), an easy calculation shows that (5.2) defines a left-symmetric product on $\mathfrak{g}$ if and only if f, g, h satisfy the following conditions:

$$f(x)g(z) + g(e_n)h(x, z) = 0, \tag{5.8}$$

$$f(x)g(y) - f(y)g(x) + (h(x, y) - h(y, x))f(e_n) = 0, \tag{5.9}$$

$$\begin{aligned} &h(x, z)(g(y) - f(y) + h(y, e_n)) - h(y, z)(g(x) - f(x) + h(x, e_n)) \\ &+ (h(x, y) - h(y, x))h(e_n, z) = 0. \end{aligned} \tag{5.10}$$

Actually, the above equalities hold for

$$X = \{(x, y, z) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \,|\, x, y, z, c \text{ are linearly independent}\}.$$

Since the equalities are polynomials, it is easy to see that they hold for any x, y, z. Furthermore, by (5.8) and (5.9), we have

$$(h(x, y) - h(y, x))(f(e_n) - g(e_n)) = 0. \tag{5.11}$$

Case A: $f = g = 0$. Then (5.8) and (5.9) hold, and (5.10) is reduced to the following equation

$$h(x, z)h(y, e_n) - h(y, z)h(x, e_n) + (h(x, y) - h(y, x))h(e_n, z) = 0. \tag{5.12}$$

Letting $z = e_n$ and $x = e_n$ respectively, we obtain

$$\begin{cases} (1) \quad (h(x, y) - h(y, x))h_{nn} = 0, \\ (2) \quad h(e_n, y)h(e_n, z) - h(y, z)h_{nn} = 0. \end{cases} \tag{5.13}$$

Case AI: $h_{nn} \neq 0$. We may assume that $h_{nn} = 1$. Then h is symmetric and $h(x, y) = h(e_n, y)h(x, e_n) = h(e_n, y)h(e_n, x)$ for any $x, y \in A$. Choosing a basis $\{e_1, \ldots, e_n\}$ in A such that $h_{ni} = h_{in} = \delta_{in}$, we have

$$A_1 : (h_{ij}) = E_{nn}.$$

Case AII: $h_{nn} = 0$. Then $h(e_n, x) = 0$ for any $x \in A$ by (5.13)-(2). Thus (5.12) is reduced to the following

$$h(x,z)h(y,e_n) - h(y,z)h(x,e_n) = 0. \tag{5.14}$$

It is clear that (5.14) holds if $h(x, e_n) = 0$. This implies that e_n is in the center of A and $x * y \in \operatorname{span}\{e_n\}$ for any $x, y \in A$. Therefore A is 2-step nilpotent and we have

$$A_2(H) : (h_{ij}) = \begin{pmatrix} H & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix},$$

where $H \in \mathbb{F}^{(n-1)\times(n-1)}$, and that $A_2(H_1)$ is isomorphic to $A_2(H_2)$ if and only if $H_1 = TH_2T'$ for some invertible matrix T.

Now assume that $h(x, e_n) \neq 0$ for some $x \in A$. We can choose a basis $\{e_1, \ldots, e_n\}$ such that $h_{in} = \delta_{i1}$. From (5.14), we obtain

$$h(x,y) = h(e_1,y)h(x,e_n)$$

for any $x, y \in A$, which implies that $h(x, y) = 0$ for any $x \in \{e_2, \ldots, e_n\}$. Replacing e_i by $e_i - h_{1i}e_n$, $i = 1, \ldots, n-1$, we may assume $h_{1i} = \delta_{in}$. Hence, we obtain

$$A_3 : (h_{ij}) = E_{1n}.$$

Case B: $f = 0$, $g \neq 0$. Then (5.9) holds. From (5.8) and $h \neq 0$, we get $g(e_n) = 0$. Thus there is only one non-trivial equation

$$\begin{aligned} &h(x,z)(g(y) + h(y,e_n)) - h(y,z)(g(x) + h(x,e_n)) \\ &+ (h(x,y) - h(y,x))h(e_n,z) = 0. \end{aligned} \tag{5.15}$$

By substituting $z = e_n$ and $y = e_n$ respectively, we obtain

$$\begin{cases} (1) \quad h(x,e_n)g(y) - h(y,e_n)g(x) + (h(x,y) - h(y,x))h_{nn} = 0, \\ (2) \quad h_{nn}h(x,z) - h(e_n,z)(g(x) + h(e_n,x)) = 0. \end{cases} \tag{5.16}$$

Case BI: $h_{nn} \neq 0$. Then we may assume that $h_{nn} = 1$ and hence we conclude that $h(x,y) = h(e_n,y)(g(x)+h(e_n,x))$. Choosing a basis $\{e_1, \ldots, e_n\}$ such that $g(e_i) = \delta_{i1}$ and $h_{ni} = \delta_{in}$, we see that

$$B_1 : (h_{ij}) = E_{1n} + E_{nn}.$$

It is easy to check that (5.15) holds.

Case BII: $h_{nn} = 0$. Then (5.16) are reduced to the following ones:

$$\begin{cases} (1) \quad h(x,e_n)g(y) - h(y,e_n)g(x) = 0, \\ (2) \quad h(e_n,z)(g(x) + h(e_n,x)) = 0. \end{cases} \tag{5.17}$$

Therefore, we know that $h(e_n, z) \equiv 0$ or $g(x) + h(e_n, x) \equiv 0$ for any $x, z \in A$.

Case BII-1: $h(e_n, z) \equiv 0$. Choosing a basis $\{e_1, \ldots, e_n\}$ such that $g(e_i) = \delta_{i1}$, we get $h_{in} = h_{1n}g(e_i)$.

First assume that $g(e_1) + h_{1n} = 0$, i.e., $h_{1n} = -1$. Then (5.15) holds. Replacing e_i by $e_i + h_{1i}e_n, i = 1, \ldots, n-1$, we get

$$B_2(\alpha, H) : (h_{ij}) = \begin{pmatrix} 0 & \mathbf{0} & -1 \\ \alpha & H & \mathbf{0} \\ 0 & \mathbf{0} & 0 \end{pmatrix},$$

where $\alpha \in \mathbb{F}^{n-2}$ and $H \in \mathbb{F}^{(n-2)\times(n-2)}$, and that $B_2(\alpha_1, H_1)$ is isomorphic to $B_2(\alpha_2, H_2)$ if and only if $TH_2T' = H_1$ and $T\alpha_2 = \alpha_1$ for some invertible matrix T.

Next assume that $g(e_1) + h_{1n} \neq 0$. Then, by (5.15),we have

$$h(x, z) = \frac{h(e_1, z)(g(x) + h(x, e_n))}{g(e_1) + h_{1n}},$$

which immediately implies that $h_{ij} = 0$ for any $i > 1$. Choosing a suitable basis, we have

$$B_3(\lambda)\ (\lambda \neq 0) : (h_{ij}) = \lambda E_{1n}.$$

$$B_4(k)\ (k < n) : (h_{ij}) = E_{1k}.$$

Case BII-2: $h(e_n, z) \not\equiv 0$. Then we get $g(x) + h(e_n, x) \equiv 0$ by (5.17)-(2). Choosing a basis $\{e_1, \ldots, e_n\}$ such that $h_{ni} = -g(e_i) = -\delta_{i1}$, we see that $h(x, e_n) = h_{1n}g(x)$ from (5.17)-(1). Substituting it into (5.15), we obtain

$$(5.18) \quad (1 + h_{1n})(h(x, z)g(y) - h(y, z)g(x)) + (h(x, y) - h(y, x))h(e_n, z) = 0.$$

If $h_{1n} = -1$, then we can show that $h(x, y)$ is a symmetric bilinear function by taking $z = e_1$. Choosing suitable $e_1, \ldots, e_{n-1}$, we get

$$B_5(r)\ (1 < r < n) : (h_{ij}) = -E_{1n} - E_{n1} + \sum_{i=2}^{r} E_{ii}.$$

If $h_{1n} \neq -1$, then by (5.18), we have, for any $z = e_i$, $i > 1$,

$$(5.19) \qquad h(x, e_i)g(y) - h(y, e_i)g(x) = 0.$$

Taking $x = e_1$, we have $h(y, e_i) = h(e_1, e_i)g(y)$ for any $y \in A$, which implies that $h(y, z) = 0$ for any $y, z \in \{e_2, \ldots, e_n\}$. Using (5.18) again, we get that $h_{1i} = -h_{1n}h_{i1}$ for $i \geq 2$.

If $h_{1n} = 1$, then, replacing e_i by $e_i + h_{i1}e_n, i = 2, \ldots, n-1$, we have

$$(h_{ij}) = h_{11}E_{11} + E_{1n} - E_{n1}.$$

Now if $h_{11} \neq 0$, replacing h by $\frac{h}{h_{11}}$ and e_n by $h_{11}e_n$ if necessary, we then get

$$B_6 : (h_{ij}) = E_{11} + E_{1n} - E_{n1}.$$

Otherwise,

$$B_7(1) : (h_{ij}) = E_{1n} - E_{n1}.$$

If $h_{1n} \neq 1$, then replacing e_1 by $e_1 + \frac{h_{11}}{1-h_{1n}}e_n$ and e_i by $e_i + h_{i1}e_n$, $i = 2, \ldots, n-1$, we obtain

$$B_7(\lambda)\ (\lambda \neq 1) : (h_{ij}) = \lambda E_{1n} - E_{n1}.$$

Case C: $f \neq 0$, $g = 0$. Then (5.8) holds and we immediately get the following equations from (5.9) and (5.10):

$$f(e_n)(h(x,y) - h(y,x)) = 0, \tag{5.20}$$

$$\begin{aligned} &h(x,z)(h(y,e_n) - f(y)) - h(y,z)(h(x,e_n) - f(x)) \\ &+ (h(x,y) - h(y,x))h(e_n,z) = 0. \end{aligned} \tag{5.21}$$

Hence $f(e_n) = 0$ or $h(x,y) = h(y,x)$ for any $x, y \in A$.

Case CI: $f(e_n) = 0$. Setting $z = e_n$ and $y = e_n$ respectively, we have

$$\begin{cases} (1) & h(y,e_n)f(x) - h(x,e_n)f(y) + (h(x,y) - h(y,x))h_{nn} = 0, \\ (2) & h_{nn}h(x,z) + h(e_n,z)(f(x) - h(e_n,x)) = 0. \end{cases} \tag{5.22}$$

If $h_{nn} \neq 0$, then we may assume that $h_{nn} = 1$. Hence we have $h(x,y) = (h(e_n,x) - f(x))h(e_n,y)$. Choosing a basis $\{e_1,\ldots,e_n\}$ such that $f(e_i) = \delta_{i1}$ and $h_{ni} = \delta_{in}$, we see that

$$C_1 : (h_{ij}) = -E_{1n} + E_{nn}.$$

If $h_{nn} = 0$, then (5.22) are reduced to the following:

$$\begin{cases} (1) & h(y,e_n)f(x) - h(x,e_n)f(y) = 0, \\ (2) & h(e_n,z)(f(x) - h(e_n,x)) = 0. \end{cases} \tag{5.23}$$

First assume that $h(e_n,x) = 0$. Choosing a basis $\{e_1,\ldots,e_n\}$ such that $f(e_i) = \delta_{i1}$, we see that $h_{in} = h_{1n}\delta_{i1}$.

Case CI-1: $h_{1n} - f(e_1) = 0$. Then (5.21) holds. Replacing e_i by $e_i - h_{1i}e_n, i < n$, we have $h_{1i} = 0$. Thus we obtain

$$C_2(\alpha, H) = (h_{ij}) = \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \alpha & H & \mathbf{0} \\ 0 & \mathbf{0} & 0 \end{pmatrix},$$

where $\alpha \in \mathbb{F}^{n-2}$ and $H \in \mathbb{F}^{(n-2)\times(n-2)}$, and $C_2(\alpha_1, H_1)$ is isomorphic to $C_2(\alpha_2, H_2)$ if and only if $TH_2T' = H_1$ and $T\alpha_2 = \alpha_1$ for some invertible matrix T.

Case CI-2: $h_{1n} - f(e_1) \neq 0$. Then we have $h(x,y) = \frac{h(e_1,y)}{h_{1n}-1}(h(x,e_n) - f(x))$ from (5.21). Clearly, $h_{ij} = \frac{h_{1j}}{h_{1n}-1}(h_{in} - 1)\delta_{i1}$.

If $h_{1n} \neq 0$, replacing e_i by $e_i - \frac{h_{1i}}{h_{1n}}e_n, i = 1,\ldots,n-1$, we obtain

$$C_3(\lambda),\ (\lambda \neq 0,1) : (h_{ij}) = \lambda E_{1n}.$$

If $h_{1n} = 0$, choosing suitable basis, we obtain that

$$C_4(k),\ (k < n) : (h_{ij}) = E_{1k}.$$

Now assume that $h(e_n,x) \neq 0$ for some $x \in A$. Then we get that $f(y) = h(e_n,y)$ for any $y \in A$ by (5.23)-(2). Choosing a basis $\{e_1,\ldots,e_n\}$ such that $h_{ni} = f(e_i) = \delta_{i1}$, we get $h(x,e_n) = f(x)h_{1n}$. By substituting it into (5.21), we have

$$(h_{1n} - 1)(f(y)h(x,z) - f(x)h(y,z)) + (h(x,y) - h(y,x))h(e_n,z) = 0 \tag{5.24}$$

for any $x, y, z \in A$.

Case CI-3: $h_{1n} = 1$. Taking $z = e_1$, we get that h is a symmetric bilinear form from (5.24). Changing $e_1, \ldots, e_{n-1}$ when necessary, we get

$$C_5(r),\ (1 < r < n) : E_{1n} + E_{n1} + \sum_{i=2}^{r} E_{ii}.$$

Case CI-4: $h_{1n} \neq 1$. For any $z = e_i, i > 1$, we have

$$f(y)h(x, z) - f(x)h(y, z) = 0.$$

This implies that $h(y, z) = 0$ for any $y, z \in \{e_2, \ldots, e_n\}$. By (5.24) again, we get that $h_{1i} = h_{1n}h_{i1}, i = 2, \ldots, n-1$.

If $h_{1n} = -1$, then, replacing e_i by $e_i - h_{1i}e_n, i = 2, \ldots, n-1$, we have $(h_{ij}) = h_{11}E_{11} + E_{n1} - E_{1n}$. Now if $h_{11} \neq 0$, then, replacing h by $\frac{h}{h_{11}}$ and e_n by $h_{11}e_n$ if necessary, we get

$$C_6 : (h_{ij}) = E_{11} + E_{1n} - E_{n1}.$$

Otherwise,

$$C_7(-1) : (h_{ij}) = E_{1n} - E_{n1}.$$

If $h_{1n} \neq -1$, then, replacing e_1 by $e_1 - \frac{h_{11}}{1+h_{1n}}e_n$ and e_i by $e_i - \frac{h_{1i}}{h_{1n}}e_n, i = 2, \ldots, n-1$, we have

$$C_7(\lambda),\ (\lambda \neq 0, -1) : (h_{ij}) = E_{n1} + \lambda E_{1n}.$$

Case CII: $f(e_n) \neq 0$. Then $h(x, y)$ is a symmetric bilinear form by (5.20). And (5.22) are reduced to the following:

$$\begin{cases} (1) \quad h(x,z)(h(y,e_n) - f(y)) - h(y,z)(h(x,e_n) - f(x)) = 0, \\ (2) \quad h(y,e_n)f(x) - h(x,e_n)f(y) = 0. \end{cases} \tag{5.25}$$

Choosing a basis $\{e_1, \ldots, e_n\}$ such that $f(e_i) = \delta_{in}$, we get

$$h_{in} = h_{nn}f(e_i) = h_{nn}\delta_{in}.$$

Case CII-1: $h(y, e_n) - f(y) \equiv 0$. Then we have $f(e_i) = h_{in} = \delta_{in}$. Therefore we can change the basis $\{e_1, \ldots, e_n\}$ when necessary to get

$$C_8(r),\ (r < n) : (h_{ij}) = \sum_{i=1}^{r} E_{ii} + E_{nn}.$$

Case CII-2: $h_{nn} - f(e_n) \neq 0$. Then $h(x, y) = \frac{h(e_n, y)}{h_{nn} - f(e_n)}(h(x, e_n) - f(x))$ for any $x, y \in A$. It suffices to show that $h(x, y) = 0$ for any $x \in \{e_1, \ldots, e_{n-1}\}$. Then we get

$$C_9(\lambda),\ (\lambda \neq 0, 1) : (h_{ij}) = \lambda E_{nn}.$$

Case D: $f \neq 0$ and $g \neq 0$. Then $g(e_n) \neq 0$ by (5.8). It follows that $h(x, y) = -\frac{f(x)g(y)}{g(e_n)}$ for any $x, y \in A$. Choosing a basis $\{e_1, \ldots, e_n\}$ such that $g(e_i) = \delta_{in}$, by (5.9) and (5.10), we get

$$(f(x)g(y) - f(y)g(x))(1 - f(e_n)) = 0, \tag{5.26}$$

Case DI: $f(x)g(y) - f(y)g(x) \equiv 0$. Then we have $f(x) = f(e_n)g(x)$. It implies that $f(e_i) = f(e_n)\delta_{in}$ and $h_{ij} = -f(e_n)g(e_i)g(e_j) = -f(e_n)\delta_{in}\delta_{jn}$. Hence, we have

$$D_1(\lambda),\ (\lambda \neq 0) : (h_{ij}) = \lambda E_{nn},\ -f(e_n) = \lambda.$$

Case DII: $f(x)g(y) - f(y)g(x) \not\equiv 0$. Then we have $f(e_n) = 1$ and $f(e_k) \neq g(e_k) = 0$ for some $k < n$. Choose a basis $\{e_1, \ldots, e_n\}$ such that $f(e_i) = \delta_{in} + \delta_{i1}$. Then $h_{ij} = -f(e_i)g(e_j) = -(\delta_{in} + \delta_{i1})\delta_{jn}$. Thus we get

$$D_2 : (h_{ij}) = -E_{1n} - E_{nn}.$$ □

Acknowledgements. The authors thank Professor Joseph A. Wolf for useful suggestion and great encouragement. The authors also thank Professor Shaoqiang Deng and Dr. Mingjing Zhang for a thorough reading of the manuscript and for many useful comments. This work was supported by the National Natural Science Foundation of China (Grant No. 11571182) and National Science Foundation for Young Scientists of China (Grant No. 11301287 and Grant No. 11401425) and China Scholarship Council.

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