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ON WEAKLY 2-ABSORBING PRIMARY SUBMODULES OF MODULES OVER COMMUTATIVE RINGS

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Abstract. Assume that M is an R-module where R is a commutative ring. A proper submodule N of M is called a weakly 2-absorbing primary submodule of M if $0 \neq abm \in N$ for any $a, b \in R$ and $m \in M$, then $ab \in (N : M)$ or $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$. In this paper, we extended the concept of weakly 2-absorbing primary ideals of commutative rings to weakly 2-absorbing primary submodules of modules. Among many results, we show that if N is a weakly 2-absorbing primary submodule of M and it satisfies certain condition $0 \neq I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and submodule K of M, then $I_1I_2 \subseteq (N : M)$ or $I_1K \subseteq M\text{-rad}(N)$ or $I_2K \subseteq M\text{-rad}(N)$.

1. Introduction

Throughout this paper, we suppose that all rings are commutative with $1 \neq 0$. Let M be an R-module. A submodule N of M is called a proper submodule if $N \neq M$. Let N be a proper submodule of M. $(N : M)$ is the set of all $r \in R$ such that $rM \subseteq N$ for any submodule N of M. Then the radical of N, denoted by $M\text{-rad}(N)$, is defined as the intersection of all prime submodules of M containing N. If M has no prime submodule containing N, then $M\text{-rad}(N) = M$.

Following the concept of 2-absorbing ideals of commutative rings as in [2] and [10] and the concept of weakly prime ideals of commutative rings as in [8], Badawi and Darani introduced the concept of weakly 2-absorbing ideals in commutative rings as in [3]. Afterwards, in [5], Badawi, Tekir and Yetkin introduced the concept of 2-absorbing primary ideals as a generalization of primary ideals. Also, the concept of weakly 2-absorbing primary ideals, which is a generalization of weakly primary as in [1], was studied extensively by Badawi, Tekir and Yetkin, see [6]. A proper ideal I of R is called 2-absorbing (weakly 2-absorbing) primary ideal if $abc \in I$ $(0 \neq abc \in I)$ for any $a, b, c \in R$,

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then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Recently, Mostafanasab and Darani generalized the concept of ϕ-2-absorbing primary ideals of commutative rings as in [4] to ϕ-n-absorbing primary ideals in commutative rings, see [14].

The concept of 2-absorbing submodule and weakly 2-absorbing submodules, generalizations of prime submodules and weakly prime submodules, respectively, were studied inclusively by Moradi, Azizi and other authours, see for example, [7]-[12]. A proper submodule N of M is called 2-absorbing (weakly 2-absorbing) submodule if $abm \in N$ ($0 \neq abm \in N$) for some $a, b \in R$ and $m \in M$, then $ab \in (N : M)$ or $am \in N$ or $bm \in N$. For general informations of ideals and submodules, we refer the reader to [9] and [15].

In this study, we investigate weakly 2-absorbing primary submodules. Recall that a proper submodule N of M is said to be 2-absorbing (weakly 2-absorbing) primary submodule of M if $abm \in N$ ($0 \neq abm \in N$) for any $a, b \in R$ and $m \in M$, then $ab \in (N : M)$ or $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$.

Our main aim is to answer the question: If N is a weakly 2-absorbing primary submodule of an R-module M and $0 \neq I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M, does it follow that $I_1I_2 \subseteq (N : M)$ or $I_1K \subseteq M\text{-rad}(N)$ or $I_2K \subseteq M\text{-rad}(N)$? (see, Theorem 2 and Theorem 3). Among some results of this paper, it is shown that (Theorem 7) if $M\text{-rad}(0)$ is a prime submodule, then N is a weakly 2-absorbing primary submodule if and only if N is a 2-absorbing primary submodule. In Theorem 8 (Theorem 9), it is seen that if N is a weakly 2-absorbing primary submodule, then $(N : M)$ is a weakly 2-absorbing primary ideal (hence if N is a weakly 2-absorbing primary submodule, then $(N : M)$ is a weakly 2-absorbing primary ideal). In Theorem 11, it is obtained that if $N = N_1 \times N_2$ is a weakly 2-absorbing primary submodule, then $N = 0$ or N is 2-absorbing primary.

2. On weakly 2-absorbing primary submodules

Definition 1. A proper submodule N of an R-module M is called a weakly 2-absorbing primary submodule of M if $0 \neq abm \in N$ for any $a, b \in R$ and $m \in M$, then $ab \in (N : M)$ or $am \in M\text{-rad}(N)$or $bm \in M\text{-rad}(N)$.

Proposition 1. *Let N be a weakly 2-absorbing primary submodule of an R-module M. Assume that K is a submodule of M with $N \subsetneqq K$. Then N is a weakly 2-absorbing primary submodule of K.*

Proof. Let $a, b \in R$ and $k \in K$ with $0 \neq abk \in N$. Then $ab \in (N : M)$ or $ak \in M\text{-rad}(N)$ or $bk \in M\text{-rad}(N)$ as N is a weakly 2-absorbing primary. Thus $ab \in (N : K)$ or $ak \in K\text{-rad}(N)$ or $bk \in K\text{-rad}(N)$ since $(N : M) \subseteq (N : K)$. □

The following result is an analogue of [6, Theorem 2.18].

Proposition 2. *Let N, K be submodules of an R-module M with $K \subseteq N$. If N is a weakly 2-absorbing primary submodule of M, then N/K is a weakly*

2-absorbing primary submodule of M/K. The converse is true when K is a weakly 2-absorbing primary submodule.

Proof. Assume that N is a weakly 2-absorbing primary submodule of M. Let $a, b \in R$ and $m + K \in M/K$ where $0_{M/K} \neq ab(m + K) \in N/K$. Since $ab(m + K) \neq 0_{M/K}$, we get $abm \in N$ and $abm \notin K$. If $abm = 0$, we obtain $abm + K = 0_{M/K}$. So $abm \neq 0$. Thus $ab \in (N : M)$ or $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$ as N is weakly 2-absorbing primary. Consequently, we get $ab \in (N/K : M/K)$ or $am+K = a(m+K) \in M\text{-rad}(N)/K = M/K\text{-rad}(N/K)$ or $bm + K = b(m + K) \in M\text{-rad}(N)/K = M/K\text{-rad}(N/K)$. Conversely, let K be a weakly 2-absorbing primary submodule. Assume that N/K is a weakly 2-absorbing primary submodule of M/K. Let $a, b \in R$ and $m \in M$ where $0 \neq abm \in N$. Then we have $abm + K \in N/K$. If $abm + K = 0_{M/K}$, then $abm \in K$. Thus $ab \in (K : M)$ or $am \in M\text{-rad}(K)$ or $bm \in M\text{-rad}(K)$, since K is weakly 2-absorbing primary. Therefore, $ab \in (N : M)$ or $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$, since $K \subseteq N$. Let $abm+K = ab(m+K) \neq 0_{M/K}$. Then $ab \in (N/K : M/K)$ or $a(m+K) \in M/K\text{-rad}(N/K)$ or $b(m+K) \in M/K\text{-rad}(N/K)$. Thus $ab \in (N : M)$ or $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$. □

The following result is an analogue of [6, Theorem 2.20].

Proposition 3. *Let M be an R-module and S be a multiplicatively closed subset of R. If N is a weakly 2-absorbing primary submodule of M where $(N : M) \cap S = \emptyset$, then $S^{-1}N$ is a weakly 2-absorbing primary submodule of $S^{-1}M$.*

Proof. Let $0 \neq \frac{r}{s}\frac{t}{k}\frac{m}{l} = \frac{rtm}{skl} \in S^{-1}N$ where $r, t \in R$, $s, k, l \in S$ and $m \in M$. Then there is an element u of S such that $0 \neq urtm \in N$. Hence we get $urt \in (N : M)$ or $urm \in M\text{-rad}(N)$ or $tm \in M\text{-rad}(N)$ since N is weakly 2-absorbing primary. Then $\frac{r}{s}\frac{t}{k} = \frac{urt}{usk} \in S^{-1}(N : M) \subseteq (S^{-1}N : S^{-1}M)$ or $\frac{r}{s}\frac{m}{l} = \frac{urm}{usl} \in S^{-1}(M\text{-rad}(N)) = S^{-1}M\text{-rad}(S^{-1}N)$ or $\frac{t}{k}\frac{m}{l} = \frac{tm}{kl} \in S^{-1}(M\text{-rad}(N)) = S^{-1}M\text{-rad}(S^{-1}N)$. □

Definition 2. Let N be a weakly 2-absorbing primary submodule of M. (a, b, m) is called a triple-zero of N if $abm = 0$, $ab \notin (N : M)$, $am \notin M\text{-rad}(N)$ and $bm \notin M\text{-rad}(N)$.

Note that if N is a weakly 2-absorbing primary submodule of M and there is no triple-zero of N, then N is a 2-absorbing primary submodule of M.

Proposition 4. *Let N be a weakly 2-absorbing primary submodule of M and K be a proper submodule of M with $K \subseteq N$. Then for any $a, b \in R$ and $m \in M$, (a, b, m) is a triple-zero of N if and only if $(a, b, m+K)$ is a triple-zero of N/K.*

Proof. Let (a, b, m) be a triple-zero of N for some $a, b \in R$ and $m \in M$. Then $abm = 0$, $ab \notin (N : M)$, $am \notin M\text{-rad}(N)$ and $bm \notin M\text{-rad}(N)$. By Proposition 2, we get that N/K is a weakly 2-absorbing primary submodule of M/K. Thus $ab(m + K) = K$, $ab \notin (N/K : M/K)$, $a(m + K) \notin M\text{-rad}(N)/K$

and $b(m+K) \notin M\text{-rad}(N)/K$. Hence $(a, b, m+K)$ is a triple-zero of N/K. Conversely, assume that $(a, b, m+K)$ is a triple-zero of N/K. Suppose that $abm \neq 0$. Then $abm \in N$ since $ab(m+K) = K$. Thus $ab \in (N : M)$ or $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$ as N is weakly 2-absorbing primary, a contradiction. So it must be $abm = 0$. Consequently, (a, b, m) is a triple-zero of N. □

The following result is an analogue of [6, Theorem 2.9].

Theorem 1. *Let N be weakly 2-absorbing primary submodule of M and (a, b, m) be a triple-zero of N for some $a, b \in R$ and $m \in M$. Then the followings hold.*

(1) $abN = am(N : M) = bm(N : M) = 0$.
(2) $a(N : M)N = b(N : M)N = m(N : M)^2 = 0$.

Proof. Suppose that (a, b, m) is a triple-zero of N for some $a, b \in R$ and $m \in M$.

(1) Assume that $abN \neq 0$. Then there is an element $n \in N$ where $abn \neq 0$. Thus $ab(m+n) = abm + abn = abn \neq 0$. So $a(m+n) \in M\text{-rad}(N)$ or $b(m+n) \in M\text{-rad}(N)$ since $ab \notin (N : M)$ and N is weakly 2-absorbing primary. Hence $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$, which is a contradiction. Now, we suppose that $am(N : M) \neq 0$. Thus there exists an element $r \in (N : M)$ where $arm \neq 0$. Hence $a(r+b)m = arm + abm = arm \neq 0$ and since $am \notin M\text{-rad}(N)$, $bm \in M\text{-rad}(N)$ or $ab \in (N : M)$, a contradiction. Similarly, it can be easily seen that $bm(N : M) = 0$.

(2) Assume that $a(N : M)N \neq 0$. Then there are $r \in (N : M)$, $n \in N$ such that $arn \neq 0$. By (1), we get $a(b+r)(m+n) = arn \neq 0$ and so $a(b+r) \in (N : M)$ or $a(m+n) \in M\text{-rad}(N)$ or $(b+r)(m+n) \in M\text{-rad}(N)$. Therefore, we obtain $ab \in (N : M)$ or $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$, a contradiction. In a similar way, one can easily seen that $b(N : M)N = 0$. Let $m(N : M)^2 \neq 0$. Thus there exist $r, s \in (N : M)$ where $mrs \neq 0$. By (1), we get $(a+r)(b+s)m = rsm \neq 0$. Thus we have $(a+r)m \in M\text{-rad}(N)$ or $(b+s)m \in M\text{-rad}(N)$ or $(a+r)(b+s) \in (N : M)$ and so we get $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$ or $ab \in (N : M)$, a contradiction. Consequently, $m(N : M)^2 = 0$. □

The following two results are an analogue of [6, Theorem 2.10].

Lemma 1. *Assume that N is a weakly 2-absorbing primary submodule of an R-module M that is not 2-absorbing primary. Then $(N : M)^2N = 0$. In particular, $(N : M)^3 \subseteq Ann(M)$.*

Proof. Suppose that N is a weakly 2-absorbing primary submodule of an R-module M that is not 2-absorbing primary. Then there is a triple-zero (a, b, m) of N for some $a, b \in R$ and $m \in M$. Assume that $(N : M)^2N \neq 0$. Thus there exist $r, s \in (N : M)$ and $n \in N$ with $rsn \neq 0$. By Theorem 1, we get $(a+r)(b+s)(n+m) = rsn \neq 0$. Then we have $(a+r)(b+s) \in (N : M)$ or $(a+r)(n+m) \in M\text{-rad}(N)$ or $(b+s)(n+m) \in M\text{-rad}(N)$ and so $ab \in (N : M)$ or $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$, which is a contradiction. Thus we get $(N:M)^2N=0$. We get $(N:M)^3 \subseteq ((N:M)^2N:M)=(0:M)=Ann(M)$. ☐

Proposition 5. *Let M be a multiplication R-module and N be a weakly 2-absorbing primary submodule of M that is not 2-absorbing primary. Then $N^3=0$.*

Proof. We have that $(N:M)M=N$ since M is multiplication module. Then $N^3=(N:M)^3M=(N:M)^2N=0$. Consequently, $N^3=0$. ☐

Definition 3. Let N be a weakly 2-absorbing primary submodule of an R-module M and let $0 \neq I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M. N is called free triple-zero in regard to I_1, I_2, K if (a,b,m) is not a triple-zero of N for every $a \in I_1, b \in I_2$ and $m \in K$.

The following result and its proof are analogues of [6, Theorem 2.29] and its proof.

Lemma 2. *Let N be a weakly 2-absorbing primary submodule of M. Assume that $abK \subseteq N$ for some $a, b \in R$ and some submodule K of M where (a,b,m) is not a triple-zero of N for every $m \in K$. If $ab \notin (N:M)$, then $aK \subseteq M$-rad(N) or $bK \subseteq M$-rad(N).*

Proof. Assume that $aK \nsubseteq M$-rad(N) and $bK \nsubseteq M$-rad(N). Then there are $x, y \in K$ such that $ax \notin M$-rad(N) and $by \notin M$-rad(N). We get $bx \in M$-rad(N) since N is a weakly 2-absorbing primary submodule, (a,b,x) is not a triple-zero of N, $ab \notin (N:M)$ and $ax \notin M$-rad(N). In a similar way, $ay \in M$-rad(N). Now, we obtain $a(x+y) \in M$-rad(N) or $b(x+y) \in M$-rad(N) since $(a,b,x+y)$ is not a triple-zero of N, $ab(x+y) \in N$ and $ab \notin (N:M)$. Assume that $a(x+y)=ax+ay \in M$-rad(N). As $ay \in M$-rad(N), we get $ax \in M$-rad(N), which is a contradiction. Assume that $b(x+y)=bx+by \in M$-rad(N). As $bx \in M$-rad(N), we get $by \in M$-rad(N), a contradiction again. Hence we obtain that $aK \subseteq M$-rad(N) or $bK \subseteq M$-rad(N). ☐

Let N be a weakly 2-absorbing primary submodule of an R-module M and $I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M where N is free triple-zero in regard to I_1, I_2, K. Note that if $a \in I_1,\ b \in I_2$ and $m \in K$, then $ab \in (N:M)$ or $am \in M$-rad(N) or $bm \in M$-rad(N).

The following result and its proof are analogues of [6, Theorem 2.30] and its proof.

Theorem 2. *Assume that N is a weakly 2-absorbing primary submodule of an R-module M and $0 \neq I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M where N is free triple-zero in regard to I_1, I_2, K. Then $I_1I_2 \subseteq (N:M)$ or $I_1K \subseteq M$-rad(N) or $I_2K \subseteq M$-rad(N).*

Proof. Let N be a weakly 2-absorbing primary submodule of M and $0 \neq I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M where

N is free triple-zero in regard to I_1, I_2, K. Suppose that $I_1I_2 \nsubseteq (N : M)$. Now, we show that $I_1K \subseteq M$-rad(N) or $I_2K \subseteq M$-rad(N). Assume that $I_1K \nsubseteq M$-rad(N) and $I_2K \nsubseteq M$-rad(N). Then $xK \nsubseteq M$-rad(N) and $yK \nsubseteq M$-rad(N) where $x \in I_1$ and $y \in I_2$. By Lemma 2, we get $xy \in (N : M)$ since $xyK \subseteq N$, $xK \nsubseteq M$-rad(N) and $yK \nsubseteq M$-rad(N). By our assumption, there are $a \in I_1$ and $b \in I_2$ such that $ab \notin (N : M)$. By Lemma 2, we get $aK \subseteq M$-rad(N) or $bK \subseteq M$-rad(N) as $abK \subseteq N$ and $ab \notin (N : M)$. We investigate three cases. **First case:** Assume that $aK \subseteq M$-rad(N) and $bK \nsubseteq M$-rad(N). Since $xbK \subseteq N$, $xK \nsubseteq M$-rad(N) and $bK \nsubseteq M$-rad(N), then we get $xb \in (N : M)$, by Lemma 2. We have $(x+a)K \nsubseteq M$-rad(N) as $(x+a)bK \subseteq N$, $xK \nsubseteq M$-rad(N) and $aK \subseteq M$-rad(N). Since $bK \nsubseteq M$-rad(N) and $(x+a)K \nsubseteq M$-rad(N), then we obtain $b(x + a) \in (N : M)$, by Lemma 2. Thus since $b(x + a) = xb + ab \in (N : M)$ and $xb \in (N : M)$, then $ab \in (N : M)$, which is a contradiction. **Second case:** Assume that $aK \nsubseteq M$-rad(N) and $bK \subseteq M$-rad(N). It is easily shown similarly to the first case. **Third case:** Suppose that $aK \subseteq M$-rad(N) and $bK \subseteq M$-rad(N). Then $(y + b)K \nsubseteq M$-rad(N) as $yK \nsubseteq M$-rad(N) and $bK \subseteq M$-rad(N). By Lemma 2, $x(y + b) \in (N : M)$ since $x(y + b)K \subseteq N$, $xK \nsubseteq M$-rad(N) and $(y + b)K \nsubseteq M$-rad(N). Then $xb \in (N : M)$ since $x(y+b) \in (N : M)$ and $xy \in (N : M)$. As $aK \subseteq M$-rad(N) and $xK \nsubseteq M$-rad(N), then $(x + a)K \nsubseteq M$-rad(N). As $(x + a)yK \subseteq N$, $yK \nsubseteq M$-rad(N) and $(x+a)K \nsubseteq M$-rad(N), then $(x+a)y = xy+ay \in (N : M)$ by Lemma 2. As $xy \in (N : M)$ and $ay + xy \in (N : M)$, then $ay \in (N : M)$. By Lemma 2, we get $(x + a)(y + b) = xy + xb + ay + ab \in (N : M)$ since $(x + a)(y + b)K \subseteq N$, $(x + a)K \nsubseteq M$-rad(N) and $(y + b)K \nsubseteq M$-rad(N). As $xb, ay, xy \in (N : M)$, then $xy + xb + ay \in (N : M)$. Thus, $ab \in (N : M)$ since $xy + xb + ay + ab \in (N : M)$ and $xy + xb + ay \in (N : M)$, a contradiction. Hence $I_1K \subseteq M$-rad(N) or $I_2K \subseteq M$-rad(N). □

Lemma 3. *Let N be a weakly 2-absorbing primary submodule of M. If $abK \subseteq N$ and $0 \neq 2abK$ for some submodule K of M and for some $a, b \in R$, then $ab \in (N : M)$ or $aK \subseteq M$-rad(N) or $bK \subseteq M$-rad(N).*

Proof. Assume that $ab \notin (N : M)$. Now, we show that $K \subseteq (M$-rad$(N) : a) \cup (M$-rad$(N) : b)$. Let $x \in K$. If $0 \neq abx$, then $ax \in M$-rad(N) or $bx \in M$-rad(N) since $ab \notin (N : M)$. Thus $x \in (M$-rad$(N) : a) \cup (M$-rad$(N) : b)$. Suppose that $abx = 0$. As $0 \neq 2abK$, there is $y \in K$ such that $2aby \neq 0$. Then we get $0 \neq aby \in N$. So $ay \in M$-rad(N) or $by \in M$-rad(N) since N is weakly 2-absorbing primary. Let $z = x + y$. Then $0 \neq abz \in N$. Since $ab \notin (N : M)$, then $az \in M$-rad(N) or $bz \in M$-rad(N). Now, we consider three cases. **First case:** Assume that $ay \in M$-rad(N) and $by \notin M$-rad(N). Let $ax \notin M$-rad(N). Then $az \notin M$-rad(N). Thus $bz \in M$-rad(N). So $a(z + x) \notin M$-rad(N) and $b(z + x) \notin M$-rad(N). Hence $0 = ab(z + x) = 2abz$ since N is a weakly 2-absorbing primary submodule and $ab \notin (N : M)$. This is a contradiction. Consequently, $ax \in M$-rad(N). **Second case:** Assume that $ay \notin M$-rad(N) and $by \in M$-rad(N). It is proved by a similar way to first case. **Third case:**

Assume that $ay \in M$-rad(N) and $by \in M$-rad(N). Since $az \in M$-rad(N) or $bz \in M$-rad(N), $ax \in M$-rad(N) or $bx \in M$-rad(N). □

Lemma 4. *Let N be a weakly 2-absorbing primary submodule of an R-module M. If $aJK \subseteq N$ and $0 \neq 4aJK$ for some submodule K of M, for any ideal J of R and for some $a \in R$, then $aJ \subseteq (N : M)$ or $aK \subseteq M$-rad(N) or $JK \subseteq M$-rad(N).*

Proof. Suppose that $aJ \nsubseteq (N : M)$. Then there is $j \in J$ such that $aj \notin (N : M)$. Our claim is that there is $b \in J$ such that $0 \neq 4abK$ and $ab \notin (N : M)$. As $0 \neq 4aJK$, then there is $j' \in J$ such that $0 \neq 4aj'K$. Assume that $aj' \notin (N : M)$ and $0 \neq 4ajK$. Then we have the result for $b = j'$ or $b = j$. Now, let $aj' \in (N : M)$ and $4ajK = 0$. Then $0 \neq 4a(j + j')K \subseteq N$ and $a(j + j') \notin (N : M)$ since $aj \notin (N : M)$ and $aj' \in (N : M)$. Hence we get $b = j + j' \in J$ such that $0 \neq 4abK \subseteq N$ and $ab \notin (N : M)$. Thus $0 \neq 2abK$ and so $K \subseteq (M\text{-rad}(N) : a) \cup (M\text{-rad}(N) : b)$ by Lemma 3. If $K \subseteq (M\text{-rad}(N) : a)$, the proof is completed. Suppose that $K \nsubseteq (M\text{-rad}(N) : a)$. Then $K \subseteq (M\text{-rad}(N) : b)$, that is, $bK \subseteq M$-rad(N). Let $c \in J$. Assume that $0 \neq 2acK$. By Lemma 3, $ac \in (N : M)$ or $cK \subseteq M$-rad(N) since $K \nsubseteq (M\text{-rad}(N) : a)$. Then $c \in ((N : M) : a) \cup (M\text{-rad}(N) : K)$. Now, let $2acK = 0$. Then $0 \neq 2a(b+c)K \subseteq N$. By Lemma 3, we have $a(b+c) \in (N : M)$ or $(b+c)K \subseteq M$-rad(N) since $K \nsubseteq (M\text{-rad}(N) : a)$. Thus we get $b + c \in ((N : M) : a) \cup (M\text{-rad}(N) : K)$. If $b + c \in (M\text{-rad}(N) : K)$, then $c \in (M\text{-rad}(N) : K)$. Hence $JK \subseteq M$-rad(N). Let $b + c \in ((N : M) : a)\backslash(M\text{-rad}(N) : K)$. Note that $2a(b + c + b)K = 4abK \neq 0$ and $2a(b + c + b)K \subseteq N$. Since $ab \notin (N : M)$ and $a(b+c) \in (N : M)$, $a(b+c+b) \notin (N : M)$. Thus by Lemma 3, $K \subseteq (M\text{-rad}(N) : a) \cup (M\text{-rad}(N) : (b + c + b))$. As $b + c \notin (M\text{-rad}(N) : K)$ and $b \in (M\text{-rad}(N) : K)$, then we get $(b + c + b) \notin (M\text{-rad}(N) : K)$ and so $K \subseteq (M\text{-rad}(N) : a)$, a contradiction. Thus $b + c \in (M\text{-rad}(N) : K)$. Since $b \in (M\text{-rad}(N) : K)$, $c \in (M\text{-rad}(N) : K)$. Hence $J \subseteq ((N : M) : a) \cup (M\text{-rad}(N) : K)$. Consequently $J \subseteq (M\text{-rad}(N) : K)$ since $aJ \nsubseteq (N : M)$. □

Theorem 3. *Let I_1, I_2 be ideals of R and N, K be submodules of an R-module M. Assume that N is a weakly 2-absorbing primary submodule. If $0 \neq I_1I_2K$ and $0 \neq 8(I_1I_2 + (I_1 + I_2)(N : K))(K + N)$, then $I_1I_2 \subseteq (N : M)$ or $I_1K \subseteq M$-rad(N) or $I_2K \subseteq M$-rad(N).*

Proof. It is clear from Lemma 4 and [12, Theorem 2.3]. □

Theorem 4. *Let M be a finitely generated multiplication R-module whose every proper submodule is weakly 2-absorbing primary. Then there exist at most three maximal ideals of R containing $Ann(M)$.*

Proof. Assume that R has four maximal ideals $\Im_1, \Im_2, \Im_3$ and $\Im_4$ containing $Ann(M)$. Let $K = \Im_1 \cap \Im_2 \cap \Im_3$ and $N = KM$. Note that $\Im_iM \neq M$ for every index i. Indeed, if $\Im_iM = M$ for some index i, then there is an element $r \in \Im_i$ such that $r - 1 \in Ann(M) \subseteq \Im_i$, contradiction. It is easily seen that

$\Im_i = (\Im_i M : M)$ as $\Im_i \subseteq (\Im_i M : M)$. We get $K \subseteq (N : M) \subseteq \bigcap_{i=1}^{3}(\Im_i M : M) = K$. Thus $\sqrt{(N:M)} = \sqrt{K} = \sqrt{\Im_1} \cap \sqrt{\Im_2} \cap \sqrt{\Im_3}$. Hence we get that $(N : M)$ is not a 2-absorbing primary ideal by [14, Corollary 2.7]. Then N is not a 2-absorbing primary submodule by [13, Theorem 2.6]. By Lemma 1, $(N : M)^3 \subseteq Ann(M) \subseteq \Im_4$. Then $\Im_i = \Im_4$ for $i \in \{1, 2, 3\}$, a contradiction. Therefore, there are at most three maximal ideals of R containing $Ann(M)$. □

Corollary 1. *Let M be a finitely generated multiplication R-module whose every proper submodule is weakly 2-absorbing primary. Then $(J(R))^3M = 0$.*

Proof. We have that $K^3 \subseteq Ann(M)$ by the proof of Theorem 4. Thus we obtain $(J(R))^3M = 0$. □

Let M be a multiplication R-module and K, L be R-submodules of M. Then there exist ideals I and J of R such that $K = IM$ and $L = JM$. Hence $KL = IJM = IL$ and $KL = IJM = KJ$. In particular, we obtain that $KM = IM = K$, $LM = JM = L$ and $Km = KRm$ for every $m \in M$. Therefore, $Km = IRm = Im$.

Lemma 5. *Let M be a multiplication R-module. If $(N : M)$ is a weakly primary ideal of R, then N is a weakly primary submodule of M.*

Proof. Assume that $0 \neq am \in N$ for some $a \in R$ and $m \in M$ with $a \notin \sqrt{(N:M)}$. Since M is multiplication, there exists an ideal I of R such that $m = Rm = IM$, then $0 \neq RaIM \subseteq N$. Since $(N : M)$ is weakly primary and $a \notin \sqrt{(N:M)}$, we have $((N : M) : Ra) = (N : M)$ or $((N : M) : Ra) = (0 : Ra)$, by [1, Proposition 2.1]. We claim that $Ra \subseteq (N : M)$ or $IM \subseteq N$. Suppose that $Ra \nsubseteq (N : M)$. Then since $0 \neq RaIM$, we get that $I \subseteq (N : RaM) = ((N : M) : Ra) = (N : M)$. Hence $IM \subseteq N$ and so $m \in N$, as needed. □

Corollary 2. *Let M be a finitely generated multiplication R-module. Suppose that every proper ideal of R is weakly primary such that not primary ideal. Then $(J(R))^3M = 0$.*

Proof. Assume that $(N : M)$ is a proper weakly primary ideal of R. Thus $(N : M)M = N$ is a weakly 2-absorbing primary submodule. By Corollary 1, we get $(J(R))^3M = 0$. □

The following result is an analogue of [6, Theorem 2.21].

Lemma 6. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where $0 \neq M_1$ is a multiplication R_1-module and $0 \neq M_2$ is a multiplication R_2-module. Then the followings hold:*

(1) *The following statements are equivalent:*
   i. *$N_1 \times M_2$ is a weakly 2-absorbing primary submodule of $M_1 \times M_2$;*
   ii. *$N_1 \times M_2$ is a 2-absorbing primary submodule of $M_1 \times M_2$;*

iii. *N_1 is a 2-absorbing primary submodule of M_1.*

(2) *The following statements are equivalent:*

i. *$M_1 \times N_2$ is a weakly 2-absorbing primary submodule of $M_1 \times M_2$;*

ii. *$M_1 \times N_2$ is a 2-absorbing primary submodule of $M_1 \times M_2$;*

iii. *N_2 is a 2-absorbing primary submodule of M_2.*

Proof. (1) (i)⇒(ii): Assume that $N_1 \times M_2$ is not a 2-absorbing primary submodule of $M_1 \times M_2$. By Lemma 1, $(0,0) = ((N_1 \times M_2) : (M_1 \times M_2))^2(N_1 \times M_2) = ((N_1 : M_1) \times (M_1 : M_2))^2(N_1 \times M_2) = (N_1 : M_1)^2 N_1 \times M_2$. Then $M_2 = 0$, a contradiction.

(ii)⇒(i): Is obvious.

(ii)⇒(iii): It is clear.

(2) The other part of the lemma can be seen in a similar way to the proof of the first part. □

Lemma 7. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where $0 \neq M_1$ is a multiplication R_1-module and $0 \neq M_2$ is a multiplication R_2-module. If $N_1 \times N_2$ is a weakly 2-absorbing primary submodule of M, $N_1 \neq M_1$, $N_2 \neq 0$ and $N_2 \neq M_2$, then N_1 is a weakly 2-absorbing primary submodule of M_1.*

Proof. Assume that $N_1 \neq M_1$, $N_2 \neq 0$ and $N_2 \neq M_2$. Let $a_1, b_1 \in R_1$ and $x_1 \in M_1$ where $a_1 b_1 x_1 \in N_1$ and let $0 \neq x_2 \in N_2$. Then $(0,0) \neq (a_1,1)(b_1,1)(x_1,x_2) \in N_1 \times N_2$. Since $N_1 \times N_2$ is a weakly 2-absorbing primary submodule, then $(a_1,1)(b_1,1) \in ((N_1 \times N_2) : M)$ or $(a_1,1)(x_1,x_2) \in M\text{-rad}(N_1 \times N_2) = M_1\text{-rad}(N_1) \times M_2\text{-rad}(N_2)$ or $(b_1,1)(x_1,x_2) \in M\text{-rad}(N_1 \times N_2) = M_1\text{-rad}(N_1) \times M_2\text{-rad}(N_2)$. As $1 \notin (N_2 : M_2)$, then we get $(a_1,1)(x_1,x_2) \in M_1\text{-rad}(N_1) \times M_2\text{-rad}(N_2)$ or $(b_1,1)(x_1,x_2) \in M_1\text{-rad}(N_1) \times M_2\text{-rad}(N_2)$. Thus $a_1 x_1 \in M_1\text{-rad}(N_1)$ or $b_1 x_1 \in M_1\text{-rad}(N_1)$. The proof is completed. □

By [15], it is said that a commutative ring R is a *u*-ring if an ideal of R contained in a finite union of ideals must be contained in one of those ideals; and a *um*-ring is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them.

Theorem 5. *Let N be a weakly 2-absorbing primary submodule of M. Then the following statements hold:*

(1) *If $ab \notin (N : M)$ for some $a, b \in R$, then $(N : ab) \subseteq (M\text{-}rad(N) : a) \cup (M\text{-}rad(N) : b) \cup (0 : ab)$.*

(2) *Let R be a um-ring. If $ab \notin (N : M)$ for some $a, b \in R$, then $(N : ab) \subseteq (M\text{-}rad(N) : a)$ or $(N : ab) \subseteq (M\text{-}rad(N) : b)$ or $(N : ab) = (0 : ab)$.*

Proof. Assume that N is a weakly 2-absorbing primary submodule of M.

(1) Let $m \in (N : ab)$. Assume that $abm \neq 0$. Then $abm \in N$ and thus $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$ since $ab \notin (N : M)$. Hence $m \in (M\text{-rad}(N) : a)$ or $m \in (M\text{-rad}(N) : b)$. Therefore, $m \in (M\text{-rad}(N) : a) \cup (M\text{-rad}(N) : b)$. Assume that $abm = 0$. Then $m \in (0 : ab)$. Consequently, $m \in (M\text{-rad}(N) : a) \cup (M\text{-rad}(N) : b) \cup (0 : ab)$.

(2) Assume that R is a um-ring. It is easily obtained by part (1). □

Theorem 6. *Let N be a weakly 2-absorbing primary submodule of M. Then the following statements hold:*

(1) *If $abm \notin N$ for some $a, b \in R$ and $m \in M$, then $(N : abm) \subseteq (M\text{-}rad(N) : am) \cup (M\text{-}rad(N) : bm) \cup (0 : abm)$.*
(2) *Let R be a u-ring. If $abm \notin N$ for some $a, b \in R$ and $m \in M$, then $(N : abm) \subseteq (M\text{-}rad(N) : am)$ or $(N : abm) \subseteq (M\text{-}rad(N) : bm)$ or $(N : abm) = (0 : abm)$.*

Proof. Assume that N is a weakly 2-absorbing primary submodule of M.

(1) Let $r \in (N : abm)$. Assume that $rabm \neq 0$. Then $rabm \in N$ and thus $ram \in M\text{-rad}(N)$ or $rbm \in M\text{-rad}(N)$ since $ab \notin (N : M)$. Hence $r \in (M\text{-rad}(N) : am)$ or $r \in (M\text{-rad}(N) : bm)$. Therefore, $r \in (M\text{-rad}(N) : am) \cup (M\text{-rad}(N) : bm)$. Assume that $rabm = 0$. Then $r \in (0 : abm)$. Consequently, $r \in (M\text{-rad}(N) : am) \cup (M\text{-rad}(N) : bm) \cup (0 : abm)$.

(2) Assume that R is a u-ring. The result is easily seen by part (1). □

Proposition 6. *Let M be a faithful multiplication R-module and N be a submodule of M. If N is a weakly 2-absorbing primary submodule but it is not 2-absorbing primary, then $N \subseteq M\text{-}rad(0)$. In addition $M\text{-}rad(N) = M\text{-}rad(0)$.*

Proof. By Lemma 1, $(N : M)^3 \subseteq Ann(M)$. Since M is a faithful module, $(N : M)^3 = 0$. Now suppose that $a \in (N : M)$. Then $a^3 = 0$ and so $a \in \sqrt{0}$. Hence $(N : M) \subseteq \sqrt{0}$ and thus $N = (N : M)M \subseteq M\text{-rad}(0)$. In addition, by Lemma 1, $(N : M)^2N = 0$. Then $(N : M)^3 = (N : M)^2(N : M) \subseteq ((N : M)^2N : M) = Ann(M)$ and so $(N : M) \subseteq \sqrt{Ann(M)}$. Thus $\sqrt{(N : M)} = \sqrt{Ann(M)}$. Hence $M\text{-rad}(N) = \sqrt{(N : M)}M = \sqrt{Ann(M)}M = M\text{-rad}(0)$. □

Theorem 7. *Let M be an R-module and N be a submodule of M. Suppose that $M\text{-}rad(0)$ is a prime submodule. Then N is a weakly 2-absorbing primary submodule if and only if N is a 2-absorbing primary submodule.*

Proof. Assume that N is weakly 2-absorbing primary. Suppose that $abm \in N$ for some $a, b \in R$ and $m \in M$. If $0 \neq abm \in N$, then either $ab \in (N : M)$ or $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$. If $abm = 0$, we may suppose that $ab \notin (N : M)$. Since that M-rad(0) is a prime submodule, we can conclude that $a \in \left(M\text{-rad}(0) :_R M\right)$ or $bm \in M\text{-rad}(0)$. Since $M\text{-rad}(0) \subseteq M\text{-rad}(N)$, we obtain that $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$. Therefore N is 2-absorbing primary. The converse is obviously. □

Theorem 8. *Let M be a multiplication R-module and N be a submodule of M. Suppose that M is a P-primary module. If N is a weakly 2-absorbing primary submodule, then $(N : M)$ is a weakly 2-absorbing primary ideal.*

Proof. Assume that $0 \neq abc \in (N : M)$ for some $a, b, c \in R$ with $ab \notin (N : M)$. Then $abcm \in N$ for each $m \in M$. If $abcm = 0$, then $abc \in (0 :_R M) = P$.

Since M is a P-primary module, we conclude that $c^n \in P \subseteq (N : M)$ for some positive integer n. Thus $ac \in \sqrt{(N : M)}$, as needed. Suppose that $0 \neq abcm \in N$. Since N is weakly 2-absorbing primary, we can conclude that $acm \in M\text{-rad}(N)$ $\big($so $ac \in \big((M\text{-rad}(N) : M)\big)$ or $bcm \in M\text{-rad}(N)$ $\big($so $bc \in \big((M\text{-rad}(N) : M)\big)$. Hence $ac \in \sqrt{(N : M)}$ or $bc \in \sqrt{(N : M)}$. Therefore $(N : M)$ is a weakly 2-absorbing primary ideal. □

Theorem 9. *Let M be a multiplication R-module and N be a submodule of M. Suppose that M is a P-primary module. If N is a weakly 2-absorbing primary submodule, then $\sqrt{(N : M)}$ is a weakly 2-absorbing ideal.*

Proof. Assume that $0 \neq abc \in \sqrt{(N : M)}$ with $ab \notin \sqrt{(N : M)}$ for some $a, b, c \in R$. Then $(abc)^n m \in N$ for some positive integer n and every $m \in M$. If $(abc)^n m = 0$ since M is a P-primary module, then $abc \in (0 :_R m) = (0 :_R M) = P \subseteq (N : M)$. Hence $ac \in \sqrt{(N : M)}$, as needed. Suppose that $0 \neq (abc)^n m = a^n b^n (c^n m) \in N$. Since N is a weakly 2-absorbing primary submodule and $ab \notin \sqrt{(N : M)}$, we conclude that $a^n (c^n m) \in M\text{-rad}(N)$ $\big($so $a^n c^n \in (M\text{-rad}(N) : M)\big)$ or $b^n (c^n m) \in M\text{-rad}(N)$ $\big($so $b^n c^n \in (M\text{-rad}(N) : M)\big)$. Since $(M\text{-rad}(N) : M) = \sqrt{(N : M)}$, we can conclude that $ac \in \sqrt{(N : M)}$ or $bc \in \sqrt{(N : M)}$. Therefore $\sqrt{(N : M)}$ is a weakly 2-absorbing ideal. □

Theorem 10. *Let M be an R-module and N be a submodule of M. If N is a weakly 2-absorbing submodule, then $(N : m)$ is a weakly 2-absorbing ideal for every $m \in M \backslash N$ with $Ann(m) = 0$.*

Proof. Assume that $0 \neq abc \in (N : m)$ for some $a, b, c \in R$. Then $abcm \in N$. If $abcm = 0$, then $abc \in (0 : Rm) = 0$ and we are done. Suppose that $0 \neq abcm \in N$. Since N is a weakly 2-absorbing submodule, we conclude that $ab \in (N :_R M)$ $\big($so $ab \in (N : m)\big)$ or $bcm \in N$ $\big($so $bc \in (N : m)\big)$ or $acm \in N$ $\big($so $ac \in (N : m)\big)$. Therefore $(N : m)$ is a weakly 2-absorbing ideal for every $m \in M \backslash N$. □

We invite the reader to compare the following two results with [6, Theorem 2.23].

Theorem 11. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ be a finitely generated multiplication R-module where M_1 and M_2 are multiplication R_1-module and R_2-module, respectively. If $N = N_1 \times N_2$ is a weakly 2-absorbing primary submodule, then $N = 0$ or N is 2-absorbing primary.*

Proof. Assume that $0 \neq N = N_1 \times N_2$ and show that N is a 2-absorbing primary submodule. Without loss of generality we may suppose that $N_2 \neq 0$. Then there exists a nonzero element $y \in N_2$. Suppose that $a \in (N_1 : M_1)$ and $x \in M_1$. Then $0 \neq (a, 1)(1, 1)(x, y) \in N = N_1 \times N_2$. Since N is a weakly 2-absorbing primary submodule, we conclude that $(a, 1)(1, 1) \in (N_1 \times N_2 : M)$ or $(a, 1)(x, y) \in M_1\text{-rad}(N_1) \times M_2\text{-rad}(N_2)$ or $(1, 1)(x, y) \in M_1\text{-rad}(N_1) \times$

M_2-rad(N_2). Then $1 \in (N_2 : M_2)$ or $ax \in M_1$-rad(N_1) or $y \in M_2$-rad(N_2). Suppose that $1 \in (N_2 : M_2)$. If $N_1 = 0$, then $0 \times M_2 \subseteq N$ and so we may assume that $N = N_1 \times M_2$. Let $ax \in M_1$-rad(N_1). Here we will have two cases; if $x \in N_1$, then since $N_2 \neq 0$, we can assume that $0 \times M_2 \subseteq N$ and so $N_1 \times M_2 = N$. If $x \notin N_1$ $\big(x \in M_1 \setminus N_1\big)$, then since $N_1 \neq M_1$ and $N_2 \neq 0$, we may suppose that $N = N_1 \times M_2$. Then we show that if $N = N_1 \times M_2$, then N_1 is 2-absorbing primary and if $N = M_1 \times N_2$, then N_2 is 2-absorbing primary. For beginning we assume that $N = N_1 \times M_2$ and show that N_1 is 2-absorbing primary. Let $rsm \in N_1$ for some $r, s \in R_1$ and $m \in M_1$. Then $(0,0) \neq (r,1)(s,1)(m,y) \in N_1 \times M_2$. Thus either $(r,1)(s,1) \in (N_1 \times N_2 :_R M)$ or $(r,1)(m,y) \in M_1$-rad$(N_1) \times M_2$-rad(M_2) or $(s,1)(m,y) \in M_1$-rad$(N_1) \times M_2$-rad(M_2) and so either $rs \in N_1$ or $rm \in M_1$-rad(N_1) or $sm \in M_1$-rad(N_1). Therefore $N = N_1 \times N_2$ is a 2-absorbing primary submodule, by Lemma 6. By similar way we can show that N_2 is a 2-absorbing primary submodule and so $N = M_1 \times N_2$ is 2-absorbing primary, by Lemma 6. Now we show that N_1 and N_2 are primary submodules. Assume that $N_2 \neq M_2$. Let $t \in R_1$ and $n \in M_1$ such that $tn \in N_1$ with $0 \neq n' \in N_2$. Then $(0,0) \neq (t,1)(1,1)(n,n') \in N_1 \times N_2$. Since $N_1 \times N_2$ is a weakly 2-absorbing primary submodule, we conclude that $(t,1)(1,1) \in (N_1 \times N_2 :_R M)$ or $(t,1)(n,n') \in M_1$-rad$(N_1) \times M_2$-rad(N_2) or $(1,1)(n,n') \in M_1$-rad$(N_1) \times M_2$-rad(N_2). But $1 \notin (N_2 : M_2)$. Then $tn \in M_1$-rad(N_1) and so N_1 is primary. Similarly we can show that N_2 is a primary submodule. Therefore N is a 2-absorbing primary submodule, by [13, Theorem 2.28]. □

Theorem 12. *Let M_1 and M_2 be multiplication modules. Suppose that $N_1 \neq M_1$ and $M_2 \neq 0$. The submodule $N_1 \times 0$ is a weakly 2-absorbing primary submodule if one of the following statements hold:*

1. *N_1 is a weakly primary submodule of M_1 and 0 is a primary submodule of M_2 and M_1-rad$(N_1) \neq 0$;*
2. *N_1 is a weakly primary submodule of M_1 and 0 is a primary submodule of M_2 and M_1-rad$(N_1) = 0$;*
3. *$N_1 = 0$.*

Proof. Assume that $(0,0) \neq (a,b)(c,d)(x,y) \in N_1 \times 0$ with $(a,b)(c,d) \notin (N_1 \times 0 : M)$. Then $0 \neq acx \in N_1$ and $bdy = 0$. Since N_1 is weakly primary, we have $a \in \sqrt{(N_1 : M_1)}$ or $cx \in N_1$. Since 0 is a primary submodule, we get that $b \in \sqrt{(0 : M_2)}$ or $dy = 0$. Then $ax \in M_1$-rad(N_1) or $cx \in M_1$-rad(N_1) and $by \in M_2$-rad(0) or $dy \in M_2$-rad(0). Hence $(a,b)(x,y) \in M_1$-rad$(N_1) \times M_2$-rad(0) or $(c,d)(x,y) \in M_1$-rad$(N_1) \times M_2$-rad(0), as needed. Now suppose that (2) holds. Then since N_1 is weakly primary and $0 \neq acx \in N_1$, we get that $a \in \sqrt{(N_1 : M_1)}$ or $cx \in N_1$. If $a \in \sqrt{(N_1 : M_1)}$, then $ax \in M_1$-rad$(N_1) = 0$. Thus as $cx \in M_1$-rad$(N_1) = 0$, $acx = 0$ which is a contradiction. Then neither $(a,b)(x,y) \in M_1$-rad$(N_1) \times M_2$-rad(0) nor

$(c,d)(x,y) \in M_1\text{-rad}(N_1) \times M_2\text{-rad}(0)$. Therefore $N_1 \times 0$ is weakly 2-absorbing primary. The latest statement is obviously. □

Example 1. Suppose that $M = \mathbb{Z} \times \mathbb{Z}$ is an $R = \mathbb{Z} \times \mathbb{Z}$-module and $N = p\mathbb{Z} \times \{0\}$ is a submodule of M where $p\mathbb{Z}$ is a weakly primary submodule and $\{0\}$ is weakly primary. Then $\mathbb{Z}\text{-rad}(p\mathbb{Z}) = p\mathbb{Z}$ and $(N : M) = 0$. Assume that $(0,0) \neq (p,1)(1,0)(1,1) \in \mathbb{Z} \times \{0\}$. Then neither $(p,1)(1,0) \in (N : M)$ nor $(p,1)(1,1) \in M\text{-rad}(N)$ nor $(1,0)(1,1) \in M\text{-rad}(N)$. Hence N is not weakly 2-absorbing primary. Notice that M is not a multiplication module.

Proposition 7. *Let M be a multiplication R-module and $N_1, \ldots, N_n$ be weakly 2-absorbing primary submodules with M-rad$(N_i) = P$ where P is a prime submodule. Then $N = \bigcap_{i=1}^{n} N_i$ is weakly 2-absorbing primary.*

Proof. Assume that $0 \neq abm \in N$ for some $a, b \in R$ and $m \in M$ with $ab \notin (N : M)$. By [5, Propositin 2.14], $M\text{-rad}(N) = \bigcap_{i=1}^{n} M\text{-rad}(N_i)$. Then $ab \notin (N_i : M)$ for some $1 \leq i \leq n$. Since N_i is a weakly 2-absorbing primary submodule, $am \in M\text{-rad}(N_i) = P$ or $bm \in M\text{-rad}(N_i) = P$. Hence $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$, as needed. □

Definition 4. Let N be a weakly 2-absorbing primary submodule of an R-module M and $M\text{-rad}(N) = P$, by Proposition 7. We say that N is a P-weakly 2-absorbing primary submodule.

Let R be a ring with identity and M be an R-module. Then $R(M) = R(+)M$ with multiplication $(a,m)(b,n) = (ab, an + bm)$ and with addition $(a,m) + (b,n) = (a+b, m+n)$ is a commutative ring with identity and $0(+)M$ is a nilpotent ideal of index 2. The ring $R(+)M$ is said to be the *idealization* of M or *trivial extension* of R by M. We view R as a subring of $R(+)M$ via $r \to (r,0)$. An ideal H is said to be *homogeneous* if $H = I(+)N$ for some ideal I of R and some submodule N of M; whence $IM \subseteq N$, [9] and [11, Sec, 25]. Let R_1 and R_2 be commutative rings, M_1 and M_2 be R-modules. Then $(R_1 \times R_2)(+)(M_1 \times M_2) \approx (R_1(+)M_1) \times (R_2(+)M_2)$, by [9, Theorem 4.4]. Now we use it to characterize one of the result of weakly 2-absorbing primary ideals by the idealization method.

Proposition 8. *Let R_1 and R_2 be commutative rings, M_1 be R_1-module and M_2 be R-module, respectively. Suppose that $H = I(+)M_1$ is an ideal of $R_1(M_1) = R_1(+)M_1$ and $J = h(+)M_2$ is an ideal of $R_2(M_2) = R_2(+)M_2$. Then the following statement are equivalent*:

(1) *$H \times R_2(M_2)$ is a 2-absorbing (resp., weakly 2-absorbing) primary ideal of $R_1(M_1) \times R_2(M_2)$;*

i. *$R_1(M_1) \times J$ is a 2-absorbing (resp., weakly 2-absorbing) primary ideal of $R_1(M_1) \times R_2(M_2)$;*

(2) *H is a 2-absorbing (resp., weakly 2-absorbing) primary ideal of $R_1(M_1)$;*

ii. *J is a 2-absorbing (resp., weakly 2-absorbing) primary ideal of $R_2(M_2)$;*

Proof. (1) $\Rightarrow$ (2) Assume that $(a,x)(b,y)(c,z) \in H = I(+)M_1$ for some $a,b,c \in R$ and $x,y,z \in M_1$. Thus $abc \in I$. Since $H \times R_2(M_2) = (I(+)M_1) \times (R_2(+)M_2) = (I_1 \times R_2)(+)(M_1 \times M_2)$ is a 2-absorbing primary ideal, $H = I(+)M_1$ is 2-absorbing primary and hence I is a 2-absorbing primary ideal of R_1, by [6, Theorem 2.21]. Then $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Therefore $(a,x)(b,y) \in I(+)M_1$ or $(b,y)(c,z) \in \sqrt{I(+)M_1} = \sqrt{I}(+)M_1$ or $(a,x)(c,z) \in \sqrt{I(+)M_1} = \sqrt{I}(+)M_1$, as needed.

(2) $\Rightarrow$ (1) Let H be 2-absorbing primary. Assume that $(u_1,v_1)(u_2,v_2)(u_3,v_3) \in H \times R_2(M_2)$ such that $u_1,u_2,u_3 \in H$ and $v_1,v_2,v_3 \in R_2(M_2)$. Then $u_1u_2u_3 \in H = I(+)M_1$ where $u_1 = (a,x)$, $u_2 = (b,y)$ and $u_3 = (c,z)$. Since H is a 2-absorbing primary ideal, I is 2-absorbing primary, by [6, Theorem 2.21]. Then $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Hence $(a,x)(b,y) = u_1u_2 \in I(+)M_1$ or $(b,y)(c,z) = u_2u_3 \in \sqrt{I(+)M_1} = \sqrt{I}(+)M_1$ or $(a,x)(c,z) = u_1u_3 \in \sqrt{I(+)M_1} = \sqrt{I}(+)M_1$. Therefore $(u_1,v_1)(u_2,v_2) \in H \times R_2(M_2)$ or $(u_2,v_2)(u_3,v_3) \in \sqrt{H \times R_2(M_2)} = (\sqrt{I}(+)M_1) \times (R_2(+)M_2)$ or $(u_1,v_1)(u_3,v_3) \in \sqrt{H \times R_2(M_2)} = (\sqrt{I}(+)M_1) \times (R_2(+)M_2)$. Then $H \times R_2(M_2)$ is a 2-absorbing primary ideal of $R_1(M_1) \times R_2(M_2)$.

The proof of (i) if and only if (ii) is clear by similar arguments as previously shown, and hence we omit the proof. □

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