

## UPPER BOUNDS OF SECOND HANKEL DETERMINANT FOR UNIVERSALLY PRESTARLIKE FUNCTIONS

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ABSTRACT. In [12,13] the researchers introduced universally convex, universally starlike and universally prestarlike functions in the slit domain  $\mathbb{C} \setminus [1, \infty)$ . These papers extended the corresponding notions from the unit disc to other discs and half-planes containing the origin. In this paper, we introduce universally prestarlike generalized functions of order  $\alpha$  with  $\alpha \leq 1$  and we obtain upper bounds of the second Hankel determinant  $|a_2a_4 - a_3^2|$  for such functions.

### 1. Introduction

Let  $\mathcal{H}(\Omega)$  denote the set of all analytic functions in a domain  $\Omega$ . Suppose  $\Omega$  contains the origin and  $\mathcal{H}_0(\Omega)$  stands for the set of all functions  $f \in \mathcal{H}(\Omega)$  with  $f(0) = 1$  and also let

$$\mathcal{H}_1(\Omega) = \{zf : f \in \mathcal{H}_0(\Omega)\}.$$

If  $\Omega = \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$  is the unit disc, we write  $\mathcal{H} \equiv \mathcal{H}(\mathbb{U})$ ,  $\mathcal{H}_0 \equiv \mathcal{H}_0(\mathbb{U})$  and  $\mathcal{H}_1 \equiv \mathcal{H}_1(\mathbb{U})$ . Let the Hadamard (or convolution) product of two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{U}$$

in  $\mathcal{H}_0(\Omega)$  is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

A function  $f \in \mathcal{H}_1$  is called a starlike function of order  $\alpha$  ( $0 \leq \alpha \leq 1$ ) if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad (z \in \mathbb{U})$$

and the set of such functions is denoted by  $\mathcal{S}_\alpha$ .

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Due to Ruscheweyh [10], for  $f \in \mathcal{H}_1$ , let us denote by  $\mathcal{R}_\alpha$ , the set of all prestarlike functions of order  $\alpha$  ( $\alpha \leq 1$ ) in  $\mathbb{U}$  satisfying the criteria

$$\begin{cases} \frac{z}{(1-z)^{2-2\alpha}} * f \in \mathcal{S}_\alpha, & \alpha < 1, \\ \Re\left(\frac{f(z)}{z}\right) > \frac{1}{2}, & \alpha = 1, z \in \mathbb{U}, \end{cases}$$

where

$$\frac{z}{(1-z)^{2-2\alpha}} = z + \sum_{n=2}^{\infty} \mathcal{C}(\alpha, n) z^n$$

is a well-known extremal function in  $\mathcal{S}_\alpha$  and

$$\mathcal{C}(\alpha, n) = \frac{\prod_{k=2}^n (k - 2\alpha)}{(n-1)!}; \quad (n \in \mathbb{N} \setminus \{1\}, \mathbb{N} := \{1, 2, 3, \dots\}).$$

Note that  $\mathcal{C}(\alpha, n)$  is a decreasing function of  $\alpha$  with

$$\lim_{n \rightarrow \infty} \mathcal{C}(\alpha, n) = \begin{cases} \infty & \text{if } \alpha < \frac{1}{2}, \\ 1 & \text{if } \alpha = \frac{1}{2}, \\ 0 & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

While working with prestarlike functions and convolutions, the following notation turned out to be useful:

$$(D^n f)(z) = \frac{z}{(1-z)^n} * f,$$

where  $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$  and therefore we have  $D^{n+1} f = \frac{z}{n!} (z^{n-1} f)^{(n)}$  for  $n \in \mathbb{N}_0$ . Using this operator we find that a function  $f \in \mathcal{H}_1$  is prestarlike of order  $\alpha \leq 1$  if and only if

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \in \mathcal{P},$$

where

$$\mathcal{P} = \{g \in \mathcal{H}_0 : \Re(g(z)) > \frac{1}{2}, z \in \mathbb{U}\}$$

or, equivalently, by Herglotz formula,

$$g \in \mathcal{P} \Leftrightarrow g(z) = \int_0^1 \frac{d\mu(t)}{1 - e^{-it}z},$$

where  $\mu$  is a probability measure on  $[0, 2\pi]$ .

The notion of prestarlike functions of order  $\alpha$  has recently been extended from the unit disc  $\mathbb{U}$  to other discs and half-planes containing the origin (see [11–13]). Define one such disc  $\Omega_{\gamma, \rho}$  by

$$\Omega_{\gamma, \rho} = \{\omega_{\gamma, \rho}(z) : z \in \mathbb{U}\},$$

where  $\gamma \in \mathbb{C} \setminus \{0\}$  and  $\rho \in [0, 1]$  are two unique parameters and  $\omega_{\gamma, \rho}(z) = \frac{\gamma z}{1 - \rho z}$ . Note that  $1 \notin \Omega_{\gamma, \rho}$  if and only if  $|\gamma + \rho| \leq 1$ . For  $\alpha \leq 1$ , and for some admissible pair  $(\gamma, \rho)$ , we define

$$\mathcal{R}_\alpha(\Omega_{\gamma, \rho}) = \{f \in \mathcal{H}_1(\Omega_{\gamma, \rho}) : \frac{1}{\gamma} f(\omega_{\gamma, \rho}(z)) \in \mathcal{R}_\alpha\},$$

where  $\mathcal{H}_1(\Omega_{\gamma, \rho}) = \{zf : f \in \mathcal{H}_0(\Omega_{\gamma, \rho}) \text{ with } f(0) = 1\}$ . A function  $f$  in  $\mathcal{R}_\alpha(\Omega_{\gamma, \rho})$  is called prestarlike of order  $\alpha$  in  $\Omega_{\gamma, \rho}$  (see [12]).

**Definition** ([13]). Let  $\alpha \leq 1$  and  $\Lambda = \mathbb{C} \setminus [1, \infty)$ . A function  $f \in \mathcal{H}_1(\Lambda)$  is called universally prestarlike of order  $\alpha$  in  $\Lambda$  if and only if  $f$  is prestarlike of order  $\alpha$  in all sets  $\omega_{\gamma, \rho}$  with  $|\gamma + \rho| \leq 1$ . Denote the set of all universally prestarlike functions in  $\Lambda$  by  $\mathcal{R}_\alpha^u$ .

Due to Ma-Minda [8] we state the following subordination principle:

**Definition.** Suppose  $\phi$  is an analytic function such that

- (1)  $\Re(\phi) > 0$  in  $\mathbb{U}$ ,
- (2)  $\phi(0) = 1, \phi'(0) > 0$ ,
- (3)  $\phi$  maps  $\mathbb{U}$  onto a region starlike with respect to 1 and symmetric with respect to the real axis.

For  $\alpha \leq 1$  and a function  $f \in \mathcal{H}_1(\Lambda)$ , we let  $\mathcal{R}_\alpha^u(\phi)$  be the generalized class of universally prestarlike functions satisfying the condition

$$(1) \quad \frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \prec \phi(z),$$

where  $\prec$  denotes the subordination and  $\phi$  is an analytic function given by Definition 1. Note that for different choices of  $\phi$ , the class  $\mathcal{R}_\alpha^u(\phi)$  gives rise to several known and unknown classes of universally prestarlike functions of order  $\alpha$  as given in the following example.

**Example 1.1.** If  $\alpha \leq 1$ , and  $f \in \mathcal{H}_1(\Lambda)$ , then

$$(2) \quad f \in \mathcal{R}_\alpha^u(A, B) \iff \frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \prec \frac{1 + Az}{1 + Bz}, \quad (-1 \leq B < A \leq 1)$$

and

$$(3) \quad f \in \mathcal{R}_\alpha^u(\beta) \iff \frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \prec \frac{1 + (1 - 2\beta)z}{1 - z}, \quad (0 \leq \beta < 1).$$

In particular  $\mathcal{R}_{\frac{1}{2}}^u(1, -1) = \mathcal{S}^*$  is the class of starlike univalent functions.

Recall that the Hankel determinants  $H_q(n)$  ( $n = 1, 2, 3, \dots; q = 1, 2, \dots$ ) of the functions  $f(z) = \sum_{n=1}^{\infty} a_n z^n$ ,  $a_1 = 1$  are defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In particular,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1 a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

For more details on Hankel determinants, one may refer to the papers [4–7, 9, 17].

Though there has been an increasing interest to study the functional  $H_2(1)$  (that is,  $a_1 a_3 - a_2^2$ ) for certain classes of universally prestarlike functions (see [14–16]) and in particular, the Fekete and Szegő estimates of  $|a_3 - \mu a_2^2|$  (see [2]), the study of the functional  $H_2(2)$  (that is,  $a_2 a_4 - a_3^2$ ) for universally prestarlike functions is not yet known. The main purpose of this paper is to obtain the upper bounds of Hankel determinant  $|a_2 a_4 - a_3^2|$  for functions  $f \in \mathcal{R}_\alpha^\mu(\phi)$ .

### 1.1. Preliminary results

To prove our main results, we state the following lemmas.

**Lemma 1.2** (see [1, p. 41]). *Let  $\mathbf{P}$  be the class of all analytic functions  $p$  of the form*

$$(4) \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

satisfying  $\Re(p(z)) > 0$  ( $z \in \mathbb{U}$ ) and  $p(0) = 1$ . Then

$$|p_n| \leq 2 \quad (n = 1, 2, 3, \dots).$$

*This inequality is sharp for each  $n$ . In particular, equality holds for all  $n$  for the function*

$$p(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n.$$

**Lemma 1.3** (see [6]). *If the function  $p \in \mathbf{P}$  is given by (4), then*

$$(5) \quad 2p_2 = p_1^2 + x(4 - p_1^2),$$

$$(6) \quad 4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some  $x, z$  with  $|x| \leq 1$ ,  $|z| \leq 1$  and  $p_1 \in [0, 2]$ .

**Lemma 1.4** ([3]). *The power series for a function  $p$  given in (4) converges in  $\mathbb{U}$  to a function in  $\mathbf{P}$  if and only if the Toeplitz determinants*

$$(7) \quad D_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 2 & p_1 & \cdots & p_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \dots$$

and  $p_{-k} = \overline{p_k}$ , are all nonnegative. They are strictly positive except for

$$p(z) = \sum_{k=1}^m \rho_k p_0(e^{it_k z}), \quad \rho_k > 0, \quad t_k \text{ real}$$

and  $t_k \neq t_j$  for  $k \neq j$ ; in this case  $D_n > 0$  for  $n < m - 1$  and  $D_n = 0$  for  $n \geq m$ .

This necessary and sufficient condition is due to Caratheodory and Toeplitz and can be found in [3].

## 2. Coefficient bounds for the function class $\mathcal{R}_\alpha^u(\phi)$

In this section we obtain the upper bounds of the Hankel determinant

$$|a_2 a_4 - a_3^2|$$

for  $f \in \mathcal{R}_\alpha^u(\phi)$ . Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1-tz},$$

where

$$a_k = \int_0^1 t^k d\mu(t),$$

$\mu(t)$  is a probability measure on  $[0, 1]$ .

**Theorem 2.1.** Let  $f \in \mathcal{R}_\alpha^u(\phi)$  be given by

$$(8) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (a_0 = 0 \text{ and } a_1 = 1)$$

and suppose  $\phi$ , defined by Definition 1, is of the form

$$(9) \quad \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0).$$

(i) If  $B_1, B_2$  and  $B_3$  satisfy the conditions

$$(2 - 2\alpha)|B_2| \leq B_1 - (1 - \alpha)(1 - 2\alpha)B_1^2,$$

$$|2(3 - 2\alpha)B_1 B_3 - (2 - 2\alpha)^2 B_1^4 - (4 - 2\alpha)B_2^2| \\ + (2 - 2\alpha)(1 - 2\alpha)B_1^2|B_2| - (4 - 2\alpha)B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2 a_4 - a_3^2| \leq \frac{B_1^2}{(3 - 2\alpha)^2}.$$

(ii) If  $B_1, B_2$  and  $B_3$  satisfy the conditions

$$(2 - 2\alpha)|B_2| \geq B_1 - (1 - \alpha)(1 - 2\alpha)B_1^2,$$

$$|2(3 - 2\alpha)B_1 B_3 - (2 - 2\alpha)^2 B_1^4 - (4 - 2\alpha)B_2^2| - (2 - 2\alpha)B_1|B_2| \\ - (3 - 2\alpha)B_1^2 + (1 - \alpha)(1 - 2\alpha)\{2B_1^2|B_2| + B_1^3\} \geq 0,$$

(or) the conditions

$$(2-2\alpha)|B_2| \leq B_1 - (1-\alpha)(1-2\alpha)B_1^2,$$

$$|(3-2\alpha)B_1B_3 - 2(1-\alpha)^2B_1^4 - (2-\alpha)B_2^2| + (1-\alpha)(1-2\alpha)B_1^2|B_2| - (2-\alpha)B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$(10) \quad |a_2a_4 - a_3^2| \leq \frac{1}{(3-2\alpha)^2(4-2\alpha)} [2(1-\alpha)^2B_1^4 + (2-\alpha)|B_2|^2 - (3-2\alpha)B_1|B_3| - (1-\alpha)(1-2\alpha)B_1^2|B_2|].$$

(iii) If  $B_1, B_2$  and  $B_3$  satisfy the conditions

$$(2-2\alpha)|B_2| > B_1 - (1-\alpha)(1-2\alpha)B_1^2,$$

$$|4(3-2\alpha)B_1B_3 - 2(2-2\alpha)^2B_1^4 - 2(4-2\alpha)B_2^2| - 2(2-2\alpha)B_1|B_2| - 2(3-2\alpha)B_1^2 + (2-2\alpha)(1-2\alpha)B_1^2(2|B_2| + B_1) \leq 0,$$

then the second Hankel determinant satisfies,

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{(3-2\alpha)^2(4-2\alpha)} \left( \frac{M}{N} \right),$$

where,

$$M = 8(3-2\alpha)B_1[(2-2\alpha)B_2 - (4-2\alpha)|B_3|] - 4(2-2\alpha)(1-2\alpha)B_1^2[2|B_2| + (3-2\alpha)B_1] + 16|B_2|^2(3-2\alpha) - 4B_1^2(4\alpha^2 - 12\alpha + 9) + (2-2\alpha)^2B_1^4(15 - 8\alpha - 4\alpha^2).$$

$$N = 4(2-2\alpha)B_1^2[(2-2\alpha)B_1^2 - 1] - 8B_1[(3-2\alpha)|B_3| + (2-2\alpha)|B_2|] - 4(2-2\alpha)(1-2\alpha)B_1^2(|B_2| - B_1) - 4(4-2\alpha)|B_2|^2.$$

*Proof.* Since  $f \in \mathcal{R}_\alpha^u(\phi)$ , there exists a Schwarz function  $\omega$ , analytic in  $\mathbb{U}$  with  $\omega(0) = 0$  and  $|\omega(z)| < 1$  in  $\mathbb{U}$  such that

$$(11) \quad \frac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} = \phi(\omega(z)).$$

Define the function  $P_1$  by

$$P_1(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$$

Since  $\omega$  is a Schwarz function, we see that  $\Re(P_1(z)) \geq 0$  and  $P_1(0) = 1$  and therefore  $P_1 \in \mathbf{P}$ . It follows that

$$(12) \quad \omega(z) = \frac{P_1(z) - 1}{P_1(z) + 1} = \frac{1}{2} \left[ p_1z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \left( p_3 - p_1p_2 + \frac{p_1^3}{4} \right) z^3 + \dots \right].$$

Then, by a simple computation we get

$$\phi(\omega(z)) = 1 + \frac{B_1p_1}{2}z + \left[ \frac{B_1}{2} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4}B_2p_1^2 \right] z^2$$

$$(13) \quad + \left[ \frac{B_1}{2} \left( p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) + \frac{B_2 p_1}{2} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{B_3 p_1^3}{8} \right] z^3 + \dots \\ \equiv 1 + b_1 z + b_2 z^2 + b_3 z^3 + \dots$$

and therefore

$$(14) \quad b_1 = \frac{B_1 p_1}{2},$$

$$(15) \quad b_2 = \frac{B_1}{2} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2,$$

$$(16) \quad b_3 = \frac{B_1}{2} \left( p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) + \frac{B_2 p_1}{2} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{B_3 p_1^3}{8}.$$

On the other hand, in view of (11) and (13), we have

$$(17) \quad 1 + \sum_{n=1}^{\infty} b_n z^n = \frac{D^{3-2\alpha} f(z)}{D^{2-2\alpha} f(z)} = \frac{z + \sum_{n=2}^{\infty} C_2(\alpha, n) a_n z^n}{z + \sum_{n=2}^{\infty} C_1(\alpha, n) a_n z^n},$$

where

$$C_1(\alpha, n) = \frac{\prod_{k=2}^n (k - 2\alpha)}{(n-1)!}, \quad C_2(\alpha, n) = \frac{\prod_{k=2}^n (k + 1 - 2\alpha)}{(n-1)!}.$$

Equating the coefficients of  $z$ ,  $z^2$  and  $z^3$  in (17), we obtain

$$(18) \quad b_1 = [C_2(\alpha, 2) - C_1(\alpha, 2)] a_2,$$

$$(19) \quad b_2 = [C_2(\alpha, 3) - C_1(\alpha, 3)] a_3 + [C_1(\alpha, 2) a_2]^2 - [C_1(\alpha, 2) C_2(\alpha, 2)] a_2^2,$$

and

$$(20) \quad b_3 = [C_2(\alpha, 4) - C_1(\alpha, 4)] a_4 \\ + [2C_1(\alpha, 2) C_1(\alpha, 3) - C_2(\alpha, 3) C_1(\alpha, 2) - C_2(\alpha, 2) C_1(\alpha, 3)] a_2 a_3 \\ + C_2(\alpha, 2) [C_1(\alpha, 2) a_2]^2 a_2 - [C_1(\alpha, 2) a_2]^3.$$

Simplifying (18), (19) and (20) we have

$$(21) \quad a_2 = b_1, \quad a_3 = \frac{b_2 + (2 - 2\alpha) b_1^2}{(3 - 2\alpha)},$$

and

$$(22) \quad a_4 = \frac{2b_3}{(3 - 2\alpha)(4 - 2\alpha)} + \frac{3(2 - 2\alpha) b_1 b_2}{(3 - 2\alpha)(4 - 2\alpha)} - \frac{(2 - 2\alpha)^2 b_1^3}{(3 - 2\alpha)(4 - 2\alpha)}.$$

Using the equations (14), (15) and (16) in (21) and (22), it follows that

$$a_2 = \frac{B_1 p_1}{2},$$

$$a_3 = \frac{1}{(3 - 2\alpha)} \left[ \frac{B_1}{2} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{B_2 p_1^2}{4} + (2 - 2\alpha) \frac{B_1^2 p_1^2}{4} \right],$$

$$a_4 = \frac{1}{8(3 - 2\alpha)(4 - 2\alpha)} \left[ 8B_1 p_3 - 2 \{ 4(B_1 - B_2) - 3(2 - 2\alpha) B_1^2 \} p_1 p_2 \right]$$

$$+ \{2(B_1 - 2B_2 + B_3) - 3(2 - 2\alpha)B_1^2 + 3(2 - 2\alpha)B_1B_2 \\ + (2 - 2\alpha)^2B_1^3\} p_1^3 \Big].$$

Thus we establish that the estimate of the second Hankel determinant is given by

$$a_2a_4 - a_3^2 = \frac{1}{\mathcal{D}(\alpha)} \Big[ \{(2 - 2\alpha)B_1(B_1 - 2B_2) - (2 - 2\alpha)(1 - 2\alpha)B_1^2(B_1 - B_2) \\ - (2 - 2\alpha)^2B_1^4 + 2(3 - 2\alpha)B_1B_3 - (4 - 2\alpha)B_2^2\} p_1^4 \\ - 2(2 - 2\alpha) \{2B_1(B_1 - B_2) - (1 - 2\alpha)B_1^3\} p_1^2p_2 \\ - 4(4 - 2\alpha)B_1^2p_2^2 + 8(3 - 2\alpha)B_1^2p_1p_3 \Big], \quad (23)$$

where

$$\mathcal{D}(\alpha) = 16(3 - 2\alpha)^2(4 - 2\alpha).$$

Using Lemma 1.3 in (23), we have

$$|a_2a_4 - a_3^2| = \frac{1}{\mathcal{D}(\alpha)} \Big| [2(3 - 2\alpha)B_1B_3 + (2 - 2\alpha)(1 - 2\alpha)B_1^2B_2 \\ - (2 - 2\alpha)^2B_1^4 - (4 - 2\alpha)B_2^2] p_1^4 \\ + (2 - 2\alpha) [2B_1B_2 + (1 - 2\alpha)B_1^3] (4 - p_1^2)p_1^2x \\ - \{(2 - 2\alpha)p_1^2 + 4(4 - 2\alpha)\} B_1^2(4 - p_1^2)x^2 \\ + 4(3 - 2\alpha)B_1^2(4 - p_1^2)p_1(1 - |x|^2)z \Big|. \quad (24)$$

Letting  $|p_1| = \xi$  and in view of Lemma 1.2, we may assume without restriction that  $\xi \in [0, 2]$ . Thus, applying the triangle inequality in (24) with  $\delta = |x| \leq 1$  and  $|z| \leq 1$ , we obtain

$$|a_2a_4 - a_3^2| \leq \frac{1}{\mathcal{D}(\alpha)} \Big[ |-(2 - 2\alpha)^2B_1^4 + 2(3 - 2\alpha)B_1B_3 - (4 - 2\alpha)B_2^2 \\ + (2 - 2\alpha)(1 - 2\alpha)B_1^2B_2| \xi^4 \\ + (2 - 2\alpha) |2B_1B_2 + (1 - 2\alpha)B_1^3| (4 - \xi^2)\xi^2\delta \\ + \{(2 - 2\alpha)\xi^2 + 4(4 - 2\alpha)\} B_1^2(4 - \xi^2)\delta^2 \\ + 4(3 - 2\alpha)B_1^2\xi(4 - \xi^2)(1 - \delta^2) \Big] = \mathcal{F}(\xi, \delta).$$

Or equivalently

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{\mathcal{D}(\alpha)} \Big[ |-(2 - 2\alpha)^2B_1^3 + 2(3 - 2\alpha)B_3 - (4 - 2\alpha)\frac{B_2^2}{B_1} \\ + (2 - 2\alpha)(1 - 2\alpha)B_1B_2| \xi^4 \\ + (2 - 2\alpha) |2B_2 + (1 - 2\alpha)B_1^2| (4 - \xi^2)\xi^2\delta \\ + \{(2 - 2\alpha)\xi^2 + 4(4 - 2\alpha)\} B_1(4 - \xi^2)\delta^2$$



$$+ 4(3 - 2\alpha)B_1\xi(4 - \xi^2)(1 - \delta^2)] = \mathcal{F}(\xi, \delta).$$

Note that for  $(\xi, \delta) \in [0, 2) \times [0, 1]$ , differentiating  $\mathcal{F}(\xi, \delta)$ , partially with respect to  $\delta$  yields

$$(25) \quad \frac{\partial \mathcal{F}}{\partial \delta} = \frac{B_1}{\mathcal{D}(\alpha)} \left[ (2 - 2\alpha) |2B_2 + (1 - 2\alpha)B_1^2| (4 - \xi^2)\xi^2 + 2 \{ (2 - 2\alpha)\xi^2 - 4(3 - 2\alpha)\xi + 4(4 - 2\alpha) \} B_1(4 - \xi^2)\delta \right].$$

Equivalently

$$\frac{\partial \mathcal{F}}{\partial \delta} = \frac{B_1}{\mathcal{D}(\alpha)} \left[ (2 - 2\alpha) |2B_2 + (1 - 2\alpha)B_1^2| (4 - \xi^2)\xi^2 + 2 \{ 2(2 - \xi)[2 + (1 - \alpha)(2 - \xi)] \} B_1(4 - \xi^2)\delta \right].$$

It is obvious that  $2(2 - \xi)[2 + (1 - \alpha)(2 - \xi)]$ , the coefficient term of  $\delta$  in (25) is always a positive real number for all  $(\xi, \delta) \in [0, 2) \times [0, 1]$ . Hence it follows that the expression (25) is always positive for  $\delta > 0$  and  $\alpha \leq 1$ , which implies that  $\mathcal{F}(\xi, \delta)$  is an increasing function of  $\delta$ . Therefore, there exists no point of maximum in the interior of the closed region  $[0, 2) \times [0, 1]$ . Moreover for fixed  $\xi \in [0, 2)$ , we have

$$\max \mathcal{F}(\xi, \delta) = \mathcal{F}(\xi, 1) = \mathcal{G}(\xi).$$

On simplification we find that

$$(26) \quad \mathcal{F}(\xi, 1) = \mathcal{G}(\xi) = \frac{B_1}{\mathcal{D}(\alpha)} [Pt^2 + Qt + R],$$

where

$$(27) \quad P = \left| -(2 - 2\alpha)^2 B_1^3 + 2(3 - 2\alpha)B_3 - (4 - 2\alpha) \frac{B_2^2}{B_1} + (2 - 2\alpha)(1 - 2\alpha)B_1B_2 \right| - (2 - 2\alpha)B_1,$$

$$(28) \quad Q = 4(2 - 2\alpha) |2B_2 + (1 - 2\alpha)B_1^2| - 8B_1,$$

$$(29) \quad R = 16(4 - 2\alpha)B_1,$$

and  $t = \xi^2$ . Since

$$\max_{0 \leq t \leq 4} (Pt^2 + Qt + R) = \begin{cases} R, & Q \leq 0, P \leq -\frac{Q}{4}, \\ 16P + 4Q + R, & Q \geq 0, P \geq -\frac{Q}{8} \text{ or } Q \leq 0, P \geq -\frac{Q}{4}, \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq -\frac{Q}{8}, \end{cases}$$

we have

$$|a_2a_4 - a_3^2| \leq \begin{cases} R, & Q \leq 0, P \leq -\frac{Q}{4}, \\ 16P + 4Q + R, & Q \geq 0, P \geq -\frac{Q}{8} \text{ or } Q \leq 0, P \geq -\frac{Q}{4}, \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq -\frac{Q}{8}, \end{cases}$$

where  $P$ ,  $Q$  and  $R$  are given in (27), (28) and (29). This completes the proof of the theorem.  $\square$

*Remark 2.2.* We note that by taking  $\alpha = 1/2$  in Theorem 2.1 we obtain the corresponding results in [5].

### 2.1. Concluding remarks

As a special case of Theorem 2.1, let  $\phi$  be

$$\phi(z) = \frac{1 + Az}{1 + Bz}, \quad (-1 \leq B < A \leq 1).$$

This gives

$$\phi(z) = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 + \dots,$$

so that  $B_1 = (A - B)$ ,  $B_2 = -B(A - B)$  and  $B_3 = B^2(A - B)$  one can state the Hankel determinant inequality for the subclasses defined in Example 1.1.

Letting  $A = 1 - 2\beta$  and  $B = -1$  in (2.1), we have

$$\begin{aligned} \phi(z) &= \frac{1 + (1 - 2\beta)z}{1 - z} \\ (30) \quad &= 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + 2(1 - \beta)z^3 + \dots \quad (0 \leq \beta < 1). \end{aligned}$$

Comparing with (9) we have  $B_1 = B_2 = B_3 = 2(1 - \beta)$ . Thus Theorem 2.1 yields the Hankel inequality for  $f \in \mathcal{R}_\alpha^u(\beta)$ .

Further, by taking  $\beta = 0$ , in (30), we let

$$\phi(z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + 2z^3 + \dots.$$

Thus by comparing with (9) we note that,  $B_1 = B_2 = B_3 = 2$  and making use of Theorem 2.1 one can easily state the Hankel inequality for  $f \in \mathcal{R}_\alpha^u(\frac{1+z}{1-z})$ .

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